

Price Heterogeneity as a source of Heterogenous Demand*

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Abstract

We explore heterogenous prices as a source of heterogenous or stochastic demand. Heterogenous prices could arise either because there is actual price variation among consumers or because consumers (mis)perceive prices differently. Our main result says the following: if heterogenous prices have a distribution among consumers that is (in a sense) stable across observations, then a model where consumers have a common utility function but face heterogenous prices has precisely the same implications as a heterogenous preference/random utility model (with no price heterogeneity).

Keywords. random utility, stochastic demand, augmented utility, misperceived prices, reference prices, equivalence scales, Afriat's Theorem.

1 Introduction

Why is demand heterogenous? Two consumers with the same budget and facing the same prices would often choose to spend their money differently. The same consumer, when

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presented with the same budgetary constraints, may also choose different purchases at different times. The most common and obvious way of modelling heterogenous behavior across consumers (or stochastic behavior in the case of a single consumer) is to assume that demand is governed by a distribution of preferences (rather than a single preference). When the distribution of preferences is stable across multiple observations, this leads to a random utility model of stochastic choice (see, for example, McFadden and Richter (1990), Kitamura and Stoye (2018), and Deb et al. (2018)). But preference heterogeneity is not the only source of variation in choice behavior. More recently, the literature on stochastic choice has highlighted the presence of other sources of heterogenous choice. For example, choice could be stochastic due to random consideration/attention sets (Manzini and Mariotti (2014), Brady and Rehbeck (2016), Cattaneo et al. (2020)) or due to deliberate randomization (Fudenberg et al. (2015), Cerreia-Vioglio et al. (2019)). In this paper, we wish to highlight another reason why choices can be heterogenous: the presence of heterogenous prices across consumers. Price heterogeneity could occur for at least two reasons.

Firstly, consumers could *actually* face different prices due to price dispersion (Stigler (1961)), which arises because they live in different locations and/or shop at different stores, and it is not always worthwhile for them to search for the cheapest price. Related to this, empirical demand analysis often involves the aggregation of highly granular demand information into broad categories (such as ‘food’ or ‘clothing’); the price index constructed for a given category of goods is not tailored to each consumer and so may not accurately represent the prices encountered by a given consumer for goods in that category.

Secondly, even when consumers face the same prices, they may still (mis)perceive the same prices differently, leading to different demand behavior. Price misperception could occur because consumers are inattentive to prices (Chetty et al. (2009), Gabaix (2014), Matějka (2015)) or because they over- or under-estimate their usage of a prepaid good or service (DellaVigna and Malmendier (2006)). A related phenomenon is that the demand for a product could be affected by its ‘reference price’ (see Koszegi and Rabin (2006), Heidhues and Köszegi (2008), Bordalo et al. (2013)). Thus, the same price can be perceived as high or low depending on the reference price and the variation in reference prices across consumers leads to heterogenous demand behavior.¹

¹ There is also a large literature in marketing on reference-price purchasing, e.g., see Monroe (1973),

In order to investigate the testable implications of price heterogeneity, we assume that the analyst observes the expenditure allocations (across K goods) of N consumers at different price indices. Formally, a data set with T observations and can be represented by

$$\mathcal{E} = \{((\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t)\}_{t \in T},$$

where $\mathbf{e}^{i,t} = (e_k^{i,t})_{k \in K}$ is the expenditure allocation of consumer i and $\bar{\mathbf{p}}^t = (\bar{p}_k^t)_{k \in K}$ are the prices of the K goods at observation t .² Ignoring the possibility of price heterogeneity for now, consumer i 's consumption at observation t is the bundle $\mathbf{x}^{i,t}$, where the k th entry, $x_k^{i,t} = e_k^{i,t}/\bar{p}_k^t$. For each consumer i , Afriat's Theorem (Afriat (1967), Varian (1982)) allows us to check whether the consumer's behavior is consistent with utility maximization, i.e., whether there is a utility function u^i such that $u^i(\mathbf{x}^{i,t}) \geq u^i(\mathbf{x})$ for all \mathbf{x} such that $\bar{\mathbf{p}}^t \cdot \mathbf{x} \leq \sum_{k \in K} e_k^{i,t}$. Indeed, Afriat's Theorem tells us that this holds if and only if the implied consumption bundles of consumer i , $\{\mathbf{x}^{i,t}\}_{t \in T}$, along with the observed prices $\{\bar{\mathbf{p}}^t\}_{t \in T}$, satisfy a property called the generalized axiom of revealed preference (or GARP, for short). If this holds for each agent i , then we say that \mathcal{E} can be *rationalized by heterogenous preferences*.

The problem we pose is different. Instead of allowing consumers to have different utility functions, we require them to have the *same* utility function but allow each consumer to face idiosyncratic (whether real or perceived) prices $\mathbf{p}^{i,t}$. So we are, in a sense, inverting the requirements in a heterogenous preference model. We say that the data set \mathcal{E} can be *rationalized by heterogenous prices* if there is a common utility function u such that consumer i 's expenditure allocations at different observations maximize u , given the idiosyncratic prices $\{\mathbf{p}^{i,t}\}_{t \in T}$ encountered by consumer i . In formal terms, $u(\hat{\mathbf{x}}^{i,t}) \geq u(\mathbf{x})$, where $\hat{x}_k^{i,t} = e_k^{i,t}/p_k^{i,t}$ and \mathbf{x} satisfies $\mathbf{p}^{i,t} \cdot \mathbf{x} \leq \sum_{k \in K} e_k^{i,t}$. The interpretation of the bundles $\{\hat{\mathbf{x}}^{i,t}\}_{t \in T}$ depends on the interpretation we give to $\{\mathbf{p}\}^{i,t}$.

In the case where $\{\mathbf{p}^{i,t}\}_{t \in T}$ are interpreted as the true prices faced by consumer i , then $\hat{\mathbf{x}}^{i,t}$ is the true bundle purchased by this consumer at observation t ; this bundle is not observed by the researcher, who only observes i 's expenditure allocation $\mathbf{e}^{i,t}$ and the price index $\bar{\mathbf{p}}^t$. Under this interpretation, if \mathcal{E} can be rationalized by heterogenous prices, then all consumers share the same utility function and each of them is rational, in the sense that

Winer (1986), and Mazumdar, Raj, and Sinha (2005).

²Note that T (similarly N and K) denotes both a set and the number of elements in the set.

there are idiosyncratic prices at which each consumer's observed expenditure allocations are optimal. In the case where $\{\mathbf{p}^{i,t}\}_{t \in T}$ are interpreted as consumer i 's misperceived prices, then $\hat{\mathbf{x}}^{i,t}$ is the bundle that this consumer *thinks* she is consuming when making her expenditure allocation decision across goods. We are assuming that the expenditure allocation $\mathbf{e}^{i,t}$ is the one consumer i implements, so that the consumer's actual consumption at observation t is $\mathbf{x}^{i,t}$. Under this interpretation, if \mathcal{E} can be rationalized by heterogenous prices, then all consumers share the same utility function and each consumer's expenditure allocations are also rational, once her misperception of prices is taken into account.

It is intuitively clear that our notion of rationalization, in itself, is too permissive to lead to meaningful implications on data. What is needed are sensible restrictions on how heterogenous prices are distributed across consumers.

In Section 3, we require the heterogenous prices to be *correct on average*, in the sense that, at each observation t and for each good k , we have $(\sum_{i \in N} p_k^{i,t})/N = \bar{p}_k^t$. Beyond this requirement, we allow the distribution of heterogenous prices to vary freely between observations. In this case, we show that *any* data set \mathcal{E} can be rationalized by heterogenous prices. In particular, price heterogeneity in this sense is more powerful than preference heterogeneity in its ability to explain observed behavior, since each consumer's expenditure allocation (along with the prevailing price indices) must satisfy GARP in the latter model.

In Section 4 we impose a stronger restriction on price heterogeneity. We assume that price heterogeneity has a *stable structure* across observations. This assumption requires that the distribution of prices for good k (across consumers) at some observation t is the same as the distribution of prices for good k at observation t' , after normalizing by the average price of k at each observation. When this condition is imposed on the price distributions, we find that stable price heterogeneity leads to the *same* restrictions on \mathcal{E} as preference heterogeneity. In other words, *a data set \mathcal{E} can be rationalized by heterogenous preferences if and only if it can be rationalized by heterogenous prices with a stable structure.*³

³ Our result also establishes another equivalence statement which may be interesting in its own right. We show that if \mathcal{E} can be rationalized by heterogenous preferences, then all the preferences can be drawn from a family of utility functions related to each other via equivalence scales. Equivalence scales are widely used in empirical models of demand to capture (among other things) the effects of household composition (see, for example, Lewbel (1989), Blundell and Lewbel (1991), and Deaton (1997)) and in models of market demand they can be used to endow a parametric structure to preference heterogeneity (see, for example, Grandmont (1992) and Quah (1997)).

In Section 5 we study the impact of price heterogeneity in a different model of demand. Instead of a consumer who maximizes utility subject to a budget constraint, we posit a consumer who maximizes a utility function where expenditure enters directly. Formally, the utility of a bundle \mathbf{x} acquired at cost e , is $V(\mathbf{x}, -e)$, where we require V to be strictly increasing in the last argument (in other words, higher expenditure gives strictly lower utility). Following Deb et al. (2018), we refer to V as an (expenditure-)augmented utility function. As a special case, V can have the quasilinear form, where $V(x, -e) = U(x) - e$ for some function U .

At prices \mathbf{p} the consumer chooses the bundle \mathbf{x} to maximize $V(\mathbf{x}, -\mathbf{p} \cdot \mathbf{x})$. In this model, the consumer does not have a hard constraint on her expenditure, but she is deterred from buying an arbitrarily large bundle because of the disutility that higher expenditure incurs. An augmented utility model is particularly suited for the study of demand under price misperception because it allows for any bundle to be purchased, even a bundle chosen under misperceived prices; on the other hand, in a model with constrained optimization, such a bundle will not be implementable if it violates the budget constraint.⁴

Given a data set \mathcal{E} , we could ask whether \mathcal{E} is rationalizable by heterogeneous augmented utility functions, assuming that (at each observation) prices are homogeneous across consumers. Deb et al. (2018) provides an analogous version of Afriat’s Theorem that allows us to characterize those data sets \mathcal{E} which can be rationalized in this sense. It would also be natural to ask what conditions on \mathcal{E} would guarantee that it can be rationalized by a model where all consumers have the same augmented utility function but are optimizing according to misperceived prices. Formally, we require an augmented utility function V and heterogeneous prices $\{\mathbf{p}^{i,t}\}_{t \in T}$ such that $\mathbf{x}^{i,t}$ (consumer i ’s actual consumption bundle given the true prices $\bar{\mathbf{p}}^t$) maximizes $V(\mathbf{x}, -\mathbf{p}^{i,t} \cdot \mathbf{x})$ at each observation t (i.e., augmented utility at misperceived prices). We show that a result similar to that presented in Section 4 holds: a data set \mathcal{E} can be rationalized by heterogeneous augmented utility functions if and only if it can be rationalized by heterogeneous prices with a stable structure.

⁴ Our solution to this problem is to assume that the consumer always implements her *expenditure* allocation decision. For another solution to this problem, see Gabaix (2014).

2 Model and basic definitions

There are N consumers who purchase bundles of K goods. We denote by $e_k^{i,t}$ the amount that consumer i spends on good k at observation t . We assume that $e_k^{i,t} \geq 0$ and $\sum_{k \in K} e_k^{i,t} > 0$, i.e., consumer i 's total expenditure is always strictly positive though expenditure on a given good may equal zero. These expenditure allocations are observed by an analyst. Consumer i at observation t bases his expenditure allocation decision on the prices $\mathbf{p}^{i,t} = (p_k^{i,t})_{k \in K}$. Notice that these prices can be idiosyncratic; we assume that they are *not* observed by the analyst. However, the analyst does observe \bar{p}_k^t , which is an aggregate (according to some formula known to the analyst) of the prices of good k at observation t . Assuming that the analyst makes $T < \infty$ observations, we may denote the data set collected in the form

$$\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}, \quad (1)$$

where $\mathbf{e}^{i,t} = (e_k^{i,t})_{k \in K}$ and $\bar{\mathbf{p}}^t = (\bar{p}_k^t)_{k \in K}$. (Notice that we have abused notation by using T to denote both the largest number of observations and the set $\{1, 2, \dots, T\}$, and similarly with K and N .) We assume that $p_k^{i,t} > 0$ and $\bar{p}_k^t > 0$, i.e., for every good and at every observation, the prices faced by the consumers, and the price index are *strictly* positive.

There are two possible interpretations of the idiosyncratic prices:

(1) Observational Errors The analyst's price index $\bar{\mathbf{p}}^t$ represents a simplification of a more complex environment; $\mathbf{p}^{i,t}$ are the true prices faced by consumer i and they could be different from $\bar{\mathbf{p}}^t$ because of geographical variation in prices, variation in prices within a particular observation period t , or unobserved costs which are incurred by the consumer when purchasing a particular good.

