Inference in auctions with many bidders using transaction prices *

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Abstract

This paper studies inference in first-price or second-price sealed-bid auctions with a large number of bidders that have symmetric independent private values. In this context, we propose an asymptotic framework where the number of bidders diverges, while the number of auctions remains fixed. This framework allows us to conduct asymptotically exact inference on several important features of the model based exclusively on observations of the transaction prices. In particular, we study inference on the winner's expected utility, the seller's expected revenue, and the tail properties of the valuation distribution. Our simulations show that our inference method delivers excellent finite-sample performance. We illustrate our inference method with an application to car license auctions in Hong Kong.

Keywords: auctions, hypothesis testing, extreme value theory, tail index.

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1 Introduction

This paper considers inference in first-price or second-price sealed-bid auctions with symmetric independent private values (IPV). Econometric analysis of such auctions has spawned an extensive literature, which typically assumes that the researcher observes a sample of bids from a large number of independent and identically distributed (i.i.d.) auctions. See Athey and Haile (2002); Haile and Tamer (2003); Athey and Haile (2007); Guerre, Perrigne, and Vuong (2000), among many others. In contrast, this paper considers inference in a setting in which the researcher observes a sample from a small number of auctions, each with a large number of bidders. In accordance to this, we propose an asymptotic framework in which the number of bidders diverges, while the number of auctions remains fixed. This setup allows us to conduct inference on several important features of the model based exclusively on observations of the transaction price from each auction. In particular, we study inference on the winner's expected utility, the seller's expected revenue, and the tail properties of the valuation distribution.

Under the classic framework with a finite number of bidders and a large number of auctions, the existing literature mainly focuses on identifying and estimating the valuation distribution. Within our sampling framework, the valuation distribution is not necessarily identified. However, we can still conduct statistical inference by exploiting the asymptotic device that the number of bidders is large. Consider the second-price auction with IPV for example. In this case, the transaction price is the second largest order statistic of the bidders' valuations. Then under extreme value (EV) theory, the distribution of the transaction price converges to the so-called EV distribution, which is uniquely characterized by a single parameter ξ after location and scale normalization. This parameter is referred to as the tail index and characterizes the tail heaviness of the valuation distribution. Given this convergence, we essentially observe a finite number of transaction prices randomly generated from the EV distribution. Therefore, a likelihood-based inference about ξ is feasible and becomes asymptotically exact when the number of bidders diverges. We formalize this idea in Section 3.1. In addition, Section 4.1 considers the corresponding analysis for first-price auctions.

The key condition for EV theory is that the valuation distribution is within the domain of attraction of the EV distribution. This is a very mild condition and is satisfied by many commonly used distributions such as Pareto, Student-t, Gaussian, and uniform distribution. See Section 2 for more details. Together with the EV distribution approximation, the parameter ξ can be informative about many features of the auction fundamentals. We study the following three in specific. First, the existing literature commonly assumes that the valuation distribution has a bounded support with a bounded density near the right end-point (e.g., Maskin and Riley, 1984; Guerre et al., 2000; Guerre and Luo, 2022). Lemma 3.1 in Section 3.2 shows that these assumptions imply that $\xi = -1$. Then testing such a hypothesis is equivalent to testing a necessary condition for the commonly adopted assumptions.

Second, we study the expected utility of the winner, which is defined as the expected value of the winner's valuation minus her bid. In the second-price auction, the expected utility is equivalent to the expected value of the largest valuation minus the second-largest one. When the sample size is finite, this quantity depends on the whole valuation distribution and hence can only be learned if multiple bids are observed. In our asymptotic framework, both the largest and the second largest order statistics converge to the right end-point of the support, which can be infinite if ξ is non-negative. In Section 3.3, we show that the expected utility can be written as a known function of ξ after location and scale transformation. Treating ξ as a nuisance parameter, we then construct a confidence interval for the expected utility that is uniformly valid over ξ. Moreover, the same derivation holds in the first-price auctions given the well-known revenue equivalence result (e.g., Krishna, 2009, Chapter 2), and hence inference can be implemented similarly. See Section 4.2 for more details.

Third, we study the expected revenue of the seller, which again can be expressed as a function of ξ after a suitable location and scale normalization. Section 3.4 presents such derivation and constructs a confidence interval with the asymptotically correct coverage. In addition, we also study the optimal reserve price that maximizes the seller's expected revenue. We note that the optimal reserve price is constant regardless of the number of bidders in the classic setup where the seller's own value of the good is constant (Riley and Samuelson, 1981, Proposition 3). To recover a non-degenerate effect on the expected revenue, we consider that the seller's own valuation is of the same order as the transaction price. Then the optimal reserve price can be derived as a function of ξ. We collect this result as Lemma 3.2.

