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Neoclassical Growth with Limited Commitment

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Abstract

This paper characterizes the stationary equilibrium of a continuous-time neoclassical production economy with capital accumulation in which agents can insure against idiosyncratic income risk by trading agent-shock contingent assets, subject to limited commitment constraints that rule out selling these assets short. For a general N-state Poisson labor productivity process we characterize the optimal consumption-asset allocation, the stationary asset distribution as well as the aggregate supply of capital by the household sector. For the special case in which production is Cobb-Douglas, agent labor productivity takes two values, one of which is zero, and agents have log-utility, we solve the equilibrium interest rate, capital stock and the consumption distribution in closed form. The paper therefore provides a tractable alternative to the standard incomplete markets general equilibrium model as in Aiyagari (1994).

JEL Codes: E21, D11, D91, G22
Keywords: Idiosyncratic Risk, Limited Commitment, Stationary Equilibrium

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1 Introduction

In this paper, we provide a fully micro-founded, analytically tractable general equilibrium macroeconomic model of neoclassical investment, production, and the cross-sectional consumption distribution in which the limits to insurance of idiosyncratic income risk are explicitly derived from a limited commitment friction.

This model seeks to integrate two foundational literatures on macroeconomics with household heterogeneity. The first strand has developed the standard incomplete markets (SIM) model with uninsurable idiosyncratic income shocks and neoclassical production, see Bewley (1986), Imrohoroglu (1989), Uhlig (1990), Huggett (1993) and Aiyagari (1994). In that model, agents can trade assets to self-insure against income fluctuations, but the payout of these assets by assumption is not contingent on an agent's individual income realization, thereby ruling out explicit insurance against income risk. The second branch is the broad literature on endogenously incomplete markets, and recursive contracts to solve them, that permit explicit insurance but the extent of it is restricted by informational or contract enforcement frictions. Specifically, we follow Alvarez and Jermann (2000) and Krueger and Uhlig (2006) and allow agents to trade assets that pay out contingent on agent-specific shocks but are subject to limited commitment: whereas the financial intermediaries (e.g., insurance companies) selling these assets are fully committed to making state-contingent payments, the agent is not. As a consequence and the assumed lack of punishment from default agents cannot sell these assets short, limiting the degree of insurance they can obtain. As in our previous paper, the contracts are front-loaded: when income is high, the agent purchases insurance that finances consumption in excess of income down the road should income change. Here we integrate this friction in a continuous time, general equilibrium neoclassical production economy and fully characterize stationary equilibria.

The result is a macroeconomic model with agent heterogeneity that links the accumulation of the aggregate capital stock to the insurance contracts of agents.[1] We assume that agents have CRRA utility in consumption. Given the aggregate interest rate r and implied wage w, we analytically characterize the optimal consumption and capital allocation choices and the resulting aggregate capital supply when income follows a general N-state Poisson process. We show how to calculate the equilibrium interest rate by solving a one-

[1] In practice, capital held for financing insurance commitments is a substantial part of the capital stock. In our model, we make the extreme assumption that this accounts for all of it. We would argue that SIM models also assume that (self-)insurance against idiosyncratic income fluctuations accounts for the entire holdings of capital: agents with constant income and the same discount factor would not accumulate capital.

dimensional nonlinear equation in r after normalizing capital supply and demand by the aggregate wage. For the special case of two income states, one of which is zero, we characterize the equilibrium interest rate and all equilibrium entities in closed form, including comparative statics with respect to the model parameters determining income risk, preferences, and technology. We use this special case throughout to illustrate how the general theory works and to show how a unique, or multiple-state equilibria can arise.

Our results for the special two-income state case and the full characterization of the resulting equilibrium can be seen as the counterpart to the characterization of the two-state continuous-time SIM model in Achdou et al. (2022). Like us, they characterize the equilibrium by two key differential equations: one governing the optimal solution of the consumption (self-)insurance problem and one characterizing the associated stationary distribution. They derive an analytical characterization of the wealth distribution, given the savings function. The latter cannot be determined in closed form there (although partially characterized); for the two-state case, we achieve a full characterization of the stationary distribution in this paper and thus can proceed all the way to provide closed-form solutions for all equilibrium objects. Methodologically, the papers complement each other by characterizing equilibria in the same physical environment but under two fundamentally different market structures. Our results for the N-state case and the full analytical characterization, given the equilibrium interest rate, go beyond Achdou et al. (2022). They open the door to quantitative applications in a rich environment but with considerable analytical transparency regarding the solution and limited needs for numerical solutions. We then study one special case analytically and one quantitatively to showcase our approach. Thus, the paper provides a quantitative workhorse model and alternative to the celebrated SIM.

Our paper builds on the substantial literature on limited commitment, including Thomas and Worrall (1988), Kehoe and Levine (1993), Phelan (1995), Kocherlakota, Broer (2013), Golosov et al. (2016), Abraham and Laczo (2018), Sargent et al. (2021), and specifically shares insights with the theoretical analyses in Krueger and Perri (2006, 2011), Zhang (2013), Grochulski and Zhang (2012), and Miao and Zhang (2015), but for a general N-state continuous time Poisson process. Our approach is related in spirit to recent work by Dàvila and Schaab (2023), Alvarez and Lippi (2022), and Alvarez, Lippi and Souganidis (2022). We provide a general equilibrium treatment, as do Hellwig and Lorenzoni (2009), Martins-da-Rocha and Santos (2019), and Gottardi and Kubler (2015).

Our theory complements recent advances regarding the empirical properties of household consumption. There is now considerable evidence that individual consumption smoothing is larger than what standard approaches of self-insurance via asset savings would generate. In a benchmark contribution, Blundell, Pistaferri and Preston (2008) have shown that there is very considerable consumption insurance even of permanent income shocks, a finding that is difficult to rationalize within the standard SIM, see Kaplan and Violante (2011). Using improved methods and data as well as alternative approaches, these results have been largely confirmed by the more recent literature such as Arellano, Blundell and Bonhomme (2017), Eika et al. (2020), Chatterjee, Morley and Sigh (2020), Braxton et al. (2021), Commault (2022), and Balke and Lamadon (2022) for the labor market, as well as Hofmann and Browne (2013), Ghili, Handel, Hendel and Whinston (2023) and Atal, Fang, Karlsson and Siebarth (2023) for the private health insurance market. Thus, alternatives to the conventional self-insurance approach are needed, which our paper provides.

As in Harris and Holmstrom (1982), one interpretation of the consumption insurance allocation in this paper is that firms insure workers against idiosyncratic productivity fluctuations. This perspective is pursued in Guiso, Pistaferri, and Schivardi (2005) and Balke and Lamadon (2022). Saporta-Eksten (2016) shows that wages are lower after a spell of unemployment, which he interprets as a loss in productivity. In the context of our model, this observation can be rationalized as part of optimal consumption insurance.

2 The Model

Time is continuous, and the economy is populated by a continuum of infinitely lived individuals of mass 1. These individuals value consumption streams. Aggregate output is produced with capital and labor and can be used for consumption and investment.

2.1 Technology

The unique final output good is produced by a perfectly competitive sector of firms that use labor and capital as input. The production function $F(K, L)$ for $K \geq 0, L \geq 0$ is assumed to be strictly concave, have constant returns to scale, be strictly increasing in each argument, satisfy $F(0, 0) = 0$ and be twice continuously differentiable. Production firms seek to maximize profits, taking as given the market spot wage w per efficiency unit of labor and the market rental rate per unit of capital. Capital accumulation is linear, and capital depreciates at rate δ. There is a resulting equilibrium rate of return (equal to the real interest rate) r for investing in capital. We drop time subscripts t to economize on notation whenever possible since we shall concern ourselves only with stationary equilibria in which aggregate variables such as the factor prices (w, r) are constant and where $w > 0$.

2.2 Preferences and Endowments

Agents have a strictly increasing, strictly concave, twice continuously differentiable CRRA period utility function $u(c)$, with risk aversion parameter σ, and discount the future at rate $\rho > 0$. The expected lifetime utility of a newborn agent is given by

$$E\left[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\sigma}}{1-\sigma}dt\right].$$

where it is understood that $\sigma = 1$ represents the log-case.

Individuals face idiosyncratic income risk. Specifically, each agent can be in one of N states $x \in X = \{1, \ldots, N\}$, with associated idiosyncratic labor productivity level $\mathbf{z}(x) \geq 0$.[2] For a fixed aggregate equilibrium wage w per labor efficiency units, individual labor income in state x is then $w\mathbf{z}(x)$, and we will use the terms (labor) productivity and income interchangeably. Let $\alpha_{x,x'}$ be the transition rate from x to x', with $\alpha_{x,x} = -\sum_{x' \neq x} \alpha_{x,x'}$ and collect the transition rates in the $N \times N$ matrix A. We assume that for every x', there is some $x \neq x'$, so that $\alpha_{x,x'} \neq 0$, i.e., every state can be reached from some other state. Transitions are assumed to be independent across individuals. Associated with A is a stationary distribution $\bar{\mu} = [\bar{\mu}_1, \ldots, \bar{\mu}_N]'$, an $N \times 1$-dimensional vector satisfying

$$A'\bar{\mu} = 0 \qquad \text{and} \qquad \sum_{x \in X} \bar{\mu}(x) = 1 \tag{1}$$

We also assume that the stationary distribution is unique, that all individuals draw their initial productivity from $\bar{\mu}$ and that the idiosyncratic shock process satisfies

$$\sum_{x \in X} \mathbf{z}(x)\bar{\mu}(x) = 1 \tag{2}$$

so that aggregate labor input is equal to $L = 1$ in every period.[3]

[2] We denote real-valued functions of x with round brackets, while subscript-x denotes vectors of length $x-1$ or matrices of size $(x-1)\times(x-1)$. For example, z_x is the $(x-1)$-dimensional vector $[\mathbf{z}(1), \ldots, \mathbf{z}(x-1)]'$. We use function-of-$x$ notation to denote entries of a vector, as in this example, as well as entries of a matrix, except denoting $\alpha_{x,x'}$ using sub-indices. We also use sub-index notation to denote functions of time.

[3] Uniqueness of $\bar{\mu}$ can be assured under standard assumptions on A, for example, that all elements of A are strictly positive. The assumption that all states can be reached assures that $\bar{\mu}(x) > 0$ for all x.The

2.3 Financial Markets

Households seek insurance against their idiosyncratic risk. For motivation, we envision a competitive sector of intermediaries who are willing to provide insurance at actuarially fair rates. These intermediaries will invest the insurance payments in agent-specific accounts in units of capital k, earning the market interest rate r. They will then make payments from this capital account in the insurance case, i.e., if the current state x of the agent state changes to a new state x' and consumption exceeds income. This may require changing the account amount from k to $k(x')$. We, therefore, need to keep track of the capital account and its relation to consumption and income. It is given by the budget constraint

$$c + \dot{k} + \sum_{x' \neq x} \alpha_{x,x'}(k(x') - k) = rk + w\mathbf{z}(x) \tag{3}$$

This constraint takes into account that insurance is actuarially fair so that the outlay for the account change $k(x') - k$ equals $\alpha_{x,x'}(k(x') - k)$. In contrast to a SIM model, capital is contingent on the agent-specific state. While we imagine the intermediaries to be fully committed to making the necessary payments, we assume that the commitment by the agent is limited and that they are free to switch intermediaries at any point and without any penalty for not making promised payments. Therefore, agents cannot spend based on such promises. This results in the constraint

$$k(x') \geq 0 \tag{4}$$

Furthermore, we need to ensure that capital does not become negative in the absence of a state transition. This is achieved by requiring that

$$\dot{k} \geq 0 \text{ if } k = 0 \tag{5}$$

Competition among intermediaries results in agent utility maximization, subject to these constraints. The agent Hamilton-Jacobi-Bellman equation can then be stated as

$$\rho U(k, x) = \max_{c \geq 0, \dot{k}, (k(x'))_{x' \in X}} \left\{ u(c) + U'(k, x)\dot{k} + \sum_{x' \neq x} \alpha_{x,x'}(U(k(x'), x') - U(k, x)) \right\} \tag{6}$$

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idiosyncratic productivity states $\mathbf{z}(x)$ can always be scaled such that (2) is satisfied.

where the maximization is subject to the budget constraint (3) and the limited commitment constraints (4) and (5). A direct interpretation is that agents maximize utility when trading claims contingent on the agent-specific states at actuarially fair prices, subject to these constraints. We take this formulation to be the starting point of our analysis.[4]

3 The Optimal Consumption-Asset Allocation

3.1 General Properties of Optimal Allocations

We now characterize the optimal consumption-saving allocation. To that end, it is helpful to move from recursive to time domain since the time dependence of allocation comes through the evolution of the individual capital account k and the state x when focusing on steady states (and thus on constant wages and interest rates). Written as a function of time, the budget constraint (3) reads

$$c_t + \dot{k}_t + \sum_{x' \neq x} \alpha_{x,x'}(k_t(x') - k_t) = rk_t + w\mathbf{z}(x) \tag{7}$$

where $k_t(x')$ is the date-t state-contingent capital stock going forward from state x'. It is also equal to the expected net present value of the future consumption stream net of income when the current state is x'.

Intuitively, agents with positive capital and no state transitions obey a standard complete markets Euler equation, and consumption is continuous when a state transition occurs. Consumption might jump upon a state transition, but only if the associated state-contingent capital $k'(x')$ is zero (i.e. if the limited commitment constraint binds). For a given rate of

[4] An alternative and (as shown in Krueger and Uhlig, 2006) equivalent formulation of the limited commitment friction without punishment for default is to explicitly introduce competitive cost-minimizing financial intermediaries that offer long-term consumption insurance contracts. These contracts stipulate full income-history contingent consumption payments in exchange for delivering all labor income to the intermediaries. One-sided limited commitment then means that intermediaries can fully commit to long-term contracts, but individuals cannot. That is, in every instant, after having observed current labor productivity, the individual can leave her current contract and sign up with an alternative intermediary at no punishment, obtaining in equilibrium the highest lifetime utility contract that allows an intermediary to break even. Here, we focus on formulating the model with financial markets in the spirit of Alvarez and Jermann (2000). The tight borrowing constraints at zero are precisely the borrowing limits they call "not too tight", given that there is no punishment from default here. As a third alternative, we could also motivate ours as an intermediate model located in between the SIM model with tight borrowing constraints and the complete markets model with a full set of state-contingent claims and natural state contingent borrowing constraints.

return r on capital, define the growth rate $g = g(r)$ per

$$g = \frac{\rho - r}{\sigma} \tag{8}$$

This growth rate g will be the common growth rate of consumption of all agents whose limited commitment constraint is not binding. Formally:

Proposition 1. *Let $w > 0$ and r be given. A solution to the HJB equation has the following properties:*

1. *For a agent with $k > 0$,* (6) *implies*

   $$\frac{\dot{c}_t}{c_t} = g. \tag{9}$$

   If $k'(x') > 0$, then consumption after the state transition is unchanged,

   $$c(k'(x'), x') = c(k, x). \tag{10}$$

   If $k'(x') = 0$, then

   $$c(k'(x'), x') \geq c(k, x) \tag{11}$$

2. *The decision rules for consumption $c(k; x)$ is strictly increasing in k. The decision rule for $k(x'; k, x)$ is weakly increasing in k and strictly increasing wherever it is positive.*

3. *$U(k, x)$ is strictly concave in k.*

4. *For $k = 0$, the HJB equation (6) implies*

   $$\dot{k}_t = 0 \text{ and } \dot{c}_t = 0 \tag{12}$$

Proof. See Appendix A. □

3.2 Explicit Characterization of the Optimal Consumption and Capital Allocation

On the basis of the previous proposition, we can provide a full characterization of the optimal consumption allocation under the assumption that $r \leq \rho$. The next proposition characterizes the optimal consumption contract by N consumption levels $\mathbf{c}(x), x \in X$, so that consumption either drifts down at rate g or jumps up to $\mathbf{c}(x')$, if a state transition to x' occurs and $\mathbf{c}(x')$ is higher than the pre-jump consumption level.

If consumption is higher than labor income, it needs to be financed with capital (income). In particular, suppose that $c_t = \mathbf{c}(x)$. Capital reserves $k_x(x') > 0$ have to be created for all transitions from state x to states x' with $\mathbf{c}(x') < \mathbf{c}(x)$, while an upward jump in consumption resets the allocation at zero capital (and the limited commitment constraint is binding).[5] For a given current state x, the state-contingent capital stocks for states $x' < x$ form an $x-1$-dimensional vector $k_x = [k_x(1), \ldots, k_x(x-1)]'$ which we need to characterize as part of the optimal allocation. This characterization proceeds by first calculating the amount of capital $d_x = [d_x(1), \ldots, d_x(x-1)]'$ needed to finance the gap between consumption and labor income until the endogenous time $T(x)$ when consumption drifting down from $\mathbf{c}(x)$ at rate g reaches the next consumption level $\mathbf{c}(x-1)$. The total capital saved to insure for a state transition to $x' < x$ is then the appropriately discounted sum of these capital differences. For a given $g = g(r)$, a full solution of the agent problem is then determined by $(\mathbf{c}(x), T(x), d_x, k_x)$ for all $x \in X$. The following proposition provides a complete and explicit characterization of these entities.