(2) Price Misperception Consumer i makes his expenditure allocation decision based on an idiosyncratic perception of prices at observation t . The prices $\mathbf{p}^{i,t}$ could be incorrect, or misperceived, in the sense that they are different from the true prices $\bar{\mathbf{p}}^t$, which (unlike in the first interpretation) are perfectly observed by the analyst. Price misperception could occur because consumers are inattentive to prices (Chetty et al. (2009), Gabaix (2014), Matějka (2015)), for example, failing to incorporate sales taxes or shipping charges. Sometimes the phenomenon could also be exacerbated by firms engaging in price shrouding (Gabaix and Laibson (2006), Brown et al. (2010)). Phenomena where people misperceive

their usage of a prepaid good/service (such as a gym membership) could also be considered as a form of price misperception (see DellaVigna and Malmendier (2006)).⁵

We would like to find conditions under which the data set \mathcal{E} has the following properties: (i) at each observation, the heterogenous prices across consumers are consistent with the observed price indices in some sense and (ii) there is a single utility function that could explain the observed expenditure patterns. To explain this more precisely, some definitions are in order.

For given expenditure and price vectors $\mathbf{e} \in \mathbb{R}_+^K$ and $\mathbf{p} \in \mathbb{R}_{++}^K$, we denote the implied consumption bundle by $\mathbf{x}(\mathbf{e}, \mathbf{p})$; formally, $\mathbf{x}(\mathbf{e}, \mathbf{p}) \equiv (e_k/p_k)_{k \in K}$. The bundles which cost as much or less than $\mathbf{x}(\mathbf{e}, \mathbf{p})$ is given by the *budget set*

$$\mathcal{B}(\mathbf{p}, m) = \{\mathbf{x} \in \mathbb{R}_+^K \mid \mathbf{p} \cdot \mathbf{x} \leq m\},$$

where $m = \sum_{k \in K} e_k$ is the total expenditure.

Utility Maximization. Let $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ be a collection of heterogenous prices. We say that \mathcal{P} **rationalizes** $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ if there is a utility function $u : \mathbb{R}_+^K \rightarrow \mathbb{R}$ such that for each $(i, t) \in N \times T$,

$$u(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})) \geq u(\mathbf{x}) \text{ for any } \mathbf{x} \in \mathcal{B}(\mathbf{p}^{i,t}, m^{i,t}), \quad (2)$$

$$\text{where } m^{i,t} = \sum_{k \in K} e_k^{i,t}.$$

In other words, there is a utility function (common to all consumers) such that each consumer's demand at observation t (implied by the observed expenditure shares and the idiosyncratic prices) gives greater utility than any bundle that costs as much or less. The data set \mathcal{E} is **rationalizable with heterogenous prices** if there are heterogenous prices \mathcal{P} that rationalize \mathcal{E} , typically subject to some consistency condition on \mathcal{P} which we shall discuss later.

Notice that in this notion of rationalization, there is *heterogeneity in prices across consumers* but no heterogeneity in preferences. The precise interpretation of $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})$ depends on our interpretation of idiosyncratic prices.

⁵ Price misperception is a form of bounded rationality, but it is formally different from models (such as Masatlioglu et al. (2012) and Manzini and Mariotti (2014)) where agents only consider some of the options open to them. The 'consideration set' in these models is always a subset of the budget set, whereas misperceived prices lead to a misperceived budget that is typically *not* a subset of the true budget set.

In the first (Errors) interpretation of idiosyncratic prices, these prices are true prices and so $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})$ is the actual bundle purchased by the consumer, while (2) guarantees that each consumer is truly optimizing at the utility function u .

In the second (Misperceptions) interpretation, the bundle $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})$ is what consumer i expects to be able to buy at his perceived prices and (2) guarantees that, with utility function u , consumer i thinks that his expenditure allocation decision is optimal, given his perception of prices. Since our model assumes that the allocated expenditures are *observed*, we are assuming that the consumer implements the expenditure allocation decision, so that (with $\bar{\mathbf{p}}^t$ being the true prices) the actual level of good k consumed by i at observation t is $e_k^{i,t}/\bar{p}_k^t$. Notice that the expenditure allocation decision by the consumer will typically not be optimal given the true prices, and the consumer may even become aware of the true prices as he implements his expenditure allocation decision, but we are assuming that he does not re-optimize: he retains the expenditure allocation decision and simply adjusts by buying more or less of the good.⁶

Price Consistency. To render this model interesting, there must be some way of disciplining the sort of price heterogeneity permitted by the model. We do this via two notions of price consistency.

We call a function $W : \mathbb{R}_{++}^N \rightarrow \mathbb{R}_{++}$ a **price aggregator** if it is strictly increasing, continuous, and $W(p, \dots, p) = p$ for any $p > 0$. For example, the arithmetic average across all consumers

$$W((p^i)_{i \in N}) = \frac{\sum_{i \in N} p^i}{N} \quad (3)$$

is a price aggregator. We assume that, at each observation t and for each good k , there is an aggregator function W_k^t that is known to the analyst. In our first result (Proposition 1), we assume that the aggregator functions $\mathcal{W} = \{W_k^t\}_{k \in K, t \in T}$ satisfy one of the following regularity conditions:

- i) $\lim_{p^i \rightarrow +\infty} W_k^t(p^i, (p^j)_{j \neq i}) = +\infty$ for any $(p^j)_{j \neq i} \in \mathbb{R}_{++}^{N-1}$ and $(t, k) \in T \times K$, or
- ii) $\lim_{p^i \rightarrow 0} W_k^t(p^i, (p^j)_{j \neq i}) = 0$ for any $(p^j)_{j \neq i} \in \mathbb{R}_{++}^{N-1}$ and $(t, k) \in T \times K$.

⁶ For another way of reconciling price misperception with a binding budget constraint see Gabaix (2014). In Section 5 we present a different model of consumer demand where there is no binding budget constraint and so even a choice under misperceived prices can be implemented exactly.

Given the price aggregators $\mathcal{W} = \{W_k^t\}_{(k,t) \in K \times T}$, the idiosyncratic prices $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ are **consistent in expectation** with the price indices $\{\bar{\mathbf{p}}^t\}_{t \in T}$ if, for any $(k, t) \in K \times T$,

$$\bar{p}_k^t = W_k^t((p_k^{i,t})_{i \in N}). \quad (4)$$

In the Errors interpretation of idiosyncratic prices, \mathcal{W} could be the actual formulae used in the calculation of the observed price indices, so that (4) requires that the hypothesized \mathcal{P} agrees on average with the aggregate price data. In the Misperceptions interpretation, (4) says that the consumers are on average correct in their price perceptions (according to some suitable notion of average imposed by the modeler via \mathcal{W}).⁷

The second notion of price consistency we consider is to require \mathcal{P} to have a **stable distribution** with respect to the price indices $\{\bar{\mathbf{p}}^t\}_{t \in T}$. This means that there is $\lambda_{i,k}$ such that for any $t \in T$,

$$p_k^{i,t} = \lambda_{i,k} \bar{p}_k^t. \quad (5)$$

For each good, the distribution of heterogenous prices is stable across observations; indeed, if consumer i assigns a price to k that is 10% above \bar{p}_k^t at *some* observation t , then he will do the same at *every* observation. It is clear that a stable price distribution will also be consistent in expectation under the following assumptions: (a) W_k^t does not vary with t and thus can be denoted by W_k ; and (b) W_k is homogeneous of degree 1, with the normalization that $W_k((\lambda_{i,k})_{i \in N}) = 1$ for all k . Obviously, these conditions hold if the price aggregator is the arithmetic average (3) for every good and every observation.

Preference heterogeneity. Of course the standard way of imposing structure on a data set is to require that all agents face the same prices but allow them to have different preferences. The data set $\mathcal{E} = \{(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)\}_{(i,t) \in N \times T}$ is **rationalizable with heterogenous preferences** if there are strictly increasing and continuous utility functions $u^i : \mathbb{R}_+^K \rightarrow \mathbb{R}$ such that for each $(i, t) \in N \times T$,

$$u^i(\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)) \geq u^i(\mathbf{x}) \text{ for any } \mathbf{x} \in \mathcal{B}(\bar{\mathbf{p}}^t, m^{i,t}). \quad (6)$$

⁷ Misperception restrictions of this type are not unique to our model; indeed, our notion is akin to the ‘rational expectations’ assumption made, for example, in Koszegi and Rabin (2006, 2007).

Cross-Sectional Data Environment. As we explore the implications of price heterogeneity in this paper, we shall often compare our results with what is known in the random utility model (RUM).⁸

The starting point in these models is a data set with repeated cross sections of the demand distribution at different prices. To keep our discussion focused on the essentials, we shall assume a discrete cross-sectional data set. With minor modifications, our results carry through to the case where the demand distributions are not necessarily discrete; this is considered in detail in the Appendix. Given the discreteness assumption, the data set, consisting of T observations, can be written as

$$\mathcal{D} = \{E^t, \bar{\mathbf{p}}^t\}_{t \in T}. \quad (7)$$

At each observation t , the analyst observes the price index $\bar{\mathbf{p}}^t$ as well as N expenditure allocation decisions over K goods, which form the set E^t . It is usual in this model to assume that, for any \mathbf{e} and \mathbf{e}' in E^t , $\sum_{k=1}^K e_k = \sum_{k=1}^K e'_k = m^t$. In other words the total expenditure is the same for different elements in E^t and their implied demand bundles $\mathbf{x}(\mathbf{e}, \bar{\mathbf{p}}^t)$ and $\mathbf{x}(\mathbf{e}', \bar{\mathbf{p}}^t)$ are both bundles in the budget set

$$\mathcal{B}(\bar{\mathbf{p}}^t, m^t) = \{\mathbf{x} \in \mathbb{R}_+^K : \bar{\mathbf{p}}^t \cdot \mathbf{x} \leq m^t\}.$$

We could think of E^t *either* as the allocation decisions of a population with N consumers *or* as N allocation decisions made by the same consumer whose demand behavior is stochastic. We shall refer to a data set of the form \mathcal{D} as a **cross-sectional data set**.

A sorting function at t , σ^t , is a one-to-one map from the set $\{1, 2, \dots, N\}$ to E^t ; in other words, σ^t attaches an index to each expenditure allocation in E^t . We say that a cross sectional data set \mathcal{D} is **RUM-rationalizable** (or admits a RUM-rationalization) if there are sorting functions $\{\sigma^t\}_{t \in T}$ such that

$$\mathcal{E}(\{\sigma^t\}_{t \in T}) = \{(\sigma^t(i))_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}, \quad (8)$$

(which has the same form as (1)) is rationalizable with heterogenous preferences. In other words, \mathcal{D} is RUM-rationalizable if there is a way of sorting the expenditure allocations in

⁸ Random utility models are studied in Luce (1959), Block and Marshak (1960), McFadden (1973), Falmagne (1978), McFadden and Richter (1990), Kitamura and Stoye (2018), among others.

E^t so that the entire data set could be understood as arising from N utility-maximizing consumers, all of whom face prices $\bar{\mathbf{p}}^t$ at each observation t , but have possibly heterogeneous preferences. We can compare this notion of rationalization with that under the **random price model** (RPM). The cross sectional data set \mathcal{D} is **RPM-rationalizable** there are sorting functions $\{\sigma^t\}_{t \in T}$ such that $\mathcal{E}(\{\sigma^t\}_{t \in T})$ (as defined by (8)) is rationalizable by heterogeneous prices, subject to some consistency condition on the heterogeneous prices.

3 Price Heterogeneity with Consistent Expectations

In this section we show that price heterogeneity, even when it is required to be consistent in expectation, is powerful enough to explain almost any data set. Our first result states this claim formally.

PROPOSITION 1. *Let $\mathcal{W} = \{W_k^t\}_{k \in K, t \in T}$ be a collection of aggregator functions and $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ be a data set, with $e_k^{i,t} > 0$ for all $(i, t) \in N \times T$ and $k = 1, 2$. Then there are heterogeneous prices $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$, which are consistent in expectation and satisfy $p_k^{i,t} = \bar{p}_k^t$ for all $k > 2$, that rationalize \mathcal{E} .*

In empirical studies of demand, heterogeneity in preferences is often employed as a way of capturing heterogeneity in observed behavior. This proposition highlights the possibility of an alternative way to explain demand heterogeneity. It states that if there are two goods for which demand is strictly positive at all observations and for all agents, then price heterogeneity in *just those two goods* can rationalize any pattern of expenditure, even if consistency in expectation is imposed. In the Errors interpretation of heterogeneous prices, the heterogeneity in demand arises from heterogeneity in prices that the analyst could not observe because of imperfections in the data collected. In the Misperceptions interpretation of heterogeneous prices, the heterogeneity in demand arises from mistakes in consumers' decision-making. Either way, heterogeneous prices could explain heterogeneous demand behavior. Of course, whether such an explanation is a plausible one would depend on the specific case being considered – we are not suggesting that it is always and everywhere a better explanation, only that it is a potentially powerful tool for explaining demand heterogeneity.

In the context of a cross-sectional data set $\mathcal{D} = \{E^t, \bar{\mathbf{p}}^t\}_{t \in T}$, it is well-known that RUM imposes observable restrictions on \mathcal{D} (see the example in Section 4). On the other hand,

Proposition 1 guarantees that *any* data set \mathcal{D} with the property that $e_k^t > 0$ for $k = 1, 2$ and all $\mathbf{e}^t \in E^t$ is RPM-rationalizable by heterogenous prices with consistent price expectations.⁹ This is good news if the objective is to have a fully flexible model that could capture any set of observed demand behavior on \mathcal{D} . Of course, any explanation of the data with such a method will have implications on the unobserved heterogeneity in prices, and it would then be matter of (context-dependent) judgment whether the unobserved heterogeneity required to explain the data is in fact reasonable. Another approach is to impose stronger restrictions on the types of price heterogeneity permitted, which will temper the explanatory power of the heterogenous-prices model; this approach is explored in Section 4.