The asymptotic framework with many bidders has been extensively employed in the economic theory literature (e.g., Hong and Shum, 2004; Virág, 2013; Di Tillio, Ottaviani, and Sørensen, 2021) but less so in econometrics. Hong, Paarsch, and Xu (2014) study the asymptotic distribution of the transaction price in a clock model of a multi-unit, oral, ascending-price auction as the numbers of bidders and units increase. William and Zachariadis (2021) study the asymptotic distribution of the transaction price in Buyer's Bid Double Auction. In comparison, much of the existing econometric literature has focused on identifying and estimating the valuation distribution. When the number of bidders is small such as in timber auctions, knowing the whole valuation distribution is indeed essential to learn about other objects of interest. The seminal work by Athey and Haile (2002) derives general results about identifying the valuation distribution. Haile and Tamer (2003) study English auctions and derive bounds on the valuation distribution and other objects of interest with minimum structural assumption. Chesher and Rosen (2017) further extend these bounds to the non-IPV setup. Aradillas-Lopez, Gandhi, and Quint (2013) nonparametric identify bounds on seller profit and bidder surplus with variation in the number of bidders across auctions. Brendstrup and Paarsch (2006) and Komarova (2013) derive nonparametric identification of the valuation distribution with the transaction price and the winner's identity. Brendstrup and Paarsch (2007) study multi-object English auctions and derive semiparametric identification with the winning bids under the Archimedean copulas assumption.

When the number of bidders is large, we show that the tail of the valuation distribution can be sufficient to characterize some key features of the auctions as discussed above. Therefore, it provides a convenient device to learn about these features without knowing the valuation distribution. Consequently, a major benefit of our approach is that it requires minimum information from the data set. This is particularly useful when the number of auctions is relatively small and/or the number of bidders is difficult to obtain. In particular, the number of all bidders could be larger than the number of observed bids when there is a binding reserve price (e.g., Hickman, Hubbard, and Paarsch, 2017) or the bidders selectively enter the auction (e.g., Li and Zheng, 2009; Gentry and Li, 2014). To solve this issue, the existing methods typically require multiple bids or additional information. For example, Li (2005) studies first-price auctions with both entry and binding reservation prices and estimates the valuation distribution with the observed bids and the number of actual bidders. Adams (2007) studies eBay auctions and requires knowing the distribution of the potential bidders. Without knowing the number of bidders, An, Hu, and Shum (2010) propose to use a proxy of the number of bidders and an instrument variable. Kim and Lee (2014), Song (2015), Mbakop (2017), and Freyberger and Larsen (2020) construct identification of the valuation distribution with two or more order statistics of bids. Shneyerov and Wong (2011) derive nonparametric identification of model primitives based on a finitely many groups of bidders. Recently, Luo and Xiao (2022) derive identification results with two consecutive order statistics and an instrument or three consecutive ones. All these methods of identification, estimation, and inference are based on the asymptotics with many auctions. We refer to Hickman, Hubbard, and Sağlam (2012) and Gentry, Hubbard, Nekipelov, and Paarsch (2018) for recent surveys.

Third and more generally, existing methods inevitably rely on order statistics to nonparametrically estimate the valuation distribution (e.g., Guerre et al., 2000; Athey and Haile, 2007) since the transaction price is typically a function of the largest or second-largest order statistics. When the number of bidders is large, the distribution of these extreme order statistics is uniquely determined by the right tail shape of the valuation distribution as implied by EV theory. In this case, nonparametric estimation of the valuation distribution might be irregular and hence performs poorly (Menzel and Morganti, 2013). In contrast, our method explicitly relies on the extreme value approximation, which does not involve estimating the whole valuation distribution. From this point of view, we would distinguish our increasing-K framework from the class one with a finite K. The simulation study in Section 5 shows that our confidence intervals perform excellently when the number of bidders is only ten. They could dominate the existing methods in terms of coverage and length even if multiple bids and the number of bidders are observed.

In terms of data structure, our paper is closest to the recent work by Guerre and Luo (2022). They focus on first-price auctions and provide a novel method to identify the valuation distribution with only transaction prices. In particular, assuming that the number of the bidders K is random with a finite support and observed by the bidders but not the econometrician, they show that the winning bid increases with K and hence the density of the winning bid exhibits discontinuities as K varies. This key feature leads to the identification of the support and distribution of K and the valuation distribution. In comparison, our paper differs from Guerre and Luo (2022) in the following perspectives. First, the discontinuity in the winning bid density does not hold in ascending auctions or when buyers do not observe K. Both of these scenarios are allowed in our context. Second, the identification strategy in Guerre and Luo (2022) relies on a finite support of K and hence its distribution is fully characterized by a finite number of probability masses. We instead focus on a large and eventually infinite K, whose distribution does not have to be identified. Finally, Guerre and Luo (2022) and many others allow for auction heterogeneity by requiring additional observations or imposing additional assumptions, while our paper focuses on the IPV setup with homogeneous auctions. Since the number of auctions is small, the homogeneity across auctions is more plausibly satisfied. We discuss potential extensions in the concluding remarks.