We need the following notation. Let $\alpha^{\min} = \min_{x<N} \alpha_{x,N}$ be the minimum hazard rate across states $x < N$ of escaping to the highest state N. Let $\mathbf{1}_x$ be the $(x-1)$-dimensional vector with only 1's, let $\mathbf{0}_x$ be the $(x-1)$-dimensional vector with only 0's, let $\mathbf{I}_x$ be the $(x-1) \times (x-1)$-dimensional identity matrix, let z_x be the $(x-1)$-dimensional vector $[\mathbf{z}(1), \ldots, \mathbf{z}(x-1)]'$ and let $\alpha_x = [\alpha_{x,1}, \ldots, \alpha_{x,x-1}]' \in \mathbf{R}^{x-1}$ be a vector of length[6] $x-1$. Define the $(x-1) \times (x-1)$-dimensional matrices A_x, B_x and C_x by $A_x(\tilde{x}, x') = \alpha_{\tilde{x},x'}$ for $\tilde{x}, x' \in \{1, \ldots, x-1\}$, $B_x = r\mathbf{I}_x - A_x$ and $C_x = (r+g)\mathbf{I}_x - A_x$. We require the following additional technical condition. It is satisfied if the matrix A_x has only positive entries off the diagonal. It is closely related to the concept of irreducibility.

Assumption 1. *For every x there is some $\bar{\epsilon} > 0$ with the property that $e^{-B_x \epsilon}$ has only nonzero entries for all $0 < \epsilon < \bar{\epsilon}$.*

Proposition 2. *Let $w > 0$ and r be given. Suppose that $-\alpha^{\min} < r \leq \rho$. Let assumption 1 be satisfied. For each state $x \in X$, let $c = \mathbf{c}(x)$ be the solution to the HJB equation* (6)

[5]Since the enumeration of states has no intrinsic importance, we can relabel them such that $\mathbf{c}(x)$ is an increasing sequence.

[6]Conventionally, the $(x-1) \times (x-1)$-dimensional identity matrix is denoted by I_{x-1}. For tightness of notation, we instead use the subscript x here, as well as for other $(x-1) \times (x-1)$-dimensional matrices.

with $k = 0$. Without loss of generality, suppose that the exogenous states are ordered such that $\mathbf{c}(x) \leq \mathbf{c}(x')$ when $x < x'$.[7] *For each $x \in X$, the consumption levels $\mathbf{c}(x)$, wait times $T(x) \in \mathbf{R}_+$ and contingent capital stocks $k_x \in \mathbf{R}_+^{x-1}$ and capital differences $d_x \in \mathbf{R}_+^{x-1}$ for $x = 1$ are given by the initialization $\mathbf{c}(1) = w\mathbf{z}(1)$ and the empty vectors $d_1 = k_1 = [\,]$, and for all states $x > 1$ solve the system of equations*[8]

$$T(x) = \frac{\log(\mathbf{c}(x)) - \log(\mathbf{c}(x-1))}{g} \in [0, \infty] \tag{13}$$

$$d_x = \mathbf{c}(x) C_x^{-1} \left(\mathbf{I}_x - e^{-C_x T(x)} \right) \mathbf{1}_x - B_x^{-1} \left(\mathbf{I}_x - e^{-B_x T(x)} \right) w z_x \tag{14}$$

$$k_x = d_x + e^{-B_x T(x)} \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \tag{15}$$

$$\mathbf{c}(x) = w\mathbf{z}(x) - \alpha_x k_x \tag{16}$$

Proof. See Appendix A □

Note that the proof establishes that the expressions in (14) and (15) are also well-defined for $T(x) = \infty$, which is important for the case $r = \rho$ and also for the example considered in the next subsection. Also note that the system of equations (13)-(16) is block-recursive in x and thus can be solved recursively by starting with the allocation for $x = 1$ and iterating forward in x.

Proposition 3. *The solution is unique.*

Proof. See Appendix A □

3.3 An Example

In this subjection, we provide an example intended to serve two purposes. First, it clarifies how to use the notation and characterization in Proposition 2 and allows us to give an intuition for the optimal solution based on closed-from formulas. Second, this example

[7] Since the enumeration of states has no intrinsic importance, we can relabel them such that $\mathbf{c}(x)$ is an increasing sequence. For a recursive algorithm, set $x = 1$ be the state resulting in the lowest income $z(x)$. Suppose the sequence of states $x = 1, \ldots, n$ and their associated consumption levels and capital reserves have already been found. Try each of the remaining states as a candidate for the state resulting in the next lowest $\mathbf{c}(x)$ and solve equations (13) to (16). Among all these candidates, pick that state x, which results in the lowest $\mathbf{c}(x)$.

[8] Note that d_1 and k_1 have dimension zero. Thus, for $x = 2$, $\begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} = [0]$ and $k_2 = d_2$ in equation (15) which is another way of starting the recursion.

delivers a closed-form solution not only of the optimal agent consumption-capital process but will also exhibit a closed-form solution for the equilibrium consumption distribution and the law of motion for the aggregate capital stock, making a complete closed-form characterization of the entire equilibrium feasible.

To this end, now assume that $X = \{1, 2\}$ and $\mathbf{z}(1) = 0$. We can interpret the state $x = 2$ as being employed and state $x = 1$ as being unemployed. Also, denote the Poisson intensity of losing a job as $\xi = \alpha_{2,1} > 0$ and the Poisson intensity of finding a job by $\nu = \alpha_{1,2} > 0$. Now consider $x = 2$. The ingredients for the characterization in Proposition 2, for state $x = 2$ are as follows. All $x-1$ entities are simply numbers (rather than vectors or matrices), and $A_x = \alpha_{1,1} = -\alpha_{1,2} = -\nu$ since all rows of the transition rate matrix A sum to zero. Then $B_x = r - \alpha_{1,1} = r + \nu$, $C_x = r + g - \alpha_{1,1} = r + g + \nu$, $\alpha_x = \alpha_{2,1} = \xi$, $1_x = 1$, $z_x = 0$. For this two-state example, $\mathbf{c}(1) = 0$ and $T(2) = \infty$, that is, consumption drifts down from $\mathbf{c}(2)$ to $\mathbf{c}(1) = 0$ at rate g asymptotically.[9] Now (15) implies that $d_2(1) = k_2(1)$ and (16) and (14) read, respectively, as

$$\mathbf{c}(2) = w\mathbf{z}(2) - \xi k_2(1) \tag{17}$$

$$k_2(1) = \frac{\mathbf{c}(2)}{r + g + \nu} \tag{18}$$

Note that (18) requires $r + g + \nu > 0$ in order for the expression to make sense. This is assured by the assumption that $-\min\{\xi, \nu\} = -\alpha^{\min} < r$ of Proposition 2. The two equations above can be easily solved explicitly as

$$\mathbf{c}(2) = \frac{r + g + \nu}{r + g + \nu + \xi} w\mathbf{z}(2) < w\mathbf{z}(2) \tag{19}$$

$$k_2(1) = \frac{1}{r + g + \nu + \xi} w\mathbf{z}(2) \tag{20}$$

We also note that for $r = \rho$ or for log-utility ($\sigma = 1$) and thus $g = \rho - r$ we have

$$\mathbf{c}(2) = \frac{\rho + \nu}{\rho + \nu + \xi} w\mathbf{z}(2) \tag{21}$$

$$k_2(1) = \frac{1}{\rho + \nu + \xi} w\mathbf{z}(2) \tag{22}$$

and thus both the share of income in the high state devoted to consumption $\mathbf{c}(2)$ as well as

[9] In the case $r = \rho$, which is encompassed in the analysis, consumption remains constant and does not drift down at all.

to the capital bought as insurance for the low state $k_2(1)$ are independent of the equilibrium interest rate r. Note that the total expense for insurance is $\xi \times k_2(1)$, which is strictly increasing in the intensity ξ with which the agent becomes unemployed (and declines with the intensity ν of finding a new job).

A visual representation of the optimal consumption dynamics is provided in Figure 1. The left panel represents the case $r = \rho$, and the right panel displays the case $r < \rho$ in which agents are impatient and, absent constraints, prefer a downward-sloping consumption time path.

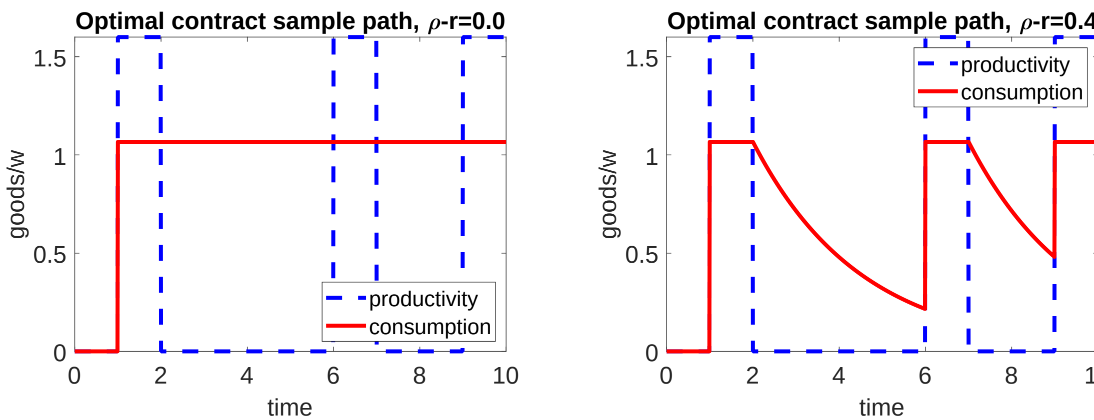


Figure 1: *These two figures show the optimal consumption dynamics for given a sample path for productivity. If the agent always had zero productivity in the past, the agent will also consume zero and hold zero contingent capital. Upon the first instance of high productivity, the agent uses the labor income to finance a jump in consumption to* **c***(2), but also to acquire the state-contingent capital position* $k_2(1)$ *which finances the optimal consumption path in the absence of labor income (i.e., then productivity falls to* **z***(1)). When* $r = \rho$ *as in the left panel, consumption is constant forever. While productivity is high, consumption is also constant for* $r < \rho$ *as shown in the right panel, and when productivity switches to zero, consumption follows the standard continuous-time Euler equation and falls at rate* g *asymptotically to zero.*

4 The Invariant Consumption Distribution

In the previous section, we have derived the optimal agent consumption allocation and shown that it is characterized by N consumption thresholds $\mathbf{c}(x)$ and wait times $T(x)$ for all $x \in X$, as well as a common downward consumption drift $-g(r) = -\frac{\rho - r}{\sigma} \leq 0$ whenever the limited commitment constraint is not binding. In this section, we will first derive the unique stationary distribution associated with this consumption process for the general case and then continue our two-state example for which a closed-form calculation of the distribution can easily be given.

4.1 Theoretical Characterization of the Distribution

Assume now that $\alpha^{\min} < r < \rho$. Let $\mu(x)$ be the mass of agents in state x and at consumption level $\mathbf{c}(x)$. Let $f_{x,\tilde{x}}(t)$ be the density of agents with current state $\tilde{x}$ whose consumption has been drifting down $t \in [0, T(x)]$ periods from $\mathbf{c}(x)$, starting at $t = 0$. For these t, consumption is equal or higher than $\mathbf{c}(x-1)$.[10] We collect the mass points and densities as

$$\mathcal{D} = ((\mu(x))_{x \in X}, (f_{x,\tilde{x}}(t))_{x,\tilde{x} \in X, t \geq 0}) \tag{23}$$

and call it the **stationary distribution** if its mass integrates to unity and it is a solution to the state and consumption transitions implied by the Markov process for the states determined by the matrix A and the consumption evolution characterized in Proposition 2. Thus, these point masses $\mu(x)$ and densities $f_{x,\tilde{x}}$ satisfy a list of conditions implied by the Kolmogorov forward equations given in Proposition 12 of Appendix B.In particular, let $f_x(t) = [f_{x,1}(t), \ldots, f_{x,x-1}(t)]'$. This vector of densities satisfies the matrix ODE

$$\dot{f}_x(t) = A'_x f_x(t). \tag{24}$$

They give rise to the following complete characterization of the stationary distribution $\mathcal{D}$ of decay times t.

Proposition 4. *Let $\bar{\mu}$ be the unconditional stationary distribution across states, solving $0 = A'\bar{\mu}$ and $\sum_x \bar{\mu}(x) = 1$ and assumed to be unique. Assume that $\alpha_{x,x} < 0$ for all x and that $\bar{\mu}_N > 0$.[11] [12] Let $f_x(t) = [f_{x,1}(t), \ldots, f_{x,x-1}(t)]'$. Then the stationary distribution $\mathcal{D}$ is unique and can be calculated recursively as follows.*

1. $\mu_N = \bar{\mu}_N$

[10]For $t > T(x)$, consumption has drifted below $\mathbf{c}(x-1)$, and we let $f_{x,\tilde{x}}(t) = 0$ for $t > T(x)$ and count the agents arriving at $t = T(x)$ towards $\mu(x-1)$ if $\tilde{x} = x-1$ or towards $f_{x-1,\tilde{x}}$ if $\tilde{x} < x-1$. Note that $f_{x,\tilde{x}}(t) = 0$, if $\tilde{x} \geq x$, since agents in state $\tilde{x} \geq x$ consume at least $\mathbf{c}(x)$.

[11]If $\alpha_{x,x} = 0$, the state x would be absorbing.

[12]It is easy to generalize the result to a case where $\bar{\mu}_N = \ldots = \bar{\mu}_{\bar{x}+1} = 0$ in which case $\mu(x) = 0$ and $f_{x,x'}(t) = 0$ for all $x > \bar{x}$.

2. *For $x = N, \dots, 2$,*

   (a) *calculate the $x-1$-dimensional vector $f_x(0) = [f_{x,1}(0), \dots, f_{x,x-1}(0)]'$ per*

$$f_{x,\tilde{x}}(0) = \begin{cases} \alpha_{x,\tilde{x}}\mu(x), & \textit{if } x = N \\ \alpha_{x,\tilde{x}}\mu(x) + f_{x+1,\tilde{x}}(T_{x+1}), & \textit{if } x < N \end{cases} \tag{25}$$

   (b) *calculate the solution $f_x(t)$ for $t \in (0, T(x)]$ to (24) as*[13]

$$f_x(t) = \exp\left(A'_x t\right) f_x(0) \tag{27}$$

   (c) *Finally,*

$$\mu(x-1) = \frac{-1}{\alpha_{x-1,x-1}}\left(f_{x,x-1}(T_x) + \sum_{\tilde{x}<x-1} \alpha_{\tilde{x},x-1}\left(\bar{\mu}_{\tilde{x}} - \sum_{x'>x-1}\int_{t=0}^{T_{x'}} f_{x',\tilde{x}}(t)dt\right)\right) \tag{28}$$

The integral terms in (28) can be calculated explicitly as

$$\int_{t=0}^{T_x} f_x(t)dt = \left(A'_x\right)^{-1}\left(\exp(A'_x T(x)) - \mathbf{I}_x\right) f_x(0) \tag{29}$$

Proof. See Appendix B □

The conditions that a stationary distribution has to satisfy have the following interpretation. Item 1. states that all individuals currently in the highest state $x = N$ (with mass $\bar{\mu}_N$) are located at the mass point μ_N and thus will have the highest consumption level $\mathbf{c}(x)$. The next condition, item 2.a, characterizes the density for the instant (i.e., $t = 0$) an individual experiences a drop in the state from x to $\tilde{x} < x$. Two groups of individuals transit here: those at the mass point $\mu(x)$ that experience a transition to $\tilde{x}$, which happens at intensity $\alpha_{x,\tilde{x}}$, and those that have continued to drift down from state $x+1$ and thus consumption $\mathbf{c}(x+1)$ for T_{x+1} units of time and have passed through $\mathbf{c}(x)$ at this very instant. From $t \in (0, T(x)]$ on the vector-valued density follows a simple matrix ordinary differential equation determined by the matrix of state transitions A_x whose solution is given in (25). Finally, the last equation characterizes the next lower mass point

[13] In particular,

$$f_x(T(x)) = \exp\left(A'_x T(x)\right) f_x(0) \tag{26}$$

$\mu_{x-1,x-1}$ and states that in the stationary distribution the outflow from this mass point, $\alpha_{x-1,x-1}\mu(x-1) = \sum_{x' \neq x-1} \alpha_{x-1,x'}\mu(x-1)$ is equal to its inflow. This inflow comes from two sources, those that have drifted down from the consumption level $\mathbf{c}(x)$ for T_x units of time (density $f_{x,x-1}(T_x)$), and then those experiencing state transitions to $x-1$ from states $\tilde{x}$ (which occur with intensity $\alpha_{\tilde{x},x-1}$). The term in brackets gives the mass of all individuals in current state $\tilde{x}$ which do not currently drift down from state even higher than $x-1$.[14]

Since there is a one-to-one mapping between the time t consumption has drifted down from one of the thresholds $\mathbf{c}(x)$ upon a transition to a lower state and the level of consumption at this time, the previous characterization of the decay time distribution then implies the cross-sectional consumption distribution for a given interest rate r. We characterize this distribution in the next proposition.

Proposition 5. *For $c \in (\mathbf{c}(x-1), \mathbf{c}(x))$, define $t(c) = (\log(\mathbf{c}(x)) - \log(c))/g$. The probability density $\phi(c)$ for consumption $c \in [\mathbf{c}(1), \mathbf{c}(N)]$, permitting mass points, is given by*

$$\phi_r(c) = \begin{cases} \frac{1}{gc} \sum_{x'<x} f_{x,x'}(t(c)) & \text{if} \quad c \in (\mathbf{c}(x-1), \mathbf{c}(x)) \\ \mu(x)\boldsymbol{\delta}_c & \text{if} \quad c = \mathbf{c}(x) \end{cases} \tag{30}$$

where $\boldsymbol{\delta}_c$ indicates a Dirac mass point at c.