3.1 Proof of Proposition 1 and related results

The proof of Proposition 1, as well as other results in this paper, relies on Afriat's Theorem (Afriat (1967), Varian (1982)). The theorem considers a data set with T observations, where observation t consists of a bundle $\mathbf{x}^t \in \mathbb{R}_+^K$ purchased when the prices are $\mathbf{p}^t \in \mathbb{R}_{++}^K$. Thus the data set may be written as $\mathcal{O} = \{(\mathbf{x}^t, \mathbf{p}^t)\}_{t \in T}$. A strictly increasing and continuous utility function $u : \mathbb{R}_+^K \rightarrow \mathbb{R}$ is said to rationalize \mathcal{O} if

$$u(\mathbf{x}^t) \geq u(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{B}(\mathbf{p}^t, \mathbf{p}^t \cdot \mathbf{x}^t).$$

Afriat's Theorem characterizes rationalizable data sets through a property called the **generalized axiom of revealed preference**, or **GARP** for short.¹⁰

GARP is basically a type of no-cycling condition on $\mathcal{X} = \{\mathbf{x}^t\}_{t \in T}$, the set of observed demand bundles. Let $\mathbf{x}^t, \mathbf{x}^s$ be two elements in \mathcal{X} . We say \mathbf{x}^t is **directly revealed preferred to \mathbf{x}^s** , denoted by $\mathbf{x}^t \succsim_D \mathbf{x}^s$, if $\mathbf{p}^t \cdot \mathbf{x}^t \geq \mathbf{p}^t \cdot \mathbf{x}^s$. Similarly, \mathbf{x}^t is **directly revealed strictly preferred to \mathbf{x}^s** , denoted by $\mathbf{x}^t \succ_D \mathbf{x}^s$, if $\mathbf{p}^t \cdot \mathbf{x}^t > \mathbf{p}^t \cdot \mathbf{x}^s$. We say \mathbf{x}^t is **revealed preferred to \mathbf{x}^s** , denoted by $\mathbf{x}^t \succsim_R \mathbf{x}^s$, if there is a sequence $\{\mathbf{x}^{t_l}\}_{l=1}^L$ such that $\mathbf{x}^{t_1} = \mathbf{x}^t$, $\mathbf{x}^{t_L} = \mathbf{x}^s$, and $\mathbf{x}^{t_l} \succsim_D \mathbf{x}^{t_{l+1}}$ for each $l < L$. GARP requires that for any $s, t \in T$, $\mathbf{x}^t \succsim_R \mathbf{x}^s$ implies $\mathbf{x}^s \not\prec_D \mathbf{x}^t$.

In essence the proof of Proposition 1 consists of constructing heterogenous prices for goods 1 and 2 that are consistent in expectation and have the property that the collection

⁹ Indeed, with *any* set of sorting functions, $\{\sigma^t\}_{t \in T}$, Proposition 1 guarantees that $\mathcal{E}(\{\sigma^t\}_{t \in T}) = \{(\sigma^t(i))_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ is rationalizable with heterogenous prices.

¹⁰ To be precise, \mathcal{O} satisfies GARP if it can be rationalized by a locally nonsatiated preference. Conversely, when \mathcal{O} obeys GARP, then it is rationalizable by a strictly increasing, continuous, and concave utility function.

of observations

$$\mathcal{O}^* = \{(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}), \mathbf{p}^{i,t})\}_{(i,t) \in N \times T} \quad (9)$$

obeys GARP. This is a notional data set with $N \times T$ observations, where each observation consists of a bundle and a price vector, and is precisely of the type where one may apply Afriat's Theorem. The theorem guarantees that there is a utility function that rationalizes this \mathcal{O}^* (equivalently, rationalizes \mathcal{E} with heterogenous prices) if \mathcal{O}^* satisfies GARP. In our proof, we choose the idiosyncratic prices $p_k^{i,t}$, for $k = 1, 2$, carefully so that the direct revealed preference relation \succeq_D on $\mathcal{X}^* = \{\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})\}_{(i,t) \in N \times T}$ has the following property:

$$\text{if } \mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \succeq_D \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s}) \text{ then either } i > j \text{ or } i = j \text{ and } t \geq s. \quad (10)$$

In other words, if $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})$ is directly revealed preferred to $\mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s})$, then (i, t) dominates (j, s) in the lexicographic order on $N \times T$.¹¹ This in turn guarantees that if $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \succeq_R \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s})$, then either $i > j$ or $i = j$ and $t \geq s$. Thus it is impossible for $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \succeq_R \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s})$ and for $\mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s}) \succ_D \mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})$, which means that GARP holds. The details of the proof can be found in the Appendix.

Relationship to Varian (1988). This paper shows that GARP becomes a vacuous condition in certain circumstances when prices or demand are only partially observed. To be specific, suppose that at each observation t , the prices of all K goods, \mathbf{p}^t , are observed, but only the consumer's demand for goods $2, 3, \dots, K$, which we denote by $\mathbf{x}_{-1}^t = (x_2^t, \dots, x_K^t)$, are observed. In this case, there would always exist $\hat{x}_1^t \in \mathbb{R}_+$ (a hypothetical demand for good 1 at observation t) such that $\{(\mathbf{x}^t, \mathbf{p}^t)\}_{t \in T}$ obeys GARP, where $\mathbf{x}^t = (\hat{x}_1^t, \mathbf{x}_{-1}^t)$. In other words, when the demand of one good is missing from the data, rationalizability can never be rejected. The proof provided by Varian involves picking a sequence x_1^t (for $t = 1, 2, \dots$) in such a way that it is impossible for \mathbf{x}^t to be revealed preferred to $\mathbf{x}^{t'}$ if $t < t'$.

While not explicitly considered in Varian (1988), it is clear that Varian's result has a variation of the following form (which is also clear from our proof of Proposition 1): *suppose each observation t consists of the prices and demand for goods $2, 3, \dots, K$ (denoted by \mathbf{p}_{-1}^t and \mathbf{x}_{-1}^t respectively) as well as the expenditure on good 1, e_1^t , where $e_1^t > 0$ for all t ; then there is \hat{p}_1^t and \hat{x}_1^t such that $\hat{p}_1^t \hat{x}_1^t = e_1^t$ and, with $\mathbf{x}^t = (\hat{x}_1^t, \mathbf{x}_{-1}^t)$ and $\mathbf{p}^t = (\hat{p}_1^t, \mathbf{p}_{-1}^t)$, the data set $\{(\mathbf{x}^t, \mathbf{p}^t)\}_{t \in T}$ obeys GARP (and is thus rationalizable).*

¹¹ The lexicographic order ranks (i, t) higher than (j, s) if either $i > j$ or $i = j$ and $t > s$.

Compared to this result, the setup of Proposition 1 is more complicated because $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ collects observations from multiple agents and the average price of each good (across all agents) is observed. Thus, while for a *given* agent i we are free to ‘break up’ the expenditure on good 1, $e_1^{i,t}$, in any way between the price and demand for good 1, a higher idiosyncratic price of good 1 for one agent has to be balanced by a lower price of good 1 for another agent. With this added consistency requirement, the rationalizability of \mathcal{E} can only be guaranteed if there is price heterogeneity in at least two goods. In the Appendix, we provide an example of a data set \mathcal{E} which cannot be rationalized with idiosyncratic prices in only one good, even if agents are permitted to have distinct preferences.

3.2 Demand Disaggregation

We now apply Proposition 1 to a situation where total demand and expenditure allocations are observed but not individual demands. As in the previous section, there are N consumers who purchase bundles of K goods. At each observation t , the analyst observes $e_k^{i,t}$, consumer i ’s expenditure on good k , and \bar{x}_k^t the total amount of good k that the N consumers purchase. Hence, the data set collected may be written as

$$\mathcal{E}^* = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{x}}^t\}_{t \in T},$$

where $\bar{\mathbf{x}}^t = (\bar{x}_k^t)_{k \in K} \gg 0$. Note that neither individual demands, $x_k^{i,t}$, nor idiosyncratic prices, $p_k^{i,t}$, are observed. The issue then is whether it is possible to disaggregate the total demand $\bar{\mathbf{x}}^t$ at each observation in a way that guarantees that each consumer is utility-maximizing, allowing for heterogenous prices. To be precise, we say that individual demands $\{\mathbf{x}^{i,t}\}_{i \in N, t \in T}$ **disaggregate** $\{\bar{\mathbf{x}}^t\}_{t \in T}$ if

$$\sum_{i \in N} \mathbf{x}^{i,t} = \bar{\mathbf{x}}^t \text{ for each } t \in T.$$

The individual demands $\{\mathbf{x}^{i,t}\}_{(i,t) \in N \times T}$ **rationalize** \mathcal{E}^* if the implied heterogenous prices $\{\mathbf{p}^{i,t} = (e_k^{i,t}/x_k^{i,t})_{k \in K}\}_{(i,t) \in N \times T}$ rationalize $\{\mathbf{e}^{i,t}\}_{(i,t) \in N \times T}$; i.e., there is a utility function $u : \mathbb{R}_+^K \rightarrow \mathbb{R}$ such that for each $(i, t) \in N \times T$,

$$u(\mathbf{x}^{i,t}) \geq u(\mathbf{x}) \text{ for any } \mathbf{x} \in \mathcal{B}(\mathbf{p}^{i,t}, m^{i,t}).$$

As idiosyncratic prices allow for two possible interpretations, individual demands can be interpreted in two ways. In the first (Error) interpretation, $\mathbf{x}^{i,t}$ are the true consumption

bundles by consumer i which are not observed by the analyst. In the second (Misperception) interpretation, consumer i makes his expenditure allocation decision based on an idiosyncratic perception of prices at observation t ; then $\mathbf{x}^{i,t}$ is the consumer's targeted demand, which could be different from what the agent actually ends up consuming, which is $z_k^{i,t} = \bar{x}_k^t e_k^{i,t} / (\sum_{i \in N} e_k^{i,t})$.

The next result states that it is always possible to disaggregate demand in a rationalizable way, provided there is price-heterogeneity for two goods.

PROPOSITION 2. *Let $\mathcal{E}^* = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{x}}^t\}_{t \in T}$ be a data set with $e_k^{i,t} > 0$ for any $(i, t) \in N \times T$ and $k = 1, 2$. Then, allowing for price heterogeneity for goods $k = 1, 2$, there is a disaggregation of $\bar{\mathbf{x}}^t$ that rationalizes \mathcal{E}^* .*

Proof. To apply Proposition 1, let $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ where $\bar{p}_k^t \equiv M_k^t / \bar{x}_k^t$ and $M_k^t \equiv \sum_{i \in N} e_k^{i,t}$. Let W_k^t be a weighted harmonic average function given by

$$W_k^t((p^i)_{i \in N}) = \frac{M_k^t}{\sum_{i \in N} e_k^{i,t} / p^i}.$$

Note that W_k^t is an aggregator function and $\{W_k^t\}$ satisfy the second property of aggregator functions defined in Section 2. Then by Proposition 1, there are heterogenous prices $\mathcal{P} = \{\mathbf{p}^{i,t}\}$, which are consistent in expectation (i.e., $\bar{p}_k^t = W_k^t((p_k^{i,t})_{i \in N})$) and $p_k^{i,t} = \bar{p}_k^t$ for any $k > 2$, that rationalizes \mathcal{E} . Let $x_k^{i,t} \equiv e_k^{i,t} / p_k^{i,t}$. Then individual demands $\{\mathbf{x}^{i,t}\}_{(i,t) \in N \times T}$ rationalizes \mathcal{E}^* since \mathcal{P} rationalizes \mathcal{E} . Moreover, for any $k > 2$, $x_k^{i,t} = z_k^{i,t}$ since we have $p_k^{i,t} = \bar{p}_k^t$. Finally, note that $\{\mathbf{x}^{i,t}\}_{(i,t) \in N \times T}$ disaggregate $\{\bar{\mathbf{x}}^t\}_{t \in T}$ since

$$\sum_{i \in N} x_k^{i,t} = \sum_{i \in N} e_k^{i,t} / p_k^{i,t} = M_k^t \frac{\sum_{i \in N} e_k^{i,t} / p_k^{i,t}}{M_k^t} = \frac{M_k^t}{W_k^t((p_k^{i,t})_{i \in N})} = \frac{M_k^t}{\bar{p}_k^t} = \bar{x}_k^t.$$

□

Relationship to Brown and Matzkin (1996). Brown and Matzkin (1996) show that the Walrasian model of an exchange economy is refutable when the economy's total endowment, the equilibrium prices and the equilibrium income distribution are observable. Using our notation and setup, their problem could be formulated in the following way.