To illustrate the empirical relevance, we apply the proposed method to the Hong Kong car license auctions from 1997 to 2008. This auction is the standard ascending price auction and is equivalent to the second-price auction under the symmetric IPV assumption. Each auction is for one license plate. Except for the transaction price and the digits on the license plate, we do not observe other information from the auction. Moreover, after dropping special plates that contain favorable digit combinations, the number of auctions/license plates is only approximately 30 each year. Given these restrictions, the existing methods based on the classic asymptotic framework cannot be applied as the valuation distribution cannot be identified or consistently estimated (cf. Athey and Haile, 2002). Applying our proposed method, we first find a considerable winner's expected utility. The middle values of the confidence intervals range from 1000 to 6800 Hong Kong dollars across years. Second, there is a sharp difference in the winner's utility before and after the year 2006, which reflects the structural change that special plates became available for auction in that year. Third, we test whether the commonly imposed conditions about the valuation distribution are satisfied and reject the null hypothesis in two out of 12 years.

The rest of the paper is organized as follows. Section 2 sets up the new asymptotic framework with many bidders and reviews EV theory. Section 3 considers second-price auctions and presents the new inference method. In particular, we introduce the auction format and derive the asymptotic distribution of the transaction prices under the new asymptotic framework in Section 3.1. Then we construct tests and confidence intervals for the tail index, the winner's expected utility, and the seller's expected revenue in Sections 3.2 to 3.4, respectively. Section 4 extends the analysis to the first-price auctions. Section 5 conducts simulation studies, and Section 6 studies the car licence plate application. Section 7 concludes with some remarks with computational details and all the proofs in the Appendix.

2 Asymptotic framework with many bidders

We consider inference in the context of sealed-bid auctions of a single object for sale. We assume that the data are composed of the transaction prices of $n \geq 3$ independent and identically distributed (i.i.d.) realizations of these auctions, given by $\{P_i : i = 1, \ldots, n\}$. One distinctive feature of our methodology is that we will not require n to diverge to infinity.

For each auction $j = 1, \ldots, n$, the setup is as follows. There is a single object for sale, and K_j potential buyers are bidding for the object. These bidders have independent private values $\{V_{i,j} : i = 1, \ldots, K_j\}$ distributed according to a common cumulative distribution function (CDF) F_V with support on $(-\infty, v^*]$, where $v^* = \infty$ is allowed. We assume that F_V is strictly increasing on $(-\infty, v^*]$ and admits a continuous probability density function (PDF) $f_V = F_V'$. As usual, bidders are assumed risk neutral and to maximize expected profits without facing any liquidity or budget constrains. We assume that F_V and K_j are common knowledge to all bidders, but unknown to the researcher. Our inference methods rely on the asymptotics with diverging numbers of bidders K_j and a finite number of auctions n. To this end, we assume that $K \equiv \min\{K_1, \ldots, K_n\} \to \infty$ and $K_i/K \to 1$ for each $i = 1, \ldots, n$. That is, all n auctions are assumed to have approximately the same "large" number of bidders.[1]

We make an additional mild assumption about the valuation distribution F_V. We assume that this distribution is in the domain of attraction of the EV distribution G_ξ, where ξ denotes the tail index. Formally, this means that there is a sequence of normalizing constants $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$ such that, for

[1]Allowing for an unknown number of bidders and $K_i/K \not\to 1$ for some $i = 1, \ldots, n$ significantly complicates our inference problem. In particular, inference in that setup would require the researcher to observe more than the transaction prices on each auction, which goes against the main premise of this paper.

all x that is a continuity point of G_ξ,

$$\lim_{K\to\infty} \left(F_V(a_K x + b_K)\right)^K \quad = \quad G_\xi(x). \tag{2.1}$$

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$$\lim_{K\to\infty} \left(F_V(a_K x + b_K)\right)^K \quad = \quad G_\xi(x). \tag{2.2}$$

See David and Nagaraja (2004, Chapter 10) or de Haan and Ferreira (2006, Chapter 1) for recent expositions on this topic, including sufficient conditions for (2.2). In particular, David and Nagaraja (2004, Theorem 10.5.2) provides sufficient conditions for (2.2) when F_V is absolutely continuous, which is the relevant case for this paper. Under (2.2), standard asymptotic results imply that G_ξ can only be one of three types: Fréchet (if $\xi > 0$), Gumbel (if $\xi = 0$), or Weibull (if $\xi < 0$). Up to location and scale transformation, these three types can be unified as the generalized EV distribution:

$$G_\xi(x) = \begin{cases} \exp\left(-(1+\xi x)^{-1/\xi}\right) & 1 + \xi x > 0,\ \xi \neq 0 \\ \exp\left(-\exp(-x)\right) & x \in \mathbb{R},\ \xi = 0. \end{cases} \tag{2.3}$$

See, e.g., David and Nagaraja (2004, Theorem 10.5.1) or de Haan and Ferreira (2006, Theorem 1.1.3).