For $x \in X$, define $D_x = g\mathbf{I}_x - A'_x$. Aggregate consumption is given by

$$C = \mathbf{c}(1)\mu_1 + \sum_{x>1} \mathbf{c}(x) \left(\mu(x) + \mathbf{1}'_x D_x^{-1} \left(\mathbf{I}_x - \exp(-D_x T(x)) \right) f_x(0) \right) \tag{31}$$

Proof. This is a direct consequence of the characterization of the distribution of consumption decay times in Proposition 4 and a change of variables from t to c through the mapping tc, see Appendix B for the details. □

So far, we have treated r as a fixed parameter. We seek to understand how the solution varies with r. All objects calculated in propositions 2, 4 and 5 are functions of r.[15] In particular, let us explicitly denote the dependence of aggregate consumption $C(r)$ on r, where $C(r) = C$ is given in equation (31).

In the next subsection, we continue the two-state example from Section 3.3 to show how Propositions 4 and 5 work in practice.

[14] The others simply continue to drift down upon making the state transition to $x-1$ rather than enter the mass point $\mu(x-1)$.

[15] Proposition 3 guarantees that they are indeed functions, not correspondences.

4.2 The Example Continued

For the two-state example from Section 3.3, the calculation of the stationary decay-time and consumption distributions is straightforward and deliver the consumption distribution in closed form. We can directly apply Proposition 4 with the highest state with a mass point being $N = 2$, and thus item 1 of Proposition 4 implies

$$\mu_2 = \bar{\mu}_2 = \frac{\nu}{\xi + \nu}. \tag{32}$$

Thus, all individuals in state 2 consume $c = \mathbf{c}(2)$, per Proposition 5.

The remainder of the decay-time distribution $f_2(t) = f_{2,1}(t)$ for $x = 2$ follows directly from parts 2.a and 2.b of Proposition 4. Since for this example the matrix $A_2 = \alpha_{1,1} = -\nu$ is just a number, $\alpha_{2,1} = \xi$ and $T(2) = \infty$ (see Section 3.3), we immediately have that

$$f_{2,1}(0) = \alpha_{2,1}\mu_2 = \frac{\xi\nu}{\xi + \nu} \tag{33}$$

$$f_{2,1}(t) = \frac{\xi\nu}{\xi + \nu} e^{-\nu t}, \qquad t \in (0, \infty). \tag{34}$$

Note that $\int_0^\infty f_{2,1}(t)dt = \frac{\xi}{\xi+\nu} = \bar{\mu}_1$, and thus the decay-time distribution in (34) accounts for the entire mass of low-productivity individuals.

Translated into the consumption distribution, equation (30) in Proposition 5 implies that $t(c) = (\log(\mathbf{c}(2)) - \log(c))/g$ and the consumption probability density function for all $c \in (0, \mathbf{c}(2))$ is given by

$$\phi_r(c) = \frac{1}{gc}\frac{\xi\nu}{\xi + \nu}e^{-\nu t(c)} = \frac{1}{gc}\frac{\xi\nu}{\xi + \nu}e^{-\frac{\nu}{g}[\log(\mathbf{c}(2)) - \log(c)]} = \frac{\xi\nu\mathbf{c}(2)^{-\frac{\nu}{g}}c^{\frac{\nu}{g}-1}}{g(\xi + \nu)} \tag{35}$$

and thus the consumption distribution in this example has a mass point at $\mathbf{c}(2)$ and a Pareto density with shape parameter $\frac{\nu}{g} - 1$ on the interval $(0, \mathbf{c}(2))$ below it.

Part 2.c of Proposition 4 immediately implies that $\mu_1 = 0$, that is, there is no mass point for state $x = 1$, which is intuitive since consumption reaches $c = \mathbf{c}(1) = 0$ only asymptotically. The normalization in equation (2), that aggregate labor $L = \bar{\mu}_2\mathbf{z}(2) = \frac{\nu}{\xi+\nu}\mathbf{z}(2) = 1$ and $\mathbf{z}(1) = 0$ implies $\mathbf{z}(2) = (\xi + \nu)/\nu$. Plug this into equation (19) to obtain

$$\mathbf{c}(2) = \frac{r + g + \nu}{r + g + \nu + \xi}\frac{\xi + \nu}{\nu}w$$

The last part of Proposition 5, with the matrix $D_2 = g + \nu$ becoming a scalar, allows us to calculate aggregate consumption as a function of the interest rate, as[16]

$$\begin{aligned} C(r) &= 0 + \mathbf{c}(2) \left(\mu_2 + \frac{1}{g+\nu} f_{2,1}(0) \right) = \mathbf{c}(2) \frac{\nu}{\xi + \nu} \left(\frac{g+\nu+\xi}{g+\nu} \right) \\ &= \left(1 + \frac{r\xi}{(g+\nu)(r+g+\nu+\xi)} \right) w \end{aligned} \tag{36}$$

For $\sigma = 1$ (log-utility, and thus $g = \rho - r$), aggregate consumption becomes

$$C(r) = \left(1 + \frac{r\xi}{(\rho+\nu-r)(\rho+\nu+\xi)} \right) w \tag{37}$$

5 Stationary Equilibrium

Equipped with the solution of the agent problem and the associated stationary consumption (and asset) distribution ϕ_r as well as aggregate consumption $C(r)$ derived in the previous section, we can now determine the general equilibrium interest rate and wage in the economy. In this economy, there are three markets: the labor market, the capital market, and the goods market. Aggregate labor supply, the sum of labor efficiency units of all agents, is exogenous and normalized to $L = 1$, and thus, the wage adjusts such that firms demand that labor in stationary equilibrium, which we define next.

Definition 1. *A stationary equilibrium consists of an equilibrium wage and interest rate* (w, r)*, aggregate capital* K*, and a stationary consumption probability density function* $\phi(c)$ *such that*

1. *The interest rate and wage* (r, w) *satisfy*

$$r = F_K(K, 1) - \delta \tag{38}$$

$$w = F_L(K, 1) \tag{39}$$

[16]To see this, add and subtract ξ to and from the numerator of $\mathbf{c}(2)$, write

$$\mathbf{c}(2) \frac{\nu}{\xi+\nu} \left(\frac{g+\nu+\xi}{g+\nu} \right) = \frac{(r+g+\nu+\xi\ -\xi)(g+\nu) + (r+g+\nu)\xi}{(r+g+\nu+\xi)(g+\nu)}$$

and combine terms.

2. *The goods market and the capital markets clear*

$$C(r) + \delta K = F(K, 1) \tag{40}$$

$$\frac{C(r) - w \times 1}{r} = K. \tag{41}$$

3. *The stationary consumption probability density function $\phi(c)$ is consistent with the dynamics of the optimal consumption allocation characterized in Proposition 2, that is, it satisfies Proposition 4.*

In the capital market clearing condition (41), the right-hand side $K = K^d$ is the demand for capital by the representative firm. The numerator on the left-hand side is the excess consumption, relative to labor income, of all agents, that is, the aggregate capital income required to finance that part of consumption that exceeds labor income. Dividing by the return to capital r gives the capital stock that agents need to own to deliver the required capital income. Thus we can think of

$$K^s(r) = \frac{C(r) - w(r)}{r} \tag{42}$$

as the household sector's supply of assets. By restating the capital market clearing condition as

$$K^s(r) = K^d(r)$$

where $K^s(r)$ is defined in (42) and $K^d(r)$ is defined through (38), we can provide an analysis of the existence and uniqueness of the stationary equilibrium in the (K, r) space, analogously to the well-known analysis familiar from Aiyagari (1994) for the standard incomplete markets model.

As long as $r \neq 0$, the usual logic of Walras' law applies and one of the two market clearing conditions is redundant. Equation (41) always implies (40), but the reverse is not true for $r = 0$.[17] Thus, we make use of the capital market clearing condition (41) rather than the goods market clearing condition (40) for our ensuing analysis of stationary equilibria.

[17]From Euler's theorem and equations (38) and (39) it follows that

$$w + rK = F(K, 1) - \delta K$$

Thus, (41) always implies (40). The reverse is true for $r \neq 0$. Note that this issue is not unique to our model, and is present in the Aiyagari model as well; see Proposition 7 in Auclert and Rognlie (2020).

5.1 Equilibrium Existence

We seek to establish the existence of an equilibrium with partial insurance. We will impose a simple condition ensuring that capital supply exceeds capital demand at $r = \rho$. One would also like to find a simple condition so that capital demand exceeds capital supply at some suitable lower bound. Suppose, say, that production is Cobb-Douglas, $F(K, L) = K^\theta L^{1-\theta}$. As r approaches $-\delta$ and thus $r + \delta$ approaches zero, equation (38) implies that $K^d(r) \to \infty$ and therefore $w(r) = (1-\theta)\left(K^d(r)\right)^\theta \to \infty$. Hence, per Lemma 5, the total asset supply associated with that wage then diverges, $K^s(r) \to \infty$, thwarting this strategy.

We therefore adopt the more fruitful approach of examining capital supply and demand $(K^d(r), K^s(r))$ normalized by the wage $w(r) = F_L(K^d(r), 1)$,

$$\kappa^s(r) = \frac{K^s(r)}{w(r)} \text{ and } \kappa^d(r) = \frac{K^d(r)}{w(r)}. \tag{43}$$

and characterize it in the following proposition.[18] Define the following bounds $\underline{r}$ and $\bar{r}$ for the interest rate such that both (normalized) capital demand and supply are well-defined for interest rates $r \in (\underline{r}, \bar{r})$ in between these bounds:

$$\underline{r} = \max\{-\alpha^{\min}, \lim_{K\to\infty} F_K(K,1) - \delta\} \text{ and } \bar{r} = \min\{\rho, \lim_{K\to 0} F_K(K,1) - \delta\} \tag{44}$$

As we show in section 6, r will not exceed ρ, since capital supply becomes infinitely elastic at $r = \rho$.

Proposition 6. *Normalized capital supply $\kappa^s(r)$ and normalized capital demand $\kappa^d(r)$ are well-defined, continuous and strictly positive functions of $r \in (\underline{r}, \bar{r})$.*

Proof. For normalized capital supply $\kappa^s(r)$, Lemma 5 in Appendix D.1 establishes that aggregate consumption $C(r)$ is differentiable and equal to $w(r)$ at $r = 0$. The existence and continuity of a well-defined $\kappa^s(r)$ function follows from L'Hospital's rule at $r = 0$ and is straightforward otherwise. Thus $\kappa^s(r)$ has the stated properties on $(\alpha^{\min}, \rho) \supseteq (\underline{r}, \bar{r})$.

For normalized capital demand $\kappa^d(r)$, observe that the marginal product of capital is a continuously differentiable and strictly decreasing function of K, mapping $K \in (0, \infty)$ onto $(\lim_{K\to\infty} F_K(K,1), \lim_{K\to 0} F_K(K,1)) \supseteq (\underline{r} + \delta, \bar{r} + \delta)$. Since the marginal product

[18] We believe that this approach of analyzing the model is fruitful more generally for any model with standard neoclassical production, including the Aiyagari (1994) model and the competitive equilibrium of the standard representative agent model.

of labor $F_L(K,1)$ is positive and continuous for all positive K, the function $\kappa^d(r)$ has the properties stipulated in Proposition 6. □

A rate of return r^* gives rise to a stationary equilibrium if

$$\kappa^s(r^*) = \kappa^d(r^*) \tag{45}$$

In order to ensure the existence of equilibrium, we need the following boundary conditions.[19]

Assumption 2. *Let* $\liminf_{r\to\underline{r}} \kappa^s(r) < \limsup_{r\to\underline{r}} \kappa^d(r)$ *and* $\limsup_{r\to\bar{r}} \kappa^s(r) > \liminf_{r\to\bar{r}} \kappa^d(r)$.

The following proposition is then a trivial consequence of Proposition 6 and the intermediate value theorem.

Proposition 7. *Suppose assumption 2 is satisfied. Then, a stationary equilibrium with an interest* $r^* \in (-\delta, \rho)$ *exists.*

Assumption 2 involves the endogenous entities $(\kappa^s(r), (\kappa^d(r))$ at the boundaries $(\underline{r}, \bar{r})$. If one is willing to put further structure on the production function and agent preferences and endowments, then it can be replaced with conditions on exogenous parameters only. In particular, consider a CES production function $F(K,L) = \left(\theta K^{1-\frac{1}{\eta}} + (1-\theta)L^{1-\frac{1}{\eta}}\right)^{\frac{\eta}{\eta-1}}$ with elasticity of substitution $\eta \in (0,\infty)$. This includes the Cobb-Douglas specification $F(K,L) = K^\theta L^{1-\theta}$ as a special case for $\eta = 1$. We show in appendix C.2 that normalized capital demand becomes

$$\kappa^d(r) = \frac{\theta}{(r+\delta)\left[\left(\frac{r+\delta}{\theta}\right)^{\eta-1} - \theta\right]} \tag{46}$$

If the elasticity of substitution is as high or higher than in the Cobb-Douglas case, $\eta \geq 1$, then $\kappa^d(r)$ is strictly decreasing and continuously differentiable. It is defined on $r \in (\theta^{\frac{\eta}{\eta-1}} - \delta, \infty)$ for $\eta > 1$ and $r = (-\delta, \infty)$ for $\eta = 1$, and diverges, as r approaches the lower bound of that interval. If $-\alpha^{\min}$ is lower than that lower bound, then the first half of Assumption 2 is automatically satisfied.[20]

[19] The use of $\liminf$ and $\limsup$ in the assumption is sufficient for the existence result of Proposition 7 and avoids a discussion of the existence of the associated limits.

[20] For $\eta \in (0,1)$, normalized capital demand $\kappa^d(r)$ is defined on $r \in (-\delta, \theta^{\frac{\eta}{\eta-1}})$. We show in appendix C.2 that in this case $\kappa^d(r)$ has an upward-sloping part. Indeed, for the limit case of $\eta = 0$ (Leontieff production

In the next subsection, we continue our example and show that for this example, the second half of Assumption 2 can also be replaced by an assumption on exogenous parameters characterizing the extent of income risk (and the other parameters of the model). Section 6 considers the case when the second inequality in Assumption 2 is reversed, and a stationary equilibrium with full consumption insurance can emerge.

5.2 The Example Continued

The properties of normalized capital supply can be examined explicitly in the two-state example of Sections 3.3 and 4.2. Equation (36) immediately implies that for this example

$$\frac{C(r)}{w(r)} = 1 + \frac{r\xi}{(g(r)+\nu)(r+g(r)+\nu+\xi)} \tag{47}$$

where we recall that the growth rate $g(r)$ (and thus the decay rate of consumption $-g$) is given per equation (8) by $g(r) = \frac{\rho - r}{\sigma}$. With a Cobb-Douglas production function and thus equation (46), and with $\kappa^s(r) = (C(r)/w(r) - 1)/r$, the capital market clearing condition can be stated explicitly as

$$\kappa^d(r) = \frac{\theta}{(1-\theta)(r+\delta)} = \frac{\xi}{\left(-\frac{r}{\sigma} + \frac{\rho}{\sigma} + \nu\right)\left(\left(1 - \frac{1}{\sigma}\right)r + \frac{\rho}{\sigma} + \nu + \xi\right)} = \kappa^s(r) \tag{48}$$

where we have now written out the growth rate $g(r) = \frac{\rho - r}{\sigma}$. This is a quadratic equation and can have no, one or two solutions in the interval $(-\delta, \rho)$.

It is easy to see that the following assumption, stated purely in terms of the exogenous parameters of the model, implies Assumption 2 with $\underline{r} = -\delta$ and $\bar{r} = \rho$.

Assumption 3. *The production function takes a Cobb-Douglas form. The parameters characterizing the production technology* (θ, δ)*, agent preferences* (ρ) *and idiosyncratic risk* (ν, ξ) *satisfy* $\alpha^{\min} = \min\{\nu, \xi\} > \delta$ *and*

$$\kappa^d(\rho) = \frac{\theta}{(1-\theta)(\rho+\delta)} < \frac{\xi}{\nu(\rho+\nu+\xi)} = \kappa^s(\rho) \tag{49}$$

function), $\kappa^d(r)$ is upward-sloping on the entire interval $r \in (-\delta, 1-\delta)$ where it is defined. In terms of general properties outside the CES case, we establish in Appendix C.1 that $\kappa^d(r)$ is strictly decreasing if F_K is strictly convex. Note that these results and issues arise in *any* models employing a neoclassical production function, including the standard representative agent model as well as the Aiyagari (1994) model. This might explain why the literature typically assumes that the production function is Cobb-Douglas.

We will now show that if, in addition to this assumption, the intertemporal elasticity of substitution $1/\sigma$ is sufficiently high (σ is sufficiently low), then capital supply is upward sloping in the interest rate and the partial insurance steady state is *unique*. In contrast, if σ is sufficiently large, then $\kappa^s(r)$ can have downward-sloping segments and the possibility of multiple partial insurance steady states emerges.

5.2.1 Logarithmic Utility ($\sigma = 1$): Uniqueness and Comparative Statics

If $\sigma = 1$, then the equilibrium condition (48) becomes linear in the interest rate, and the unique partial insurance stationary equilibrium can be characterized in closed form.