The data set consists of T observations where, where each observation t consists of aggregate demand $\bar{\mathbf{x}}^t$, the prevailing price vector $\bar{\mathbf{p}}^t$, and the distribution of expenditure across agents, $(m^{i,t})_{i \in N}$. Formally, the data set is $\mathcal{E}^{**} = \{(m^{i,t})_{i \in N}, \bar{\mathbf{x}}^t, \bar{\mathbf{p}}^t\}_{t \in T}$. Brown and Matzkin

(1996) show that there exists \mathcal{E}^{**} where $\{\bar{\mathbf{x}}^t\}_{t \in T}$ cannot be disaggregated into $\{\mathbf{x}^{i,t}\}_{i \in N, t \in T}$ such that agent i is maximizing some increasing utility function u^i , i.e., $u^i(\mathbf{x}^{i,t}) \geq u^i(\mathbf{x}^i)$ for all $\mathbf{x}^i \in \mathcal{B}(\bar{\mathbf{p}}^t, m^{i,t})$ (for all $i \in N$). The refutability of utility-maximization in this setting is surprising given that only aggregate demands are observed, but its conclusion is sensitive to the assumption of homogenous prices. Indeed, the data set we consider, \mathcal{E}^* , contains more information than \mathcal{E}^{**} , since it provides the breakdown of expenditure on each good for each agent (and not the just the agent's total expenditure $m^{i,t}$) and (implicitly) the average prices, with $\bar{p}_k^t = (\sum_{i \in N} e^{i,t})/\bar{x}_k^t$. Nonetheless, applying Proposition 2 to \mathcal{E}^{**} , we would conclude that it can always be rationalized (in fact, by a single utility function), provided we allow for price heterogeneity in two goods.

4 Stable Price Heterogeneity and Preference Heterogeneity

In the previous section, we analyzed price heterogeneity with consistent expectations and found that it gives rise to a very flexible model that imposes no restrictions on observed demand. In this section, we explore the implications of stable price heterogeneity.

4.1 Preference heterogeneity when prices are heterogenous

In the result below, we ask what happens when there is stable price heterogeneity in a market where consumers have heterogenous preferences.

PROPOSITION 3. *For any stable price distribution $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$, the data set $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ is rationalizable with heterogenous preferences if and only if, for each $i \in N$, $\{(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})\}_{t \in T}$ is rationalizable; i.e., there is a strictly increasing and continuous utility function $u^i : \mathbb{R}_+^K \rightarrow \mathbb{R}$ such that for each $t \in T$,*

$$u^i(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})) \geq u^i(\mathbf{x}) \text{ for any } \mathbf{x} \in \mathcal{B}(\mathbf{p}^{i,t}, m^{i,t}).$$

To interpret this result, we first consider the case where prices vary across consumers (*in fact*) with $\bar{\mathbf{p}}^t$ being the average price vector at observation t . Then Proposition 3 tells us that, provided each agent's demand is rationalizable and price heterogeneity is stable, the average price $\bar{\mathbf{p}}^t$ is a good substitute for the true price $\mathbf{p}^{i,t}$, in the sense that it does not destroy the rationalizability of the data. In the second interpretation, where $\mathbf{p}^{i,t}$ is agent

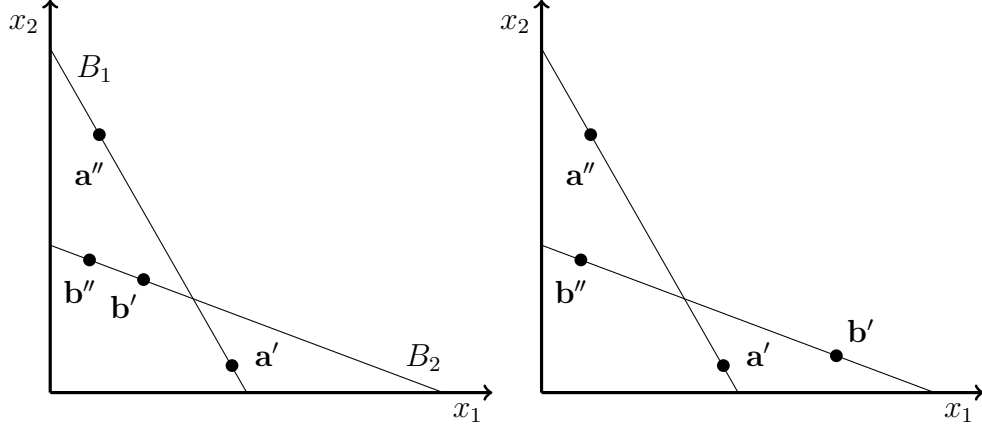


Figure 1: Observable restrictions in the Random Utility Model

The cross-sectional data set depicted on the right, with $E^1 = \{\mathbf{a}', \mathbf{a}''\}$ and $E^2 = \{\mathbf{b}', \mathbf{b}''\}$ is RUM-rationalizable since one could pair up \mathbf{a}' with \mathbf{b}' and \mathbf{a}'' with \mathbf{b}'' and each of the agent-level data sets obeys GARP. On the other hand, the data set on the left is not RUM-rationalizable because it is impossible to sort the elements of E^1 and E^2 in such a way that both agent-level data sets obey GARP.

i 's perceived price, this proposition tells us that stable price misperception is undetectable: the agent's behavior is indistinguishable from someone who perceives prices correctly and is maximizing utility.

Cross-Sectional Data Environment. Proposition 3 has the following immediate corollary for cross-sectional data sets: *for any stable price distribution $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$, the cross sectional data set $\mathcal{D} = \{E^t, \bar{\mathbf{p}}^t\}_{t \in T}$ is RUM-rationalizable if and only if there are sorting functions $\{\sigma^t\}_{t \in T}$ and strictly increasing and continuous utility functions $u^i : \mathbb{R}_+^K \rightarrow \mathbb{R}$ such that for each $i \in N$ and $t \in T$,*

$$u^i(\mathbf{x}(\sigma^t(i), \mathbf{p}^{i,t})) \geq u^i(\mathbf{x}) \text{ for any } \mathbf{x} \in \mathcal{B}(\mathbf{p}^{i,t}, m^t). \quad (11)$$

In other words, the insertion of stable price heterogeneity does not destroy RUM-rationalizability: if $\mathcal{D} = \{E^t, \bar{\mathbf{p}}^t\}_{t \in T}$ is a cross-sectional data set collected from a population of consumers who maximize (heterogenous) increasing and continuous utility functions under prices with stable heterogeneity, then \mathcal{D} will be RUM-rationalizable. It follows that the observable implications of the random utility model (such as that depicted in Figure 1) remain precisely the same even if there is stable price heterogeneity among agents in the population.

The proof of Proposition 3 is a straightforward application of Afriat's Theorem. It is based on the observation that distorting prices of good k by real numbers $\lambda_{i,k}$, which are

independent of t , does not (in a sense) change an agent's revealed preference relations.

Proof of Proposition 3. Take any stable price distribution $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ where $p_k^{i,t} = \lambda_{i,k} \bar{p}_k^t$ for some $\{\lambda_{i,k}\}_{(i,k) \in N \times K}$. By Afriat's Theorem, $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ is rationalizable with heterogenous preferences if and only if $\mathcal{O}^i = \{(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}), \mathbf{p}^{i,t})\}_{t \in T}$ satisfies GARP for each $i \in N$. Similarly, $\{(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})\}_{(i,t) \in N \times T}$ is rationalizable with heterogenous preferences if and only if $\bar{\mathcal{O}}^i = \{(\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t), \bar{\mathbf{p}}^t)\}_{t \in T}$ satisfies GARP for each $i \in N$. Hence, we shall show that for any $i \in N$, \mathcal{O}^i satisfies GARP if and only if $\bar{\mathcal{O}}^i$ satisfies GARP. To show this, note that the revealed preference relations on \mathcal{O}^i and $\bar{\mathcal{O}}^i$ are the 'same' in the following sense: for any $s, t \in T$, $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})$ is directly revealed preferred to $\mathbf{x}(\mathbf{e}^{i,s}, \mathbf{p}^{i,s})$ if and only if $\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)$ is directly revealed preferred to $\mathbf{x}(\mathbf{e}^{i,s}, \bar{\mathbf{p}}^s)$. This is simply because, by the definition of revealed preference relations, $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \succeq_D \mathbf{x}(\mathbf{e}^{i,s}, \mathbf{p}^{i,s})$ is equivalent to

$$\mathbf{p}^{i,t} \cdot \mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) = m^{i,t} \geq \mathbf{p}^{i,t} \cdot \mathbf{x}(\mathbf{e}^{i,s}, \mathbf{p}^{i,s}) = \sum_{k \in K} p_k^{i,s} \frac{e_k^{i,t}}{p_k^{i,t}} = \sum_{k \in K} \lambda_{i,k} \bar{p}_k^s \frac{e_k^{i,t}}{\lambda_{i,k} \bar{p}_k^t} = \sum_{k \in K} \bar{p}_k^s \frac{e_k^{i,t}}{\bar{p}_k^t}$$

while $\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t) \succeq_D \mathbf{x}(\mathbf{e}^{i,s}, \bar{\mathbf{p}}^s)$ is also equivalent to

$$\bar{\mathbf{p}}^t \cdot \mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t) = m^{i,t} \geq \bar{\mathbf{p}}^t \cdot \mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^s) = \mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) = \sum_{k \in K} \bar{p}_k^s \frac{e_k^{i,t}}{\bar{p}_k^t}.$$

Hence, for each $i \in N$, \mathcal{O}^i satisfies GARP if and only if $\bar{\mathcal{O}}^i$ satisfies GARP. \square

4.2 Stable Price Heterogeneity and Equivalence Scale Utilities

Our main objective in this section is to establish a threefold equivalence. Our first equivalence is to show that if $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ is rationalizable with heterogenous preferences, then it suffices to choose these preferences from a semiparametric family of utility functions where (in sense we shall make specific) any two utility functions in the family are related by a linear transformation. The second equivalence is that the rationalizability of \mathcal{E} with heterogenous preferences is equivalent to its rationalizability with heterogenous prices with a stable distribution.

Take any non-empty subset $R \subseteq K$. For any utility function $U : \mathbb{R}_+^K \rightarrow \mathbb{R}$ and $\beta \in \mathbb{R}_+$, let

$$U_R^\beta(\mathbf{x}) \equiv U(\beta \mathbf{x}_R, \mathbf{x}_{-R}) \quad \text{for any } \mathbf{x} \in \mathbb{R}_+^K.$$

We refer to U^β as a *scale transformation* of U . A collection \mathcal{U} of utility functions form an **R -scale family** if there is a strictly increasing and concave utility function U such that $\mathcal{U} = \{U^\beta : \beta \in \mathbb{R}_{++}\}$. Scale transformations first appeared in the econometric literature on demand estimation where it is used to capture (among other things) the effects of household composition on demand (see, for example, Barten (1964), Prais and Houthakker (1971), and Muellbauer (1980)). They have also been widely used in the study of demand aggregation where they are useful in obtaining properties on *market* demand which are not necessarily present at the agent-level demand correspondence (see Mas-Colell and Neufeind (1977), Dierker et al. (1984), Grandmont (1987, 1992), and Quah (1997)).

The basic reason for the use of scale transformations is that there is a convenient relationship between the demand generated by U and that generated by U^β . Let $\mathbf{x}(\mathbf{p}, m)$ and $\mathbf{x}^\beta(\mathbf{p}, m)$ be their demand sets at price \mathbf{p} and income/expenditure m , i.e., $\mathbf{x}(\mathbf{p}, m) = \operatorname{argmax}\{U(\mathbf{x}) | \mathbf{x} \in \mathcal{B}(\mathbf{p}, m)\}$ and $\mathbf{x}^\beta(\mathbf{p}, m) = \operatorname{argmax}\{U_R^\beta(\mathbf{x}) | \mathbf{x} \in \mathcal{B}(\mathbf{p}, m)\}$. Then it is straightforward to check that \mathbf{x} and \mathbf{x}^β are related in the following way:

$$\mathbf{x}^\beta(\mathbf{p}, m) = \left(\frac{\mathbf{x}_R(\mathbf{p}, \beta m)}{\beta}, \mathbf{x}_{-R}(\mathbf{p}, \beta m) \right).$$

Note that in the case where $R = K$, we have $\mathbf{x}^\beta(\mathbf{p}, m) = \mathbf{x}(\mathbf{p}, \beta m)/\beta$.

We have explained in Section 2 what it means for a data set $\mathcal{E} = ((\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t)_{t \in T}$ to be rationalizable with heterogenous preferences. We say that \mathcal{E} is rationalizable with heterogenous preferences **drawn from an R -scale family** if the preferences needed to explain \mathcal{E} can all be induced by utility functions drawn from an R -scale family; more formally, there are real numbers $\beta_1, \dots, \beta_N \in \mathbb{R}_{++}$ and a strictly increasing and concave utility function U such that for each $(i, t) \in N \times T$,

$$U_R^{\beta_i}(\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)) \geq U_R^{\beta_i}(\mathbf{x}) \text{ for any } \mathbf{x} \in \mathcal{B}(\bar{\mathbf{p}}^t, m^{i,t}). \quad (12)$$

The next proposition states the threefold equivalence between three distinct notions of rationalizability.

PROPOSITION 4. *Let $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ be a data set and suppose that for some $R \subseteq K$, we have $\mathbf{e}_R^{i,t} > 0$ for each $(i, t) \in N \times T$. Then the following are equivalent.*

- (i) \mathcal{E} is rationalizable with heterogenous preferences.

- (ii) \mathcal{E} is rationalizable with heterogenous preferences drawn from an R -scale family.
- (iii) Let W be any linear homogeneous aggregator function. Then there are real numbers $\lambda_1, \dots, \lambda_N > 0$ such that the stable price distribution $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ with $\mathbf{p}^{i,t} = (\lambda_i \bar{\mathbf{p}}_R^t, \bar{\mathbf{p}}_{-R}^t)$ is consistent in expectation (with respect to W) and rationalizes \mathcal{E} .