Condition (2.2) is a very mild restriction, as is satisfied by commonly used valuation distributions F_V. First, the case with $\xi > 0$ covers distributions with unbounded support (i.e., $v^* = \infty$) and polynomial decaying (i.e., "heavy") right tail. In this case, moments of order less than $1/\xi$ exist and moments of order greater than $1/\xi$ do not (e.g, see de Haan and Ferreira (2006, Exercise 1.16)). Then, the restriction to $\xi \leq 1/2$ implies that F_V has finite second moments. Examples include Pareto, Student's t, and F distributions. Second, the case with $\xi = 0$ also covers distributions with unbounded support (i.e., $v^* = \infty$) but with exponential decaying (i.e.,"light") right tail and bounded moments of any order. This case includes distributions such as Normal and log-Normal distributions. Finally, the case $\xi < 0$ covers distributions with $v^* < \infty$ (i.e., bounded support), including Beta, Uniform, and triangular distributions.

For the purpose of this paper, the significance of (2.2) is that it allows us to characterize the joint distribution of the extreme ordered valuations for all auctions as the number of bidders diverges. We now introduce the relevant notation to this end. For each auction $j = 1, \ldots, n$, let $\{V_{(i),j} : i = 1, \ldots, K_j\}$ denote the order statistics of the valuations $\{V_{i,j} : i = 1, \ldots, K_j\}$ expressed in decreasing order, i.e., $V_{(1),j} \geq V_{(2),j} \geq \cdots \geq V_{(K_j),j}$. With this notation in place, Lemma 2.1 provides the joint distribution of the extreme order statistics for all auctions.

Lemma 2.1. *Assume* (2.2) *holds. For any* $n \in \mathbb{N}$*, and as* $K \to \infty$*,*

$$\left\{ \left(\frac{V_{(1),j} - b_K}{a_K}, \ \frac{V_{(2),j} - b_K}{a_K}, \ \ldots \ , \ \frac{V_{(d),j} - b_K}{a_K} \right) : j = 1, \ldots, n \right\} \overset{d}{\to}$$
$$\left\{ \left(H_\xi(E_{1,j}), \ H_\xi(E_{1,j} + E_{2,j}), \ \ldots \ , \ H_\xi\left(\sum\nolimits_{s=1}^{d} E_{s,j} \right) \right) : j = 1, \ldots, n \right\}, \qquad (2.4)$$

where $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$ *are the normalizing constants in* (2.2)*,* $\{E_{s,j} : s = 1, \ldots, d, \ j = 1, \ldots, n\}$ *are i.i.d. standard exponential random variables, and*

$$H_\xi(x) \ \equiv \ \begin{cases} (x^{-\xi} - 1)/\xi & \text{if } \xi \neq 0, \\ -\ln(x) & \text{if } \xi = 0. \end{cases}$$

Lemma 2.1 characterizes the asymptotic distribution of the largest order statistics. In this paper, we consider first-price and second-price auction formats, which involve only $V_{(1),j}$ and $V_{(2),j}$, respectively. We assume that bidders follow a symmetric equilibrium strategy, which allows us to fully characterize the mapping between private valuations and equilibrium bids and observed transaction prices $\{P_j : j = 1, \ldots, n\}$. This allows us to fully characterize the joint distribution of prices as a function the tail index ξ. In turn, there are several hypotheses of interest that are exclusively a function of the tail index ξ. By linking these two ideas, our framework allows us to develop inference for these hypotheses based on the observed transaction prices.

3 Second-price auctions

3.1 Auction format

We first consider second-price sealed-bid actions, where the highest bidder gets the object and pays the second-highest bid. By standard arguments, the weakly dominant strategy is that each bidder bids her valuation. See, e.g., Krishna (2009, Proposition 2.1). As a consequence, the observed transaction price in auction $j = 1, \ldots, n$ is equal to the second-highest bid, i.e.,

$$P_j = V_{(2),j}. \qquad (3.1)$$

By Lemma 2.1 and (3.1), we conclude that $K \to \infty$,

$$\left\{ \frac{P_j - b_K}{a_K} : j = 1, \ldots, n \right\} \overset{d}{\to} \left\{ H_\xi(E_{1,j} + E_{2,j}) : j = 1, \ldots, n \right\}, \qquad (3.2)$$

where $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$, $\{(E_{1,j}, E_{2,j}) : \ j = 1, \ldots, n\}$, and H_ξ as in Lemma 2.1.