Proposition 8. *Suppose $\sigma = 1$.*

1. *Suppose Assumption 2 is satisfied and that normalized capital demand $\kappa^d(r)$ is downward sloping. Then the equilibrium is unique.*

2. *Now suppose Assumption 3 is satisfied. Then the unique equilibrium interest rate $r^* \in (-\delta, \rho)$ is given by*

$$r^* = \frac{\theta(\nu + \rho + \xi)(\nu + \rho) - \xi\delta(1 - \theta)}{\xi + \theta(\nu + \rho)} \tag{50}$$

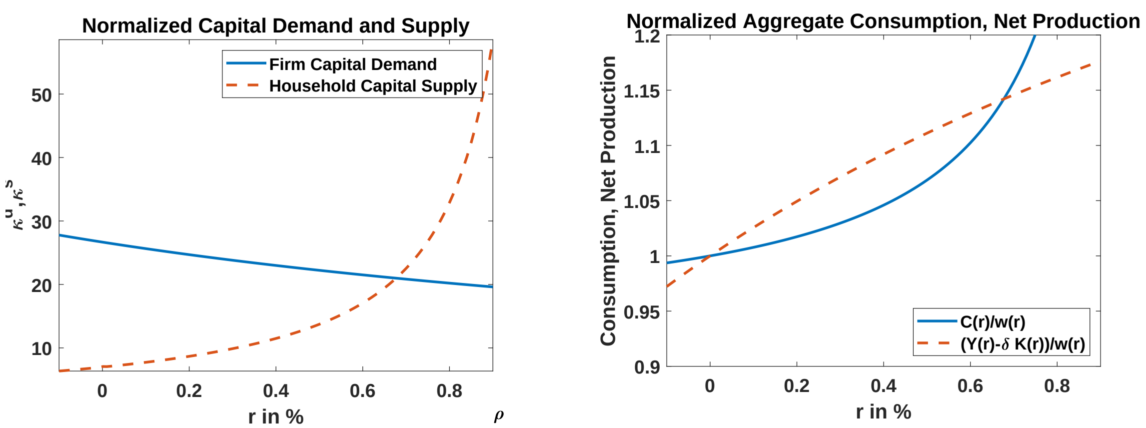
r^* is strictly increasing in $\rho + \nu$ and θ and strictly decreasing in ξ and δ. The equilibrium capital stock K^* is strictly increasing in ξ and strictly decreasing in $\rho + \nu$ and in δ. The stationary consumption distribution has a mass point and a truncated Pareto distribution with Pareto coefficient $\kappa = \frac{\nu}{\rho - r^*} - 1$ below the mass point.

Proof. With $\sigma = 1$, equation (47) implies that normalized capital supply is given by

$$\kappa^s(r) = \frac{\xi}{(\rho - r + \nu)(\rho + \nu + \xi)} \tag{51}$$

and is strictly increasing in r. Thus, the equilibrium must be unique. Equation (50) follows from solving (the now linear) equation (48), when $\sigma = 1$. The comparative static properties for the equilibrium interest rate follow directly from its closed-form expression, and the comparative statics results for the equilibrium capital stock follow from the fact that it is a decreasing function of r^*. The statements about the consumption distribution follow directly from equation (35). □

The finding that the equilibrium capital stock increases and the interest rate falls with an increase in the risk of income falling, ξ, indicates the presence of precautionary saving in our model. To see this, note that the variance of income is given by $Var(z) = \bar{\mu}_1(0-1)^2 + \bar{\mu}_2(\zeta - 1)^2 = \frac{\xi}{\nu}$ where we used the normalization $\bar{\mu}_2\zeta = 1$ that average labor productivity equals 1. Increasing ξ, holding ν fixed and increasing ζ such that average productivity remains at one therefore constitutes a mean-preserving spread increasing income risk. In response agents save more individually[21] and the aggregate capital stock rises as a result. That is, there is precautionary saving on the micro and macro level in our model, but it takes the form of state-contingent saving due to the market structure we have assumed.



(a) Capital Demand $\kappa^d(r)$ and Supply $\kappa^s(r)$ as Function of the Interest Rate r

(b) Goods Demand $C(r)/w(r)$ and Net Supply $(Y(r) - \delta K(r))/w(r)$ as a Function of r

Figure 2: *The left panel shows wage-normalized capital demand $\kappa^d(r)$ and capital supply by the household sector $\kappa^s(r)$, as a function of the interest rate. The figure is drawn with Assumption 3 in place, guaranteeing a unique stationary equilibrium interest rate $r^* < \rho$. The right panel plots consumption demand $C(r)$ by the household sector and net goods supply. There are two intersections: one at the stationary equilibrium interest rate r^* and one at $r = 0$; but at the latter interest rate $(r = 0)$, the capital market does not clear.*

The unique equilibrium is represented graphically in Figures 2a and 2b. There is a unique equilibrium with an interest rate $r < \rho$ that clears *both* the capital market (Figure 2a) and the goods market (Figure 2b).

[21] Capital saved for the transition to the low state (see equation (22)) is given by $k_2(1) = \frac{1}{\xi+\nu+\rho}\zeta w = \frac{w}{\nu(1+\rho/(\xi+\nu))}$ which is increasing in ξ.

5.2.2 Multiple Partial Insurance Steady State Equilibria

If normalized capital demand $\kappa^d(r)$ is downward sloping, as it is for Cobb-Douglas production, the CES specification for $\eta \geq 1$ and a multitude of other production functions, the key to establishing the existence of a unique partial insurance steady state is an upward-sloping normalized capital supply function.

Inspection of the asset supply function on the right-hand side of (47) shows this to be the case if $\sigma \leq 1$. In contrast, as σ approaches infinity and the IES converges to zero, the lifetime utility function becomes Leontieff, and the asset supply function is downward-sloping, raising the possibility of multiple partial insurance stationary equilibria. The next proposition summarizes the various possibilities for $\sigma \neq 1$. For simplicity, we impose a Cobb-Douglas production function on the capital demand side.

Proposition 9. *Let Assumption 3 be satisfied.*

1. *If $\sigma < 1$, then $\kappa^s(r)$ is strictly increasing on $r \in (-\delta, \rho)$. There exists a unique stationary equilibrium with interest rate $r \in (-\delta, \rho)$.*

2. *Let $\sigma > 1$ and let $\frac{\sigma\nu+\rho}{\sigma-1} > \delta$ be satisfied.*[22] *There exists at least one stationary equilibrium with $r \in (-\delta, \rho)$.*

   (a) *Suppose $\sigma \in (1, 2]$ and let $\xi \geq \delta$ be satisfied.*[23] *Then $\kappa^s(r)$ is increasing on $r \in [-\delta, \rho)$ and the stationary equilibrium with interest rate $r \in (-\delta, \rho)$ remains unique.*

   (b) *There exist parameter combinations with $2 < \sigma < \infty$ such that $\kappa^s(r)$ has decreasing parts on $[-\delta, \rho)$ and that there are two stationary equilibria with $r \in (-\delta, \rho)$ solving the quadratic capital market clearing condition* (48).

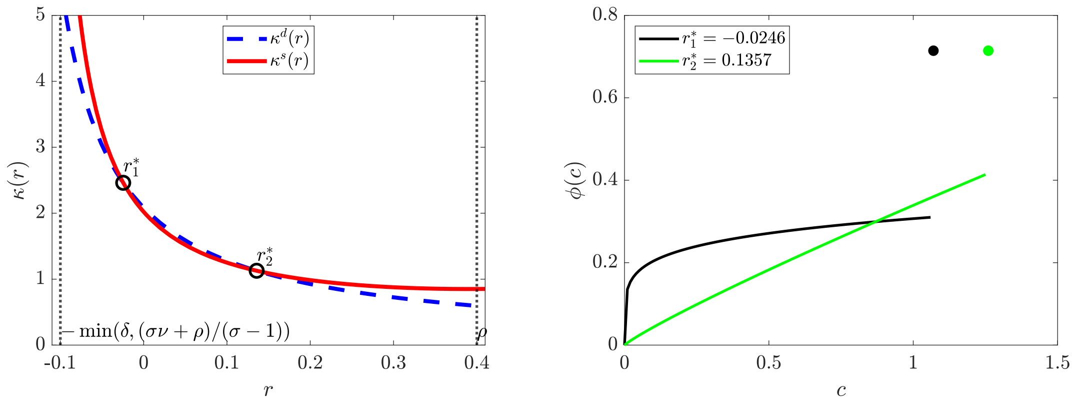
Proof. See Online Appendix E. For the last part, see the example in Figure 3. □

This proposition shows that for wide parameter combinations, the uniqueness of equilibrium can be guaranteed (parts 1 and 2a). It also identifies (in part 2b) the range of parameters where two stationary equilibria can emerge. This scenario is depicted in Figure 3.

[22]This condition ensures that the effective discount rate $r + \nu + g(r)$ used to determine $\mathbf{c}(2)$ is positive even at $r = -\delta$, and thus $\mathbf{c}(2)$ is finite at that interest rate and at all higher interest rates.

[23]This condition ensures that $\kappa^s(r)$ is increasing at $r = -\delta$.

Figure 3: Two equilibria with partial insurance when $\sigma > 2$.



(a) Capital Market Clearing

(b) Equilibrium Consumption Distributions

This figure plots an example of two equilibria, both with partial insurance, under parameter values $\sigma = 10, \theta = 0.25, \delta = 0.16, \nu = 0.05, \xi = 0.02, \rho = 0.4$. The two equilibrium interest rates are given by $r_1^* = -0.0246, r_2^* = 0.1357$. Left panel: solid line represents the capital supply curve $\kappa^s(r)$, dashed line represents the capital demand curve $\kappa^d(r)$. The right panel displays the two equilibrium consumption distributions, including the mass point for each of them.

6 Stationary Equilibrium with Full Insurance

Suppose that $\bar{r} = \rho$, but that the second part of assumption 2 is violated, i.e., suppose $\limsup_{r \to \rho} \kappa^s(r) \leq \liminf_{r \to \rho} \kappa^d(r)$. Since there is full consumption insurance when $r = \rho$, it follows that the capital supply needed to provide this full insurance is insufficient to meet capital demand from the production side. As a result, agents hold capital for conventional consumption, smoothing savings motives, and not just as an insurance cushion. Capital supply becomes infinitely elastic at $r = \rho$, just as in the steady state of the standard representative agent neoclassical growth model.

Consider an agent indexed by $j \in [0, 1]$ in state x. With full insurance, consumption is constant at some level c_j. There is no disinvestment, $\dot{k}_{j,t,x} = 0$, and hence, there is a constant capital level $k_j(x)$ for every level of productivity x, with the flow of interest payments financing the gap between income and consumption. Budget constraint (7) reads

$$c_j + \sum_{x' \neq x} \alpha_{x,x'}(k_j(x') - k_j(x)) = \rho k_j(x) + w\mathbf{z}(x) \tag{52}$$

or

$$(\rho \mathbf{I}_{N+1} - A)\mathbf{k}_j = c_j - w\mathbf{z} \tag{53}$$

where $\mathbf{I}_{N+1}$ is the identity matrix of size $N \times N$ and where $\mathbf{k}_j = [k_j(1), \ldots, k_j(N)]$ is the vector of capital stocks held by agent j, conditional on each income state.[24] Equation (53) can be solved for the capital levels $\mathbf{k}_j$, provided the wage w and consumption c_j are known. The wage w follows directly from the production side at $r = \rho$. As for the consumption level, note that $c_j \geq \mathbf{c}(N)$, where the latter is the lowest consumption level in state N compatible with $r = \rho$ and as calculated in proposition 2. This follows because agents will eventually reach state N, with zero capital and permanent consumption of at least $\mathbf{c}(N)$, even if that agent starts off with zero capital in some other state. The proof of the following proposition in appendix E then implies that $\mathbf{k}_j$ is non-negative for any agent j.

Proposition 10. *Suppose that $\bar{r} = \rho$ and $\limsup_{r \to \rho} \kappa^s(r) \leq \liminf_{r \to \rho} \kappa^d(r)$. Then there is a stationary equilibrium with $r = \rho$ in which every agent $j \in [0, 1]$ consumes a constant amount $c_j \geq \mathbf{c}(N)$. Average consumption $\bar{c}$ is given by sum of the flow income from capital and wages*

$$\bar{c} = \rho K^d(\rho) + w(\rho) \tag{54}$$

where $K = K^d(\rho)$ solves (38) at $r = \rho$ and where $w(\rho)$ follows from (39) at $K = K^d(\rho)$. Individual capital holdings $\mathbf{k}_j$ satisfy equation (53)*. If $c_j = \bar{c}$ for all agents, the distribution of the agents over the point masses $(x, \bar{k}_j(x))$ is given by the stationary distribution $\bar{\mu}$ for A, where $\bar{\mathbf{k}}$ solves (53) for $c_j = \bar{c}$.[25] For arbitrary consumption distributions $c_j \geq \mathbf{c}(N)$, $\bar{\mathbf{k}}$ is the average of the capital holdings across agents.*

The proof is in appendix E. In principle, nothing guarantees that the vector of capital holdings defined in (53) satisfies $k_j(x) \geq 0$ for all x. The point of the proof is to show that this is precisely what the assumed inequality $\lim_{r \to \rho} \kappa^s(r) \leq \kappa^d(\rho)$ guarantees.

Note that replacing $c_j = \bar{c}$ in the agent budget constraint (53) with the goods market clearing condition (54) and taking the inner product with the stationary distribution $\bar{\mu}$ yields

$$\bar{\mu} \cdot (\rho \mathbf{I}_{N+1} - A)\bar{\mathbf{k}} \;\; = \;\; \rho K^d(\rho) + w(\rho) - w(\rho)\bar{\mu} \cdot \mathbf{z}.$$

[24] Recall that we use the notation $\mathbf{I}_x$ to denote the $(x-1) \times (x-1)$ identity matrix: thus the subscript $N+1$ here. Further, recall that $\alpha_{x,x} = -\sum_{x' \neq x} \alpha_{x,x'}$.

[25] It is not necessarily true that all agents have the same consumption: they just each have consumption of at least $\mathbf{c}(N)$ and average consumption is $\bar{c}$, but we cannot say more than that. The consumption distribution is indeterminate and depends on the (arbitrary) initial distribution of capital, exceeding $k_N(x)$ for $x < N$ or 0 for $x = N$.

Since $\bar{\mu} \cdot \mathbf{z} = 1$ by normalization and $\bar{\mu}' A = 0$ by stationarity of $\bar{\mu}$, we have

$$\bar{\mu} \cdot \bar{\mathbf{k}} = K^d(\rho) \tag{55}$$

and thus, the asset market clearing condition is satisfied at $r = \rho$ as well, a simple consequence of Walras' law.

Proposition 10 opens the door for the existence of (at least) two steady-state equilibria, one with partial insurance, the other with full insurance, by reversing the orderings at both ends in Assumption 2.

Assumption 4. *Assume that* $\limsup_{r \to \underline{r}} \kappa^s(r) > \liminf_{r \to \underline{r}} \kappa^d(r)$. *Assume that* $\bar{r} = \rho$ *and that* $\liminf_{r \to \rho} \kappa^s(r) < \limsup_{r \to \rho} \kappa^d(r)$.

Proposition 11. *Suppose assumption 4 is satisfied. Then there is at least one partial insurance stationary equilibrium with an interest* $r^* \in [\underline{r}, \rho)$ *and one full insurance equilibrium at* $r = \rho$.

Proof. Like Proposition 7, the existence of a partial insurance stationary equilibrium follows from the intermediate value theorem and Proposition 6. The existence of the full insurance equilibrium follows from proposition 10. □

Assumption 4 requires that $\kappa^d(r)$ is upward-sloping or that $\kappa^s(r)$ is downward-sloping for at least a certain range of the interest rate. As shown above, this cannot occur in our simple 2-state example with Cobb-Douglas production and log-utility. However, even for this example, Assumption 4 is not empty as long as σ is sufficiently large, and thus, the IES is sufficiently small. [26]

[26] Recall that normalized capital supply is $\kappa^s(r) = \xi/((g + \nu)(r + g + \nu + \xi))$ with $g = (\rho - r)/\sigma$, see equation (48). While this function is well defined for $-\nu - \xi < r \leq \rho$, examination of (18) shows that we need to keep $r > -\nu$, as $\sigma \to \infty$. Impose that $0 < \nu = \xi < \delta$. Therefore $\underline{r} = \alpha^{\min} = \nu$ is the lower bound for r in assumption 4. As $r \to -\nu$ and $\sigma \to \infty$, normalized capital supply converges to $1/\nu$, while normalized capital demand in the Cobb-Douglas case is $\theta/((1-\theta)(\delta-\nu))$. If $\nu < (1-\theta)\delta$, one can therefore find $\underline{r} > -\nu$ and σ large enough that Assumption 4 is satisfied.

These calculations also allow for a non-existence example. Assume again Cobb-Douglas production and Leontieff preferences $\sigma \to \infty$, and impose $\theta = 1/3$ as well as $\nu = \xi < \delta$ and thus $\underline{r} = -\nu$. Non-existence follows if $\kappa^s = 1/(r + 2\nu) > 2(r + \delta) = \kappa^d$ for all $r \in (-\nu, \rho]$. This is the case if $\nu < 2\delta/3$. For finite, but large σ then follows for ν sufficiently small compared to δ. Concretely, assume that $\nu = \xi = \rho < \delta/2$. With some algebra one can show that $\kappa^s > \kappa^d$ for all $r \in (-\nu, \rho]$ if $\sigma \geq 7$.

7 Quantitative Exploration

The previous sections characterized partial- and full insurance stationary equilibria theoretically. We now demonstrate that our model is amenable to the same quantitative analysis as the standard incomplete markets (SIM) model. For a plausible calibration of idiosyncratic risk consistent with micro data, it delivers a unique partial insurance interest rate and consumption distribution that can be quantitatively compared to the SIM in continuous time, as explored recently in, e.g., Kaplan, Nikolakoudis and Violante (2023). To do so, we first discuss the calibration of the model, with focus on the idiosyncratic productivity process. We then show the stationary consumption distribution and contrast the capital market equilibrium in our model with that in the standard incomplete market model.