Note: Since our maintained assumption is that \mathcal{E} satisfies $\sum_{k \in K} e_k^{i,t} > 0$ for all (i, t) , we could always choose $R = K$, but R could be strictly smaller. If (say) good 1 satisfies $e_1^{i,t} > 0$ for all (i, t) , then we could choose $R = \{1\}$. Statement (ii) then refers to rationalizability with preference heterogeneity obtained by scaling good 1, while statement (iii) refers to rationalizability with heterogeneity in just the price of good 1; each of these statements is equivalent to rationalizability with heterogenous preferences.

In the literature on demand aggregation, it is fairly common to assume that all agents in a market (or some segment of the market) have preferences that belong to an R -scale family (see, for example, Grandmont (1987, 1992) and Quah (1997)). The equivalence of statements (i) and (ii) in this proposition tells us that this modeling practice is (in essence) without loss of generality, since any data set \mathcal{E} that is rationalizable with heterogenous preferences can also be rationalized by preferences drawn from an R -scale family. Of course, in that literature, further conditions are imposed on the distribution of the scales (that is, the distribution of β in our notation) and it is these further distributional assumptions that drive the strong conclusions on market demand obtained in that literature.¹²

Recall that Proposition 1 tells us that *any* data set \mathcal{E} can be explained by heterogenous prices, even when this heterogeneity is required to satisfy a consistency condition. The equivalence of (i) and (iii) in Proposition 4 tells us that the imposition of stable heterogeneity on prices disciplines the implications of heterogenous prices; in fact, the explanatory power of such a model (in the sense of the data sets that it could explain) coincides precisely with that of a model with heterogenous preferences. In the case where there is a good – say, good 1 – where the expenditure is strictly positive for all agents and at all observations, then (stable) price heterogeneity for that good alone is sufficient to capture the effects of random preferences. Lastly, note that the combination of Propositions 3 and 4 means that if preference

¹² For example, sufficient dispersion in the distribution of β leads to the approximate linearity of the market demand function with respect to total expenditure/income (see Grandmont (1992) and Quah (1997)).

heterogeneity and stable price heterogeneity are both present in a given environment, the data it generates would be equivalent to one where there is only preference heterogeneity or only price heterogeneity.

The next result is an obvious corollary of Proposition 4 obtained by transposing the proposition to the setting of a cross sectional data set. The punchline is that *RUM-rationalizability is equivalent to RPM-rationalizability with stable heterogenous prices*. In the Appendix, we provide a more general formulation of this equivalence in which the cross-sectional data set is allowed to be non-discrete.

PROPOSITION 5. *Let $\mathcal{D} = \{E^t, \bar{\mathbf{p}}^t\}_{t \in T}$ be a cross-sectional data set and suppose that for some $R \subseteq K$, we have $\mathbf{e}_R^{i,t} > 0$ for each $(i, t) \in N \times T$. Then the following are equivalent.*

- (i) \mathcal{D} is RUM-rationalizable.
- (ii) \mathcal{D} is RUM-rationalizable with heterogenous preferences drawn from an R -scale family.
- (iii) Let W be any linear homogeneous aggregator function. Then \mathcal{D} is RPM-rationalizable with a price distribution \mathcal{P} that is stable and consistent in expectation (with respect to W), with heterogenous prices only for the goods in R .

Proof sketch of Proposition 4. The detailed proof is found in the Appendix, but the following quick sketch may be helpful.

Obviously (ii) implies (i). To show that (iii) implies (ii), let $R \subseteq K$ with $\mathbf{e}_R^{i,t} > 0$ for each $(i, t) \in N \times T$. Then there are real numbers $\lambda_1, \dots, \lambda_N > 0$ and an increasing and continuous utility function U such that, with $\mathbf{p}^{i,t} = (\lambda_i \bar{\mathbf{p}}_R^t, \bar{\mathbf{p}}_{-R}^t)$,

$$U(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})) \geq U(\mathbf{x}) \text{ for any } \mathbf{x} \in \mathcal{B}(\mathbf{p}^{i,t}, m^{i,t}) \quad (13)$$

for all (i, t) . We show that this implies that

$$U_R^{\frac{1}{\lambda_i}}(\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)) \geq U_R^{\frac{1}{\lambda_i}}(\tilde{\mathbf{x}}) \text{ for any } \tilde{\mathbf{x}} \in \mathcal{B}(\bar{\mathbf{p}}^t, m^{i,t})$$

and thus \mathcal{E} is R -equivalence scale rationalizable. It remains to show that (i) implies (iii). Suppose \mathcal{E} is rationalizable with heterogenous preferences. We need to find real numbers $\lambda_1, \dots, \lambda_N > 0$ such that the data set $\mathcal{O}^* = \{(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}), \mathbf{p}^{i,t})\}_{(i,t) \in N \times T}$ where $\mathbf{p}^{i,t} = (\lambda_i \bar{\mathbf{p}}_R^t, \bar{\mathbf{p}}_{-R}^t)$ obeys GARP. Obviously, $\mathcal{O}^* = \bigcup_{i \in N} \mathcal{O}^i$, where $\mathcal{O}^i = \{(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}), \mathbf{p}^{i,t})\}_{t \in T}$.

Note that \mathcal{O}^i obeys GARP for any $\lambda_i > 0$; this follows from Proposition 3 and the fact that \mathcal{E} is rationalizable with heterogeneous preferences. We then show that it is possible to choose λ_{i+1} sufficiently smaller than λ_i so that, for any two consumers i and j with $j > i$, their budget sets are sufficiently distinct so that no bundle chosen by i is revealed preferred to a bundle chosen by j . We could then conclude that \mathcal{O}^* obeys GARP. \square

5 Expenditure-Augmented Utility and Stable Price Heterogeneity

Let $\mathcal{O} = \{(\mathbf{x}^t, \mathbf{p}^t)\}_{t \in T}$ be the data set collected from a single consumer, where \mathbf{x}^t is the bundle purchased at the price \mathbf{p}^t . In Section 3.1 we recounted Afriat's Theorem, which provides necessary and sufficient conditions for \mathcal{O} to be rationalizable, in the sense that there is a utility function such that the chosen bundle \mathbf{x}^t is weakly preferred to alternative bundles in the budget set $\mathcal{B}(\mathbf{p}^t, \mathbf{p}^t \cdot \mathbf{x}^t)$. Afriat's Theorem addresses the problem of rationalizability in the context of the constrained-optimization model of consumer demand, but there are other ways in which consumer demand can be modeled.

In partial equilibrium settings, where the set of goods being studied is just a subset of all the goods consumed by a consumer, it is common to model demand with a utility function of the quasilinear form; the consumer's choice is then the bundle $\mathbf{x} \in \mathbb{R}_+^K$ that maximizes $\bar{U}(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$, when \mathbf{p} is the price vector for the K goods. More generally, we can model the consumer's choice as arising from the unconstrained maximization of a utility function where expenditure enters as an argument of the utility function. Formally, the utility of acquiring the bundle \mathbf{x} at cost $e \geq 0$ is $V(\mathbf{x}, -e)$ where $V : \mathbb{R}_+^K \times \mathbb{R}_- \rightarrow \mathbb{R}$ is required to be strictly increasing in the last argument (so that increasing expenditure strictly reduces utility). Quasilinear utility-maximization would then correspond to the special case where $V(\mathbf{x}, -e) = \bar{U}(\mathbf{x}) - e$. We refer to V as an **expenditure-augmented utility function** or simply an augmented utility function.

Consumer demand arising from the maximization of such a utility function is studied in Deb et al. (2018). This study also provides necessary and sufficient conditions under which a data set \mathcal{O} is augmented utility-rationalizable, in the sense that there is an augmented

utility function V for which

$$V(\mathbf{x}^t, -\mathbf{p}^t \cdot \mathbf{x}^t) \geq V(\mathbf{x}, -\mathbf{p}^t \cdot \mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{R}_+^K. \text{ }^{13}$$

In this section we study the implications of price heterogeneity in the augmented utility model (AUM). We shall focus on two notions of price heterogeneity, both of which are inspired by the behavioral economics literature. The first notion is price heterogeneity arising from **inattention to prices**. While we have already introduced this idea in the previous sections, we would argue that the augmented utility model is a particularly convenient setting in which to study this phenomenon: since this model is not a constrained-optimization model, we can assume that a consumer who has been inattentive to prices and who has chosen a bundle based on a wrong impression of prices will simply *purchase the bundle he intended*; in contrast, in a constrained-optimization model, such an assumption cannot be made because a consumer who chooses a bundle based on the wrong prices could violate the budget constraint.

For example, imagine a consumer i who on a shopping trip to a grocery store chooses a bundle that maximizes an augmented utility function. However, consumer i is not fully attentive to prices and so chooses the bundle \mathbf{x} that maximizes $V(\mathbf{x}, -\mathbf{p}^i \cdot \mathbf{x})$, where \mathbf{p}^i is the consumer's impression of prices, which may differ from the true prices \mathbf{p} . We assume that if $\tilde{\mathbf{x}}$ is the outcome of this maximization problem, then $\tilde{\mathbf{x}}$ is the bundle bought. The consumer may realize at some point that he has made a mistake and total expenditure is different from what he expected, but since groceries form just a subset of the goods purchased by this consumer, it is plausible to assume that he could adjust for this mistake when purchasing other goods in the future.

The second notion of price heterogeneity we have in mind arises from **reference price dependence**, of the type investigated in Koszegi and Rabin (2006) and Heidhues and Kőszegi (2008). In this case, the consumer observes prices perfectly but has some expectation (or a reference) of what prices should be and his demand is affected by the extent to which the actual prices he encounters differ from his reference. Heterogenous reference prices is then a source of heterogeneity in demand.

¹³The characterization of those data sets \mathcal{O} that are quasilinear-rationalizable, in the sense that there is \bar{U} such that $\bar{U}(\mathbf{x}^t) - \mathbf{p}^t \cdot \mathbf{x}^t \geq \bar{U}(\mathbf{x}) - \mathbf{p}^t \cdot \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}_+^K$, can be found in Brown and Calsamiglia (2007).

5.1 AU-rationalizability with heterogenous prices

We now explain how we can formally incorporate both price inattention and reference price dependence into the augmented utility model. We assume that a consumer i has in mind a price vector \mathbf{p}^i , which may be different from the true price vector \mathbf{p} . We interpret \mathbf{p}^i as either what the consumer thinks are the prevailing prices (which could be wrong because he is inattentive to prices) or as the consumer's reference prices. Let $e = \mathbf{p} \cdot \mathbf{x}$ and $e^i = \mathbf{p}^i \cdot \mathbf{x}$. We assume that the consumer behaves as though he attributes to the bundle \mathbf{x} an expenditure level that depends on e and e^i ; formally, the expenditure associated with the bundle \mathbf{x} is $\phi(\mathbf{p} \cdot \mathbf{x}, \mathbf{p}^i \cdot \mathbf{x})$, where $\phi : \mathbb{R}_{++}^2 \rightarrow \mathbb{R}_+$ has as its argument e and e^i . We refer to ϕ as the **behavioral expenditure function**. The consumer i chooses \mathbf{x} to maximize

$$V(\mathbf{x}, -\phi(\mathbf{p} \cdot \mathbf{x}, \mathbf{p}^i \cdot \mathbf{x})),$$

where V is the augmented utility function.

For example, if we are modeling consumers who are inattentive to prices, we let $\phi(e, e^i) = e^i$, so the consumer chooses \mathbf{x} to maximize $V(\mathbf{x}, -\mathbf{p}^i \cdot \mathbf{x})$. If we are modeling price reference dependence, we could let $\phi(e, e^i) = ef(e/e^i)$, where $f(1) = 1$. The effective expenditure attributed a bundle \mathbf{x} is then $\mathbf{p} \cdot \mathbf{x} f((\mathbf{p} \cdot \mathbf{x})/(\mathbf{p}^i \cdot \mathbf{x}))$. If f is increasing then the consumer penalizes a bundle \mathbf{x} when its actual cost $\mathbf{p} \cdot \mathbf{x}$ exceeds the benchmark cost $\mathbf{p}^i \cdot \mathbf{x}$ in the sense that the consumer attributes to it an expenditure $\mathbf{p} \cdot \mathbf{x} f((\mathbf{p} \cdot \mathbf{x})/(\mathbf{p}^i \cdot \mathbf{x}))$ that is greater than the actual expenditure $\mathbf{p} \cdot \mathbf{x}$. Conversely, a bundle that costs less than its benchmark is attributed a cost that is even lower than the actual cost.