If the normalizing constants $\{(a_K, b_K) \in \mathbb{R}_{++} \times \mathbb{R} : K \in \mathbb{N}\}$ were known, (3.2) could be used to conduct inference on functions of the EV index ξ. Unfortunately, these constants are unknown, and depend implicitly on the underlying distribution of valuations. To sidestep this issue, we consider the following self-normalized statistics: for $j = 1, \ldots, n^* \equiv n - 2$,

$$P_j^* \equiv \frac{P_j - P_n}{P_{n-1} - P_n}. \tag{3.3}$$

Note that (3.2) implies that observed prices are continuously distributed, which guarantees that the denominator in (3.3) is non-zero almost surely. Therefore,

$$\left\{P_j^* : j = 1, \ldots, n^*\right\} \xrightarrow{d} \left\{\frac{H_\xi(E_{1,j} + E_{2,j}) - H_\xi(E_{1,n} + E_{2,n})}{H_\xi(E_{1,n-1} + E_{2,n-1}) - H_\xi(E_{1,n} + E_{2,n})} : j = 1, \ldots, n^*\right\}, \tag{3.4}$$

where $\{(E_{1,j}, E_{2,j}) : j = 1, \ldots, n\}$ and H_ξ as in Lemma 2.1. As anticipated, (3.4) reveals that the asymptotic distribution of $\{P_j^* : j = 1, \ldots, n^*\}$, which further provides useful information about the tail index ξ.

Given the data (i.e., $\{P_j^* : j = 1, ..., n^*\}$), we now propose inference about the value of ξ, the expected utility of the winner, and the seller's expected revenue in second-price auctions in the following three subsections.

3.2 Inference about the tail index

Given Lemma 2.1 and (3.4), the asymptotic distribution of $\left(P_j - b_K\right)/a_K$ becomes that of $H_\xi\left(E_{1,j} + E_{2,j}\right)$ for all $j = 1, ..., n$. For notation simplicity, we introduce the short-hand notation that $Z_j = H_\xi\left(E_{1,j} + E_{2,j}\right)$ for $j = \{1, ..., n\}$ and

$$(P_1^*, ..., P_{n^*}^*) \xrightarrow{d} \left(\frac{Z_1 - Z_n}{Z_{n-1} - Z_n}, ..., \frac{Z_{n-2} - Z_n}{Z_{n-1} - Z_n}, 1, 0\right) \equiv (Z_1^*, ..., Z_{n^*}^*).$$

Moreover, since $\{Z_j : j = 1, ..., n\}$ are independent, we can order them such that $\{Z_{n-1} \geq Z_1 \geq Z_2 \geq \ldots \geq Z_{n-2} \geq Z_n\}$, whose joint density is $n! \prod_{j=1}^n f_{Z|\xi}(z_j)$. This is the same density of the un-ordered $\{Z_j : j = 1, ..., n\}$, that is, $\prod_{j=1}^n f_{Z|\xi}(z_j)$ multiplied by the constant $n!$ (e.g., Krishna, 2009, p.283). By doing so, we have that $1 \geq P_1^* \geq \ldots \geq P_{n^*}^* \geq 0$, which simplifies the numerical computation.

Using change of variables and the fact that

$$f_{Z|\xi}(x) = \left\{ \begin{array}{cc} (1 + \xi x)^{-\frac{2+\xi}{\xi}} \exp\left(-(1 + \xi x)^{-1/\xi}\right) & \text{if } \xi \neq 0 \\ \exp(-2x) \exp\left(-\exp(-x)\right) & \text{if } \xi = 0 \end{array} \right\},$$

we obtain that

$$\begin{aligned} & f_{Z_1^*, ..., Z_{n^*}^*|\xi}(z_1^*, ..., z_{n^*}^*) \\ & = n!\Gamma(2n) \int_0^{b(\xi)} s^{n-2} \exp\left(\begin{array}{c} -2n \log\left(\sum_{j=1}^{n^*} \left(1 + z_j^* \xi s\right)^{-1/\xi} + (1 + \xi s)^{-1/\xi}\right) \\ -\left(1 + \frac{2}{\xi}\right)\left(\sum_{j=1}^{n^*} \log\left(1 + z_j^* \xi s\right) + \log(1 + \xi s)\right) \end{array}\right) ds, \end{aligned} \tag{3.5}$$

where $\Gamma(\cdot)$ is the Gamma function, $b(\xi) = -1/\xi$ for $\xi < 0$ and $b(\xi) = \infty$ otherwise. This density can be numerically calculated by Gaussian Quadrature. The case with $\xi = 0$ can be computed as

$\lim_{\xi \to 0} f_{Z_1^*,...,Z_n^*|\xi}(z_1^*,...,z_n^*)$ given that $f_{Z_1^*,...,Z_n^*|\xi}$ is continuous in ξ.

Now we are ready to conduct inference about ξ. Consider the following testing problem with the competing hypotheses

$$\mathsf{H}_0 : \xi = \xi_0 \text{ against } \mathsf{H}_1 : \xi = \xi_1.$$

By Neyman-Pearson Lemma, the optimal test with observations $(Z_1^*,...,Z_n^*)$ that asymptotically controls size is the likelihood ratio test, that is,

$$\varphi(z_1^*,...,z_n^*) = 1\left[\frac{f_{Z_1^*,...,Z_n^*|\xi_1}(z_1^*,...,z_n^*)}{f_{Z_1^*,...,Z_n^*|\xi_0}(z_1^*,...,z_n^*)} > \mathrm{cv}(\xi_0,\xi_1,\alpha,n)\right],$$

where $\mathrm{cv}(\xi_0,\xi_1,\alpha,n)$ denotes the critical value depending on (ξ_0,ξ_1), the number of auctions n, and the significance level α.