7.1 Calibration

For the calibration, we adopt the five-state process used by Kaplan et al. (2023), but augment it by a sixth state, referred to below as the superstar state, see Table 1. With this

Table 1: Parameterization of the Quantitative Model

Parameter	Interpretation	Value
θ	Capital Share	40%
δ	Depreciation Rate	2.25%
σ	Risk Aversion	1
ρ	Time Discount Rate	1%
ν	Poisson Rate of Moving into Top	0.001
ξ	Poisson Rate of Moving out of Top	0.1
$\mathbf{z}$	Labor Productivity States	(0.5,0.65,0.81,0.96,1.12,20)
$\bar{\mu}$	Labor Productivity Distribution	(0.07,0.24,0.37,0.24,0.07,0.01)

The table contains the parameterization of the model at a quarterly frequency. The last two rows contain the idiosyncratic labor productivity states $\mathbf{z}(.)$ as well as the associated stationary distribution $\bar{\mu}$ over these states. The complete matrix of Poisson transition rates is contained in Appendix F

superstar state, the insights from the simple two-state example above carry over to the quantitative version here: essentially, the agent switches back and forth between the very high income and low incomes, setting aside insurance in the former against the transition to the latter. Specifically, we choose the highest state in such a way that the share of the population in that state is 1% and that their share of labor income is 20% (see, e.g., Piketty and

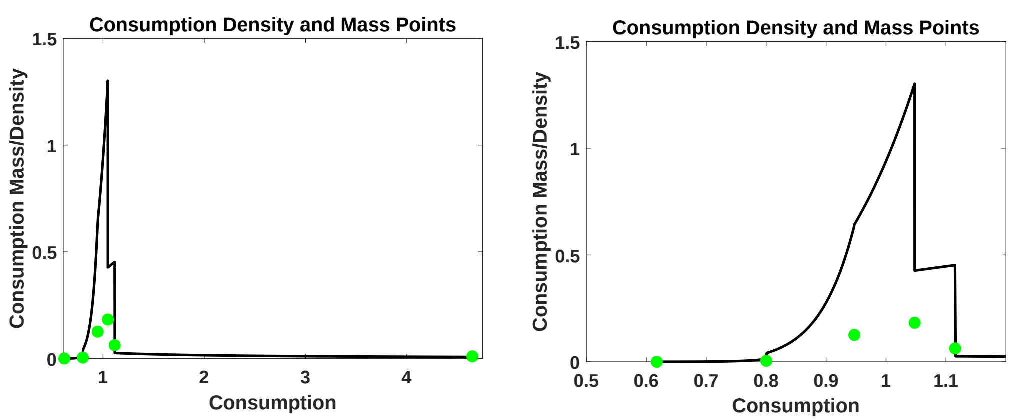
Saez, 2003). Since average labor productivity is normalized to 1, we have $0.01 * \mathbf{z}(6) = 0.2$ which implies $\mathbf{z}(6) = 20$. Analogous to our two-state example, let ξ and ν denote the Poisson intensity of leaving or arriving at the high state.[27] Since the share of highly productive agents is $0.01 = \frac{\nu}{\xi+\nu} = \frac{1}{1+\xi/\nu}$, this implies that we need $\frac{\xi}{\nu} = 99$. This leaves us with one degree of freedom determining the persistence (or expected duration, $1/\xi$) of remaining in the superstar state. For a quarterly frequency of the calibration and assuming that duration to be 10 quarters, we obtain $\xi = 0.1$ and thus $\nu = 0.1/99 = 0.001$.

For the remaining parameters, we follow Kaplan et al. (2023) and set the capital share to $\theta = 0.4$ and the quarterly depreciation rate to $\delta = 2.5\%$. Risk aversion is $\sigma = 1$, and the quarterly time discount rate is $\rho = 1\%$.

7.2 Stationary Consumption Distribution and Capital Market

Figure 4 shows the consumption distribution. As Proposition 5 implies, there are $N = 6$ mass points. The highest mass point contains 1% of the population at the consumption level $\mathbf{c}(6)/w = 4.65 = 0.23 * \mathbf{z}(6)$. Thus, agents in the highest income state set aside more than three-quarters of their income as insurance payments against an income change.

Figure 4: Consumption Distribution: Quantitative Version Limited Commitment Model



(a) Equilibrium Consumption Distribution (b) Consumption Distribution: Zoom

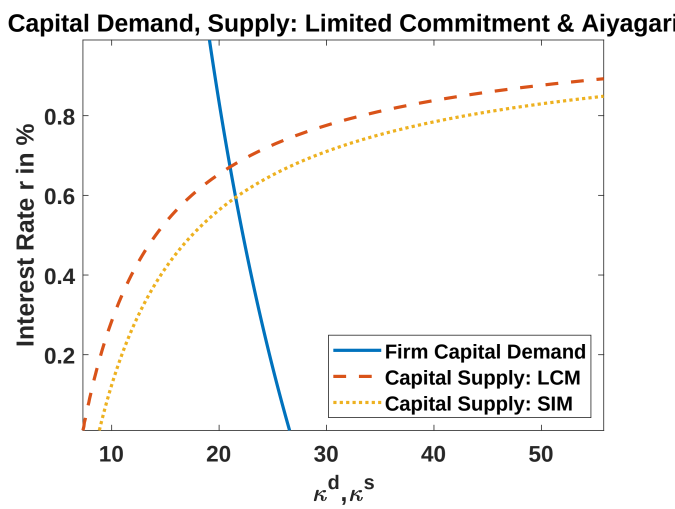
This figure displays the stationary consumption distribution, normalized by the wage. We chose $r \approx 0$ because for low r, the mass points, depicted as green solid points, are more clearly visible. The left panel shows the entire distribution with its long right tail, and the right panel zooms in on the middle of the distribution.

[27] Formally, $\xi = \sum_{x<6} \alpha_{6,x}$ and $\nu = \sum_{x<6} \alpha_{x,6} \bar{\mu}(x)$.

The density in between the mass points is provided in Proposition 5, exploiting the matrix exponential formula (27). It is determined both by the common consumption decay rate of all unconstrained individuals $(r - \rho)$ as well as the outflow rates into higher states (the $\alpha_{x,\tilde{x}}$), resulting from the matrix exponential differential equation. It therefore displays exponential decay (at a rate that varies across the different segments).

Figure 5 depicts the capital market equilibrium for our limited commitment model (LCM) and compares it to the standard incomplete markets (SIM) model. With the assumed Cobb-Douglas production function, normalized capital κ^d is downward-sloping, see equation (48), and is the same for both models. For our calibration normalized capital supply κ^s is strictly upward sloping in the limited commitment model. Assumption 2 is satisfied. Thus, there is a unique equilibrium interest rate r^*, which takes the (quarterly) value of $r^* = 0.68\%$, smaller than the quarterly discount rate of $\rho = 1\%$.

Figure 5: Capital Market Equilibrium in the Limited Commitment Model and the Standard Incomplete Markets Model with Neoclassical Production



This figure displays the equilibrium in the capital market in the limited commitment model and SIM. The normalized capital demand schedule (blue solid, downward sloping schedule) is identical in both models. The normalized capital supply from the household sector, for a given interest rate, is larger in the SIM (yellow line) than in the LCM (red line). Therefore, the interest rate is lower and the capital stock is higher in the SIM than in the LCM, which in turn features a lower interest rate and higher capital than the SIM.

Capital supply for the SIM model needs to be calculated numerically, using standard techniques, and is likewise upward sloping.[28] We observe that capital supply for the SIM model is larger for each interest rate than in our model. As a consequence, the stationary

[28] We thank Greg Kaplan for providing us with the code for the SIM with $N > 2$ states.

equilibrium interest rate, unique in both economies, is strictly smaller (and thus the equilibrium capital stock is strictly larger) in the SIM model[29] than in our economy, and is, in turn, smaller in both models than the subjective time discount factor $\rho = 1\%$. As summarized in Table 2, the equilibrium interest rates for our benchmark calibration are $r^*_{LCM} = 0.68\%$ and $r^*_{SIM} = 0.6\%$. Finally, as the interest rate approaches the time discount factor from below, asset supply in the SIM model diverges to infinity (as is well-known), whereas it remains finite in the limited commitment economy.

7.3 Comparative Statics

So far, we set the parameter $\xi = 0.1$, implying an expected time of remaining in the superstar state of 10 quarters, or 2.5 years. We now vary this parameter, with the objective of not only providing sensitivity analysis but also showing how the equilibrium interest rate and associated capital stock respond to a change in the extent of labor income risk as well as its persistence. Raising ξ but holding ξ/ν constant keeps the cross-sectional distribution over states constant, but it decreases the persistence of both remaining in the superstar state and remaining in one of the 5 "normal" states.[30] By contrast, only raising ξ and keeping ν constant implies a mean-preserving spread, as in the simple model with two states.[31]

Table 2 shows the outcome of both experiments. The interest rate in our model is always larger and the associated equilibrium capital stock smaller than in the SIM. Making

Table 2: Comparative Statics

Parameter	Bench.	Low Pers.	High Pers.	MPS
ξ	0.10	0.2475	0.025	0.20
ν	0.001	0.0025	0.00025	0.001
r^*_{LCM}	0.675%	0.775%	0.615%	0.635%
r^*_{SIM}	0.595%	0.695%	0.565%	0.545%

The table summarizes the equilibrium r^* for different parameterizations of the idiosyncratic income process.

[29]While the claim that this is always the case seems intuitive, it is not easy to prove. In a nutshell, our environment allows agents to redistribute savings from states where they have a low marginal value of wealth to states where this value is high. The effect on the marginal value of overall capital can then turn either way.

[30]In particular, the mass of agents in the high income state remains at 1% and thus there is no need to adjust the highest state **z(6).**

[31]This increase $\frac{\xi}{\nu}$ and makes the top group smaller. As a consequence, we increase $\mathbf{z}(6)$ and make the top group income-richer, such that average productivity remains at one.

the superstar state less persistent without changing the cross-sectional labor productivity distribution lowers precautionary saving in both models, increases the equilibrium interest rate and decreases the equilibrium capital stock in both models. By contrast, a mean preserving spread (MPS) increases precautionary saving in both models and leads to a reduction in the equilibrium interest rate and an increase in the steady state capital stock in the economy (as we showed analytically for our model with two states).

8 Conclusion

The standard incomplete markets general equilibrium model of Aiyagari (1994) has become a workhorse model for a substantial literature, but does not address the source for market incompleteness and results in less consumption smoothing than documented by the empirical literature. In this paper, we therefore propose an alternative model in which market incompleteness arises endogenously due to limited commitment. The resulting model is analytically tractable yet as amenable to quantitative analysis as the benchmark SIM.

For a general continuous-time N-state Poisson labor productivity process, we have characterized the optimal consumption-asset allocation, the stationary asset distribution, as well as the aggregate supply of capital. For a specific example in which labor productivity takes two values, one of which is zero and when agents have log-utility and production is Cobb-Douglas, the entire stationary equilibrium can be computed in closed form. In contrast, multiple steady states can arise for large risk aversion. We have analyzed a calibrated version of our model, using six income states, and shown numerically that the nominal interest rate is higher and less sensitive to comparative static changes in parameters than in the SIM model. Our paper, therefore, provides a tractable alternative to the standard incomplete markets general equilibrium model as in Aiyagari (1994).

In this paper we have focused on stationary equilibria, sidestepping the question of whether this stationary equilibrium is reached from a given initial aggregate stock, and what the qualitative properties of the transition path are. We pursue this analysis for our two-sate example in Krueger, Li and Uhlig (2024). Similarly, thus far we have focused on an environment that has idiosyncratic but no aggregate shocks. We study a discrete-time version of our model with aggregate shocks and its asset pricing implications in Ando, Krueger and Uhlig (2023).

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Appendix

A Characterizing the optimal contract: lemmas and proofs.

Proof of Proposition 1. 1. Write the Lagrangian

$$\begin{aligned} L &= u(c) + U'(k,x)\dot{k} + \sum_{x' \neq x} \alpha_{x,x'}(U(k(x'),x') - U(k,x)) \\ &\quad + \lambda \left(rk + w\mathbf{z}(x) - c - \dot{k} - \sum_{x' \neq x} \alpha_{x,x'}(k(x') - k) \right) \\ &\quad - 1_{k=0}\mu_x \dot{k} - \sum_{x' \neq x} \alpha_{x,x'}\mu_{x'}k(x') \end{aligned}$$

The first-order conditions are

$$\frac{\partial L}{\partial c} : \quad u'(c) \quad = \lambda \tag{56}$$

$$\frac{\partial L}{\partial \dot{k}} : \quad U'(k,x) \quad = \lambda - 1_{k=0}\mu_x \tag{57}$$

$$\frac{\partial L}{\partial k(x')} : \quad U'(k(x'),x') \quad = \lambda - \mu_{x'}, \text{ all } x' \tag{58}$$

with the additional complementary slackness conditions

$$1_{k=0} \min\{\mu_x, \dot{k}\} = 0 \text{ and for all } x' : \min\{\mu_{x'}, k(x')\} = 0 \tag{59}$$

as well as the envelope condition

$$\begin{aligned} \rho U'(k,x) &= \frac{\partial L}{\partial k} \\ &= U''(k,x)\dot{k} + r\lambda - \sum_{x' \neq x} \alpha_{x,x'}(U'(k,x) - \lambda) \end{aligned}$$

or

$$\left(\rho - r\lambda + \sum_{x' \neq x} \alpha_{x,x'} \right) (U'(k,x) - \lambda) = U''(k,x)\dot{k} \tag{60}$$

For $k > 0$, (56), (57) and (59) imply

$$U'(k, x) = \lambda = u'(c) \tag{61}$$

With equation (58) and (59), we then get

$$u'(c(k'(x')) = U'(k'(x'), x') = u'(c) \text{ for all } k'(x') > 0 \tag{62}$$

showing (10). Suppose by contradiction, that $u'(c(k, x)) < u'(c(k(x'), x'))$ for some state x'. This cannot be optimal since a small increase of $k(x')$ and thus a small increase in $c(k(x'), x')$ at the cost of a small decrease in $c(k, x)$ would improve overall utility. We therefore obtain that $u'(c(k, x)) \geq u'(c(k(x'), x'))$, in particular for states x', for which $k(x') = 0$. The statement (11) now follows from the strict concavity of $u(\cdot)$.

Rewriting (61) as a function of time and taking the derivative with respect to time, we get

$$U''(k_t, x_t)\dot{k}_t = \dot{\lambda}_t = u''(c_t)\dot{c}_t \tag{63}$$

Rewriting (63) and combining it with (60) and (61) for $u(c) = c^{1-\sigma}/(1 - \sigma)$ yields

$$\frac{\dot{c}_t}{c_t} = \frac{\dot{\lambda}_t}{\lambda_t} \frac{u'(c_t)}{cu''(c_t)} = \frac{\rho - r}{\sigma} \tag{64}$$

and thus (9).

2. This follows because any allocation that can be afforded for k can also be afforded for $\tilde{k} > k$.

3. This is a standard and straightforward argument. Consider two values for k, say $k_A \neq k_B$ and some $\lambda \in (0, 1)$. The λ-convex combination of the solutions for k_A and k_B is feasible at the λ-convex combination of k_A and k_B and thus provides a lower bound for $U(k_\lambda, x)$. This lower bound is strictly higher than the convex combination of $U(k_A, x)$ and $U(k_B, x)$ since $u(\cdot)$ is strictly convex and c is strictly increasing in k.

4. A formal proof is via Lemma 8 in the Online Appendix. Here, we provide a somewhat heuristic argument instead. If the constraint (5) is binding, then $\dot{k} = 0$, $U(k_t, x)$

is a constant function of time and thus so is c_t, establishing the claim. Suppose, thus, by contradiction, that the constraint is not binding and that $\dot{k}_t > 0$. In that case, we have (61) as well (64). Consider now a small time interval δ later. At that point, $k_{t+\delta} \approx \dot{k}\delta > 0$ as well as $c_{t+\delta} \approx c_t(1-\delta g) < c_t$. We still have (61). Noting that $U(\cdot, x)$ is strictly concave with the previous part, we have

$$U'(0,x) > U'(k_{t+\delta}, x) = u'(c_{t+\delta}) < u'(c_t) \tag{65}$$

in contradiction to (61).

□

Lemma 1. *For this lemma*[32]*, denote the spectral radius of a matrix M as $\rho(M)$.*

1. $e^{-B_x s} \geq 0$ *and* $e^{-C_x s} \geq 0$*. If, additionally, assumption 1 holds, then* $e^{-B_x s}$ *and* $e^{-C_x s}$ *have only strictly positive entries for all* $s > 0$*.*
2. *The spectral radius of* $e^{-B_x s}$ *satisfies* $e^{-(r+\alpha^{\max}(x))s} \leq \rho(e^{-B_x s}) \leq e^{-(r+\alpha^{\min})s} \leq e^{-rs}$*.*
3. *If* $\alpha^{\min} = \alpha^{\max}(x)$*, then* $\mathbf{1}_x$ *is an eigenvector of* B_x *and* $e^{-B_x s}$ *with eigenvalue* $r+\alpha^{\min}$ *and* $e^{-(r+\alpha^{\min})s}$*.*
4. *With assumption 1, there is an eigenvector* $\mathbf{e}_x$ *to* e^{-B_x} *and the largest eigenvalue* $\rho(e^{-B_x}) > 0$*, which has only strictly positive entries. It furthermore is the eigenvector to* $e^{-B_x s}$ *for all* $s \geq 0$ *to the largest eigenvalue* $\left(\rho(e^{-B_x})\right)^s > 0$*.*
5. *With assumption 1, let* $y \geq 0$ *be a* $(x-1)$*-dimensional vector with only non-negative entries, such that* $y(j) \leq M\mathbf{e}_x(j)$ *for some constant* $M > 0$ *and all* $j = 1, \ldots, x-1$*. Suppose that* $-\alpha^{\min} < r$*. Then*

$$0 \leq e^{-B_x s} y \leq M e^{-(r+\alpha^{\min})s} \mathbf{e}_x \to 0 \text{ as } s \to 0 \tag{66}$$

Proof. 1. Note[33] that $-B_x = -r\mathbf{I}_x + A_x$ only has non-negative entries off the diagonal. For sufficiently small $\epsilon > 0$, $e^{-B_x \epsilon} = \mathbf{I}_x - \epsilon B_x + o(\epsilon)$ has therefore only non-negative entries since the diagonal is dominated by $\mathbf{I}_x$ and the off-diagonal is dominated by

[32] Outside this lemma, ρ denotes the utility discount factor.

