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(Near) Substitute Preferences and Equilibria with Indivisibilities

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Abstract

An obstacle to using market mechanisms to allocate indivisible goods is the non-existence of competitive equilibria (CE). To surmount this Arrow and Hahn proposed the notion of social-approximate equilibria: a price vector and corresponding excess demands that are 'small'. We identify social approximate equilibria where the excess demand, good-by-good, is bounded by a parameter that depends on preferences only and not the size of the economy. This parameter measures the degree of departure from substitute preferences. As a special case, we identify a class called geometric substitutes that guarantees the existence of competitive equilibria in non-quasi-linear settings. It strictly generalizes prior conditions such as single improvement, no complementarities, gross substitutes and net substitutes.

1 Introduction

The use of market mechanisms for allocating resources like school slots and courses is hindered by the non-existence of competitive equilibria (CE) when the goods to be allocated are

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indivisible. One can ensure existence by restricting agents' preferences (e.g., Gul and Stacchetti (1999)) or assuming the setting is large enough to render the indivisibilities negligible (e.g., Azevedo et al. (2013)). Unfortunately, many applications are neither large enough nor satisfy the preference restrictions.

To accommodate non-existence, Arrow and Hahn (1971) proposed the notion of social-approximate equilibria: a price vector and associated demands that 'approximately' clear the market in that the excess demand for goods is 'small'. See, for example, Starr (1969), Dierker (1971) and Mas-Colell (1977). The bounds on excess demand offered in these papers depend on the number of goods or agents or both, which is unsatisfactory. They also do not contain the existence results as a special case.

This paper identifies social approximate equilibria where the gap between supply and demand, good-by-good, is bounded by a parameter Δ that depends only on agent preferences and, *not* the size of the economy. The parameter Δ is a bound on the ℓ_1 norm of the edges of the convex hull of the demand correspondence. This ostensibly abstract quantity can be interpreted as the maximum number of goods involved in a utility-improving exchange. Thus, it is a measure of the degree of preference complementarity, hence, the name Δ-substitutes. When all preferences satisfy Δ-substitutes, we exhibit prices where the excess demand good-by-good is at most $\Delta - 1$. This is accomplished using a strengthening of the Shapley-Folkman-Starr lemma (Starr (1969)) that is of independent interest.

While the violation of market-clearing is unavoidable, it opens up new possibilities for market design using pseudo-markets. The magnitude of the excess demand, $\Delta - 1$, is the 'shadow' cost of dividing the indivisible. If a 'planner' knows the excess demand a priori, she can withhold that amount to 'add back' to ensure that each agent's demand is satisfied. Hence, 'burning' some of the supply to ensure feasibility.

We emphasize that Δ-substitute preferences are not restricted to being quasi-linear, so they accommodate financing constraints and other income effects, which can be important. Obvious examples of Δ-substitute preferences are those that are satiated outside of a finite region. Examples are Shapley and Shubik (1971), Quinzii (1984), where agents consume a single good; and Budish (2011) in which there is an upper bound on the size of bundles consumed. This paper contains other less obvious but natural instances of Δ-substitute preferences.

Within the class of Δ-substitute preferences, we identify a subclass, we call geometric substitutes, that guarantees the existence of CE. We show that geometric substitutes strictly subsumes many prior preference restrictions that guarantee the existence of CE. For example, gross substitutes (Kelso Jr and Crawford (1982)), M-natural concave valuations (Murota and Tamura (2003)), the single improvement property (Gul and Stacchetti (1999)) and net substitutes (Baldwin et al. (2020)). There is also a connection between our results and Danilov et al. (2001) that we discuss in more detail in Section 5.

Our results go beyond the existence of (approximate) CE. We show that any CE in probability shares can be implemented as a lottery over (approximate) CE allocations. In this way, one can satisfy ex-ante, properties unattainable in any ex-post CE. This generalizes Hylland and Zeckhauser (1979) and Gul et al. (2019) and has a wide range of applications in market design as discussed in Section 6.

We summarize the relationship between this paper and prior work next.

1.1 Prior Work

Prior work deals with the non-existence of CE in two distinct ways. First, by restricting agent's preferences, for example, quasi-linearity and gross substitutes (see Kelso Jr and Crawford (1982)). Subsequent work by Danilov et al. (2001) and Baldwin et al. (2020) relaxed the quasi-linearity assumption. They introduce a condition that the second calls net substitutes, which guarantees the existence of a CE in the presence of income effects and contains gross substitutes as a special case. Section 3 introduces geometric substitutes preferences which is a strict generalization of the net substitutes condition. A detailed discussion of the relationship between geometric substitutes and other sufficient conditions for existence of CE is deferred to that section.

The second is to determine prices that 'approximately' clear the market. These social-approximate equilibria are preference independent and rely on the economy growing to infinity to ensure that the excess demand is negligible. See, for example, Starr (1969), Dierker (1971), Mas-Colell (1977), and Azevedo et al. (2013).[1]

Our paper establishes a link between the magnitude of excess demand in a social-approximate equilibrium to a parameter, Δ, that characterizes agent preferences. Δ can be interpreted as the degree of complementarity exhibited by the preferences. The smaller Δ is, the more limited the complementarities. Gross substitutes preferences, for example, are a *strict* subset of Δ substitute preferences for $\Delta = 2$. We exhibit a social-approximate equilibrium in which the excess demand is at most $\Delta - 1$ for each good. Budish (2011) also exhibits a social approximate equilibrium in which the magnitude of the excess demand depends on the underlying preferences. We differ from that in two ways. First, our result holds for a class of preferences that strictly subsumes the ones considered in Budish (2011). Second, Budish (2011) bounds the excess demand in terms of the Euclidean distance between demand and supply, whereas we offer a bound that holds good by good. (See Section 6.2.) To the best of our knowledge, our paper is the first to derive a scale-free bound on excess demand of CE. Bounds based on prior approaches depends on either the number of goods or the number of agents.[2]

Our results apply to resource allocation problems both with and without monetary transfers. For example, if agents' von Neuman-Morgenstern preferences satisfy Δ-substitutes, our methods allow one to implement a CE in probability shares as a lottery over allocations in which the excess demand good-by-good is at most $\Delta - 1$, while maintaining *any* linear ex-ante constraints including budget constraints. This generalizes Gul et al. (2019) which

[1]There is also a focus on finding an acceptable approximation of a CE outcome in terms of cardinal measures of welfare that scale appropriately with the size of the economy. See for example Akbarpour and Nikzad (2020), Cole and Rastogi (2007), Milgrom and Watt (2021) and Feldman and Lucier (2014).

[2]The bound in Nguyen and Vohra (2018) is also independent of the market size, but it is valid for a strictly smaller class of preferences and for stable solutions rather than CE.

only considers budget constraints and assumes preferences satisfy gross substitutes.

The next section introduces notation and is followed by Section 3 which introduces geometric substitute preferences and characterizes them in terms of a single improvement property. Section 4 defines Δ-substitute preferences and characterizes them in terms of a generalization of the single improvement property. This section also summarizes the main results on social approximate equilibria. Section 5 outlines the ideas used to obtain the main results and the connection to the Shapley-Folkman-Starr lemma. Section 6 summarizes applications to pseudo-markets. The appendix contains the proofs.

2 Notation & Preliminaries

The economy has m indivisible goods and one divisible good (money). Let M denote a set of m indivisible goods. A bundle of goods is denoted by a binary vector $x \in \{0,1\}^m$ whose i^{th} component, denoted x_i, indicates whether good $i \in M$ is in the bundle. Thus, we assume that each agent is interested in consuming at most one copy of each good. Call this **single copy demand**.[3] Section 4.1 shows how to reduce the case of multiple copy demand to the case of single copy demand.

Let N denote the set of agents. Given a bundle x, and a quantity $w \in \mathbb{R}$ of a divisible good which we interpret as money, the utility of each agent $j \in N$ for the bundle x and the amount of money w is denoted $U_j(x, w)$.[4] We assume $U_j(x, w)$ is continuous, non-decreasing in w, and $U_j(\vec{0}, 0) = 0$.

Each agent j is endowed with b_j units of money only.[5] Their utility for a bundle x which costs t is $U_j(x, b_j - t)$. Associated with each $j \in N$ is a finite set of bundles $X_j \subset \{0,1\}^m$ that they can feasibly consume, their feasible bundles. The bundle $\vec{0}$ is always assumed to be feasible. If $x \notin X_j$, then, $U_j(x, b_j - t) = -\infty$ for all t. As is standard (see, for example,

[3] Also called unit demand. However, this term is also used for the case where agents demand at most one unit of any good, so we avoid it.

[4] These are called expenditure augmented utilities in Deb et al. (2021).

[5] Our results extend to the case where agents are endowed with indivisible goods as well as 'bads'.

Fleiner et al. (2019)) we assume a bounded willingness to pay condition. Specifically, there is a monetary amount B_j such that $U_j(x, b_j - B_j) = -\infty$ for all $x \in X_j$.[6] While we state our results in terms of utility functions that depend on money, we show in Section 6.2 how to accommodate artificial currency.

Let $p \in \mathbb{R}^m_+$ be a price vector where p_i is the unit price of good $i \in M$. The utility of agent j for bundle x at price p will be $U_j(x, b_j - p \cdot x)$. Given a price vector p, agent j's **choice correspondence**, denoted $Ch_j(p)$, is defined as follows:

$$Ch_j(p) = \arg\max\{U_j(x, b_j - p \cdot x) : x \in X_j\}.$$

Denote the convex hull of $Ch_j(p)$ by $conv(Ch_j(p))$. Under single copy demand, no interior point of $conv(Ch_j(p))$ is contained in $Ch_j(p)$.

Let $(x - y)^+$ denote the vector whose i^{th} component is $\max\{x_i - y_i, 0\}$. The ℓ_1 norm of vectors will play an important role, to see why, consider two bundles x and y. Then,

$$||x - y||_1 = \vec{1} \cdot (x - y)^+ + \vec{1} \cdot (y - x)^+.$$

The term on the right hand side can be interpreted as the total number of items that must be swapped to get from bundle x to bundle y.

Let $n = |N|$, $s_i \in \mathbb{Z}_+$ denote the supply of good $i \in M$, $\vec{s} \in \mathbb{Z}^m_+$ the supply vector and $\vec{b}$ the vector of cash endowments. An economy is the collection $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$.

Definition 2.1 *A* <u>*competitive equilibrium*</u> *for the economy* $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ *is a price vector* p *and demands* $x^j \in Ch_j(p)$ *for all* $j \in N$ *such that* $\sum_{j\in N} x^j \leq s$ *with equality for each* $i \in M$ *for which* $p_i \neq 0$.

Definition 2.2 *A* <u>*α-approximate competitive equilibrium*</u> *for the economy* $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$

[6] Quasi-linear preferences, where $U_j(x, b_j - t) = v_j(x) + b_j - t$ for some valuation function $v_j(\cdot)$ satisfy these conditions. Hard budget constraints can be approximated by allowing $U_j(x, b_j - t)$ to approach $-\infty$ as t approaches the budget.

is a competitive equilibrium for the economy $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s'}\}$*, where* $|s_i - s'_i| \leq \alpha$ *for every good* $i \in M$*.*

As mentioned in the introduction, the key is to analyze preferences using the edges of the convex hull of the demand correspondence. To this end, we recall some terminology from convex geometry.