In practice, we are often more interested in a hypothesis test with a composite alternative hypothesis. Given a composite alternative space $\Xi \subset \mathbb{R}\backslash\{\xi_0\}$, this test is given by

$$\mathsf{H}_0 : \xi = \xi_0 \text{ against } \mathsf{H}_1 : \xi \in \Xi.$$

To this end, we consider tests that maximize the weighted average power criterion (e.g. Andrews and Ploberger, 1994). In particular, for any probability measure $W(\cdot)$ on Ξ, we construct the following generalized likelihood ratio test

$$\varphi(z_1^*,...,z_n^*) = 1\left[\frac{\int_\Xi f_{Z_1^*,...,Z_n^*|\xi}(z_1^*,...,z_n^*)\,dW(\xi)}{f_{Z_1^*,...,Z_n^*|\xi_0}(z_1^*,...,z_n^*)} > \mathrm{cv}(\xi_0,W,\alpha,n)\right],$$

where the critical value now depends on ξ_0, α, n, and $W(\cdot)$. The weighting function $W(\cdot)$ transforms the composite alternative into a simple one so that the above test maximizes the W-weighted average power

$$\int_\Xi \mathbb{E}_\xi\left[\varphi(Z_1^*,...,Z_n^*)\right]dW(\xi),$$

where $\mathbb{E}_\xi[\cdot]$ is the expectation w.r.t. the density $f_{Z_1^*,...,Z_n^*|\xi}$. In practice, $W(\cdot)$ is chosen by the econometrician to reflect the importance attached to the ability of the test to reject for certain alternatives (e.g. Müller, 2011, pp. 400-401). In our auction problems, we set $\Xi = [-1, 0.5]$ and $W(\cdot)$ as the uniform measure. We rule out $\xi > 0.5$ to guarantee that the variance of valuation is finite. We rule out $\xi < -1$ since the density of the valuation will diverge to infinity near the right end-point (see Section 3.2 ahead). Therefore, our test is simplified as

$$\varphi(z_1^*,...,z_n^*) = 1\left[\frac{\int_{[-1,0.5]} f_{Z_1^*,...,Z_n^*|\xi}(z_1^*,...,z_n^*)\,d\xi}{f_{Z_1^*,...,Z_n^*|\xi_0}(z_1^*,...,z_n^*)} > \mathrm{cv}(\xi_0,\alpha,n)\right], \quad (3.6)$$

where the critical value is simulated as the $1-\alpha$ quantile of the likelihood ratios with random draws from the limiting density $f_{Z|\xi_0}$. Then by the continuous mapping theorem, this test controls size at least asymptotically such that $\lim_{K\to\infty} \mathbb{E}\left[\varphi(P_1^*,...,P_n^*)\right] = \mathbb{E}\left[\varphi(Z_1^*,...,Z_n^*)\right] \le \alpha$ for any fixed $n \ge 3$. By inverting (3.6), we can obtain the corresponding $1-\alpha$ level confidence interval for ξ.

We close this subsection with a remark about the special case with $\xi = -1$, which characterizes a set of commonly adopted regularity assumptions in the existing literature. Specifically, von Mises' condition (cf. Embrechts, Klüpperberg, and Mikosch, 1997, Chapter 3.3) implies that there are only four cases for the limit of the valuation distribution $f_V(v)$ as $v \to v^* \equiv \sup\{v : F_V(v) < 1\}$, namely

Case 1 $\xi \geq 0$, $v^* \leq \infty$ and $f_V(v) \to 0$ as $v \to v^*$;

Case 2 $\xi \in (-1, 0)$, $v^* < \infty$ and $f_V(v) \to 0$ as $v \to v^*$;

Case 3 $\xi = -1$, $v^* < \infty$ and $f_V(\cdot)$ is uniformly bounded away from 0 and ∞;

Case 4 $\xi < -1$, $v^* < \infty$ and $f_V(\cdot) \to \infty$ as $v \to v^*$.

The existing methods to analyze auction data sets typically assume that (i) the valuation has a bounded support and (ii) the density of V is uniformly bounded away from 0 and ∞ near the upper bound. If both assumptions are satisfied, the tail index has to be -1 as in Case 3 above. We formalize this result as the following lemma.

Lemma 3.1. *Suppose the following conditions holds that*

(i) $v^* < \infty$; *(ii)* $0 < \underline{C} < \inf_{v\in[v^*-\varepsilon,v^*]} f_V(v) \leq \sup_{v\in[v^*-\varepsilon,v^*]} f_V(v) < \bar{C}$ *for some positive constants* $\varepsilon, \underline{C}, \bar{C}$; *and (iii)* $f_V(\cdot)$ *is continuous, then* F_V *is in the domain of attraction of* G_ξ *with* $\xi = -1$.