[33] The source for this part of the proof is an answer on math.stackexchange.com.

A_x. Pick such an ϵ. For arbitrary s, use $e^{-B_x s} = \left(e^{-B_x \epsilon}\right)^{s/\epsilon}$. The argument for C_x is exactly the same since $g \geq 0$. The argument that $e^{-B_x s}$ has only strictly positive entries under assumption 1 follows, since $e^{-B_x s} = \left(e^{-B_x \epsilon}\right)^n$ for $\epsilon = s/n$, where n is a sufficiently large natural number. It then also follows for $e^{-C_x s} = e^{-gs} e^{-B_x s}$.

2. Recall that $\sum_{x'<x} \alpha_{\tilde{x},x'} = -\sum_{x' \geq x} \alpha_{\tilde{x},x'}$. Thus, $\max_{\tilde{x}<x} \sum_{x'<x} A_x(\tilde{x}, x') = -\alpha^{\min}$ and likewise for the minimum. With that and for any $\epsilon \geq 0$, the row sums of $\mathbf{I}_x - \epsilon B_x$ are between $1 - \epsilon(r + \alpha^{\max}(x))$ and $1 - \epsilon(r + \alpha^{\min})$. Let $\Delta > 0$. Since $e^{-B_x \epsilon} = \mathbf{I}_x - \epsilon B_x + o(\epsilon)$, there is thus $\bar{\epsilon} > 0$, so that the sums of any row of $e^{-B_x \epsilon}$ are between $1 - (r + \alpha^{\max}(x) + \Delta)\epsilon$ and $1 - (r + \alpha^{\min} - \Delta)\epsilon$ for any $0 < \epsilon < \bar{\epsilon}$. Theorem 8.1.22 in Horn-Johnson (1985) implies that $1 - (r + \alpha^{\max}(x) + \Delta)\epsilon \leq \rho(e^{-B_x \epsilon}) \leq 1 - (r + \alpha^{\min} - \Delta)\epsilon$. Thus $(1 - (r + \alpha^{\max}(x) + \Delta)\epsilon)^{s/\epsilon} \leq \rho(e^{-B_x s}) \leq (1 - (r + \alpha^{\min} - \Delta)\epsilon)^{s/\epsilon}$. Letting $\epsilon \to 0$ delivers that $e^{-(r+\alpha^{\max}(x)+\Delta)s} \leq \rho(e^{-B_x s}) \leq e^{-(r+\alpha^{\min}-\Delta)s}$. Since $\Delta > 0$ can be arbitrarily small and since $\sum_{x' \geq x} \alpha_{\tilde{x},x'} \geq 0$ for $\tilde{x} < x$, the result about the spectral radius follows.

3. This follows from direct calculation for $B_x \mathbf{1}_x$ and then for $e^{-B_x s} \mathbf{1} x = \sum_{j=0}^{\infty} (-sB_x)^j \mathbf{1}_x / j!$.

4. Assumption 1 implies that that $e^{-B_x s}$ is irreducible. The existence of $\mathbf{e}_x$ is a consequence of the Perron-Frobenius theorem applied to e^{-B_x}. Let $n > 0$ and $m > 0$ be two natural numbers. Let $s = n/m$. Then

$$\left(e^{-B_x s}\right)^m \mathbf{e}_x = \left(e^{-B_x}\right)^n \mathbf{e}_x = \left(\rho(e^{-B_x})\right)^n \mathbf{e}_x$$

The result now follows from the fact that $e^{-B_x s}$ has only strictly positive entries, which rules out periodicity, i.e., $\mathbf{e}_x$ must be an eigenvector of $e^{-B_x s}$. By continuity, the result then holds not just for all rational but also for all real $s > 0$.

5. The first inequality follows from the first part of this lemma. For the second, use the first and the third part of the lemma and calculate

$$e^{-B_x s} y \leq M e^{-B_x s} v \leq M e^{-(r+\alpha^{\min})s} \mathbf{e}_x$$

The convergence to zero follows because $r + \alpha^{\min} > 0$ by assumption.

□

Proof of Proposition 2. Suppose we are in some state $\tilde{x}$ at t. Rewrite the budget constraint

(7) as

$$\dot{k}_t(\tilde{x}) - rk_t(\tilde{x}) + \sum_{x'} \alpha_{\tilde{x},x'} k_t(x') = w\mathbf{z}(x) - c_t(\tilde{x}) \tag{67}$$

where we now explicitly denote the current state $\tilde{x}$ as argument for k_t, $\dot{k}_t$ and c_t and where we have exploited that $\alpha_{\tilde{x},\tilde{x}} = -\sum_{x' \neq \tilde{x}} \alpha_{\tilde{x},x'}$, aside from moving terms from one side of the equation to the other. We proceed recursively. At state $x = 1$, $c = \mathbf{c}(1) = w\mathbf{z}(1)$ and the net costs are zero. Define $d_1 = k_1 = [\,]$ of dimension 0.

Consider now any state $x > 1$ and its associated consumption level $c = \mathbf{c}(x)$. Suppose that we start the consumption plan at this consumption level but for some other state $\tilde{x} < x$ at $t = 0$. Consumption will now drift down until either there is a transition to some $x' \geq x$ or until the consumption level $\mathbf{c}(x-1)$ is reached. Consumption will then continue to drift down if the current state is $x' < x - 1$: we take this into account when we aggregate costs. Let $T(x)$ be the time it takes for consumption to drift down from $\mathbf{c}(x)$ to $\mathbf{c}(x-1)$, i.e. $T(x)$ solves

$$\mathbf{c}(x-1) = e^{-gT(x)}\mathbf{c}(x)$$

Thus,

$$T(x) = \frac{\log(\mathbf{c}(x)) - \log(\mathbf{c}(x-1))}{g}$$

as in equation (13). At time $0 \leq t \leq T(x)$ and current state $\tilde{x} < x$, consumption will be

$$c_t = e^{-gt}\mathbf{c}(x), \tag{68}$$

provided no transition to some state $x' \geq x$ has yet occurred.

In (67), $k_t(x') = 0$ for all $x' \geq x$ and $t > 0$, since $c_t(x') \geq \mathbf{c}(x) > c_t$: the agent would therefore rather dis-save in order to smooth consumption, but he is prevented from doing so, due to our limited commitment assumption. Therefore, we only need to calculate the entries of the $(x-1)$-dimensional vector

$$k_{x,t} = [k_{x,t}(1), \ldots, k_{x,t}(x-1)], \tag{69}$$

where the second sub-index x indicates that we are at a consumption level c_t in the interval $c_t \in [\mathbf{c}(x-1), \mathbf{c}(x)]$. Therefore, rewrite the differential equation (67) in vector notation as

$$\dot{k}_{x,t} - B_x k_{x,t} = w\mathbf{z}_x - e^{-gt}\mathbf{c}(x)\mathbf{1}_x \tag{70}$$

with terminal condition[34]

$$k_{x,T(x)} = \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \tag{71}$$

since $k_{x-1,0} = k_{x-1}$ is needed to finance the consumption plan going forward for states $\tilde{x} < x-1$ and $c_t \leq \mathbf{c}(x-1)$. The solution is

$$k_{x,t} = d_{x,t} + e^{-B_x(T(x)-t)} \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \tag{72}$$

where the solution $d_{x,t}$ to the non-homogeneous part with terminal condition $d_{x,T(x)} = \mathbf{0}_{x-1}$ is given by[35]

$$d_{x,t} = e^{B_x t} \int_{s=t}^{T(x)} e^{-B_x s} \left(e^{-gs}\mathbf{c}(x)\mathbf{1}_x - wz_x \right) ds \tag{73}$$

$$= \mathbf{c}(x) C_x^{-1} e^{-gt} \left(\mathbf{I}_x - e^{-C_x(T(x)-t)} \right) \mathbf{1}_x - B_x^{-1} \left(\mathbf{I}_x - e^{-B_x(T(x)-t)} \right) wz_x \tag{74}$$

as one can verify directly or derive, using standard ODE calculus. The difference $d_x = d_{x,0}$ at $t = 0$ and given in equation (14) is now the $(x-1)$-dimensional vector of net costs for the piece of the consumption plan, starting at states $\tilde{x} \in \{1, \ldots, x-1\}$ and consumption level $\mathbf{c}(x)$ for the time between $t = 0$ and $t = T(x)$.

It follows from lemma 2 and equation (85) below that $k_{x,t} \geq 0$, thus satisfying the limited commitment constraint (5).

We finally need to solve for $\mathbf{c}(x)$. Observe that the budget constraint in state x and at $c = \mathbf{c}(x)$ needs to hold. It generally is given by (7). At $c = \mathbf{c}(x)$, $\dot{k}_t = 0$ and $k_t = 0$. Note that $k_{t,\tilde{x}} = 0$ for all $\tilde{x} > x$, since $\mathbf{c}(\tilde{x}) > \mathbf{c}(x)$. Note that $k_{t,\tilde{x}} = k_x(\tilde{x})$ for $\tilde{x} < x$, since $k_x(\tilde{x})$ is needed to finance the consumption plan going forward from state $\tilde{x}$ and starting consumption $\mathbf{c}(x)$. The budget constraint (7) then reads

$$0 = \mathbf{c}(x) - w\mathbf{z}(x) + \sum_{\tilde{x}<x} \alpha_{x,\tilde{x}} k_x(\tilde{x}) \tag{75}$$

[34] Thus, if $x = 2$, the terminal condition is $k_{T(2),2} = 0$.

[35] In principle, the net present value calculation of equation (73) can be done for arbitrary utility functions, except that one would then need to replace $e^{-gs}\mathbf{c}(x)$ by the appropriate path for consumption c_s at date s and starting at $\mathbf{c}(x)$, which solves the optimal consumption-savings problem at interest r. While it is unlikely that one then gets an explicit formula for the arrival time $T(x)$ of $c(s) = \mathbf{c}(x-1)$ or an explicit solution for the ODE as in the second line (74), one can still proceed to calculate these arrival times and integrals numerically. The rest of the analysis then continues to go through.

As in the proposition, let $\alpha_x = [\alpha_{x,1}, \ldots, \alpha_{x,x-1}]'$. Then, write equation (75) as equation (16). □

Note that $e^{-gs}\mathbf{c}(x) < wz(x-1)$ for $x \geq 3$ and s sufficiently close to $T(x)$, since $e^{-gT(x)}\mathbf{c}(x) = \mathbf{c}(x-1)$. Therefore, $d_{x,t}(x-1)$ in equation (32) is increasing from a negative value to zero rather than decreasing from a positive value as t approaches $T(x)$. Nonetheless, we have the following lemma. The statement may seem obvious. The proof, however, is far from it.

Lemma 2. *The solution $k_{x,t}$ to the vector ODE (70) together with (71) is strictly monotonically decreasing to $k_{x,T(x)} = \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix}$.*

Proof. Define

$$v_x = wz_x + B_x \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \tag{76}$$

Rewrite the solution for $k_{x,t}$ by combining (72) and (73) as

$$\begin{aligned} k_{x,t} &= \int_{s=t}^{T(x)} e^{-B_x(s-t)} \left(e^{-gs}\mathbf{c}(x)\mathbf{1}_x - wz_x\right) ds + e^{-B_x(T(x)-t)} \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} &(77) \\ &= \int_{s=t}^{T(x)} e^{-B_x(s-t)} \left(e^{-gs}\mathbf{c}(x)\mathbf{1}_x - v_x\right) ds + \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} &(78) \end{aligned}$$

with $k_x = k_{x,0}$. Since $e^{-gs}\mathbf{c}(x) > \mathbf{c}(x-1)$ for $s < T(x)$, it suffices to show that

$$v_x \leq \mathbf{c}(x-1)\mathbf{1}_x \tag{79}$$

We shall show this recursively. Note that this is trivially true for $x = 2$, since $v_2 = wz(1) = c(1)$. Suppose now that (79) is true up to some state x. We shall establish that

$$v_{x+1} \leq \mathbf{c}(x)\mathbf{1}_{x+1} \tag{80}$$

With the definition (76) applied to $x+1$, note that

$$v_{x+1} = wz_{x+1} + B_{x+1} \begin{bmatrix} k_x \\ 0 \end{bmatrix} \tag{81}$$

Consider first the last entry $v_{x+1}(x)$. With equation (16), this is

$$v_{x+1}(x) = wz(x) - \alpha_x k_x = c(x), \tag{82}$$

thus establishing (80) for that entry.

Next, note first that B_x is the top left $(x-1) \times (x-1)$ sub-matrix of B_{x+1}, i.e.

$$B_x = B_{x+1}(1 : x-1, 1 : x-1) \tag{83}$$

Thus, the vector of the other entries $v_{x+1}(1 : x-1)$ can be written as

$$v_{x+1}(1 : x-1) = wz_x + B_x k_x \tag{84}$$

Replace k_x with (78) for $t = 0$ and use $e^{-gT(x)}\mathbf{c}(x) = \mathbf{c}(x-1)$ to see that

$$\begin{aligned} v_{x+1}(1 : x-1) &= wz_x + B_x \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \\ &\quad + B_x \int_{s=0}^{T(x)} e^{-B_x s} \left(e^{-gs}\mathbf{c}(x)\mathbf{1}_x - v_x \right) ds \\ &= e^{-B_x T(x)} v_x + C_x \int_{s=0}^{T(x)} e^{-C_x s} ds \, \mathbf{c}(x)\mathbf{1}_x \\ &\quad - g \int_{s=0}^{T(x)} e^{-C_x s} ds \, \mathbf{c}(x)\mathbf{1}_x \\ &= \mathbf{c}(x)\mathbf{1}_x - e^{-B_x T(x)} \left(\mathbf{c}(x-1)\mathbf{1}_x - v_x \right) \\ &\quad - g \int_{s=0}^{T(x)} e^{-C_x s} ds \, \mathbf{c}(x)\mathbf{1}_x \\ &\leq \mathbf{c}(x)\mathbf{1}_x \end{aligned}$$

where the last inequality follows per the induction hypothesis (79) and because $e^{-B_x T(x)} \geq 0$ and $\int_{s=t}^{T(x)} e^{-C_x s} ds \geq 0$ per part 1 of lemma 1. □

The lemma immediately implies that the solution stated in proposition 2 satisfies

$$k_x \geq \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \geq \mathbf{0}_x \tag{85}$$

and therefore, indeed satisfies the limited commitment requirement (4). The lemma is thus needed to complete the proof of proposition 2. Proposition 2 provides a system of equations that the solution must satisfy. The system of equations has a recursive structure. Given the solution up to $x-1$, one may then seek to calculate the solution for x. Given $\mathbf{c}(x)$, the values for $T(x)$, d_x and k_x can be calculated, but there could potentially be many values for $\mathbf{c}(x)$ for which (16) is then also satisfied. The next proposition shows that this cannot be the case.

Proof of Proposition 3. The solution is unique for $x = 1$. Exploiting the block recursive structure, suppose uniqueness has been shown for $x-1$. We seek to show that there is a unique solution $\mathbf{c}(\mathbf{x})$. Suppose by contradiction that there are two solutions $\mathbf{c}^a(x) > \mathbf{c}^b(x)$. Calculate the corresponding times $T^a(x)$ and $T^b(x)$ per (13). Note that $T^a(x) > T^b(x)$. Define $t = T^a(x) - T^b(x)$ and note that

$$\mathbf{c}^b(x) = e^{-gt}\mathbf{c}^a(x) \tag{86}$$

Next, calculate k_x^a and k_x^b, using (78). We have

$$k_x^a = \int_{s=0}^{T^a(x)} e^{-B_x s}\left(e^{-gs}\mathbf{c}^a(x)\mathbf{1}_x - v_x\right) ds + \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \tag{87}$$

and, with (86),

$$\begin{aligned} k_x^b &= \int_{s=0}^{T^b(x)} e^{-B_x s}\left(e^{-gs}\mathbf{c}^b(x)\mathbf{1}_x - v_x\right) ds + \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \\ &= \int_{s=t}^{T^a(x)} e^{-B_x (s-t)}\left(e^{-gs}\mathbf{c}^a(x)\mathbf{1}_x - v_x\right) ds + \begin{bmatrix} k_{x-1} \\ 0 \end{bmatrix} \\ &= k_{x,t}^a \end{aligned}$$

Lemma 2 implies that $k_x^b < k_x^a$. Equation(16) now implies that

$$\mathbf{c}^a(x) = w\mathbf{z}(x) - \alpha_x k_x^a < w\mathbf{z}(x) - \alpha_x k_x^b = \mathbf{c}^b(x),$$

which is a contradiction. □

Solving the system of equations (13) to (16) requires numerical techniques[36]. Generally, the ordering of the states x such that $\mathbf{c}(x)$ is increasing in x will not be known a priori. The

[36]No numerical techniques are required if $x = 2$ and $z_1 = 0$. In that case, $\mathbf{c}(x-1) = 0$, $T(x) = \infty$,

block recursive structure of equations (13) to (16) in proposition 2 suggest the following **algorithm**. Pick as $x = 1$ the state which generates the lowest flow income $w\mathbf{z}(x)$. Then, recursively at each stage $j = 2, \ldots, N$, pick each of the remaining states x. For x, calculate the candidate $\mathbf{c}(x)$ per solving the system of equations (14) to (16). Among all x, pick $x = j$ to be that state, which produces the lowest candidate $\mathbf{c}(x)$ and remove it from the pool of remaining states.