A finite dimensional **polyhedron** is defined by the intersection of a finite number of half-spaces. A bounded polyhedron is called a **polytope**. It is also the convex hull of a finite set of points.

A subset F of a polytope $Q \subseteq \mathbb{R}^m$ is called a **face** of Q if there is a hyperplane $\{x : h \cdot x = \gamma\}$ such that $F = Q \cap \{x : h \cdot x = \gamma\}$ and $Q \subseteq \{x : h \cdot x \leq \gamma\}$. A face of a polytope is itself a polytope. A zero dimensional face is called an **extreme point** and a face of dimension 1 is called an **edge** of the polytope. The polytope Q is a (trivial) face of itself, where $h = \vec{0}, \gamma = 0$. The **dimension** of a face F is denoted $dim(F)$. For any $y \in Q$, there is a unique face F of Q with lowest dimension that contains y. We call F the **minimal face** containing y.

3 Geometric Substitutes

In this section, we introduce a strict generalization of gross substitutes called geometric substitutes. We show that a competitive equilibrium exists when all preferences satisfy geometric substitutes.

DEFINITION 3.1 *A utility function* U_j *satisfies* <u>*geometric substitutes*</u> *if for all price vectors* p*, the edges of* $conv(Ch_j(p))$ *are* $\{0, \pm 1\}$ *vectors with at most two non-zero entries and these being of opposite sign.*

The following result establishes the existence of CE in this environment.

Theorem 3.1 *An economy $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s}\}$ in which each U_j satisfies geometric substitutes has a competitive equilibrium.*

Proof. Theorem 3.1 is a corollary of Theorem 5.2 which is stated later.

We emphasize that neither Definition 3.1 or Theorem 3.1 assume quasi-linearity. Next, we show that geometric substitutes subsumes many preference conditions under which a CE exists. We also characterize geometric substitutes in terms of a single improvement property.

3.1 Relation with Gross Substitutes

For the remainder of this section and the next, as we are focused on the properties of utility functions, we suppress the dependence on the agent index j. We recall the definition of gross substitutes:

Definition 3.2 *A utility function satisfies gross substitutes if for all budgets b and for every pair of price vectors p and q such that $p \leq q$ and for all $x \in Ch(p)$, there exists $y \in Ch(q)$ such that $y_i \geq x_i$ for all $i \in M$ such that $q_i = p_i$.*

Geometric substitutes and gross substitutes coincide in the quasi-linear preferences domain. Now, we furnish an example of a utility function satisfying geometric but not gross substitutes.

Example 3.1 *Suppose two goods and a soft 'budget' of 1. In particular,*[7]

$$\begin{aligned} U(x_1, x_2, 1 - p \cdot x) =& v(x_1, x_2) - \epsilon \cdot (p \cdot x) + \min\{0, \log(\frac{1 - p \cdot x}{\epsilon})\} && \text{for } p \cdot x < 1 \\ & -\infty && \text{for } p \cdot x \geq 1 \end{aligned}$$

where $\epsilon = 0.01$, and $v(0,0) = 0, v(1,0) = 1, v(0,1) = 2$ and $v(1,1) = 3$.

[7]This utility function is continuous and decreasing in $p \cdot x$. As $p \cdot x$ approaches 1, it approaches $-\infty$. This captures the soft budget of the agent. The purpose of the term $-\epsilon \cdot (p \cdot x)$ is to ensure that utility is decreasing in $p \cdot x$, which is needed for Proposition 3.2. Proposition 3.1 holds without this term.

Proposition 3.1 *The utility function defined in Example 3.1 is geometric substitute but not gross substitutes.*

Proof. To show that the utility function in Example 3.1 is geometric substitutes, we argue that for any $(p_1, p_2) \geq 0$ the agent's choice correspondence never contains both $(0,0)$ and $(1,1)$. If both $(0,0)$ and $(1,1)$ were in the choice correspondence for some $(p_1, p_2) \geq 0$, the utility of bundle $(1,1)$ must be 0, and thus $p_1 + p_2 < 1$. But then, $\min\{p_1, p_2\} \leq 0.5$ in which case the utility of either the bundle $(1,0)$ or $(0,1)$ is strictly greater than 0, contradicting the assumption that $(0,0)$ was in the choice correspondence.

Hence, at any price vector, the choice correspondence can only contain pairs of bundles whose difference is a $0, \pm 1$ vector with at most two non-zero entries of opposite sign. Therefore, the utility function in Example 3.1 satisfies geometric substitutes.

Gross substitutes requires that if the price of a single good increases, the demand for the other goods does not decline. Suppose $p_1 = p_2 = 0.4$. The unique utility-maximizing bundle is $(1,1)$. Now increase p_2 to 0.6. The unique utility maximizing bundle is $(0,1)$. Thus, the utility function in Example 3.1 is not gross substitutes. ■

Proposition 3.2 *Suppose $U(x, b-t)$ is strictly decreasing in t. Then, gross substitutes is a strict subset of geometric substitutes.*

Proof. Example 3.1 shows that it suffices to prove that gross substitutes implies geometric substitutes. Assume, for a contradiction, that there exist $x, y \in Ch(p)$ such that x and y form an edge (1-dimensional face) of the convex hull of $Ch(p)$ and $||(x-y)^+||_1 \geq 2$. Because x, y is an edge of $conv(Ch(p))$, there exists a vector $v \in \mathbb{R}^m$ such that

$$v \cdot x = v \cdot y < v \cdot z \text{ for all } z \in Ch(p) \setminus \{x, y\}. \tag{1}$$

Set $p' = p + \epsilon \cdot v$, for small $\epsilon > 0$, and increase the agent's budget to $b' = b + \epsilon \cdot v \cdot x$. We choose ϵ so that $Ch(p') = \{x, y\}$. This corresponds to determining an ϵ that solves:

$$U(x, b + \epsilon \cdot v \cdot x - p' \cdot x) = U(y, b + \epsilon \cdot v \cdot x - p' \cdot y) \tag{2}$$

$$U(x, b + \epsilon \cdot v \cdot x - p' \cdot x) > U(x, b + \epsilon \cdot v \cdot x - p' \cdot z) \ \forall z \in X \setminus \{x, y\} \tag{3}$$

The (2) holds because of (1). Existence of a suitable ϵ satisfying (3) follows from continuity of U and the fact that it is strictly decreasing in transfers.

We now invoke the gross substitute property for the given choice correspondence at p'. Because $||(x-y)^+||_1 \geq 2$, assume without loss of generality that $x_1 = x_2 = 1$, $y_1 = y_2 = 0$. If we increase the price of good 1 by $\delta > 0$ to get new price q, then the price of bundle x increases, while the price of bundle y is unchanged, thus at q, the agent will only choose y, but $y_2 < x_2$ even though $q_2 = p'_2$, which contradicts the definition of gross substitutes. ■

3.2 Relation with Single Improvement and No Complementarities

Quasi-linear gross substitutes with single copy demand is equivalent to the single improvement property as well as no complementarities (see Gul and Stacchetti (1999)). This equivalence does not hold in the non-quasilinear setting.[8] We show that these properties are a special case of geometric substitutes. Together with Theorem 3.1, this shows that single improvement/ no complementarities implies the existence of CE in non-quasi-linear setting as well. To the best of our knowledge, this extension of Gul and Stacchetti (1999) to the non-quasi-linear setting is new.[9]

We use the following definition of single improvement and no complementarities that is modified to account for the cash endowment b.

Definition 3.3 *We say $U(x, b-t)$ satisfies the single improvement property if for all b and*

[8]For example, Schlegel (2018) shows that the law of aggregate demand is also needed to guarantee equivalence.

[9]It is not implied by either Echenique (2012) nor Baldwin et al. (2020).

all price vectors p if $x \notin Ch(p)$, then there exists a superior bundle y, (that is $U(x, b-p\cdot x) < U(y, b-p\cdot y)$) such that $||(x-y)^+||_1 \leq 1$ and $||(y-x)^+||_1 \leq 1$.

Definition 3.4 *We say $U(x, b-t)$ satisfies the <u>no complementarities</u> property if for all b and all price vectors p if $x, y \in Ch(p)$ and any bundle $x' \leq x$, then there exists a bundle $y' \leq y$ such that $x - x' + y' \in Ch(p)$.*

Proposition 3.3 *Let $U(x, b-t)$ be strictly decreasing in t. If U satisfies the single improvement property or no complementarities, then, it satisfies geometric substitutes.*

Proof. Suppose not. Let $x, y \in Ch(p)$ determine an edge of $conv(Ch(p))$ such that $||(x-y)^+||_1 \geq 2$. Without loss we may assume that $x_1 = x_2 = 1; y_1 = y_2 = 0$. Similar to the proof of Proposition 3.2, we can perturb the price to p' and the cash endowment of the agent so that $Ch(p') = \{x, y\}$.

If $U(.)$ satisfies the no complementarity property, we can remove good #1 from x and replace it with a subset of goods from y to obtain a new bundle in the choice correspondence. However, this contradicts the fact that $Ch(p') = \{x, y\}$.

Now suppose that $U(.)$ satisfies single improvement. Then, we can increase the price of good #1 by a small ϵ to get a new price p'' such that $x \notin Ch(p'')$ and the utility at x is greater than all other bundles except bundle y, which is the only bundle in $Ch(p'')$. In other words, y is the only bundle superior to x. This contradicts single improvement because $||(x-y)^+||_1 \geq 2$. ■

Unlike the quasi-linear case, these properties do not characterize geometric substitutes. However, we show that a modified form of these conditions does.

Definition 3.5 *A utility function $U(x, b-p\cdot x)$ satisfies the <u>generalized single-improvement</u> property if for any price vector p, any two bundles $x, y \in Ch(p)$ and any price change $\delta p \in \mathbb{R}^m$ satisfying $\delta p \cdot x > \delta p \cdot y$, there exist $a \leq (x-y)^+$ and $d \leq (y-x)^+$ such that*

1. $\vec{1} \cdot a \leq 1$ *and* $\vec{1} \cdot d \leq 1$,
2. $\delta p \cdot a > \delta p \cdot d$, *and*
3. $x - a + d \in Ch(p)$.

Generalized single improvement coincides with the definition of single improvement when preferences are quasi-linear.

THEOREM 3.2 *A preference satisfies geometric substitutes if and only if it satisfies the generalized single-improvement property.*

See Appendix A.