Given Lemma 3.1, testing

$$\mathsf{H}_0 : \xi = -1 \text{ against } \mathsf{H}_1 : \xi \in (-1, 0.5]$$

suffices to test a necessary condition of these regularity conditions. Figure 1 presents the asymptotic (as $K \to \infty$) rejection probabilities of the LR test (3.6) with data generated from the EV distribution under different values of ξ. The significance level is 5%. As expected, this test controls size under the null hypothesis and has more powers as ξ or n increase. The finite sample performance with small numbers of bidders is presented in Section 5.

3.3 Inference about winner's expected utility

This section considers inference about the winner's expected utility, that is $\mu_K \equiv \mathbb{E}[V_{(1),j} - V_{(2),j}]$. When K is finite, this quantity depends on the whole valuation distribution. But under our asymptotics with $K \to \infty$, Lemma 2.1 implies that

$$\frac{\mu_K}{a_K} \to \Gamma(1 - \xi),$$

which is now fully characterized by ξ. Since a_K is unknown, we cannot simply implement the test (3.6) for inference about μ_K.

Alternatively, we note that μ_K is invariant to location shift and shares the same scale as transaction price. Therefore, denoting $\mathbf{P} = (P_1, \ldots, P_n)$, we aim to construct a confidence interval $U(\mathbf{P}) \subset \mathbb{R}$ for μ_K that satisfies $U(a\mathbf{P} + b) = aU(\mathbf{P})$ for any constants $a \neq 0$ and b, where $aU(\mathbf{P}) = \{x : (x/a \in U(\mathbf{P})\}$.

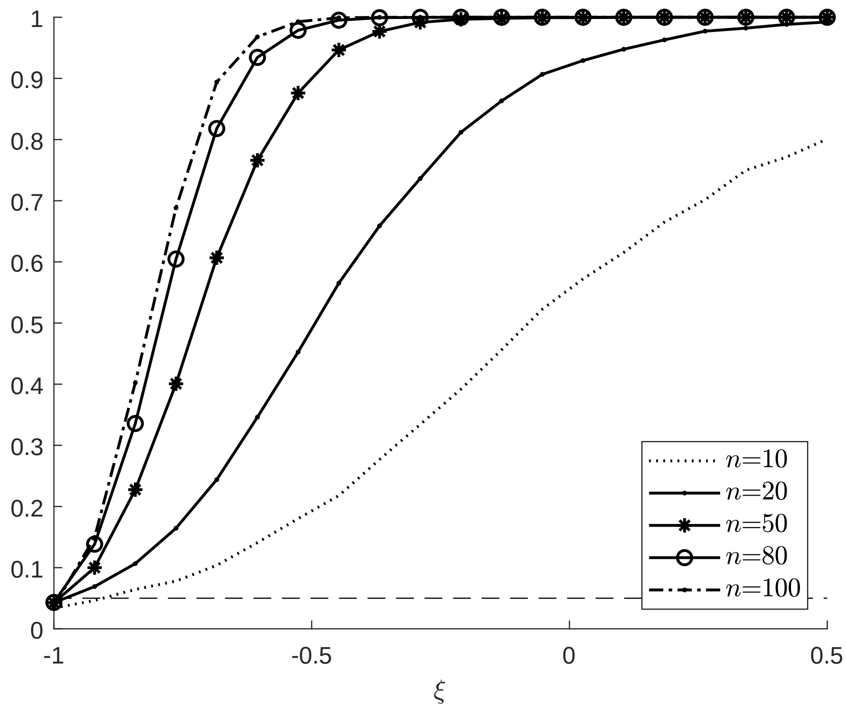

Figure 1: Asymptotic rejection probabilities of the LR test at 5% for $\xi = -1$. Based on 10,000 simulations.

To this end, we employ Lemma 2.1 and the continuous mapping theorem again to derive that

$$\left(\frac{\mu_K}{P_{n-1}-P_n},(P_1^*,\dots,P_{n^*}^*)\right)\xrightarrow{d}\left(\frac{\Gamma(1-\xi)}{Z_{n-1}-Z_n},(Z_1^*,\dots,Z_{n^*}^*)\right)\equiv\left(Y_\mu^*,\mathbf{Z}^*\right).$$

The joint density of Y_μ^* and $\mathbf{Z}^*$ can be derived similarly as in (3.5) by change of variables. We postpone the details to the Appendix for readability.