B Characterizing the consumption distribution: lemmas, propositions and proofs.

Proposition 12. *A stationary distribution $\mathcal{D}$ solves the following system of equations*

$$-\alpha_{x,x}\mu_x = f_{x+1,x}(T_{x+1}) + \sum_{\tilde{x}<x} \alpha_{\tilde{x},x} \left(\mu_{\tilde{x}} + \sum_{x':\tilde{x}<x'\leq x} \int_{t=0}^{T_{x'}} f_{x',\tilde{x}}(t)dt \right) \tag{88}$$

$$0 < t \leq T(x), \tilde{x} < x: \quad \dot{f}_{x,\tilde{x}}(t) = \sum_{x'<x} \alpha_{x',\tilde{x}} f_{x,x'}(t) \tag{89}$$

$$t = 0, \tilde{x} < x: \quad f_{x,\tilde{x}}(0) = \alpha_{x,\tilde{x}}\mu_x + f_{x+1,\tilde{x}}(T_{x+1}) \tag{90}$$

$$\textit{if } \tilde{x} \geq x \textit{ or } t > T(x) \quad f_{x,\tilde{x}}(t) = 0. \tag{91}$$

This follows from straightforward accounting of the various flows. We note that the system of ODE's in (89) can be stated more compactly as (24).

$B_x = r - \alpha_{1,1}$, $C_x = r + g - \alpha_{1,1}$, $\alpha_x = \alpha_{2,1}$, $1_x = 1$, $z_x = [0]$, $k_x = d_x$. Now (16) reads as

$$\mathbf{c}(x) = w\mathbf{z}(2) - \alpha_{2,1}\mathbf{c}(x)\frac{1}{r + g - \alpha_{1,1}}$$

which can be easily solved for $\mathbf{c}(x)$,

$$\mathbf{c}(x) = \frac{r + g - \alpha_{1,1}}{r + g - \alpha_{1,1} + \alpha_{2,1}} w\mathbf{z}(2)$$

For example, when $N = 2$ and $z(1) = 0$, and with $\zeta = z(2) = \zeta$, $\nu = \alpha_{1,2} = -\alpha_{1,1}$, $\xi = z_{2,1}$ as well as $\sigma = 1$ and thus $g = \rho - r$, we have $\mathbf{c}(1) = 0$ and

$$\mathbf{c}(2) = \frac{\rho + \nu}{\rho + \nu + \xi} w\zeta$$

Lemma 3. *Let $\bar{\mu}_x$ denote the unconditional probability of being in state x. Let $\bar{\mu} = [\bar{\mu}_1, \ldots, \bar{\mu}_N]'$.*

1. *The unconditional probabilities solve*

$$0 = A'\bar{\mu} \text{ and } \sum_x \bar{\mu}_x = 1 \tag{92}$$

2. *A distribution $\mathcal{D}$ is a stationary distribution if and only if it satisfies proposition 12 and whose unconditional probabilities $\bar{\mu}_x$ of being in state x,*

$$\bar{\mu}_x = \mu_x + \sum_{x'>x} \int_{t=0}^{T_{x'}} f_{x',x}(t)dt \tag{93}$$

   sum to unity. The unconditional probabilities then satisfy (92).

3. *Given equation (93), equation (88) is equivalent to*

$$-\alpha_{x,x}\mu_x = f_{x+1,x}(T_{x+1}) + \sum_{\tilde{x}<x} \alpha_{\tilde{x},x} \left(\bar{\mu}_{\tilde{x}} - \sum_{x'>x} \int_{t=0}^{T_{x'}} f_{x',\tilde{x}}(t)dt \right) \tag{94}$$

Proof. Equation (92) is the usual property of stationary distributions for continuous-time finite-state Markov processes. Equation (93) is accounting for all the possibilities. It conversely implies that the marginal unconditional probabilities $\bar{\mu}_x$ calculated from a stationary distribution $\mathcal{D}$ satisfy (92): beyond that restriction and proposition 12 there is nothing else to satisfy. Finally, rewrite equation (93) for $\tilde{x}$ rather than x. For any $x > \tilde{x}$, this equation then implies

$$\mu_{\tilde{x}} + \sum_{x':\tilde{x}<x'\leq x} \int_{t=0}^{T_{x'}} f_{x',\tilde{x}}(t)dt = \bar{\mu}_{\tilde{x}} - \sum_{x'>x} \int_{t=0}^{T_{x'}} f_{x',\tilde{x}}(t)dt \tag{95}$$

Plugging this into equation (88) delivers (94) and vice versa. □

Proof of Proposition 4. 1. Note that $\bar{\mu}_x \geq \mu_x$ per (93), since $f_{x',x}(t) \geq 0$. Thus, if $\bar{\mu}_x = 0$, then $\mu_x = 0$, since $\mu_x \geq 0$. Since consumption is only drifting down, it follows[37] that $f_{x,x'}(t) = 0$ for all t, all $x > \bar{x}$ and all $x' < x$.

[37] A more formal argument can be made by first establishing step 3 of the corollary.

2. Note that $f_{x',\bar{x}}(t) = 0$ for all $x' > \bar{x}$. Equation (94) at $x = \bar{x}$ then reduces to

$$0 = \alpha_{\bar{x},\bar{x}}\mu_{\bar{x}} + \sum_{\tilde{x}<\bar{x}} \alpha_{\tilde{x},\bar{x}}\bar{\mu}_{\tilde{x}} \tag{96}$$

Compare this to the equation for the unconditional probability $\bar{\mu}_{\bar{x}}$,

$$0 = \alpha_{\bar{x},\bar{x}}\bar{\mu}_{\bar{x}} + \sum_{\tilde{x}\neq\bar{x}} \alpha_{\tilde{x},\bar{x}}\bar{\mu}_{\tilde{x}} \tag{97}$$

and recall that $\alpha_{\bar{x},\bar{x}} \neq 0$ as well as $\bar{\mu}_{\tilde{x}} = 0$ for $\tilde{x} > \bar{x}$. The equation $\mu_{\bar{x}} = \bar{\mu}_{\bar{x}}$ now follows.

3. (a) Note that $f_x(0)$ can be calculated via (90), since all other terms are known per by recursivity. The result is unique.

   (b) (27) is the unique solution to (24) or, equivalently (24), given the initial condition $f_x(0)$.

   (c) Equation (28) is equation (94) stated for $x - 1$ rather than x. Note that all other terms are known by recursivity and recall that $\alpha_{x-1,x-1} < 0$ by assumption.

The resulting $\mathcal{D}$ satisfies proposition 12 as well as lemma 3 and thus is a stationary distribution satisfying (93) by construction. The calculation for $\mathcal{D}$ is unique. Thus, this is the unique stationary distribution satisfying (93) by the third part of lemma 3.

To establish (29), define

$$g_x(s) = \int_{t=0}^{s} f_x(t)dt \tag{98}$$

We seek to calculate $g_x(T(x))$. Per (24), $\dot{f}_x(t) = A'_x f_x(t)$. Thus,

$$\begin{aligned} A'_x g_x(s) &= \int_{t=0}^{s} A'_x f_x(t)dt \\ &= \int_{t=0}^{s} \dot{f}_x(t)dt \\ &= f_x(s) - f_x(0) \\ &= (\exp(A'_x s) - \mathbf{I}_x) f_x(0) \end{aligned}$$

where the last equality follows with (27). The result now obtains for $s = T(x)$. □

Proof of Proposition 5. The corollary follows from proper accounting and the consumption dynamics in proposition 2. It is clear that the mass points are as stated. For the density, calculate instead the cdf Φ first. It is given by

$$\Phi(c) = \Phi(\mathbf{c}(x-1)) + \sum_{x'<x} \int_0^{t(c)} f_{x,x'}(t)dt \tag{99}$$

The expression for the density in (30) follows directly by taking the derivative and the dependence of the upper bound of the integral on $t(c)$. With equation (30), we seek to explicitly calculate aggregate consumption

$$C_r = \int_{\mathbf{c}(1)}^{\mathbf{c}(N)} c\phi_r(c)dc \tag{100}$$

We follow a similar strategy as the proof for (29). Note that the integral expressions in (100) can be rewritten as

$$\int_{\mathbf{c}(x-1)}^{\mathbf{c}(x)} \frac{f_x(t(c))}{g} dc = \mathbf{c}(x)h_x(T(x)) \tag{101}$$

where

$$h_x(s) = \int_0^s e^{-gt} f_x(t)dt \tag{102}$$

using the transformation of variable from c to $t(c)$[38]. Recall that $\dot{f}_x(t) = A'_x f_x(t)$ per equation (24). Thus, using integration by parts as well as the explicit solution (27) to (24),

$$\begin{aligned} A'_x h_x(s) &= \int_0^s e^{-gt} A'_x f_x(t)dt \\ &= \int_0^s e^{-gt} \dot{f}_x(t)dt \\ &= e^{-gt} f_x(t) \mid_0^s + g\int_0^s e^{-gt} f_x(t)dt \\ &= \left(e^{-gs}\exp(A'_x s) - \mathbf{I}_x\right) f_x(0) + gh_x(s) \end{aligned}$$

or

$$D_x h_x(s) = (\mathbf{I}_x - \exp(-D_x s)) f_x(0) \tag{103}$$

For $s = T(x)$ and with (100), we obtain (31). □

[38]Thus, $dt = dc/(cg)$ or $\mathbf{c}(x)e^{-gt}dt = dc/g$.

C Aggregate Capital Demand

C.1 General Production Function

Proposition 13. *Suppose that F_K is strictly convex. Then normalized capital demand $\kappa^d(r)$ is strictly decreasing in r.*

Proof. Define $f(K) = F(K, 1)$ (within this proof). Due to constant returns to scale, $F(K, L) = f(K/L)L$. Equation (39) can be rewritten as $w(K) = f(K) - f'(K)K$. Due to the strict concavity of F, capital demand $K(r)$ characterized by (38) is strictly decreasing in r. Therefore κ^d is strictly decreasing if

$$g(K) = \frac{w(K)}{K} = \frac{f(K)}{K} - f'(K) \tag{104}$$

is strictly decreasing in K, since $g(K(r)) = 1/\kappa^d(r)$.

With $f(0) = 0$ and by the mean value theorem, there is some $0 < \tilde{K} < K$ so that $f(K) = Kf'(\tilde{K})$. Applying the mean value theorem to f', there is some $\hat{K}$ with $\tilde{K} < \hat{K} < K$ so that $f'(K) - f'(\tilde{K}) = (K - \tilde{K})f''(\hat{K})$. Since F_K is strictly convex, so is f', i.e., f'' is strictly increasing. Thus, $f''(\hat{K}) < f''(K)$. Combining,

$$\begin{aligned}
g'(K) &= -\frac{f(K)}{K^2} + \frac{f'(K)}{K} - f''(K) \\
&= \frac{f'(K) - f'(\tilde{K})}{K} - f''(K) \\
&= f''(\hat{K}) - f''(K) < 0
\end{aligned}$$

□

C.2 CES Production Function

The following proposition completely characterizes $\kappa^d(r)$ for a general CES production function.

Proposition 14. *Suppose that F is of the CES variety,*

$$F(K, L) = \left(\theta K^{1-\frac{1}{\eta}} + (1-\theta)L^{1-\frac{1}{\eta}}\right)^{\frac{\eta}{\eta-1}} = (\theta K^\nu + (1-\theta)L^\nu)^{\frac{1}{\nu}} \tag{105}$$

where the elasticity of substitution η satisfies $0 < \eta < \infty$ and thus $\nu \in (-\infty, 1)$.[39] *Define*

$$\breve{r} = \begin{cases} \theta^{\frac{\eta}{\eta-1}} - \delta, & \text{if} \quad \eta \neq 1 \\ -\delta, & \text{if} \quad \eta = 1 \end{cases} \tag{106}$$

Note that $\breve{r} \geq -\delta$.

1. *Capital demand $K^d(r)$ satisfying $F_K(K^d(r), 1) - \delta = r$ (and thus normalized capital demand $\kappa^d(r) = K^d(r)/w(r)$) is well-defined for the range of interest rates r:*
    (a) *For $\eta \in [1, \infty)$ the interval is given by $r \in (\breve{r}, \infty)$.*
    (b) *For $\eta \in (0, 1)$, the interval is given by $r \in (-\delta, \breve{r})$.*
2. *On the range for which $K^d(r)$ is defined, normalized capital demand is given by*

$$\kappa^d(r) = \frac{\theta}{(r+\delta)\left[\left(\frac{r+\delta}{\theta}\right)^{\eta-1} - \theta\right]} \tag{107}$$

3. *For $\eta \in [1, \infty)$, normalized capital demand is strictly decreasing in r , with $\lim_{r \to \breve{r}} \kappa^d(r) = \infty$ and $\lim_{r \to \infty} \kappa^d(r) = 0$.*
4. *For $\eta \in (0, 1)$, $\kappa^d(r)$ is downward sloping on $(-\delta, \eta^{\frac{1}{1-\eta}} \theta^{\frac{\eta}{\eta-1}} - \delta]$ and upward-sloping on $[\eta^{\frac{1}{1-\eta}} \theta^{\frac{\eta}{\eta-1}} - \delta, \breve{r})$. Since $\eta \in (0, 1)$ we have $\eta^{\frac{1}{1-\eta}} \in (0, 1)$ and thus both subintervals are nonempty. Furthermore, $\lim_{r \to -\delta} \kappa^d(r) = \infty$ and $\lim_{r \to \breve{r}} \kappa^d(r) = \infty$.*
5. *For $\eta = 0$ (Leontieff production), $\kappa^d(r)$ is strictly increasing on its entire domain $r \in (-\delta, 1 - \delta)$, with $\lim_{r \to -\delta} \kappa^d(r) = 1$ and $\lim_{r \to 1-\delta} \kappa^d(r) = \infty$.*

Proof. For ease of notation define $\nu = 1 - \frac{1}{\eta} \in (-\infty, 1)$. Thus the production function is given by

$$F(K, L) = (\theta K^\nu + (1-\theta) L^\nu)^{\frac{1}{\nu}}$$

and the marginal products (in equilibrium equal to factor prices) are given by

$$F_K(K, 1) = \theta\left(\theta + (1-\theta) K^{-\nu}\right)^{\frac{1-\nu}{\nu}} = r + \delta \tag{108}$$

$$F_L(K, 1) = (1-\theta)\left(\theta K^\nu + (1-\theta)\right)^{\frac{1-\nu}{\nu}} = w \tag{109}$$

[39] For $\eta = 1$, this is the Cobb-Douglas production function $F(K, L) = K^\theta L^{1-\theta}$.

1. For the first part, we note that $K^d(r) = K$ is defined through the equation (108). First consider $\eta > 1$ and thus $\nu \in (0, 1)$. In that case $F_K(K, 1)$ is strictly decreasing and

$$\begin{aligned} \lim_{K \to 0} F_K(K,1) &= \infty \\ \lim_{K \to \infty} F_K(K,1) &= \theta^{\frac{1}{\nu}} \end{aligned}$$

Therefore, equation (108) has a solution if and only if

$$\theta^{\frac{1}{\nu}} < r + \delta$$

The unique solution $K^d(r)$ is thus well-defined on the interval $r \in (\breve{r}, \infty)$.

Now consider $0 < \eta < 1$ and thus $\nu \in (-\infty, 0)$. Then $F_K(K, 1)$ is still strictly decreasing, with

$$\begin{aligned} \lim_{K \to 0} F_K(K,1) &= \theta^{\frac{1}{\nu}} < \infty \\ \lim_{K \to \infty} F_K(K,1) &= 0 \end{aligned}$$

and equation (108) has a unique solution if and only if

$$\theta^{\frac{1}{\nu}} > r + \delta$$

Thus, for $\nu \in (-\infty, 0)$, we have that $K^d(r)$ is well-defined on $r \in (-\delta, \breve{r})$, where $\breve{r} = \theta^{\frac{1}{\nu}} - \delta > -\delta$.

Finally, for the Cobb-Douglas case $\eta = 1$ or $\nu = 0$, we have

$$F_K(K, 1) = \theta K^{\theta - 1}$$

with

$$\begin{aligned} \lim_{K \to 0} F_K(K,1) &= \infty \\ \lim_{K \to \infty} F_K(K,1) &= 0 \end{aligned}$$

Thus $K^d(r)$ is well-defined on all of $r \in (-\delta, \infty)$.

2. Now we derive $\kappa^d(r) = \frac{K^d(r)}{w(r)}$ on the interval of interest rates for which $K^d(r)$ is defined. From equations (108) and (109), we note that

$$\frac{r+\delta}{w} = \frac{F_K(K,1)}{F_L(K,1)} = \frac{\theta K^{\nu-1}}{(1-\theta)} = \frac{\theta}{1-\theta} w^{\nu-1}\kappa^{\nu-1}$$

and thus

$$\kappa = \left[\frac{\frac{\theta}{r+\delta} w^\nu}{1-\theta}\right]^{\frac{1}{1-\nu}} \tag{110}$$

We can express w^ν in terms of r from equation (109) as

$$w^\nu = (1-\theta)^\nu \left(\theta K^\nu + (1-\theta)\right)^{1-\nu} \tag{111}$$

Rewrite (108) as

$$K^\nu = \frac{(1-\theta)}{\left[\frac{r+\delta}{\theta}\right]^{\frac{\nu}{1-\nu}} - \theta}$$

Use it to substitute K^ν in equation (111) to obtain

$$\begin{aligned} w^\nu &= (1-\theta)\left(\frac{\theta}{\left[\frac{r+\delta}{\theta}\right]^{\frac{\nu}{1-\nu}} - \theta} + 1\right)^{1-\nu} \\ &= \frac{(1-\theta)\left[\frac{\theta}{r+\delta}\right]^{-\nu}}{\left(\left[\frac{r+\delta}{\theta}\right]^{\frac{\nu}{1-\nu}} - \theta\right)^{1-\nu}} \end{aligned}$$

Inserting w^ν back into equation (110) and exploiting the relationship $\frac{\nu}{1-\nu} = \eta - 1$ gives the expression (107) for $\kappa^d(r)$ in the proposition.