3.3 Relation with Net Substitutes

Baldwin et al. (2020) accommodate non-quasi-linear preferences using Hicksian demand.[10] Given a price vector p and a target level of utility u, the Hicksian demand at (p, u) is

$$D_H(p, u) = \arg\min\{px : x \in X, U(x, b - p \cdot x) \geq u\}.$$

They introduce an analog of gross substitutes for Hicksian demand called (strong) net substitutes.[11]

DEFINITION 3.6 *Suppose $U(x, b-t)$ is* strictly *decreasing in the transfer t (this ensures that Hicksian demand is well defined). $U(.)$ satisfies net substitutes if for all utility levels u and price vectors p and $\lambda > 0$ whenever $D_H(p, u) = x$ and $D_H(p + \lambda e^i, u) = x'$ where e^i is the i^{th} unit vector, we have that $x'_k \geq x_k$ for all $k \neq i$.*

Baldwin et al. (2020) show that if all preferences satisfy net substitutes, a CE exists. We show that the geometric substitutes property is more general than net substitutes. In particular,

[10] A similar approach is taken in Danilov et al. (2001).
[11] This condition is not explicitly articulated in Danilov et al. (2001).

the former is a property that holds for all prices while the latter must hold for all prices *and* utility levels.[12]

Proposition 3.4 *Suppose $U(x, b-t)$ satisfies net substitutes, then, $U(.)$ is geometric substitutes.*

Proof. Given price vector p, let $u^* = \max_{x \in X} U(x, b - p \cdot x)$. At the optimal utility level u^*, Hicksian demand coincides with the choice correspondence. That is, $D_H(p, u^*) = Ch(p)$.

Baldwin et al. (2020) show that $D_H(p, u^*)$ is a choice a correspondence of a quasi-linear utility function. Under net substitutes, it satisfies the gross substitute property. Hence, according to Proposition 3.2, $U(.)$ is geometric substitutes. ■

4 Δ-substitutes

Here we introduce a generalization of the geometric substitutes condition.

Definition 4.1 *A utility function $U(x,b-p \cdot x)$ satisfies the* <u>*Δ-substitutes*</u> *property if for any price vector p, the ℓ_1 norm of each edge of $conv(Ch(p))$ is at most Δ.*

Geometric substitutes is a special case with Δ=2. However, geometric substitutes is not simply a small bound on the edge length but a restriction on the 'sign pattern' of the edges as well.

While a CE need not exist under Δ-substitutes preferences, we show that one can perturb the supply of each good to guarantee the existence of CE. Our main result is the following.

Theorem 4.1 *If all agent's preferences possess the Δ-substitutes property, then for every supply vector s the economy $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s}\}$ has a $\Delta - 1$-approximate CE.*

Proof. This follows from the proof Theorem 5.3.

[12]In the proof below, we only invoke net-substitutes at the maximum utility level.

Notice the excess demand for *each* good is at most $\Delta - 1$, a quantity independent of both the number of agents and goods. Applications of Theorem 4.1 are discussed in Section 6. We outline the key ideas used to prove our main results in Section 5.

We give examples of natural classes of preferences that satisfy the Δ-substitutes property.

Example 4.1 *If an agent is satiated outside of a finite region, as is true in our case, an agent's preferences satisfy the Δ-substitutes property for a suitable chosen Δ. Examples are Shapley and Shubik (1971), Dierker (1971)), Quinzii (1984), Budish (2011) and Milgrom and Watt (2021). In Budish (2011), goods correspond to courses in an academic term, and there is usually an upper limit on the number of courses a student can take in the term. If that limit is Δ, i.e., no bundle can contain more than Δ items, the preferences satisfy 2Δ substitutes. This is because the l_1 distance between any two bundles of size Δ is at most 2Δ.*

Next, we give examples where agents have preferences for bundles whose size exceeds some Δ, but nevertheless satisfy Δ-substitutes.

To motivate the next definition, consider what can happen when the price of a good, i, say is increased. First, the demand for good i might decline. Second, the demand for goods that complement i may also decline. Third, the demand for goods other than its complements may increase. The conditions for the existence of a CE operate by limiting the impact that the demand for one good has on the demand of others which inspire the next definition.

DEFINITION 4.2 *A utility function $U(x, b-t)$ that is strictly decreasing in t satisfies the $\underline{\Delta\text{-bounded impact property}}$ if for every pair of price vectors (p, q) that differ in one item such that $p \leq q$, then for all $x \in Ch(p)$, there exists $y \in Ch(q)$ such that $|\{i \in M : y_i \neq x_i, q_i = p_i\}| \leq \Delta$.*

Thus, when the price of good i alone changes, the number of other goods whose demand is affected is at most Δ.

Proposition 4.1 *A utility function that is strictly decreasing in transfers and satisfies the Δ-bounded impact property is $(\Delta+1)-$substitutes.*

Proof. Suppose not. Let $x, y \in Ch(p)$ determine an edge of $conv(Ch(p))$ such that $||(x-y)||_1 \geq \Delta+2$. Without loss we may assume that $||(x-y)^+||_1 > 0$ and $x_1 = 1; y_1 = 0$.

Similar to the proof of Proposition 3.2, we can perturb the price to p' and the cash endowment of the agent so that $Ch(p') = \{x, y\}$. Consider price q obtained from p' by increasing p'_1. Because $U(.)$ is strictly decreasing in transfers, $Ch(p') = \{y\}$. However, $|\{i \in M : y_i \neq x_i, q_i = p_i\}| = ||(x-y)||_1 - 1 \geq \Delta+1$, contradicting the Δ-bounded impact property. ■

One can also generalize the single improvement property of Definition 3.3 as follows.

DEFINITION 4.3 *A utility function $U(x, b-t)$ satisfies the* <u>*Δ-improvement property*</u> *if, for all b, whenever $x \notin Ch(p)$ there is a superior bundle y such that $||x-y||_1 \leq \Delta$.*

Proposition 4.2 *A utility function that is strictly decreasing in transfers and satisfies Δ-improvement is $\Delta-$substitutes.*

Proof. The proof is similar to that of Proposition 3.3 and so omitted. ■

We illustrate Definition 4.3 with an example of preference that generalizes gross substitutes that we call **bundled gross substitutes**.

Example 4.2 *Each agent is interested in at most one copy of each good. Associated with each agent is a partition $P_1, P_2 \ldots, P_k$ of M such that $|P_r| \leq \Omega$ for all $r = 1, \ldots, k$. The partitions can vary across agents. If x is a bundle, let $x|_{P_r}$ denote the sub-bundle consisting only of goods in P_r. Suppose x is not a utility maximizing bundle at price p. Then, there is a better bundle to be had by either adding a new sub-bundle, subtracting a sub-bundle, or exchanging $x|_{P_r} \neq 0$ with a different bundle in $P_{r'}$, where $r, r' \in \{1, .., k\}$ and are not necessarily different. Thus, within P_r, there can be complementarities among the goods.* Fox *and Bajari (2013) suggests that spectrum preferences resemble bundled substitutes. Spectrum licenses within the same geographical cluster complement each other but are substitutes across different clusters. It is easy to see that bundled gross substitutes satisfies 2Ω-substitutes.*

Δ-improvement does not characterize Δ-substitutes. The following related notion does.

DEFINITION 4.4 *A utility function $U(x, b - p \cdot x)$ satisfies the generalized Δ-improvement property if for any price vector p, any two bundles $x, y \in Ch(p)$ and any price change $\delta p \in \mathbb{R}^m$ satisfying $\delta p \cdot x > \delta p \cdot y$, there exist $a \leq (x-y)^+$ and $d \leq (y-x)^+$ such that*

1. $\vec{1} \cdot a + \vec{1} \cdot d \leq \Delta$,
2. $\delta p \cdot a > \delta p \cdot d$, *and*
3. $x - a + d \in Ch(p)$.

THEOREM 4.2 *A preference satisfies the Δ-substitutes property, if and only if it satisfies the generalized Δ-improvement property.*

Proof. See Section A. ■

4.1 Multi-copy Demand

A bundle x is now any vector in Z_+^m, where x_i represents the number of copies of good $i \in M$. It would be natural to define Δ-substitute preferences in the same way, i.e., the ℓ_1 norm of each edge of $conv(Ch(p))$ is at most Δ. However, unlike single copy demand, $conv(Ch(p))$ can contain bundles that are not in $Ch(p)$. For this reason, we define Δ-substitute preferences for the multi-copy case by reduction to the single copy case. Hence, Theorems 3.1, 3.2, 4.1, 4.2 all hold for multi-copy demand. Indeed, their proofs in the appendix are stated for the multi-copy demand case.

When a bundle contains multiple copies of the same good, think of each copy of the good as being a separate good. For example, if the bundle contains three oranges, we represent that as three distinct objects called orange copy #1, orange copy #2, and orange copy #3. Therefore, any vector $x \in Z_+^m$ can be represented as a 0-1 vector, which we call its binary representation.

Let $C \in \mathbb{Z}_+$ be a constant at least as large as the maximum number of copies of a good that an agent consumes. Make C copies of each good. Let $y \in \{0,1\}^{C \cdot m}$ be a binary representation of a bundle. The total number of copies of good $i \in M$ contained in y is

$$T_i(y) = \sum_{k=C\cdot(i-1)+1}^{C\cdot i} y_k. \tag{4}$$

Thus, y is a binary representation of the bundle $(T_1(y), .., T_m(y))$.

A bundle $x \in Z_+^m$ can have multiple binary representations. Hence, to each bundle $x \in Z_+^m$ we associate a set $B(x)$ in $\{0,1\}^{C \cdot m}$ of all possible binary representations of x. Formally,

$$B(x) := \{y \in \{0,1\}^{C.m} | T_i(y) = x_i \ \forall i \in M\}. \tag{5}$$

The following is the definition of geometric substitutes for the multi-copy demand case.

Definition 4.5 *Multi-copy preferences satisfy <u>geometric substitutes</u> if for all price vectors p, the edges of $\cup_{x \in Ch(p)} B(x)$ are $\{0, \pm 1\}$ vectors with at most two non-zero entries and these being of opposite sign.*

The following is the definition of Δ-substitutes for the multi-copy demand case.

Definition 4.6 *We say that multi-copy preferences satisfy the <u>Δ-substitutes</u> property if the ℓ_1 norm of each edge of the convex hull of $\cup_{x \in Ch(p)} B(x)$ is at most Δ.*

Notice the restriction is on the convex hull $\cup_{x \in Ch(p)} B(x)$ and not $conv(Ch(p))$. The advantage of defining preferences with respect to the convex hull of $\cup_{x \in Ch(p)} B(x)$ instead of $conv(Ch(p))$ is that one doesn't need to specify whether an integer vector in the interior of $conv(Ch(p))$ lies in $Ch(p)$.

From now on, we do not restrict ourselves to single copy demand.

4.2 Relation with Demand Types

Baldwin and Klemperer (2019) proposed that the preferences of an agent over bundles of indivisible goods be characterized in terms of demand changes in response to a small generic price change.[13] The set of vectors that summarize the possible demand changes is called the **demand type** of an agent. They give a variety of definitions that under quasi-linearity are equivalent. One involves the edges of $conv(Ch(p))$. Scale the edges of $conv(Ch(p))$ so that their greatest common divisor is one and call them **primitive edge directions**. The demand type of an agent is the set of primitive edge directions of $Ch(p)$ for all price vectors p. Baldwin and Klemperer (2019) show that if utilities are quasi-linear, concave and the demand types of all agents form a unimodular vector system, then a CE exists.