For a given ϕ , we say that a collection of heterogenous prices $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ **AU-rationalizes** $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ if there is an augmented utility function V such that for each $(i, t) \in N \times T$,

$$V(\mathbf{x}^{i,t}, -\phi(\bar{\mathbf{p}}^t \cdot \mathbf{x}^{i,t}, \mathbf{p}^{i,t} \cdot \mathbf{x}^{i,t})) \geq V(\mathbf{x}, -\phi(\bar{\mathbf{p}}^t \cdot \mathbf{x}, \mathbf{p}^{i,t} \cdot \mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}_+^K,$$

where $\mathbf{x}^{i,t} = \mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)$. The main result of this section establishes the equivalence between this notion of rationalizability and **AU-rationalizability with heterogenous preferences**, which means the following: there are augmented utility functions V^1, V^2, \dots, V^N such that for each $(i, t) \in N \times T$,

$$V^i(\mathbf{x}^{i,t}, -\bar{\mathbf{p}}^t \cdot \mathbf{x}^{i,t}) \geq V^i(\mathbf{x}, -\bar{\mathbf{p}}^t \cdot \mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}_+^K.$$

We say that ϕ is *regular* if (a) $\phi(e, \alpha e)$ is strictly increasing in e for every $\alpha > 0$, (b) *either* for any compact interval I of \mathbb{R}_{++} and $a, b, y \in I$, there is $M > 0$ such that $\phi(a, x) > \phi(b, y)$ for any $x > M$ *or* for any compact interval I of \mathbb{R}_{++} and $a, b, y \in I$, there is $m > 0$ such that $\phi(a, x) > \phi(b, y)$ for any $x < m$.¹⁴ The second part of regularity requires that if x and y are distant enough, then the comparison between $\phi(a, x)$ and $\phi(b, y)$ only depends on the second argument. Recall the two examples of ϕ we have discussed above. When $\phi(e, e') = e'$, ϕ is regular. Moreover, $\phi(e, e') = e f(e/e')$ is regular when either $\lim_{x \rightarrow 0} f(x) = \infty$ or $\lim_{r \rightarrow \infty} f(r) = \infty$.

PROPOSITION 6. *For any data set $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ the following are equivalent.*

- (i) \mathcal{E} is AU-rationalizable with heterogenous preferences.
- (ii) Let W be any linear homogeneous aggregator function and ϕ be any regular behavioral expenditure function. Then there are $\lambda_1, \lambda_2, \dots, \lambda_N > 0$ such that the stable price distribution $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ with $\mathbf{p}^{i,t} = \lambda_i \bar{\mathbf{p}}^t$ is consistent in expectation (with respect to W) and AU-rationalizes \mathcal{E} .

Similar to Proposition 5, we obtain the following immediate corollary of Proposition 6 by transposing it to the setting of a cross sectional data set. We say that a cross sectional data set \mathcal{D} is **AU-rationalizable** with heterogenous preferences if there are sorting functions $\{\sigma^t\}_{t \in T}$ such that $\mathcal{E}(\{\sigma^t\}_{t \in T}) = \{(\sigma^t(i))_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ is AU-rationalizable with heterogenous preferences. We can compare this notion of AU-rationalization with that under the **random price model** (RPM). The cross sectional data set \mathcal{D} is **AU-RPM-rationalizable** if there are sorting functions $\{\sigma^t\}_{t \in T}$ such that $\mathcal{E}(\{\sigma^t\}_{t \in T})$ is AU-rationalizable.

PROPOSITION 7. *For any cross-sectional data $\mathcal{D} = \{E^t, \bar{\mathbf{p}}^t\}_{t \in T}$, the following are equivalent.*

- (i) \mathcal{D} is AU-rationalizable with heterogenous preferences.
- (ii) Let W be any linear homogeneous aggregator function and ϕ be any regular behavioral expenditure function. Then there are $\lambda_1, \dots, \lambda_N > 0$ such that the stable price distribution \mathcal{P} with $\mathbf{p}^{i,t} = \lambda_i \bar{\mathbf{p}}^t$ that is consistent in expectation (with respect to W) AU-RPM-rationalizes \mathcal{D} .

¹⁴In our proof, we only need a, b to be elements of a finite set $\{\bar{\mathbf{p}}^t \cdot \mathbf{x}^{j,s}\}_{t,s \in T, j \in N}$.

To prove Proposition 6, we use Theorem 2 of Deb et al. (2018), which characterizes a more general version of the AUM that allows for non-linear pricing. Consider a collection of functions (or price systems) $\{f^t\}_{t \in T}$ on \mathbb{R}_+^K , where $f^t(\mathbf{x})$ is interpreted as the cost of purchasing \mathbf{x} at observation t . For example, prices are linear and $\bar{\mathbf{p}}^t$ are the prevailing prices, then $f^t(\mathbf{x}) = \bar{\mathbf{p}}^t \cdot \mathbf{x}$. We say a collection of consumption bundles and price systems $\{(\mathbf{x}^t, f^t)\}_{t \in T}$ is *AU-rationalizable* if there is an augmented utility function V such that for each $t \in T$,

$$V(\mathbf{x}^t, -f^t(\mathbf{x}^t)) \geq V(\mathbf{x}, -f^t(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}_+^K.$$

Theorem 2 of Deb et al. (2018) shows that AU-rationalization is characterized by a type of no-cycling condition, which we call *generalized axiom of price preference* (GAPP). We say f^t is **directly revealed (strictly) preferred to** f^s , denoted by $f^t \succcurlyeq_P f^s$, if $f^t(\mathbf{x}^s) \leq (<)f^s(\mathbf{x}^s)$. We say f^t is **revealed preferred to** f^s , denoted by $f^t \succcurlyeq_P^* f^s$, if there is a sequence $\{f^{t_l}\}_{l=1}^L$ such that $f^{t_1} = f^t$, $f^{t_L} = f^s$, and $f^{t_l} \succcurlyeq_P f^{t_{l+1}}$ for each $l < L$. We say that $\{(\mathbf{x}^t, f^t)\}_{t \in T}$ satisfies GAPP if, for any $s, t \in T$, $f^t \succcurlyeq_P^* f^s$ implies $f^s \not\prec_P f^t$.

Proof sketch of Proposition 6. The detailed proof can be found in the Appendix, but the following quick sketch may be helpful. Fix $\lambda_1, \dots, \lambda_N$ and let $f^{i,t}(\mathbf{x}) \equiv \phi(\bar{\mathbf{p}}^t \cdot \mathbf{x}, \lambda_i \bar{\mathbf{p}}^t \cdot \mathbf{x})$. Then by Theorem 2 of Deb et al. (2018), \mathcal{P} AU-rationalizes \mathcal{E} iff $\{(\mathbf{x}^{i,t}, f^{i,t})\}_{(i,t) \in N \times T}$ satisfies GAPP. On the other hand, \mathcal{E} is AU-rationalizable with heterogenous preferences iff $\{(\mathbf{x}^{i,t}, \bar{\mathbf{p}}^{i,t})\}_{t \in T}$ satisfies GAPP for each i . It is now easy to see that (ii) implies (i) since if $\{(\mathbf{x}^{i,t}, f^{i,t})\}_{(i,t) \in N \times T}$ satisfies GAPP, then $\{(\mathbf{x}^{i,t}, f^{i,t})\}_{t \in T}$ satisfies GAPP for each i . By the first part of regularity of ϕ , $f^{i,t}$ is strictly increasing. Hence, $\{(\mathbf{x}^{i,t}, f^{i,t})\}_{t \in T}$ satisfies GAPP iff $\{(\mathbf{x}^{i,t}, \bar{\mathbf{p}}^{i,t})\}_{t \in T}$ satisfies GAPP.

To prove that (i) implies (ii), we construct $\lambda_1, \dots, \lambda_N$ such that λ_i/λ_{i+1} is sufficiently large (or small) so that $f^{j,s}$ cannot be revealed preferred to $f^{i,t}$ when $i < j$. In other words, there will be no revealed preference cycle involving observations from different agents. Therefore, checking whether $\{(\mathbf{x}^{i,t}, f^{i,t})\}_{(i,t) \in N \times T}$ satisfies GAPP is equivalent to checking whether $\{(\mathbf{x}^{i,t}, f^{i,t})\}_{t \in T}$ satisfies GAPP for each i . Hence, (i) implies (ii). \square

Appendix

Proof of Proposition 1. By Afriat's theorem, we shall construct heterogenous prices $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$, which are consistent in expectation and satisfy $p_k^{i,t} = \bar{p}_k^t$ for all $k > 2$, such that a data set $\mathcal{O}^* = \{(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}), \mathbf{p}^{i,t})\}_{(i,t) \in N \times T}$ satisfies GARP. For any $(i, t) \in N \times T$ and $k \in K$ with $k \geq 3$, let $p_k^{i,t} \equiv \bar{p}_k^t$. Hence, $W_k^t(p_k^{1,t}, \dots, p_k^{N,t}) = W_k^t(\bar{p}_k^t, \dots, \bar{p}_k^t) = \bar{p}_k^t$ when $k \geq 3$.

Let $>^*$ be a lexicographic order on $N \times T$ such that $(j, s) >^* (i, t)$ if $j > i$ or $i = j$ and $s > t$. In other words, $(i, t + 1) >^* (i, t)$ and $(i + 1, 1) >^* (i, T)$. We now construct $\{p_1^{i,t}, p_2^{i,t}\}_{(i,t) \in N \times T}$ such that the bundle $\mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s})$ for (j, s) is not affordable for (i, t) whenever $(j, s) >^* (i, t)$. That is, $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \not\leq_D \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s})$ (i.e., $\mathbf{p}^{i,t} \cdot \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s}) > \mathbf{p}^{i,t} \cdot \mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) = m^{i,t}$) when $(j, s) >^* (i, t)$.

We first construct $p_1^{i,t}$ with $i < N$ as follows. Take any $p_1 > 0$ and let $p_1^{1,1} \equiv p_1$. Moreover, take any small enough $\epsilon > 0$ and we can construct the rest of $p_1^{i,t}$ such that

$$\epsilon p_1^{i,t} \geq p_1^{j,s} \text{ whenever } (j, s) >^* (i, t).$$

In words, the price of good 1 is lower in later periods and is also lower for agents with higher indexes; i.e., $p_1^{i,t}$ is decreasing exponentially in i as well as t . In particular, set $p_1^{i,t} = p_1 \epsilon^{(i-1)T+t-1}$. Therefore, $\mathbf{x}^{j,s}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s})$ is not affordable for (i, t) when $(j, s) >^* (i, t)$ and $i, j < N$. More formally, when $i, j < N$, when ϵ is small enough we have

$$\mathbf{p}^{i,t} \cdot \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s}) = \sum_{k \in K} p_k^{i,t} \frac{e_k^{j,s}}{p_k^{j,s}} > \frac{p_1^{i,t}}{p_1^{j,s}} e_1^{j,s} \geq \frac{e_1^{j,s}}{\epsilon} > m^{i,t} \text{ since } \frac{p_1^{i,t}}{p_1^{j,s}} \geq \frac{1}{\epsilon}.$$

Second, let $p_1^{N,t}$ be the solution to $W_1^t(p_1^{1,t}, \dots, p_1^{n-1,t}, p_1^{N,t}) = \bar{p}_1^t$. Third, take any $p_2 > 0$ and let $p_2^{N,t} \equiv \epsilon^t p_2$. Similar to the previous argument, the bundle $\mathbf{x}(\mathbf{e}^{N,t+1}, \mathbf{p}^{N,t+1})$ for $(N, t + 1)$ is not affordable for (N, t) (i.e., $\mathbf{x}(\mathbf{e}^{N,t}, \mathbf{p}^{N,t}) \not\leq_D \mathbf{x}(\mathbf{e}^{N,t+1}, \mathbf{p}^{N,t+1})$) since the price of good 2 for agent N is exponentially decreasing in t . Fourth, for any (i, t) with $i < N$, let $p_2^{i,t} \equiv \tilde{p}_2^t$ where \tilde{p}_2^t solves $W_2^t(\tilde{p}_2^t, \dots, \tilde{p}_2^t, \epsilon^t p_2) = \bar{p}_2^t$.

Case 1. Suppose for any $(p^j)_{j \neq i} \in \mathbb{R}_{++}^{N-1}$ and $(t, k) \in T \times K$, $\lim_{p^i \rightarrow +\infty} W_k^t(p^i, (p^j)_{j \neq i}) = +\infty$.

In this case, the equation $W_1^t(p_1^{1,t}, \dots, p_1^{n-1,t}, p_1^{N,t}) = \bar{p}_1^t$ has a solution when p_1 is small enough so that $\bar{p}_1^t \geq p_1^{i,t} = p_1 \epsilon^{(i-1)T+t-1}$ for each $i < N$. Moreover, $W_2^t(\tilde{p}_2^t, \dots, \tilde{p}_2^t, \epsilon^t p_2) = \bar{p}_2^t$ has a solution when p_2 is small enough.

To prove GARP, it is sufficient to show that $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \not\leq_D \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s})$ whenever $(j, s) >^* (i, t)$. By the previous arguments, it is enough to consider the case in which $j = N$ and $i < N$. Since $p_2^{i,t} = \tilde{p}_2^t \geq \bar{p}_2^t$, we have $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \not\leq_D \mathbf{x}(\mathbf{e}^{N,s}, \mathbf{p}^{N,s})$; i.e.,

$$\mathbf{p}^{i,t} \cdot \mathbf{x}(\mathbf{e}^{N,s}, \mathbf{p}^{N,s}) = \sum_{k \in K} p_k^{i,t} \frac{e_k^{N,s}}{p_k^{N,s}} > \frac{p_2^{i,t}}{p_2^{N,s}} e_2^{N,s} \geq \frac{\bar{p}_2^t}{\epsilon^s p_2} e_2^{N,s} > m^{i,t} \text{ when } \epsilon \text{ and } p_2 \text{ are small enough.}$$

Case 2. Suppose for any $(p^j)_{j \neq i} \in \mathbb{R}_{++}^{N-1}$ and $(t, k) \in T \times K$, $\lim_{p^i \rightarrow +0} W_k^t(p^i, (p^j)_{j \neq i}) = 0$.