3. For the case $\eta \geq 1$ we have $\eta - 1 \geq 0$ and the properties of $\kappa^d(r)$ stated in the proposition follow from direct inspection of equation (107).

4. Suppose that $\eta \in (0,1)$ or, equivalently, $\nu < 0$. Inspecting equation (107) and noting that $\left(\frac{\breve{r}+\delta}{\theta}\right)^{\eta-1} = \theta$ yields

$$\lim_{r\to-\delta} \kappa^d(r) = \lim_{r\to\breve{r}} \kappa^d(r) = \infty$$

since $\kappa^d(r)$ is finite on $(-\delta, \breve{r})$, it follows that $\kappa^d(r)$ is non-monotone on its domain.

$\kappa^d(r)$ is decreasing, if and only if $1/\kappa^d(r)$ is increasing. The derivative $d(r)$ of

$$1/\kappa^d(r) = \left(\frac{r+\delta}{\theta}\right)^{\eta} - r - \delta \tag{112}$$

is

$$d(r) = \frac{\eta}{\theta}\left(\frac{r+\delta}{\theta}\right)^{\eta-1} - 1 \tag{113}$$

and is decreasing in r. Thus, $d(r) > 0$ and $\kappa^d(r)$ is decreasing, if and only if $r < \hat{r}$, where $\hat{r}$ solves $d(\hat{r}) = 0$, i.e.,

$$\hat{r} = \theta\left(\frac{\theta}{\eta}\right)^{\frac{1}{\eta-1}} - \delta \tag{114}$$

This delivers the downward-sloping and upward-sloping segmentation of $(-\delta, \breve{r})$, as stated in the proposition. Since $\eta \in (0,1)$ we have $\eta^{\frac{1}{1-\eta}} \in (0,1)$ and thus both intervals above are nonempty.

5. For the Leontieff case, $\eta = 0$, and thus

$$\kappa^d(r) = \frac{\theta}{(r+\delta)\left[\left(\frac{r+\delta}{\theta}\right)^{-1} - \theta\right]} = \frac{1}{1-(r+\delta)}$$

and the stated properties in the proposition directly follow.

□

D Aggregate Capital Supply

D.1 General Theoretical Properties

In this subsection, we state and prove Lemma, providing the general characterization of aggregate consumption as a function of the interest rate. We now note explicitly the dependence of the wage w on r. The following lemma is needed in preparation.

Lemma 4. *Let $-\alpha^{\min} < r < \rho$. Then*

$$k_{x,t}(x') \leq \bar{\kappa}^s w \tag{115}$$

where

$$\bar{\kappa}^s = \frac{\sigma \mathbf{z}(N)}{\rho - r} < \infty \tag{116}$$

Proof. Since $r < \rho$, the agents with $x = N$ and the highest income do not hold any capital for financing their own consumption and only set capital aside for insurance purposes in case of dropping to a lower state. Lemma 2 together with (85) guarantee that $k_{x,t}(x') \leq k_N(x')$. In order to find a bound for these values, consider instead a two-state process, where the agent oscillates between income $\mathbf{z}(N)w$ and zero and where the transition from zero back to $\mathbf{z}(N)w$ happens at rate $\nu = \alpha^{\min}$. Suppose that the consumption in the high-income state takes the same value $\mathbf{c}(N)$ as before. For that two-state process and as in equation (18) of section 3.3, the insurance capital needed to be set aside in the high-income state is

$$\tilde{k} = \int_{s=0}^{\infty} e^{-(r+g+\nu)s} ds \, \mathbf{c}(N) = \frac{\mathbf{c}(N)}{r + g + \nu} \tag{117}$$

where $g = g(r) = (\rho - r)/\sigma$. Since the agent in the original specification transits back to state N at least at rate $\alpha^{\min}$ and makes income no less than zero, regardless of the state, it follows that the agent needs to set aside less insurance capital in the original specification in state N and for any x' than in the high-income state in this "worst case scenario" two-state comparison, i.e., $k_N(x') \leq \tilde{k}$ for all x'. Since $r > -\alpha^{\min}$ and since $\mathbf{c}(N) \leq \mathbf{z}(N)$ due to $r < \rho$, the bound follows. □

Lemma 5. *1. C(r) is continuously differentiable in $r \in (-\alpha^{\min}, \rho)$.*

2. *$C(r) - w(r)$ has the same sign as r. In particular, $C(0) = w(0)$.*

3. *$-C(r)/w(r)$ converges to a strictly positive and finite value, as $r \to -\alpha^{\min}$.*

4. *$\kappa^s(r) = (C(r)/w(r) - 1)/r$ satisfies $0 \leq \kappa^s(r) \leq \bar{\kappa}^s$, where $\bar{\kappa}^s$ is given in (116).*

Proof of Lemma 5. 1. The fact that $C(r)$ is continuously differentiable follows from the implicit function theorem since all equations in propositions 2, 4 and 5 are differentiable in r as well as in the endogenous objects to be calculated and since proposition 3 and its proof guarantee the invertibility of the relevant Jacobian in the endogenous objects.

2. We have characterized the stationary distribution in terms of (x, t) in (23), where x characterizes the current consumption interval $c_t \in (\mathbf{c}(x-1), \mathbf{c}(x)]$ and t denotes the time drifting down from $\mathbf{c}(x)$, rather than the current state $\tilde{x}$ and current capital

holdings $k = k_{x,t}(\tilde{x})$. These imply the decision rules decision rules for consumption $c(\tilde{x}, k) = e^{-gt}\mathbf{c}(x)$, capital depletion $\dot{k}(\tilde{x}, k) = \dot{k}_{x,t}(\tilde{x})$ and insurance $k(x'; \tilde{x}, k) = k_{x,t}(x')$. The budget constraint (3) in terms of the decision rules in the original state space

$$c(\tilde{x}, k) + \dot{k}(\tilde{x}, k) + \sum_{x' \neq \tilde{x}} \alpha_{\tilde{x},x'}(k(x'; \tilde{x}, k) - k) = rk + w\mathbf{z}(\tilde{x}) \tag{118}$$

can therefore be rewritten as

$$e^{-gt}\mathbf{c}(x) + \dot{k}_{x,t}(\tilde{x}) + \sum_{x' \neq \tilde{x}} \alpha_{\tilde{x},x'}(k_{x,t}(x') - k) = rk_{x,t}(\tilde{x}) + w\mathbf{z}(\tilde{x}) \tag{119}$$

in terms of (x, t) as in (23) as well as the current state $\tilde{x}$. Integrate this budget constraint with the stationary distribution (23) across all $x, t, \tilde{x}$. Due to stationarity, the integrals over capital depletion terms plus insurance terms must be zero, as there cannot be capital depletion or insurance in the aggregate, i.e., these terms reflect cross-population redistributions. Note that $C(r)$ is the integral over the consumption terms $e^{-gt}\mathbf{c}(x)$. Let $K^s(r)$ denote the integral over the capital holdings $k_{x,t}(\tilde{x})$.[40] Since average labor productivity is normalized to be 1, it follows that

$$C(r) = rK^s(r) + w \tag{120}$$

By lemma 4, $k_{x,t}(\tilde{x}) \leq \bar{k}$, where $\bar{k} < \infty$ is defined in equation (116). Since $0 < k_{x,t}(\tilde{x})$ except on a null set, it follows that

$$0 < K^s(r) \leq \bar{\kappa}^s w \tag{121}$$

for all $r \in (-\alpha^{\min}, \rho)$. Equation (120) now implies the claim.

3. By the first part of the lemma, $C(r)$ and, analogously, $K^s(r)$ are differentiable functions of $r \in (-\alpha^{\min}, \rho)$. Note that the solutions for consumption and capital in propositions 2, 4, and 5 are homogeneous of degree 1 in w. Therefore, $C(r)/w(r)$ is differentiable in $r \in (-\alpha^{\min}, \rho)$, establish the claim of a finite limit. Equation (120) and the bound (121) together with the degree-1 homogeneity of $K^s(r)$ in s imply that

$$0 < -C(r)/w(r) < \delta\bar{\kappa}^s, \tag{122}$$

[40]The superscript "s" denotes that this will be capital supply; see equation (42).

4. The last part now follows immediately, completing the proof.

□

E Equilibrium and Proofs of Propositions 9 and 10

Proof of Proposition 9: The first step of the proof is to establish that the normalized capital supply function is well-defined and continuous on $r \in (-\delta, \rho)$. Recall that normalized capital supply is $\kappa^s(r) = \xi/((g+\nu)(r+g+\nu+\xi))$ with $g = (\rho - r)/\sigma$, see equation (48). It is evidently continuous and well-defined on $(-\delta, \rho)$ as long as both terms of the denominator are strictly positive. Since $r < \rho$, and thus $g > 0$, the first term in the denominator of κ^s is always strictly positive. The second term is always positive, since $g > 0$ and $r + \nu + \xi > \delta + \nu + \xi > -\min \nu, \xi + \nu + \xi > 0$ per assumption 3.

By Assumption 3, we have $\kappa^d(r = \rho) < \kappa^s(r = \rho)$. Since $\kappa^s(r = -\delta) < \infty = \kappa^d(r = -\delta)$, it follows that κ^s and κ^d intersect at least once in $(-\delta, \rho)$. This establishes the existence of a stationary equilibrium.

The uniqueness of equilibrium follows if $\kappa^s(r)$ is increasing (given that $\kappa^d(r)$ is strictly decreasing). The derivative of $\kappa^s(r)$ is given by

$$\frac{d\kappa^s(r)}{dr} = \xi \frac{\left[\frac{2}{\sigma} - 1\right]\left[\frac{\rho-r}{\sigma} + \nu\right] + \frac{\xi+r}{\sigma}}{\left[\left(\xi + \nu + \frac{\rho-r}{\sigma} + r\right)\left(\nu + \frac{\rho-r}{\sigma}\right)\right]^2}$$

A sufficient condition for this expression to be positive is $\sigma < 1$ (part 1 of the proposition) or $\sigma \in (1, 2]$ and $\xi \geq \delta$ (part 2a of the proposition). Part 2b follows from the fact that equation (48) is a quadratic equation and thus has at most two solutions (and we have already established that under the assumptions made, it has at least one solution). The numerical example in the main text shows that the statement in 2b of the proposition is not vacuous. □

Proof of Proposition 10: The proof consists of two parts. For the first, we use proposition 2 to calculate the capital vector k_N, when $r = \rho$. That proposition calculates $\mathbf{c}(N)$, when agents start with zero capital. We then show that the capital vector of an agent has to be at least as high as k_N and thus non-negative if the agent consumes at least $\mathbf{c}(N)$. For the second, we use proposition 4 to calculate the stationary distribution when $r \to \rho$ and agents in state N do not have capital. This delivers the limit capital supply $\lim_{r \to \rho} \kappa^s(r)$. We then argue that $\lim_{r \to \rho} \kappa^s(r) \leq \kappa^d(r)$ implies that all agents consume at least $\mathbf{c}(N)$.

1. Consider the results in proposition 2 for $r = \rho$. In that case, $g = 0$ and $e^{-C_x T(x)} = e^{-B_x T(x)} = 0_{x-1,x-1}$. The equations for $\mathbf{c}(N)$ and k_N read

$$k_N = d_N \;=\; \mathbf{c}(N) C_N^{-1} \mathbf{1}_N - B_N^{-1} w z_N \tag{123}$$

$$\mathbf{c}(N) \;=\; w\mathbf{z}(N) - \alpha_N k_N \tag{124}$$

Pre-multiplying equation (123) with $B_N = C_N = \rho I_N - A_N$, these equations can be written as

$$\left[\begin{array}{c|c} \rho I_N - A_N & * \\ -\alpha_N & * \end{array}\right] \begin{bmatrix} k_N \\ 0 \end{bmatrix} = \mathbf{c}(N)\mathbf{1}_{N+1} - w \begin{bmatrix} z_N \\ \mathbf{z}(N) \end{bmatrix} \tag{125}$$

where $\mathbf{1}_{N+1}$ denotes a vector of ones of length N. where "$*$" denotes that the coefficients in that last column are arbitrary, as they multiply zero. Recall that $\mathbf{c}(N)$ was defined as that level of consumption in state N, if $k = 0$. We might as well write (125) as

$$(\rho I_{N+1} - A) \begin{bmatrix} k_N \\ 0 \end{bmatrix} = \mathbf{c}(N)\mathbf{1}_{N+1} - w\mathbf{z} \tag{126}$$

where I_{N+1} is the identity matrix of size $N \times N$. Now note that

$$(A - \rho I_{N+1})\, \mathbf{1}_{N+1} = -\rho \mathbf{1}_{N+1} \tag{127}$$

Thus, for $c \geq \mathbf{c}(N)$, the vector

$$\mathbf{k} = \begin{bmatrix} k_N \\ 0 \end{bmatrix} + \frac{c - \mathbf{c}(N)}{\rho} \mathbf{1}_{N+1} \tag{128}$$

is non-negative and is the solution to (53).

2. The main purpose of this part is to establish that

$$\lim_{r \to \rho} \kappa^s(r) w(r) = \lim_{r \to \rho} K^s(r) = \begin{bmatrix} k_N, & 0 \end{bmatrix} \bar{\mu}, \tag{129}$$

which may seem rather obvious. The formal argument relies on the definition of aggregate capital supply via the calculation of the stationary distribution in proposition 4, which we shall now provide. That proposition assumes that agents in state N

do not hold capital. For $r \to \rho$, the proposition delivers $\mu_N = \bar{\mu}_N$ and

$$f_N(t) = \exp(A'_N t) \begin{bmatrix} \alpha_{N,1} \\ \vdots \\ \alpha_{N,N-1} \end{bmatrix} \bar{\mu}_N, t \in [0, \infty).$$

Equation (29) delivers $\int_0^\infty f_N(t)dt = (A'_N)^{-1} \begin{bmatrix} \alpha_{N,1}, & \cdots & \alpha_{N,N-1} \end{bmatrix}' \bar{\mu}_N$. Since $\bar{\mu}$ is the stationary distribution, rewrite the first $N-1$ rows of $0 = A'\bar{\mu}$ as

$$\begin{aligned} 0 &= A'_N \begin{bmatrix} \bar{\mu}_1 \\ \vdots \\ \bar{\mu}_{N-1} \end{bmatrix} - \begin{bmatrix} \alpha_{N,1} \\ \vdots \\ \alpha_{N,N-1} \end{bmatrix} \bar{\mu}_N \\ &= A'_N \left(\begin{bmatrix} \bar{\mu}_1 \\ \vdots \\ \bar{\mu}_{N-1} \end{bmatrix} - \int_{t=0}^\infty f_N(t)dt \right) \end{aligned}$$

Thus, (28) and (25) deliver recursively, starting from $x = N$,

$$\begin{aligned} \mu(x-1) &= \frac{-1}{\alpha_{x-1,x-1}} \sum_{\tilde{x}<x-1} \alpha_{\tilde{x},x-1} \left(\bar{\mu}_{\tilde{x}} - \int_{t=0}^\infty f_{N,\tilde{x}}(t)dt \right) \\ &= 0 \\ f_{x-1} &= \mathbf{0}_{x-1} \end{aligned}$$

completing the description of the stationary distribution $\mathcal{D}$ for $r = \rho$, when $k = 0$ for agents in state N. It implies that agents are either in state N with probability $\bar{\mu}_N$ and holding zero capital or in some state $x < N$ with probability $\int_0^\infty f_{N,x}(t)dt = \bar{\mu}(x)$, "drifting down" at zero drift from $\mathbf{c}(N)$ and holding capital $k_N(x)$. Total capital supply is therefore $K^s(\rho) = \begin{bmatrix} k_N, & 0 \end{bmatrix} \bar{\mu}$, thus finally justifying (129).

Therefore, take the inner product of (126) with the stationary distribution $\bar{\mu}$, i.e. premultiply (126) with the row vector $\bar{\mu}'$, and exploit $\bar{\mu}'A = \mathbf{0}_{N+1}$ and $\bar{\mu}'\mathbf{z} = 1$ to find

$$\rho K^s(\rho) = \bar{\mu}' \left(\rho I_{N+1} - A\right) \begin{bmatrix} k_N \\ 0 \end{bmatrix} = \mathbf{c}(N) - w$$

Compare this to equation (54), defining $\bar{c}$ from capital demand. The condition $\kappa^s(\rho) \leq$

$\kappa^d(\rho)$ or, equivalently, $K^s(\rho) \leq K^d(\rho)$ now implies that

$$\mathbf{c}(N) \leq \bar{c} \tag{130}$$

which is the desired inequality. Since agents will always end up in state N with some positive probability and have at least zero capital there, it follows that all agents consume at least $\mathbf{c}(N)$, validating the calculations and conclusions of the first part.

□

F Poisson Transition Matrix

The complete matrix $(\alpha_{x,x'})$ used in Section 7 is given by:

$$\begin{bmatrix} -0.232 & 0.060 & 0.093 & 0.060 & 0.018 & 0.001 \\ 0.018 & -0.190 & 0.093 & 0.060 & 0.018 & 0.001 \\ 0.018 & 0.060 & -0.157 & 0.060 & 0.018 & 0.001 \\ 0.018 & 0.060 & 0.093 & -0.190 & 0.018 & 0.001 \\ 0.018 & 0.060 & 0.093 & 0.060 & -0.232 & 0.001 \\ 0.020 & 0.020 & 0.020 & 0.020 & 0.020 & -0.100 \end{bmatrix}$$