The column vectors of a network matrix, which is a $0, \pm 1$ matrix with at most two non-zero entries in each column and these being of opposite sign, is a unimodular vector system. When the matrix of vectors in the demand type is a network matrix, the underlying quasi-linear preferences are gross substitutes. Hence, Baldwin and Klemperer (2019) extends Kelso Jr and Crawford (1982) but maintains quasi-linearity.

Unlike Baldwin and Klemperer (2019), our preferences depend on the edges of a different polytope: the convex hull of the choice correspondence in its binary presentation. The polytopes coincide under single copy demand. Under quasi-linearity, geometric substitutes coincides with gross substitutes and is a special case of unimodular demand types. However, our existence results extends to the non-quasi-linear setting.

5 Proof Ideas and Extension

In this section we state and sketch the proofs of two Theorems that imply Theorems 3.1 and 4.1. The proofs rely on the concept of a pseudo-equilibrium (see Milgrom and Strulovici (2009)). It is a relaxation of competitive equilibrium where agent's preferences are 'con-

[13]This idea is implicit in Danilov et al. (2001).

vexified' by replacing each agent $j \in N$'s choice correspondence with its convex hull, $conv(Ch_j(p))$. Each $x \in conv(Ch_j(p))$, because it is a convex combination of the bundles in $Ch_j(p)$, is interpreted as a lottery over those bundles, and x itself is the 'expected' bundle. Therefore, a pseudo-equilibrium allocates to each agent lotteries over bundles. Unfortunately, it is not always possible to implement these lotteries in the sense that there is a lottery over allocations consistent with the individual lotteries over bundles.[14] However, if preferences are geometric substitutes, then a pseudo-equilibrium can be implemented as a lottery over integral allocations. Furthermore, each of the integral allocations in this lottery is a CE allocation.

From the perspective of the parachutist, this will appear similar to the proof of existence of a CE in Danilov et al. (2001). However, from the perspective of the truffle hunter, there are important differences. They require that the choice correspondences be discrete convex sets; we do not. This requirement amounts to assuming concavity of each agents utility functions. Second, we provide a lottery implementation. They do not. Furthermore, we furnish, as detailed below, a social approximate CE, they do not.

Similarly, if preferences possess the Δ-substitutes property, this lottery can be implemented as a lottery over approximate equilibria, such that the ex-post violation of any supply constraint is bounded by $\Delta - 1$.[15]

Definition 5.1 *A price vector p and $x^j \in conv(Ch_j(p))$ for all $j \in N$ is called a* <u>*pseudo-equilibrium*</u> *if $\sum_{j \in N} x^j \leq s$ with equality for every good $i \in M$ with $p_i \neq 0$.*

The next theorem shows the existence of a pseudo-equilibrium.

[14] See Budish et al. (2013) for more on this issue.

[15] It is well known that when preferences violate non-satiation, CE allocations need not be Pareto optimal. This is because some agents do not purchase their least expensive optimal bundle. To ensure efficiency, attention has focused on CE with slack (eg. McLennan (2017)) or 'paper money' (eg. Kajii (1996)). The first allows for reallocating unspent wealth (slack). The second interprets the slacks (or the dividends) as paper money, which is allocated to the consumers before the market takes place. If the conditions for the existence of these efficient CE hold, they can be used in our implementation results (Theorem 5.2 and 5.3). The resulting lottery will be over allocations that are efficient CE.

THEOREM 5.1 *Let $\vec{0} \in X_j$ denote the finite set of bundles that agent $j \in N$ can feasibly consume. Each agent j's utility function $U_j(x, b - p \cdot x)$ satisfies*

- $U_j(0, 0) = 0$,
- $U_j(x, b - p \cdot x) = -\infty$ *for* $x \notin X_j$
- $U_j(x, b - p \cdot x)$ *is continuous in* $p \in \mathbb{R}^m$ *for each* $x \in X_j$ *and*
- *there exists* $B > 0$ *such that if* $p\dot{x} \geq B$, *then,* $U_j(x, b - p \cdot x) < 0$.

Then, there exists a pseudo-equilibrium.

Proof. See Appendix B.

We begin with the case of geometric substitutes. The following Theorem yields Theorem 3.1 as a corollary.

THEOREM 5.2 *Let $(p, \{x^j\}_{j=1}^n)$ be a pseudo-equilibrium of an economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ in which each U_j satisfies geometric substitutes. Then, the vector $(x^1, \ldots, x^n)$ of lotteries can be implemented as a lottery over competitive equilibrium allocations.*

Proof. See Appendix C.

A pseudo-equilibrium allocation associated with price vector p lies in the intersection of two polytopes. The first is the Cartesian product of the convex hull of choice correspondences of each of the agents. Because each $conv(Ch_j(p)$ is an integral polytope, their Cartesian product is also an integral polytope. The second polytope is defined by the requirement that demand for each good equals its supply. This polytope is also integral. Any integral vector in the intersection of the two polytopes corresponds to a competitive equilibrium allocation. To show that any pseudo-equilibrium allocation can be expressed as a convex combination of competitive equilibrium allocations, we prove that the *intersection* of these two polytopes is an integral polytope.

The following gives an approximate implementation of a pseudo-equilibrium for the case of Δ-substitutes, which yields Theorem 4.1 as a consequence.

THEOREM 5.3 *Let* $(p, \{x^j\}_{j=1}^n)$ *be a pseudo-equilibrium of an economy* $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ *in which each* U_j *satisfies the* Δ*-substitutes property. Then, the vector of lotteries* $(x^1, \ldots, x^n)$ *can be implemented as a lottery over the allocations of* $\Delta - 1$*-approximate competitive equilibria.*

Proof. See Appendix D.

The proof of Theorem 5.3 relies on an extension of the Shapley-Folkman-Starr lemma. We now outline the proof.

Assume single copy demand and suppose a pseudo-equilibrium with price vector p and demands $x^j \in conv(Ch_j(p))$ for all $j \in N$ such that $\sum_{j\in N} x^j \leq s$. For ease of exposition only, suppose $\sum_{j\in N} x^j = s$, i.e., demand meets supply. This means that the vector s is in the Minkowski sum of the sets $\{conv(Ch_j(p))\}_{j\in N}$. For a pseudo-equilibrium to be an actual equilibrium, we need s be in the Minkowski sum of $\{Ch_j(p)\}_{j\in N}$ instead. The next Lemma tells us that under certain conditions this is 'almost' true.

Call a polytope P **binary** if it is the convex hull of 0-1 vectors and denote its set of extreme points by $ext(P)$. In our case, P will correspond to the convex hull of a choice correspondence and $ext(P)$ to the choice correspondence itself.

Recall that the edges of a polytope are its one-dimensional faces and are vectors of the form $v - u$ for some (not all) pairs of extreme points v and u. The edges of a binary polytope have components in $\{0, \pm 1\}$. Call a binary polytope **Δ-uniform** if the ℓ_1 norm of each of its edges is at most Δ. The key technical result that yields Theorem 5.3 is the following.

THEOREM 5.4 *Let* $P_1, \ldots, P_n$ *be binary* Δ*-uniform polytopes in* $\mathbb{R}^m$*. Then, each* $y \in P_1 + \ldots + P_n$ *can be expressed as a convex combination of points in* $ext(P_1) \times \ldots \times ext(P_n)$*. Furthermore, for each* $(z^1, \ldots, z^n) \in ext(P_1) \times \ldots \times ext(P_n)$ *in the support of the convex combination,* $||\sum_{j=1}^n z^j - y||_\infty \leq \Delta - 1$.

Proof. See Appendix D.

Theorem 5.4 has applications beyond obtaining social approximate equilibria. One can interpret any mechanism as giving to each agent $j \in N$, depending on what they report, a 'budget' or 'option' set, denoted B_j from which they may choose. Let $P_j \subseteq B_j$ be the set of agent j's most preferred elements of B_j. We 'convexify' the sets B_j and P_j by allowing lotteries. Suppose these convexified sets have been chosen so that 'average' demand is equal to supply. Then, depending on the properties of the sets P_j, one may invoke Theorem 5.4 to implement these allocations as lotteries over approximately feasible allocations of the goods.

Next, we describe the connection between Theorem 5.4 and the Shapley-Folkman-Starr lemma.

5.1 Relation to Shapley-Folkman-Starr Lemma

This lemma can be interpreted as saying that the non-convexities in an aggregate of non-convex sets diminishes with the number of sets making up the aggregate. We state two versions of it. The first version is from Budish and Reny (2020).

THEOREM 5.5 *Let $P_1, \ldots, P_n$ be a collection of binary polytopes in $\mathbb{R}^m$ with $n > m$. Suppose the diameter of each P_i is no larger than d. For any $y \in P_1 + \ldots + P_n$ there exist vectors $z^j \in P_j$ for all j of which at least $n - m$ are in $ext(P_j)$ and*

$$||y - \sum_{j=1}^{n} z^j||_2 \leq \frac{d\sqrt{m}}{2}.$$

The second version is due to Cassels (1975).

THEOREM 5.6 *Let $P_1, \ldots, P_n$ be a collection of binary polytopes in $\mathbb{R}^m$ with $n > m$. For any $y \in P_1 + \ldots + P_n$ there exist vectors $z^j \in P_j$ for all j of which at least $n - m$ are in $ext(P_j)$ such that $||y - \sum_{j=1}^{n} z^j||_\infty \leq m$.*

If we let $P_j = conv(Ch_j(p))$ for all $j \in N$, because $s = \sum_{j \in N} x^j$ and $x^j \in P_j$ for all $j \in N$, it follows that $s \in P_1 + \ldots + P_n$. Hence, in either version of the Shapley-Folkman-Starr Lemma, we can give to at least $n - m$ agents a bundle $z^j \in ext(P_j)$ that is in their demand correspondence (not just its convex hull) in such a way that the discrepancy between supply and demand is small. The two versions obviously differ in the way the discrepancy is measured. Furthermore, neither version imposes restrictions on the P_j's.

Theorem 5.4 relaxes the requirement that $n >> m$ by imposing restrictions on the ℓ_1 norm of the edges of P_j and bounds the discrepancy in terms of the ℓ_1 norm.

6 Applications to Market Design

This section contains applications of our results to market design. Our existence theorems for (social approximate) equilibria are relevant for the design of (approximate) market-clearing mechanisms with money. We also show that our existence results are relevant in settings without money and ordinal preferences although we assume each agent's preferences depend on money. In particular, our existence results cover pseudo-market mechanisms that use artificial currency rather than coin of the realm because of fairness concerns or a desire to limit certain trades. In course allocation, for example, all students are considered to have an equal claim on available classes. In this case, the goal of a pseudo-market mechanism is to allocate the available slots fairly (see Budish (2011)). In the context of food banks (Prendergast (2017)), the goal is to reallocate food to reduce waste. Artificial currency ensures that donated food remains within the food bank network rather than sold to the 'outside.'