In this case, the equation $W_1^t(p_1^{1,t}, \dots, p_1^{n-1,t}, p_1^{N,t}) = \bar{p}_1^t$ has a solution when p_1 is large enough so that $\frac{\bar{p}_1^t}{\epsilon} \leq p_1^{i,t} = p_1 \epsilon^{(i-1)T+t-1}$ for each $i < N$. Moreover, $W_2^t(\tilde{p}_2^t, \dots, \tilde{p}_2^t, \epsilon^t p_2) = \bar{p}_2^t$ has a solution when p_2 is large enough.

To prove GARP, it is sufficient to show that $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \not\leq_D \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s})$ whenever $(j, s) >^* (i, t)$. By the previous arguments, it is enough to consider the case in which $j = N$ and $i < N$. Since $p_1^{i,t} \geq \frac{\bar{p}_1^t}{\epsilon}$ and $p_1^{N,s} \leq \bar{p}_1^s$, we have $\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) \not\leq_D \mathbf{x}(\mathbf{e}^{N,s}, \mathbf{p}^{N,s})$; i.e.,

$$\mathbf{p}^{i,t} \cdot \mathbf{x}(\mathbf{e}^{N,s}, \mathbf{p}^{N,s}) = \sum_{k \in K} p_k^{i,t} \frac{e_k^{N,s}}{p_k^{N,s}} > \frac{p_1^{i,t}}{p_1^{N,s}} e_1^{N,s} \geq \frac{\bar{p}_1^t}{\epsilon \bar{p}_1^s} e_1^{N,s} > m^{i,t}$$

when ϵ is small enough. □

Below we provide an example of a data set that cannot be rationalized by price heterogeneity in one good. This shows that, in Proposition 1, it is essential that there be at two least two goods with price heterogeneity. In fact, the example establishes a stronger conclusion: the data set it considers cannot be rationalized by price heterogeneity in a single good alone even if we allow the two agents to have distinct preferences.

Example. We assume that there are two agents, four goods, and four observations. The two agents have identical expenditure allocations and the observations are as follows.

$$\begin{aligned} \mathbf{e}^{i,1} &= (10, 1, 1, 1) & \mathbf{e}^{i,2} &= (1, 10, 1, 1) & \mathbf{e}^{i,3} &= (1, 1, 10, 1) & \mathbf{e}^{i,4} &= (1, 1, 1, 10) \\ \bar{\mathbf{p}}^1 &= (2, 1, 1, 1) & \bar{\mathbf{p}}^2 &= (1, 2, 1, 1) & \bar{\mathbf{p}}^3 &= (1, 2, 1, 1) & \bar{\mathbf{p}}^{i,2} &= (1, 1, 1, 2) \end{aligned}$$

Proposition 1 guarantees that this data can be rationalized by heterogenous prices with consistent expectations, so long as there is price heterogeneity in at least two goods. However, assuming that prices are aggregated by taking the arithmetic average, we claim that this data *cannot* be rationalized with price heterogeneity in good 1 alone, even if we allow the two agents to have different preferences. Of course, given the symmetry, it is clear that this data set cannot be rationalized by price heterogeneity in any other good alone.

Let $x_k^{i,t} = e_k^{i,t}/p_k^{i,t}$. Note that for any $i \in N$ and distinct $t, t' \in \{2, 3, 4\}$, $\mathbf{x}^{i,t}$ is directly revealed strictly preferred to $\mathbf{x}^{i,t'}$ if and only if

$$p_1^{i,t} \times \frac{1}{p_1^{i,t'}} + 1 \times 1 + 1 \times 5 + 2 \times 1 < 13$$

or $p_1^{i,t}/p_1^{i,t'} < 5$. Therefore, $\mathbf{x}^{i,t}$ is directly revealed strictly preferred to $\mathbf{x}^{i,t'}$ and vice versa if and only if

$$\frac{1}{5} < \frac{p_1^{i,t'}}{p_1^{i,t}} < 5. \quad (14)$$

Since $p_1^{1,t} + p_1^{2,t} = 1$ for any $t = 2, 3, 4$, there is an agent i and distinct $t, t' \in \{2, 3, 4\}$ such that $1 \geq p^{i,t}, p^{i,t'} \geq 1/2$. Therefore,

$$\frac{1}{2} < \frac{p_1^{i,t}}{p_1^{i,t'}} < 2$$

which means (given (14)) that agent i has violated GARP. \square

Proof of Proposition 4. Clearly, (ii) implies (i). To show that (iii) implies (ii), take any $R \subseteq K$ with $e_R^{i,t} > 0$ for each $(i, t) \in N \times T$. Suppose there are real numbers $\lambda_1, \dots, \lambda_N > 0$ such that stable price distribution $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ with $\mathbf{p}^{i,t} = (\lambda_i \bar{\mathbf{p}}_R^t, \bar{\mathbf{p}}_{-R}^t)$ rationalizes \mathcal{E} .¹⁵ Hence, there is a utility function $U : \mathbb{R}_+^K \rightarrow \mathbb{R}$ such that for each $(i, t) \in N \times T$,

$$U(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})) \geq U(\mathbf{x}) \text{ for any } \mathbf{x} \in \mathcal{B}(\mathbf{p}^{i,t}, m^{i,t}). \quad (15)$$

Since $\mathbf{x}_R(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) = \mathbf{x}_R(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)/\lambda_i$ and $\mathbf{x}_{-R}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) = \mathbf{x}_{-R}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)$, we have

$$U(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})) = U\left(\frac{\mathbf{x}_R(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)}{\lambda_i}, \mathbf{x}_{-R}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)\right) = U_R^{\frac{1}{\lambda_i}}(\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)).$$

Let $\tilde{\mathbf{x}} = (\lambda_i \mathbf{x}_R, \mathbf{x}_{-R})$. Then $U(\mathbf{x}) = U_R^{\frac{1}{\lambda_i}}(\lambda_i \mathbf{x}_R, \mathbf{x}_{-R}) = U_R^{\frac{1}{\lambda_i}}(\tilde{\mathbf{x}})$. Moreover,

$$\mathbf{x} \in \mathcal{B}(\mathbf{p}^{i,t}, m^{i,t}) \text{ if and only if } \tilde{\mathbf{x}} \in \mathcal{B}(\bar{\mathbf{p}}^t, m^{i,t})$$

since $\mathbf{p}^{i,t} \cdot \mathbf{x} = \lambda_i \bar{\mathbf{p}}_R^t \cdot \mathbf{x}_R + \bar{\mathbf{p}}_{-R}^t \cdot \mathbf{x}_{-R} = \bar{\mathbf{p}}^t \cdot \tilde{\mathbf{x}}$. Finally, the above three observations show that (15) is equivalent to

$$U_R^{\frac{1}{\lambda_i}}(\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t)) \geq U_R^{\frac{1}{\lambda_i}}(\tilde{\mathbf{x}}) \text{ for any } \tilde{\mathbf{x}} \in \mathcal{B}(\bar{\mathbf{p}}^t, m^{i,t}).$$

Therefore, \mathcal{E} is R -equivalence scale rationalizable.

¹⁵ We will not use the fact that prices are consistent in expectation.

It remains to show that (i) implies (iii). Suppose \mathcal{E} is rationalizable with heterogeneous preferences. We shall find $\lambda_1, \dots, \lambda_N > 0$ such that stable price distribution $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ with $\mathbf{p}^{i,t} = (\lambda_i \bar{\mathbf{p}}_R^t, \bar{\mathbf{p}}_{-R}^t)$ is consistent in expectation and that rationalizes \mathcal{E} .

Let $\lambda_i = \beta \epsilon^i$ where $\epsilon > 0$ and $\beta = 1/W((\epsilon^i)_{i \in N})$. By homogeneous of degree 1 of W , we have $W((\lambda_i \bar{p}_k^t)_{i \in N}) = W((\beta \epsilon^i \bar{p}_k^t)_{i \in N}) = \bar{p}_k^t \beta W((\epsilon^i)_{i \in N}) = \bar{p}_k^t$ for any $k \in R$ and $t \in T$. By the property of W , we also have $W((p_k^{i,t})_{i \in N}) = W((\bar{p}_k^t)_{i \in N}) = \bar{p}_k^t$ for any $k \notin R$ and $t \in T$. Therefore, $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ is consistent in expectation.

We now shall show that $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ rationalizes \mathcal{E} when ϵ is small enough. By Afriat's theorem, it is enough to show that a data set $\mathcal{O}^* = \{(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}), \mathbf{p}^{i,t})\}_{(i,t) \in N \times T}$ obeys GARP when ϵ is small enough. We write $\mathcal{O}^* = \bigcup_{i \in N} \mathcal{O}^i$, where $\{(\mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}), \mathbf{p}^{i,t})\}_{t \in T}$.

Let $\succsim_{\bar{D}}$ be the revealed preference relation on \mathcal{O}^* . We claim that when ϵ is sufficiently small, for any $i, j \in N$ with $i > j$ and $s, t \in T$, $\mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s}) \not\prec_{\bar{D}} \mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})$. Indeed,

$$\mathbf{p}^{j,s} \cdot \mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t}) = \frac{\lambda_j}{\lambda_i} \sum_{k \in R} \bar{p}_k^s \frac{e_k^{i,t}}{\bar{p}_k^t} + \sum_{k \notin R} \bar{p}_k^s \frac{e_k^{i,t}}{\bar{p}_k^t} \geq \frac{1}{\epsilon^{i-j}} \sum_{k \in R} \bar{p}_k^s \frac{e_k^{i,t}}{\bar{p}_k^t}$$

and since $i > j$ and $\mathbf{e}_R^{i,t} > 0$, $\mathbf{p}^{j,s} \cdot \mathbf{x}(\mathbf{e}^{i,t}, \mathbf{p}^{i,t})$ will exceed $\mathbf{p}^{j,s} \cdot \mathbf{x}(\mathbf{e}^{j,s}, \mathbf{p}^{j,s}) = \sum_{k \in K} e_k^{j,s}$ when ϵ is small enough. Therefore, there is no revealed preference cycle involving observations from \mathcal{O}^i and \mathcal{O}^j when $i \neq j$. Lastly, since \mathcal{E} is rationalizable with heterogeneous preferences, Proposition 3 guarantees that \mathcal{O}^i satisfies GARP for each $i \in N$. Therefore, \mathcal{O}^* satisfies GARP. \square

Proof of Proposition 6. It is clear from the proof sketch provided in Section 5.1 that (ii) implies (i). To prove that (i) implies (ii), we will construct $\lambda_1, \dots, \lambda_N$ such that $f^{j,s}$ cannot be revealed preferred to $f^{i,t}$ whenever $i < j$. If such construction is possible, then there will be no revealed preference cycle involving observations from different agents. Therefore, checking whether $\{(\mathbf{x}^{i,t}, f^{i,t})\}_{(i,t) \in N \times T}$ satisfies GAPP will be equivalent to checking whether $\{(\mathbf{x}^{i,t}, f^{i,t})\}_{t \in T}$ satisfies GAPP for each i . Hence, by Theorem 2 of Deb et al. (2018), (i) will imply (ii) under such construction.

Let us fix $\alpha_2, \dots, \alpha_N > 0$. Let $\lambda_i = \prod_{j=1}^i \alpha_j$ where

$$\alpha_1 = \frac{1}{W(1, \prod_{j=2}^2 \alpha_j, \dots, \prod_{j=2}^N \alpha_j)}.$$

By homogeneous of degree 1 of W , we have

$$W((\lambda_i \bar{p}_k^t)_{i \in N}) = \alpha_1 \bar{p}_k^t W\left(\left(\frac{\lambda_i}{\alpha_1}\right)_{i \in N}\right) = \alpha_1 \bar{p}_k^t W\left(1, \prod_{j=2}^2 \alpha_j, \dots, \prod_{j=2}^N \alpha_j\right) = \bar{p}_k^t$$

for any $k \in K$ and $t \in T$. Therefore, $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ is consistent in expectation.

We now shall construct $\alpha_2, \dots, \alpha_N$ for any $i, j \in N$ with $i < j$ and $s, t \in T$, $f^{j,s} \not\prec_P f^{i,t}$. Our construction is inductive: for each $j \in N$, let us fix $\alpha_1, \dots, \alpha_{j-1}$ and find α_j such that for any $i < j$ and $s, t \in T$, $f^{j,s} \not\prec_P f^{i,t}$. First, note that there is a compact interval I_j that contains all elements of sets $\{\bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}\}_{s,t \in T, i \in N}$ and $\{\prod_{l=1}^i \alpha_l \bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}\}_{s,t \in T, i < j}$. We now consider two cases:

Case 1. Suppose for any compact interval I of \mathbb{R}_{++} and $a, b, y \in I$, there is $M > 0$ such that $\phi(a, x) > \phi(b, y)$ for any $x > M$.