Hylland and Zeckhauser (1979) was the first to propose the use of CE allocations with *equal* artificial currency endowments and goods that are the probabilities of being assigned to each object.[16] The corresponding mechanism is called the CEEI mechanism, the acronym standing for competitive equilibrum with equal incomes. The outcome of the CEEI mecha-

[16] See Pratt and Zeckhauser (1990) for a discussion of an actual application.

nism is a lottery over feasible allocations. Assuming agents have von Neuman-Morgenstern preferences, the CEEI mechanism returns a lottery over allocations that is both ex-ante Pareto optimal and ex-ante envy free.

In section 6.1 we show how Theorem 5.2 and 5.3 can be used to extend the CEEI mechanism to identify lotteries that satisfy, ex-ante, a variety of side constraints. This allows one to apply the CEEI mechanism in the presence of constraints that encode bounds on the probability of receiving bundles from a particular category. Such constraints are relevant, for example, in online advertising where advertisers prefer to diversify the audiences they reach. In a school choice setting, they capture proportionality requirements to ensure diversity.[17]

Requiring agents to communicate preferences over lotteries is impractical in many contexts. Offering agents lotteries over bundles is also seen as unappealing. Motivated by course allocation, Budish (2011) proposes a modification of the CEEI mechanism that relies on ordinal preference information information only. To circumvent non-existence of a CE, agents receive approximately equal budgets. Under these conditions, a social approximate equilibrium is determined. Thus, instead of ex-ante Pareto optimality, one obtains approximate ex-post Pareto optimality. Instead of ex-ante envy freeness, one obtains approximate ex-post envy-freeness. The first follows from the existence of excess demand, while the second is a consequence of perturbing budgets.

Using the course allocation problem, we show in Section 6.2 how Theorem 4.1 can be adapted to a setting where agent preferences are ordinal and do not depend on money. Our approach gives similar results as in Budish (2011): deterministic social approximate equilibrium with perturbed budget. The main difference is in the way we measure the deviation from market clearing. We use the ℓ- infinity norm instead of the Euclidean norm as in Budish (2011). We elaborate on this difference in Section 6.2.

[17] See He et al. (2018) for a discussion of CEEI in school choice.

6.1 Side Constraints

In this section, we show how to accommodate *any* finite collection of ex-ante linear constraints on the probability shares an agent can consume. These constraints could be endowment limitations, bounds on the probability of receiving bundles of a specific category as well, as budget constraints. Such requirements could be baked into the utility function but might destroy any nice properties the utility function might have enjoyed. We show that under Δ-substitutes, a CE in probability shares can be implemented as a lottery over approximately feasible allocations. This generalizes Gul et al. (2019) who limit attention to gross substitutes preferences and budget constraints only.

Section 6 of Akbarpour and Nikzad (2020) also contains an approximate competitive equilibrium result which accommodates side constraints on agents choices such as we have here. However, the quality of the approximation is in terms of agent utility and relies on a large market assumption.

Let $v_j(x)$ be the value of agent j for bundle $x \in X_j$. We assume $0 \in X_j$ and $v_j(0) = 0$. Let $\mathcal{L}(X_j)$ denote the set of lotteries over bundles in X_j. For each lottery $y \in \mathcal{L}(X_j)$, denote by $\bar{v}_j(y)$ its expected utility. Let $\overline{y}$ be the average consumption of the lottery y. Thus, $p \cdot \overline{y}$ is the expected price of the lottery y.

Associated with each agent $j \in N$, is a $k \times m$ matrix A_j and vector $b^j \in \mathbb{R}^k_+$. We assume A_j and b^j are continuous functions of p. Agent j is restricted to choosing lotteries $y \in L(X_j)$ such that $A_j \cdot \overline{y} \leq b^j$.

Notice a budget constraint corresponds to a row of A_j being the price vector. A_j and b^j being positive independent of p capture bounds on the probability of receiving bundles from specific categories. As the entries of A_j are not required to be positive, diversity concerns expressed in as bounds on proportions can also be modeled.

Let

$$Ch^*_j(p) := \arg\max\{\bar{v}_j(y) : y \in \mathcal{L}(X_j), A_j \cdot \overline{y} \leq b^j\}.$$

Note, $Ch^*_j(p)$ is convex and compact and nonempty.[18] Furthermore, the value of $v_j(y)$ for $y \in Ch^*_j(p)$ is continuous in p. By standard arguments, there exists a competitive equilibrium allocation of lotteries.

Claim 6.1 *There exists a price vector p and a vector of lotteries $(y^1, .., y^n)$ such that $y^j \in Ch^*_j(p)$ and $\overline{y}_i \leq s_i$ for all $i \in M$ and $\overline{y}_i = s_i$ if $p_i > 0$, where $\overline{y}$ denotes the average aggregate consumption.*

The ability to implement $(y^1, .., y^n)$ depends on the quasi-linear utility $v_j(x) - p \cdot x$. In particular, if $v_j(x) - p \cdot x$ is (Δ) substitutes for all j and p, we show how to implement $(y^1, .., y^n)$ as a lottery over (approximately) feasible allocations. At the heart of the argument is the following claim which generalizes the result in Gul et al. (2019) beyond a single budget constraint to any *any* finite collection of linear constraints.

We associate with each $v_j(x)$ a quasi-linear choice correspondence:

$$Ch_j(p) := \arg\max\{v_j(x) - p \cdot x \ , x \in X_j\}.$$

Proposition 6.1 *For every price vector p, and for every agent $j \in N$, let A_j be a matrix of size $k \times m$, and $b^j \in \mathbb{R}^k_+$ (both A_j and b^j can be continuous functions of p). Then, there exists a price vector p' such that $Ch^*_j(p) = conv(Ch_j(p'))$.*

Proof. For convenience, we omit the agent index in the proof. First, we show that $Ch^*(p)$ is the set of optimal solutions to a linear program. For each $x \in X$ let $w_x \in [0, 1]$ denote the fraction of bundle x selected. The average bundle $\bar{y} := \sum_{x \in X} w_x \cdot x$.

The constraint $A \cdot \bar{y} \leq \vec{b}$ can be reformulated in terms of w_x as $A \cdot (\sum_{x \in X} w_x \cdot x) \leq \vec{b}$. Thus, the problem of selecting a utility maximizing lottery over X_j can be represented as

[18] 0 is always a feasible choice as $b^j \geq 0$.

follows:

$$\begin{aligned}
\max \quad & \sum_{x\in X} w_x \cdot v(x) \\
\text{s.t} \quad & \sum_{x\in X} w_x \leq 1 \\
& A \cdot (\sum_{x\in X} w_x \cdot x) \leq \vec{b} \\
& w_x \geq 0 \ \forall x \in X.
\end{aligned}$$

Here we assume $v(0) = 0$, which is why we write $\sum_x w_x \leq 1$ instead of $\sum_x w_x = 1$.

Let α be the dual variable associated with the first constraint and $\vec{\beta} \geq 0$ the dual variable associated with the second constraint. For every good i, denote

$$\gamma_i := \vec{\beta}^T \cdot A^i, \text{ where } A^i \text{ is the column } i \text{ of matrix } A.$$

Dual feasibility and complementary slackness imply:

$$v(x) - \sum_i \gamma_i \cdot x_i \leq \alpha \text{ for all } x \in X$$

and

$$\text{if } w_x > 0, \text{ then } v(x) - \sum_i \gamma_i \cdot x_i = \alpha.$$

Hence $w_x > 0 \ \Rightarrow \ x \in \arg\max\{v(x) - \sum_i \gamma_i \cdot x_i\} = Ch(\vec{\gamma})$.

If $\alpha > 0$, then $\sum_x w_x = 1$ and thus any optimal solution of the linear program is a lottery over $Ch_j(\vec{\gamma})$.

If $\alpha_j = 0$ any solution of the linear program with $\sum_x w_x < 1$ can be extended to a lottery over $Ch_j(\vec{\gamma})$ by setting $w_0 = 1 - \sum_x w_x > 0$. Then, $0 \in Ch_j(\vec{\gamma})$.

Furthermore, by complementary slackness, any lottery over $Ch_j(\vec{\gamma})$ is an optimal solution of the linear program. ∎

If a quasi-linear preference satisfies (Δ) substitutes, then, if the prices are scaled by a constant, (Δ) substitutes is maintained. It is clear that Proposition 6.1 together with Theorem 5.2 and Theorem 5.3, imply the following:

Proposition 6.2 *If the quasi-linear preference $v_j(x) - p \cdot x$ is substitutes, or Δ-substitutes for all j and p, the equilibrium of Claim 6.1 can be implemented as a lottery over feasible allocations, or allocations whose excess demand is at most $\Delta - 1$ good-by-good, respectively.*

6.2 Course Allocation

Motivated by the course allocation problem, Budish (2011) proposed implementing a social approximate equilibria with approximately equal incomes (A-CEEI). It does not require agents to communicate their preferences over lotteries because it offers a deterministic allocation.

In the A-CEEI mechanism, endowments of artificial currency are allocated at random rather than goods. In expectation, each agent receives the same endowment of artificial currency but ex-post, they receive different but roughly equal amounts. Using the realized endowments of artificial currency, a social approximate CE is computed. Budish (2011) bounds the resulting excess demand in terms of the *Euclidean* distance between the supply vector and the vector of demands by $\frac{\sqrt{\min\{2\Delta, m\}m}}{2}$, where Δ is the size of a maximum bundle that an agent is interested in consuming and m is the number of goods.

The A-CEEI mechanism has been implemented to assign students to courses at the Wharton School (see Budish et al. (2017)). Agents are students, and objects are courses, with the number of seats in a course being the supply of that course. The supply is upper bounded by Fire Safety regulations. Every semester students take about five courses, thus $\Delta = 5$. Because the bound on excess demand is in terms of Euclidean distance, there is no guarantee that the number of students assigned to a course will exceed the regulated limit. For this reason, the mechanism needs to be rerun several times with reduced capacities to ensure feasibility. Nguyen et al. (2016) propose an alternative based on the probabilistic serial mechanism (see Bogomolnaia and Moulin (2001)). It enjoys different efficiency and fairness properties than the A-CEEI mechanism. However, it returns an allocation in which the excess demand for each course is at most $\Delta - 1$. It has been implemented at the Technical University of Munich (see Bichler et al. (2018)). The advantage of this mechanism is that one knows a-priori how many seats in each class to 'withhold' to ensure feasibility. Theorem 4.1 yields a mechanism with the same fairness and efficiency guarantees as in Budish (2011) but with a course-by-course bound of $2\Delta - 1$ on excess demand. Furthermore, this bound holds beyond the case where agents can only consume small bundles. An argument in Nguyen et al. (2016), specific to the case where no agent demands a bundle whose size exceeds Δ, yields a bound of $\Delta - 1$.

Suppose each student j has a feasible set of bundles (course schedules) X_j and a budget of 1.[19] Let $\succeq_j$ denote the ordinal preference of agent j and let $v_j(x)$ be the utility function that represents these preferences. We assume $0 \in X_j$ and $v_j(0) = 0$. Without loss we can assume that each $v_j(x)$ is a rank score of the bundles, $r_j(x)$, consistent with $\succeq_j$ and $r_j(0) = 0$.