In this case, there is M_j such that $\phi(a, x) > \phi(b, y)$ for any $x > M_j$ and $a, b, y \in I_j$. Then we will have

$$\begin{aligned} f^{j,s}(\mathbf{x}^{i,t}) &= \phi(\bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}, \lambda_j \bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}) = \phi(\bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}, \prod_{l=1}^j \alpha_l \bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}) \\ &> f^{i,t}(\mathbf{x}^{i,t}) = \phi(\bar{\mathbf{p}}^t \cdot \mathbf{x}^{i,t}, \lambda_i \bar{\mathbf{p}}^t \cdot \mathbf{x}^{i,t}) = \phi(\bar{\mathbf{p}}^t \cdot \mathbf{x}^{i,t}, \prod_{l=1}^i \alpha_l \bar{\mathbf{p}}^t \cdot \mathbf{x}^{i,t}) \end{aligned}$$

when α_j is large enough since $\lambda_j \bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}$ will become larger than m_j .

Case 2. Suppose for any compact interval I of \mathbb{R}_{++} and $a, b, y \in I$, there is $m > 0$ such that $\phi(a, x) > \phi(b, y)$ for any $x < m$.

In this case, there is m_j such that $\phi(a, x) > \phi(b, y)$ for any $x < m_j$ and $a, b, y \in I_j$. Then we will have $f^{j,s}(\mathbf{x}^{i,t}) = \phi(\bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}, \prod_{l=1}^j \alpha_l \bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}) > f^{i,t}(\mathbf{x}^{i,t}) = \phi(\bar{\mathbf{p}}^t \cdot \mathbf{x}^{i,t}, \prod_{l=1}^i \alpha_l \bar{\mathbf{p}}^t \cdot \mathbf{x}^{i,t})$ when α_j is small enough since $\lambda_j \bar{\mathbf{p}}^s \cdot \mathbf{x}^{i,t}$ will become smaller than m_j .

□

Random Utility and Random Price Models. So far we have considered discrete cross-sectional data sets. In this section, we show an equivalence between price and preference heterogeneity in the general cross-sectional data environment, which allows for continuous probability distributions. Let $\mathcal{E}^t = \{\mathbf{e} \in \mathbb{R}_+^K : \sum_{k \in K} e_k = m^t\}$ where m^t is the income level and μ_e^t is a probability distribution over \mathcal{E}^t . Then our data set takes the form

$$\mathcal{D} = \{(\mu_e^t, \bar{\mathbf{p}}^t)\}_{t \in T}.$$

Note that \mathcal{D} generalizes the discrete cross-sectional data set introduced in Section 2. (With some abuse, we have used the same notation for both.) Let \mathcal{U} be the set of all strictly increasing, strictly concave, and continuous utility functions on \mathbb{R}_+^K . We then say that the dataset \mathcal{D} is **RUM-rationalizable** if there is a probability distribution ρ on \mathcal{U} such that for any $t \in T$ and any measurable subset $E \subseteq \mathbb{R}_+^K$,

$$\mu_e^t(E) = \rho(\{u \in \mathcal{U} : \arg \max_{\mathbf{e} \in \mathcal{E}^t} u(\mathbf{x}(\mathbf{e}, \bar{\mathbf{p}}^t)) \in E\}).$$

Take any subset $R \subseteq K$. We say that the dataset $\hat{\mathcal{D}}$ is **RPM-rationalizable** if there is a probability distribution η on \mathbb{R}_{++} , a utility function $u \in \mathcal{U}$ such that, for any $t \in T$ and any measurable subset $E \subseteq \mathbb{R}_+^K$,

$$\mu_e^t(E) = \eta(\{\lambda \in \mathbb{R}_{++} : \arg \max_{\mathbf{e} \in \mathcal{E}^t} u(\mathbf{x}(\mathbf{e}, \mathbf{p}^{\lambda, t})) \in E\}),$$

where $\mathbf{p}^{\lambda, t} \equiv (\lambda \bar{\mathbf{p}}_R^t, \bar{\mathbf{p}}_{-R}^t)$. We say that \mathcal{D} is **RPM-rationalizable with consistency in expectation** if $\int \lambda d\eta(\lambda) = 1$.

We now formally state an equivalence between price and preference heterogeneity. The Levy-Prokhorov metric is denoted by d_{LP} .¹⁶

PROPOSITION 8. *Take any data set $\hat{\mathcal{D}} = \{(\mu_e^t, \bar{\mathbf{p}}^t)\}_{t \in T}$ and subset $R \subseteq K$ with $\mu_e^t(\{\mathbf{e} \in \mathcal{E}^t : \mathbf{e}_R > 0\}) = 1$ for each $t \in T$. If $\hat{\mathcal{D}}$ is RPM-rationalizable, then it is RUM-rationalizable. Conversely, if $\hat{\mathcal{D}}$ is RUM-rationalizable, then for any $\epsilon > 0$, there are probability distributions $\hat{\mu}_e^t \in \Delta(\mathcal{E}^t)$ such that $\max_{t \in T} d_{LP}(\mu_e^t, \hat{\mu}_e^t) < \epsilon$ and $\{(\hat{\mu}_e^t, \bar{\mathbf{p}}^t)\}_{t \in T}$ is RPM-rationalizable with consistency in expectation.*

The first part of Proposition 8 states that RPM-rationalization implies RUM-rationalization. The second part states that for any RUM-rationalizable data set $\hat{\mathcal{D}}$ there is a discrete approximation of it that is RPM-rationalizable. It will be clear from the proof that when μ_e^t is discrete, we can have $\hat{\mu}_e^t = \mu_e^t$. In other words, when $\{\mu_e^t\}$ are discrete, RUM-rationalization also implies RPM-rationalization.

Proof of Proposition 8. We first show that RPM implies RUM. Suppose \mathcal{D} is RPM-rationalizable; i.e., there is a distribution η on $\Delta(\mathbb{R}_{++})$ and $u \in \mathcal{U}$ such that, for any $t \in T$

¹⁶ For any probability distributions μ and μ' , $d_{LP}(\mu, \mu') = \inf\{\epsilon > 0 : \mu'(E^\epsilon) + \epsilon > \mu(E) \text{ and } \mu(E^\epsilon) + \epsilon > \mu'(E) \text{ for any } E\}$ where $E^\epsilon = \bigcup_{x \in E} \{y \in \mathbb{R}^K : \|x - y\| < \epsilon\}$.

and any measurable subset $E \subseteq \mathbb{R}_+^K$, $\mu_e^t(E) = \eta(\{\lambda \in \mathbb{R}_{++} : \arg \max_{\mathbf{e} \in \mathcal{E}^t} u(\mathbf{x}(\mathbf{e}, \mathbf{p}^{\lambda,t})) \in E\})$. For each $\lambda > 0$ and $t \in T$, let $\mathbf{e}^{\lambda,t} \equiv \arg \max_{\mathbf{e} \in \mathcal{E}^t} u(\mathbf{x}(\mathbf{e}, \mathbf{p}^{\lambda,t}))$. Then the data set $\mathcal{O}_\lambda = \{(\mathbf{x}(\mathbf{e}^{\lambda,t}, \mathbf{p}^{\lambda,t}), \mathbf{p}^{\lambda,t})\}_{t \in T}$ is rationalized by utility function u . Then the data set $\overline{\mathcal{O}}_\lambda = \{(\mathbf{x}(\mathbf{e}^{\lambda,t}, \overline{\mathbf{p}}^t), \overline{\mathbf{p}}^t)\}_{t \in T}$ is also rationalizable since \mathcal{O}_λ obeys GARP if and only if $\overline{\mathcal{O}}_\lambda$ obeys GARP (see the the proof of Proposition 3). Hence, there is a utility function u_λ such that $u_\lambda(\mathbf{x}(\mathbf{e}^{\lambda,t}, \overline{\mathbf{p}}^t)) \geq u_\lambda(\mathbf{x})$ for every $\mathbf{x} \in \mathcal{B}(\overline{\mathbf{p}}^t, m^t)$. Let ρ be a probability distribution over \mathcal{U} such that $\rho(\mathcal{U}) = \eta(\{\lambda \in \mathbb{R}_{++} : u_\lambda \in \mathcal{U}\})$ for any measurable subset $\mathcal{U} \in \mathcal{U}$. Therefore, we have $\mu_e^t(E) = \rho(\{u_\lambda \in \mathcal{U} : \arg \max_{\mathbf{e} \in \mathcal{E}^t} u_\lambda(\mathbf{x}(\mathbf{e}, \overline{\mathbf{p}}^t)) \in E\})$ for any $t \in T$ and any measurable subset $E \subseteq \mathbb{R}_+^K$. That is, $\hat{\mathcal{D}}$ is RUM-rationalizable.

It remains to show that RUM implies approximate RPM. Our proof uses Theorem 1 of Kitamura and Stoye (2018). Suppose \mathcal{D} is RUM-rationalizable; i.e., there is $\rho \in \Delta(\mathcal{U})$ such that for any $t \in T$ and any measurable subset $E \subseteq \mathbb{R}_+^K$, $\mu_e^t(E) = \rho(\{u \in \mathcal{U} : \arg \max_{\mathbf{e} \in \mathcal{E}^t} u(\mathbf{x}(\mathbf{e}, \overline{\mathbf{p}}^t)) \in E\})$.

Let $B_t = \{\mathbf{x} \in \mathbb{R}_+^K : \mathbf{x} \cdot \overline{\mathbf{p}}^t = m^t\}$ and let $\mathcal{Y} = \{Y_1, \dots, Y_L\}$ be the coarsest partition of $\bigcup_{t \in T} B_t$ such that for any $l \leq L$ and $t \in T$, Y_l is either completely on, completely strictly above, or completely strictly below budget plane B_t . Formally, any $\mathbf{x}, \mathbf{x}' \in \bigcup_{t \in T} B_t$ are in the same element of the partition if and only if $\text{sign}(\overline{\mathbf{p}}^t \mathbf{x} - m^t) = \text{sign}(\overline{\mathbf{p}}^t \mathbf{x}' - m^t)$ for all $t \in T$. Following Kitamura and Stoye, we refer to elements of \mathcal{Y} as *patches*.

Note that each budget set B_t can be uniquely expressed as a union of patches and so we may write $B_t = \bigcup_{l=1}^{L_t} Y_{l,t}$. Let $\pi \equiv (\pi_{1,1}, \dots, \pi_{L_1,1}, \dots, \pi_{1,t}, \dots, \pi_{L_t,t}, \dots, \pi_{1,T}, \dots, \pi_{L_T,T})$ be the vector where $\pi_{l,t} = \mu^t(Y_{l,t})$ for each (l, t) . In other words, $\pi_{l,t}$ is the probability that a consumption bundle lies in $Y_{l,t}$.

Given any $\epsilon > 0$, for each $t \in T$, there is a discrete probability distribution $\hat{\mu}^t \in \Delta(\mathcal{E}^t)$ such that $\hat{\mu}^t(Y_{l,t}) = \pi_{l,t}$ for each $l \leq L_t$ and $d_{LP}(\hat{\mu}^t, \mu^t) < \epsilon$. By Theorem 1 in Kitamura and Stoye (2018), we know that that the RUM-rationalizability of \mathcal{D} only depends on π ; given our construction of $\hat{\mu}^t$, we can conclude that

$$\hat{\mathcal{D}} = \{(\hat{\mu}_e^t, \overline{\mathbf{p}}^t)\}_{t \in T}$$

is also RUM-rationalizable. Since $\hat{\mu}^1, \dots, \hat{\mu}^T$ are discrete probability distributions, there is a discrete probability distribution $\hat{\rho}$ on \mathcal{U} such that, for any $t \in T$ and any measurable subset $E \subseteq \mathbb{R}_+^K$, $\hat{\mu}_e^t(E) = \hat{\rho}(\{u \in \mathcal{U} : \arg \max_{\mathbf{e} \in \mathcal{E}^t} u(\mathbf{x}(\mathbf{e}, \overline{\mathbf{p}}^t)) \in E\})$. Let $\text{supp}(\hat{\rho}) = \hat{\mathcal{U}} =$

$\{u^1, \dots, u^N\}$.

For each $u^i \in \hat{\mathcal{U}}$ and $t \in T$, let $\mathbf{e}^{i,t} = \arg \max_{\mathbf{e} \in \mathcal{E}^t} u^i(\mathbf{x}(\mathbf{e}, \bar{\mathbf{p}}^t))$. Then $\mathcal{O}^i = \{(\mathbf{x}(\mathbf{e}^{i,t}, \bar{\mathbf{p}}^t), \bar{\mathbf{p}}^t)\}_{t \in T}$ is rationalized by u^i . In other words, the data set $\mathcal{E} = \{(\mathbf{e}^{i,t})_{i \in N}, \bar{\mathbf{p}}^t\}_{t \in T}$ is rationalizable with heterogenous preferences. By Proposition 4, there are real numbers $\lambda_1, \dots, \lambda_N > 0$ such that the stable price distribution $\mathcal{P} = \{\mathbf{p}^{i,t}\}_{(i,t) \in N \times T}$ with $\mathbf{p}^{i,t} = (\lambda_i \bar{\mathbf{p}}_R^t, \bar{\mathbf{p}}_{-R}^t)$ rationalizes \mathcal{E} and satisfies the consistency condition $\sum_{i \in N} \lambda_i \hat{\rho}^i = 1$ (where $\hat{\rho}^i$ denotes the probability of u^i under ρ). It follows that $\hat{\mathcal{D}} = \{(\hat{\mu}_e^t, \bar{\mathbf{p}}^t)\}_{t \in T}$ is RPM-rationalizable, where the probability of λ_i under η is $\hat{\rho}^i$. \square

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