Recall that our Theorems on $\Delta - 1$ approximate CE rely on utility functions that depend on money. In the course allocation setting, there is no money, and agents preferences depend on goods only. In order to apply our existence results, we define the following auxiliary utility function for each student. Let

$$U_j^\epsilon(x,t) := v_j(x) + \min\{0, \log\frac{1-t}{\epsilon}\} \text{ for an arbitrarily small, but positive } \epsilon. \tag{6}$$

Let $Ch_j^\epsilon(p)$ denote the choice correspondence of this auxiliary utility function. We show that for any bundle in the choice correspondence of the auxiliary utility function, there is a way to perturb the budget so that under the original ordinal preference, the bundle continues to be the optimal choice. Formally we have the following.

Claim 6.2 *Let $x \in Ch_j^\epsilon(p)$, then there exists a new budget b such that $1-\epsilon \le b < 1$ and*

$$x = max_{(\succeq_j)}\{x' \in X_j \text{ and } p \cdot x' \le b\}.$$

[19]The approach trivially extends to the case where agent's have differing budgets.

Proof. Let $t := p \cdot x$, because of the form of the utility, we know that $t < 1$.

Case 1: If $1 - \epsilon < t < 1$, define the new budget to be $b := t$. Since $x \in Ch_j^{\epsilon}(p)$, for any feasible bundle x' such that $v_j(x') > v_j(x)$ we have $p \cdot x' > p \cdot x = b$. Thus, under the new budget, the agent cannot afford x'.

Case 2: If $t \leq 1 - \epsilon$, then $U_j^{\epsilon}(x, t) = v_j(x)$. Define the new budget to be $b := 1 - \epsilon$. Because $x \in Ch_j^{\epsilon}(p)$, for any feasible bundle x' such that $v_j(x') > v_j(x)$, $p \cdot x' > 1 - \epsilon$ otherwise x' will give a strictly better utility than x. ■

To formally state our result, we modify the notion of approximate competitive equilibrium with equal budgets to account for the ℓ_∞-norm instead of the ℓ_2 norm as in Budish (2011).

Definition 6.1 *Fix an economy, the allocation $(x^1, .., x^n)$, budgets $(b_1, .., b_n)$ and item prices $(p_1, .., p_m)$ constitute an (Δ, ϵ)-approximate competitive equilibrium with equal budget if the following hold:*

- $x^j = max_{(\succeq_j)}\{x \in X_j \text{ and } p \cdot x \leq b_j\}$ *for all* $j \in N$
- $1 - \epsilon \leq b_j \leq 1$ *for all* $j \in N$
- $\max_i |z_i| \leq \Delta$, *where* $z = (z_1, .., z_m)$ *and*

  a) $z_i = \sum_j x_i^j - s_i$ *if* $p_i > 0$

  b) $z_i = \max\{0, \sum_j x_i^j - s_i\}$ *if* $p_i = 0$

Formally, we have the following result, which is a direct application of Theorem 4.1.

Proposition 6.3 *For every $\epsilon > 0$, if U_j^{ϵ} defined as in (6) is Δ-substitutes, there exists a $(\Delta - 1, \epsilon)$-approximate competitive equilibrium with equal budget.*

If no agent is interested in consuming a bundle whose size exceeds Δ, each U_j^{ϵ} is 2Δ-substitutes. Hence, we have an approximate CE with approximately equal incomes in which the excess demand for each good with a positive price is at most $2\Delta - 1$.

Using a specialized argument from Nguyen et al. (2016) for just the case where no agent wants a bundle of size more than Δ, one can improve this bound to $\Delta - 1$. Thus, unlike Budish (2011) one can guarantee that no supply constraint will be violated by more than $\Delta - 1$. Therefore, no more than $\Delta - 1$ seats need be withheld from each course. An approximate competitive equilibrium for the economy where the supply of each course is reduced by $\Delta - 1$ is computed. Seats withheld can be added back to satisfy excess demand. Hence, a single equilibrium computation suffices.

The bound on excess demand given in Budish (2011) depends on the diameter of $conv(Ch(p))$ while ours depends on the edge lengths of the same object. Two examples serve to illustrate the contrast.

First, suppose for some p, $conv(Ch(p))$ is the m-dimensional hypercube. The hypercube is an example of the underlying preferences having many near substitutes. It has maximum diameter among all binary polytopes, i.e., $\sqrt{m}$. The ℓ_1 norm of the edge lengths of a hypercube, however, are all 1, so the bound on excess demand for each good we offer is zero! This illustrates that a large diameter polytope can have small edge lengths.

Now, suppose $conv(Ch(p))$ is the convex hull of just two points: the origin and the m-vector of all 1's. This instance is unusual in that the agent's demand correspondence consists of just two bundles: everything and nothing. It suggests very strong complementarities in preferences. The diameter of $conv(Ch(p))$ is $\sqrt{m}$ while the ℓ_1 norm of the edge length is m.

When there are many near substitutes, we expect that our bound is likely to dominate. When complementarities are pronounced, we expect the ℓ_2 norm bound to dominate.

It is helpful to keep two polar cases of preferences in mind. Gross substitutes and pure complementarities, where the agent desires everything or nothing. In these polar cases, CE exist. Deviations from either result in non-existence of CE. Our results are focused on deviations from the gross substitutes case. Bounds based on the diameter would be more relevant for minor deviations from the case of pure complements.

7 Conclusion

This paper makes two contributions. The first identifies a new sufficient condition for existence of a competitive equilibria which generalizes gross substitute preferences to non quasilinear settings and subsumes other known sufficient conditions. Second, a relaxation of the sufficient condition yields a social approximate equilibrium where the mismatch between supply and demand depends on preferences rather than the size of the economy. The usefulness of this approximation is illustrated in the context of pseudo-markets.

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Appendix

A Proof of Theorem 3.2 and 4.2

We prove Theorem 4.2 for the case of multi-copy demand. The proof of Theorem 3.2 is the same and omitted.

Let P be the convex hull of $\cup_{x \in Ch(p)} B(x)$. Assuming preferences satisfy Δ-improvement we show that P is Δ-uniform. Let $C \geq \max_{i \in M} x_i$ for all $x \in Ch(p)$. Let $\bar{y}$ be an extreme point of P, let $\bar{z}^1, \ldots, \bar{z}^K \in P$ be *all* extreme points of P such that $||\bar{z}^r - y||_1 \leq \Delta$. Note, as P is binary, every integer vector in P is an extreme point.

Suppose, for a contradiction that $\bar{w} - \bar{y}$ is an edge of P such that $||\bar{w} - \bar{y}||_1 > \Delta$. Then, $\bar{w}$ cannot lie in the cone generated by $\{\bar{z}^1 - \bar{y}, \ldots, \bar{z}^K - \bar{y}\}$, i.e., there do not exist non-negative numbers $\{\lambda_i\}_{r=1}^K$ such that

$$\bar{w} - \bar{y} = \sum_{r=1}^{K} \lambda_i (\bar{z}^r - \bar{y}).$$

By Farkas' lemma, there exists a vector β such that $\beta \cdot (\bar{w} - \bar{y}) < 0$ and $\beta \cdot (\bar{z}^r - \bar{y}) \geq 0$ for all $r = 1, \ldots, K$.

Without loss of generality, we order the copies of each good i in non-decreasing order in β, i.e.,

$$\beta_{C \cdot (i-1)+1} \leq \beta_{C \cdot (i-1)+2} \leq \ldots \leq \beta_{C \cdot i}.$$

We now argue that for each good i we can assume that

$$\bar{y}_{C\cdot(i-1)+1} \geq \bar{y}_{C\cdot(i-1)+2} \geq \ldots \geq \bar{y}_{C\cdot i}.$$

If not, there exist two copies of good i: i_1 and i_2 such that $\beta_{i_1} < \beta_{i_2}$ and $\bar{y}_{i_1} = 0; \bar{y}_{i_2} = 1$. Consider the bundle $\bar{y}'$ obtained from $\bar{y}$ by switching the values of $\bar{y}_{i_1}$ and $\bar{y}_{i_2}$. Note $\bar{y}' \in P$ and $\beta \cdot (\bar{y}' - \bar{y}) = \beta_{i_1} - \beta_{i_2} < 0$. Hence, $\bar{y}'$ cannot lie in the cone generated by $\{\bar{z}^1 - y, \ldots, \bar{z}^K - y\}$, However, $2 = ||\bar{y}' - \bar{y}||_1 \leq \Delta$ which is a contradiction.

We now use the extreme point $\bar{w} \in P$ to construct a vector $\bar{w}^* \in P$ such that for each good i:

$$\bar{w}^*_{C\cdot(i-1)+1} \geq \bar{w}^*_{C\cdot(i-1)+2} \geq \ldots \geq \bar{w}^*_{C\cdot i}$$

and $\bar{w}^*$ is not in the cone generated by $\{\bar{z}^1 - \bar{y}, \ldots, \bar{z}^K - \bar{y}\}$.

If there exist two copies of good i: i_1 and i_2 such that $\beta_{i_1} < \beta_{i_2}$ and $\bar{w}_{i_1} = 0; \bar{w}_{i_2} = 1$, switch the values of $\bar{w}_{i_1}$ and $\bar{w}_{i_2}$ to obtain the bundle $\bar{w}' \in P$. Then, $\beta \cdot (\bar{w}' - \bar{w}) = \beta_{i_1} - \beta_{i_2} < 0$. Therefore,

$$\beta \cdot (\bar{w}' - \bar{y}) = \beta \cdot (\bar{w}' - \bar{w}) + \beta \cdot (\bar{w} - \bar{y}) < 0.$$

Hence, $\bar{w}'$ is not in the cone generated by $\{\bar{z}^1 - \bar{y}, \ldots, \bar{z}^K - \bar{y}\}$.

Repeat this step until we terminate in a vector $\bar{w}^*$ where for each j, the components in the interval $[C \cdot (i-1) + 1, C \cdot i]$ are arranged in non-increasing order. Note $\beta \cdot (\bar{w}^* - \bar{y}) < 0$. Hence, $\bar{w}^* \in P$ cannot lie in the cone generated by $\{\bar{z}^1 - \bar{y}, \ldots, \bar{z}^K - \bar{y}\}$ and so $||\bar{w}^* - \bar{y}||_1 > \Delta$.

Let x be such that $\bar{y} \in B(x)$ and x^* such that $\bar{w}^* \in B(x^*)$. Because $||\bar{w}^* - \bar{y}||_1 > \Delta$ and the components of $\bar{w}^*$ and $\bar{y}$ in each interval $[C \cdot (i-1) + 1, C \cdot i]$ are both arranged in non-increasing order, $||x^* - x||_1 > \Delta$.

Consider good i. If $x^*_i = x_i$ set $\tilde{\beta}_i = 0$. If $x^*_i > x_i$, set $\tilde{\beta}_i$ to be the average of the β-s of the copies of good i in $(\bar{w}^* - \bar{y})$. Similarly, if $x^*_i < x_i$, then let $\tilde{\beta}_j$ be the average of the β-s of the copies of good j in $(\bar{y} - \bar{w}^*)$. Hence,

$$\tilde{\beta} \cdot (x^* - x) = \beta \cdot (\bar{w}^* - \bar{y}) < 0.$$

We now invoke the definition of Δ-improvement. Interpret $\tilde{\beta}$ as a price change, i.e. $\delta p = \tilde{\beta}$. Then, $\delta p \cdot x > \delta p \cdot x^*$. Hence, there exist non-negative integer vectors $a \leq (x - x^*)^+$ and $b \leq (x^* - x)^+$ such that $\vec{1} \cdot a + \vec{1} \cdot b \leq \Delta$, $z^* := x - a + b \in Ch(p)$, $||z^* - x||_1 \leq \Delta$ and $(p' - p) \cdot x > (p' - p) \cdot z^*$, i.e. $\tilde{\beta} \cdot (z^* - x) < 0$.

Choose $\bar{z} \in B(z^*)$ such that

$$\bar{z}_{C \cdot (i-1)+1} \geq \bar{z}_{C \cdot (i-1)+2} \geq \ldots \geq \bar{z}_{C \cdot i}.$$

Because of this ordering

$$||\bar{z} - \bar{y}||_1 = ||z^* - x||_1 \leq \Delta.$$

As $\bar{z}$ is a binary representation of z^* and $\bar{y}$ is a binary representation of x it follows that $\beta \cdot (\bar{z} - \bar{y}) = \tilde{\beta} \cdot (z^* - x) < 0$, which is a contradiction.

We now prove the converse. Denote the convex hull of $\cup_{x \in Ch(p)} B(x)$ by P. We will show that if P is Δ-uniform, the corresponding preferences satisfy the Δ-improvement property.

For economy of exposition only, suppose $C = 1$, i.e., the agent wishes to consume at most one copy of each good. In this case the binary presentation of a vector is itself: $B(x) = x$, $\cup_{x \in Ch(p)} B(x) = Ch(p)$ and P is the convex hull of $Ch(p)$. Fix a price vector p and two bundles $x, y \in Ch(p)$. Consider a price change $\delta p \in \mathbb{R}^m$ satisfying $\delta p \cdot x > \delta p \cdot y$.

As all binary vectors of binary polytope are extreme points, $x, y \in ext(P)$. Now, $y - x$ is in the cone generated by the edges of P adjacent to x. This means that there exit a set of extreme points $\{z^1, \ldots, z^K\}$ adjacent to x and a set of *positive* numbers $\{\lambda_1, .., \lambda_K\}$ such that

$$y - x = \sum_{r=1}^{K} \lambda_r (z^r - x). \tag{7}$$

Multiplying both sides of equation (7) by δp, yields

$$0 > \delta p \cdot (y - x) = \sum_{r=1}^{K} \lambda_r \cdot \delta p \cdot (z^r - x).$$

Hence, there exists $r \in \{1, .., K\}$ such that $\delta p \cdot (z^r - x) < 0$.

Let $a := (x - z^r)^+$ and $b := (z^r - x)^+$. Because P is Δ-uniform, $\vec{1} \cdot a + \vec{1} \cdot b \leq \Delta$.

It is left to show that $a \leq (x-y)^+$ and $b \leq (y-x)^+$. Let I_0 and I_1 be the set of coordinates in which both x, y are 0 and 1, respectively. Notice $(z_i^r - x_i) \geq 0$ while $(y_i - x_i) = 0$ for all $i \in I_0$, thus, because of (7), $(z_i^r - x_i) = 0$ for all $r \in \{1, \ldots, K\}$ and $i \in I_0$. Similarly, $(z_j^r - x_j) \leq 0$ while $(y_j - x_j) = 0$ for all $i \in I_1$, and thus because of (7), $(z_j^r - x_j) = 0$ for all $r \in \{1, \ldots, K\}, j \in I_1$. This shows that $(x - z^r)^+ \leq (x - y)^+$ and $(z^r - x)^+ \leq (y - x)^+$ for all $r \in \{1, \ldots, K\}$. ■

B Proof of Theorem 5.1: Existence of a Pseudo Equilibrium

Proof. Denote by X_j the set of feasible bundles available to agent $j \in N$. Let $\mathcal{L}(X_j)$ be the set of lotteries over X_j. We construct a correspondence

$$f : [0, B]^m \times \mathcal{L}(X_1) \times \ldots \times \mathcal{L}(X_n) \rightrightarrows [0, B]^m \times \mathcal{L}(X_1) \times \ldots \times \mathcal{L}(X_n)$$

and use Kakutani's fixed point theorem to show that it has a fixed point, $(p, x^1, .., x^n)$. This fixed point will correspond to a pseudo-equilibrium.

Given $\{x^j \in \mathcal{L}(X_j)\}_{j=1}^n$, let $\overline{x} = \sum_{j=1}^n x^j$ be the aggregate consumption. The excess demand vector is $\overline{x} - s$. Let

$$g(p, x^1, .., x^n) := (p + \overline{x} - s)^+.$$

Notice, for $p \in \mathbb{R}^m_+$, $g(p, x^1, .., x^n) = p$ implies that for all $i \in M$, if $p_i > 0$, then $\overline{x}_i = s_i$ and if $p_i = 0$, then $\overline{x}_i \leq s_i$. This is exactly the condition of that price and excess demand must meet to be considered a pseudo equilibrium.

To apply Kakutani's Theorem, we need the function g to be bounded, so we modify it. As B is the bound on the willingness to pay of each agent, let

$$z_i(p, x^1, .., x^n) := \min\{(p_i + \overline{x}_i - s_i)^+, B\} \text{ for all } i \in M.$$

Each z_i is bounded. Define the following correspondence

$$f(p, x^1, .., x^n) := (z, conv(Ch_1(p)), .., conv(Ch_n(p)).$$

It is easy to see that f satisfies all the conditions of Kakutani's Theorem. Let $(p, x^1, .., x^n)$ be a fixed point of the correspondence f.

At this fixed point $x^j \in conv(Ch_j(p))$. Furthermore, because of the bounded willingness to pay assumption, if $p_i = B$, then for all $x^j \in conv(Ch_j(p))$, $x^j_i = 0$. Thus,

$$\min\{(p_i + \overline{x}_i - s_i)^+, B\} = (B - s_i)^+ < B.$$

Therefore, at the fixed point $p_i = (p_i + \overline{x}_i - s_i)^+$. This implies that $(p, x^1, .., x^n)$ is a pseudo-equilibrium. ■

A pseudo-equilibrium is an equilibrium with respect to the 'convexified' choice correspondences. As these may violate non-satiation, some competitive equilibria may be Pareto inefficient (with respect to the convexified choice correspondences). To avoid this, one can, when possible, select an **efficient disposal** equilibrium as defined in McLennan (2017).

C Proof of Theorem 5.2

C.1 Special Polytope

Call a binary polytope **special** if all its edge vectors have at most two non-zero components and these being of opposite sign.[20]

LEMMA C.1 *If $P \subset \mathbb{R}^{d_1}$ and $Q \subset \mathbb{R}^{d_2}$ are special, then $P \times Q$ is also special.*

Proof. Let $A \subset \mathbb{R}^{d_1}$, be the set of edges in P, $B \subset \mathbb{R}^{d_2}$, be the set of edges in Q, then $A \times \vec{0}^{d_1} \cup \vec{0}^{d_2} \times B$ is the set of edges of $P \times Q$. This shows that if P, Q are special, then $P \times Q$ is also special. (Here $\vec{0}^d$ denotes the 0 vector in $\mathbb{R}^d$.) ■

LEMMA C.2 *Let A be a 0-1 matrix where each column contains exactly one non-zero entry and b a non-negative integral vector. Then, the polytope $H = \{z : Az = b, 0 \leq z \leq 1\}$ is special.*

Proof. The d-dimensional simplex, $\{z : z_1 + \ldots + z_d = k,\ 0 \leq z_i \leq 1\}$ is a special polytope. The polytope H is a Cartesian product of d-dimensional simplices for various d, therefore it is also special. ■

Standard arguments (see Frank et al. (2014), for example) tell us that the intersection of two special binary polytopes has integral extreme points. For completeness, we furnish proof of this fact.

LEMMA C.3 *If P and Q are two special binary polytopes in $\mathbb{R}^d$ with $P \cap Q \neq \emptyset$, then, the extreme points of $P \cap Q$ are integral.*

Proof. By the fundamental theorem of Linear Programming, it suffices to show that for any $w \in \mathbb{R}^d$, $\{\max w \cdot z : z \in P \cap Q\}$ has an integral solution. Let z be an extreme point solution of this program. Let F_1, F_2 be the minimal faces of P and Q, respectively, that contain z.

[20] This is a subset of the class of compressed polytopes introduced in Stanley (1980).

Because z is an extreme point solution, the linear spaces spanned by the edges of F_1 and F_2 are independent.[21]

Let v_i be an extreme point of F_i and E_i the set of its edges for $i = 1, 2$. As $z \in F_1 \cap F_2$, there exist $\alpha_e \in \mathbb{R}$ for $e \in E_1 \cup E_2$ such that

$$z = v_1 + \sum_{e \in E_1} \alpha_e \cdot e = v_2 + \sum_{e \in E_2} \alpha_e \cdot e. \tag{8}$$

It remains to show that $\sum_{e \in E_1} \alpha_e \cdot e$ is integral as this implies that z is integral.

Equation (8) implies

$$v_1 - v_2 = \sum_{e \in E_2} \alpha_e \cdot e - \sum_{e \in E_1} \alpha_e \cdot e. \tag{9}$$

Because F_1, F_2 are each special polytopes, their edges $E_1 \cup E_2$ are $\{0, \pm 1\}$ vector with at most 2 nonzero components and these are of opposite signs. Hence, the matrix each of whose columns is in $E_1 \cup E_2$ is a network matrix and therefore totally unimodular. Combined with the fact that $v_1 - v_2$ is integral implies that there exists $\beta_e \in \mathbb{Z}$ for $e \in E_1 \cup E_2$ such that

$$v_1 - v_2 = \sum_{e \in E_2} \beta_e \cdot e - \sum_{e \in E_1} \beta_e \cdot e. \tag{10}$$

Equations (9) and (10) imply

$$\sum_{e \in E_1} (\alpha_e - \beta_e) \cdot e = \sum_{e \in E_2} (\alpha_e - \beta_e) \cdot e. \tag{11}$$

Now, the linear spaces spanned by E_1, E_2 are independent. Therefore, either side of equation (11) must be zero 0. Thus, $\sum_{e \in E_1} \alpha_e \cdot e = \sum_{e \in E_1} \beta_e \cdot e$, which is an integral vector as desired.

■

[21] This is because the faces F_1 and F_2 correspond to the binding constraints which must be linearly independent at an optimal solution.

C.2 Proof of Theorem 5.2

First, we identify a pseudo-equilibrium price p and corresponding allocation $\{x^1, \ldots, x^n\}$ such that $x^j \in conv(Ch_j(p))$ for all $j \in N$. It is technically convenient to assume that $\sum_{j \in N} x^j = s$. This is without loss because otherwise, the price will be 0, and we can add a dummy agent with 0 marginal utility over the good and consumes the leftover.

Let

$$P_j = conv(\cup_{x \in Ch_j(p)} B(x)),$$

where $B(x)$ is a binary presentation of x, as defined in (5). Because the preferences are geometric substitutes, the binary polytopes $P_1, \ldots, P_n$ are special. Therefore, according to Lemma C.1, $P_1 \times .. \times P_n$ is special.

Let Q denote the following polytope

$$Q := \{(z^1, .., z^n) | z^i \in [0,1]^{Cm} \text{ and } \sum_{j=1}^{n} T_i(z^j) = s_i \text{ for all } i \in M\}.$$

From the definition of $T(.)$ in (4), the i^{th} constraint that defines Q has the form:

$$\sum_{j=1}^{n} \sum_{k=C\cdot(i-1)+1}^{C\cdot i} z_k^j = s_i.$$

From this we see that the non-zero coefficients in constraint i and $i' \neq i$ do not overlap. Hence, the matrix of coefficients associated with the linear system $\sum_{j=1}^{n} T_i(z^j) = s_i$ for all $i \in M$ has all 0-1 entries with exactly one non-zero entry in each column. According to Lemma C.2, Q is a special polytope.

For each $x^j \in conv(Ch_j(p))$ there is at least one corresponding $y^j \in P_j$. Furthermore, $\sum_{j \in N} x^j = s$ implies that $\sum_{j=1}^{n} T_i(y^j) = s_i$ for all $i \in M$. Hence, $y \in P_1 \times .. \times P_n \cap Q$. According to Lemma C.3, $P_1 \times .. \times P_n \cap Q$ is an integral polytope. Because of this y can be expressed as a lottery over the integral vectors in $P_1 \times .. \times P_n \cap Q$. Each integral vector in $P_1 \times .. \times P_n \cap Q$ is a binary presentation of an allocation $(w^1, .., w^n)$ in $Ch_1(p) \times .. \times Ch_n(p)$

that satisfies $\sum_{j \in N} w^j = s$. This shows that w is a competitive equilibrium. Hence, any pseudo equilibrium can be expressed as a lotteries over a set of competitive equilibria. ■

D Proof of Theorem 5.3 and 5.4

D.1 Proof of Theorem 5.3

We first show how Theorem 5.3 follows from Theorem 5.4.

Proof of Theorem 5.3 First, we identify a pseudo-equilibrium price p and corresponding allocation $\{x^1, \ldots, x^n\}$ such that $x^j \in conv(Ch_j(p))$ for all $j \in N$. It is technically convenient to assume that $\sum_{j \in N} x^j = s$. This is without loss because we can add dummy agents who demand all goods. Let P_j denote the binary polytope $\cup_{x \in Ch_j(p)} B(x)$, where $B(x)$ is a binary presentation of x, as defined in (5).

For each $x^j \in conv(Ch_j(p))$ there is at least one corresponding $y^j \in P_j$. Furthermore, $\sum_{j \in N} x^j = s$ implies that $\sum_{j=1}^{n} T_i(y^j) = s_i$ for all $i \in M$. Hence, all the conditions needed to invoke Theorem 5.4 are satisfied. This yields Theorem 5.3. ■

D.2 Lottery implementation

The following basic result is needed for our lottery implementation result.

LEMMA D.1 *Given an $y \in \mathbb{R}^m$, and some property $(*)$, y can be expressed as a lottery over a set of vectors satisfying $(*)$ if and only if for every weight vector $w \in \mathbb{R}^m$, there exists $z \in \mathbb{R}^m$ satisfying $(*)$ and $w \cdot z \geq w \cdot y$.*

Proof. Let Z be the set of all vectors satisfying property $(*)$. Now, $y \notin conv(Z)$ if and only if there exists a hyperplane separating y from $conv(Z)$. This means that there is a $w \in \mathbb{R}^m$ such that $w \cdot y > w \cdot z$ for all $z \in Z$. ■

Any algorithm that finds a vector z satisfying $(*)$ and $w \cdot z \geq w \cdot y$ can be used to express y as a convex combination of vectors in Z (see Nguyen et al. (2016)).

D.3 Proof of Theorem 5.4

First we need some terminology. Given a binary polytope $Q \in \mathbb{R}^{Cmn}$, a coordinate $i \in \{1, \ldots, Cmn\}$ is called **fixed with respect to** Q if $x_i = y_i$ for all $x, y \in Q$. Otherwise it is called **free**. In other words, if a coordinate i is fixed with respect to Q it means that there is a constant θ such that $x_i = \theta$ for all $x \in Q$. The proof is based on the algorithm described in Figure 1.

Figure 1: ALGORITHM

Input: $\{P_j\}_{j=1}^n$ Δ-uniform binary polytopes, $y = (y^1, \ldots, y^n) \in P_1 \times \ldots \times P_n$, $\sum_{j=1}^n T_i(y^j) = s_i \in \mathbb{Z}$, and weight vector $w \in \mathbb{R}^{Cmn}$.

Output: $z \in ext(P_1) \times \ldots \times ext(P_n)$ such that $w \cdot z \geq w \cdot y$ and for all $i \in M$, $|T_i(\sum_j z^j) - s_i| \leq \Delta - 1$.

Step 0: Initiate $Q = P_1 \times \ldots \times P_n$, and let $S := M$ to be the set of active supply constraints.

Step 1: Let

$$R := \{z = (z^1, \ldots, z^n) | z^i \in [0,1]^{Cm}; \sum_{j=1}^n T_i(z^j) = s_i \text{ for all } i \in S\}.$$

Solve

$$\{\max w \cdot z | z \in Q \cap R\}. \tag{12}$$

Let $z = (z^1, \ldots, z^n)$ be an optimal extreme point solution and F be the minimal face of Q containing z.

Step 2a: If $dim(F) = 0$, that is F contains a single element, call it $(z^1, \ldots, z^n)$ and STOP.

Step 2b: Else, each constraint i such that $\sum_{j=1}^n T_i(z^j) = s_i$, can be written as $\alpha(i) \cdot z = s_i$, where $\alpha(i)$ is a $\{0,1\}$ vector.

Let a_i be the number of non-zero coordinates of $\alpha(i)$ that are free with respect to F. Among the active constraints in S, choose the constraint $i \in S$ with smallest a_i.

Update $Q := F$ and $S := S \setminus \{i\}$.

Step 3: Return to **Step 1**.

We now show that the algorithm is correct and terminates in the desired solution z. At each iteration of the algorithm, we solve a linear program (12) with the objective given by the weight vector w and one less supply constraint than the iteration before. Thus, the linear program (12) is feasible at each iteration, and the optimal objective function value is non-decreasing. Thus, the terminal solution z will satisfy $w \cdot z \geq w \cdot y$.

Claim D.1 *At each iteration of the algorithm Q is Δ uniform and the minimal face $F \subset Q$ containing the extreme point solution of (12) satisfies $dim(F) \leq |S|$.*

Proof. At the beginning of the algorithm $Q = P_1 \times .. \times P_n$. As each P_i is Δ uniform, by Lemma C.1, so is Q. In the subsequent iterations, Q is replaced by one of its faces. The edges of a face are a subset of the edges of the corresponding polytope. Thus, Q remains Δ uniform.

To show $dim(F) \leq |S|$, rewrite the constraints of the linear program (12) in matrix form:$\{Az \leq a, Bz = b\}$. Here $Az \leq a$ expresses $z \in Q$ while $Bz = b$ expresses $z \in R$. In an extreme point solution, the number of independent binding constraints is equal to the number of non-zero coordinates of z, which is Cmn. B has $|S|$ rows. Thus, the number of independent binding constraints in $Az \leq a$ is at least $Cmn - |S|$. This shows that the minimal face containing z is of dimension at most $|S|$. ■

Claim 2 shows that when there are no more supply constraints left to delete, $dim(F) = 0$ and the algorithm terminates. Hence, the algorithm terminates in at most m iterations.

To show that the violation in each supply constraint cannot exceed $\Delta - 1$, we need the following.

Claim D.2 *At each iteration, the number of free coordinates with respect to Q is at most $\Delta \cdot dim(Q)$.*

Proof. Fix an extreme point v of Q and let E_v be a maximal linearly independent set of edges of Q that are incident to v. Recall that the dimension of the space spanned by the edges incident to any extreme point is equal to the dimension of the polytope. Thus, $|E_v| = dim(Q)$.

By claim D.1, Q is binary and Δ-uniform, the components of each of the vectors in E_v belongs to $\{-1, 0, 1\}$ and the number of non-zero components in each of them is at most Δ.

By the definition of free coordinates, a non-zero component of an edge can only be at a free coordinate of Q. The reverse is also true. If j is a free coordinate, then there exists $v' \in Q$ such that the j^{th} coordinate of $v' - v$ is not 0. But $v' - v$ is in the span of E_v, thus, there must be a vector in E_v, whose j^{th} coordinate is not 0.

There are $dim(Q)$ vectors in E_v, each vector has at most Δ nonzero components. Therefore, the number of free coordinates of Q is at most $\Delta \cdot dim(Q)$. ∎

Claim D.3 *Let $z^* = (z^{*1}, .., z^{*n})$ be the algorithm's output. Then, for every good $i \in M$*

$$|\sum_{j=1}^{n} T_i(z^{*j}) - s_i| \leq \Delta - 1.$$

Proof. Fox some iteration of the algorithm where S is the set of active supply constraints, z is the extreme point solution of (12), and $F \subset Q$ is the minimal face of Q containing z.

The coordinates of the non-zero components of distinct supply constraints are disjoint. By Claim D.2 there are at most $\Delta \cdot dim(F)$ free coordinates with respect to F. By Claim D.1, $dim(F) \leq |S|$. Thus, the supply constraint selected for deletion in Step (2b) of the algorithm contains at most Δ free coordinates with respect to F.

The outcome of the algorithm belongs to a polytope that shrinks in dimension at every step of the algorithm. Thus, as the output of the algorithm, $z^* \in F$. Consider the supply constraint i to be deleted at this iteration. Because constraint i contains at most Δ free coordinates with respect to the binary polytope F and $z \in F$ as well, it follows that

$$|\sum_{j=1}^{n} T_i(z^{*j}) - \sum_{j=1}^{n} T_i(z^j)| \leq \Delta.$$

Equality can only occur if constraint i contains *exactly* Δ free coordinates and the values of these coordinates in z are either all 0 or all 1, while the opposite is true for z^*. However, this is impossible because, for example, if the values of these coordinates in z are all 0 then, these coordinates will be fixed (at 0) with respect to the minimal face containing $\{z, z^*\}$, which contradicts the fact that they are free coordinates.

Now, because $\sum_{j=1}^n T_i(z^j) = s_i$ and $\sum_{j=1}^n T_i(z^{*j})$ are both integral, we have.

$$|\sum_{j=1}^n T_i(z^{*j}) - \sum_{j=1}^n T_i(z^j)| = |\sum_{j=1}^n T_i(z^{*j}) - s_i| \leq \Delta - 1.$$

This is what we need to prove. ∎