Now we work with the asymptotic problem in which we observe $\mathbf{Z}\equiv(Z_1,\dots,Z_n)$. Note that ξ is unknown even asymptotically since we have a finite number n of auctions. Therefore, we require the correct asymptotic coverage uniformly over $\xi\in\Xi$. Specifically, using the equivariance/invariance restriction, we have that

$$\begin{aligned}\mathbb{P}(\mu_K\in U(\mathbf{P})) &= \mathbb{P}\left(\frac{\mu_K}{P_{n-1}-P_n}\in U\left(\frac{\mathbf{P}-P_n}{P_{n-1}-P_n}\right)\right)\\ &\to \mathbb{P}_\xi\left(Y_\mu^*\in U(\mathbf{Z}^*)\right),\end{aligned}$$

where the notation $\mathbb{P}_\xi$ (and $\mathbb{E}_\xi$ below) indicates that the randomness is entirely characterized by ξ asymptotically. The asymptotic problem then is to construct a scale equivariant and location invariant U that satisfies

$$\mathbb{P}_\xi\left(Y_\mu^*\in U(\mathbf{Z}^*)\right)\ge 1-\alpha \text{ for all } \xi\in\Xi. \tag{3.7}$$

Given Lemma 2.1 and the equivariance/invariance constraint, such an interval satisfies

$$\lim_{n\to\infty} \mathbb{P}(\mu_K \in U(\mathbf{P})) \geq 1-\alpha \text{ for all } \xi \in \Xi.$$

In principle, there could still be many solutions that satisfy the asymptotic size constraint. To obtain the optimal one, we consider the weighted average expected length criterion

$$\int \mathbb{E}_\xi[\text{lgth}(U(\mathbf{P}))]dW(\xi), \tag{3.8}$$

where W again denotes some weighting measure on Ξ, and $\text{lgth}(A) = \int \mathbf{1}[y \in A]dy$ for any Borel set $A \subset \mathbb{R}$.

To solve the program of minimizing (3.8) subject to (3.7) among all equivariant/invariant intervals U, we write $\mathbb{E}_\xi[\text{lgth}(U(\mathbf{Z}))] = \mathbb{E}_\xi[(Z_{n-1}-Z_n)\text{lgth}(U(\mathbf{Z}^*))] = \mathbb{E}_\xi[\kappa_\xi(\mathbf{Z}^*)\text{lgth}(U(\mathbf{Z}^*))]$ with $\kappa_\xi(\mathbf{Z}^*) = \mathbb{E}_\xi[(Z_{n-1} - Z_n)|\mathbf{Z}^*]$. Thus, our problem becomes

$$\begin{aligned} &\min\nolimits_{U(\cdot)} \textstyle\int_\Xi \mathbb{E}_\xi[\kappa_\xi(\mathbf{Z}^*)\text{lgth}(U(\mathbf{Z}^*))]dW(\xi) \\ &\text{s.t. } \mathbb{P}_\xi\left(Y_\mu^* \in U(\mathbf{Z}^*)\right) \geq 1-\alpha \text{ for all } \xi \in \Xi. \end{aligned} \tag{3.9}$$

Note that any solution to (3.9)) also provides the form of U, that is, $U(\mathbf{Z}) = (Z_{n-1} - Z_n)U(\mathbf{Z}^*)$. So once $U(\cdot)$ is determined, the confidence interval can be constructed in practice by plugging in $(P_{n-1} - P_n)U(\mathbf{P}^*)$.

To make further progress in solving (3.9), we follow Müller and Wang (2017) to write the problem in the following Lagrangian form:

$$\min_{U(\cdot)} \int_\Xi \mathbb{E}_\xi[\kappa_\xi(\mathbf{Z}^*)\text{lgth}(U(\mathbf{Z}^*))]dW(\xi) + \int_\Xi \mathbb{P}_\xi\left(Y_\mu^* \in U(\mathbf{Z}^*)\right)d\Lambda(\xi),$$

where the non-negative measure Λ denotes the Lagrangian weights that guarantee the asymptotic size constraint. These weights can be considered as the least favorable approximation in the inference problem with a nuisance parameter under the null hypothesis (e.g., Lehmann and Romano, 2005, Chapter 3). By writing the expectations above as integrals over the densities $f_{\mathbf{Z}^*|\xi}$ of $\mathbf{Z}^*$ and $f_{Y_\mu^*,\mathbf{Z}^*|\xi}$ of $(Y_\mu^*, \mathbf{Z}^*)$, the solution of the above problem is given by

$$U(\mathbf{z}^*) = \left\{y : \int_\Xi \kappa_\xi(\mathbf{z}^*) f_{\mathbf{Z}^*|\xi}(\mathbf{z}^*)dW(\xi) < \int_\Xi f_{Y_\mu^*,\mathbf{Z}^*|\xi}(y, \mathbf{z}^*)d\Lambda(\xi)\right\}. \tag{3.10}$$

The integrals can be numerically calculated by Gaussian quadrature, and then the only remaining challenge is to find some suitable Lagrangian weights Λ. We solve this challenge by the numerical approach developed in Elliott, Müller, and Watson (2015). Then by construction, the confidence interval (3.10) is nearly optimal in the sense that it is a level 1-α equivariant set whose the W-weighted average length (3.8) is no more than 1% longer than any other equivariant set satisfying the coverage (3.7). See Appendix A.1 for further details. The MATLAB program and the weights Λ are available at the author's website. Of course Λ only needs to be computed once and then is ready to use for practitioners. Figure 2 depicts the results for $n = 30$ and 50. Then the most time-consuming part in solving the program (3.9) is the numerical integration, which costs only a few seconds in a modern PC. Further details are provided in Appendix A.1.

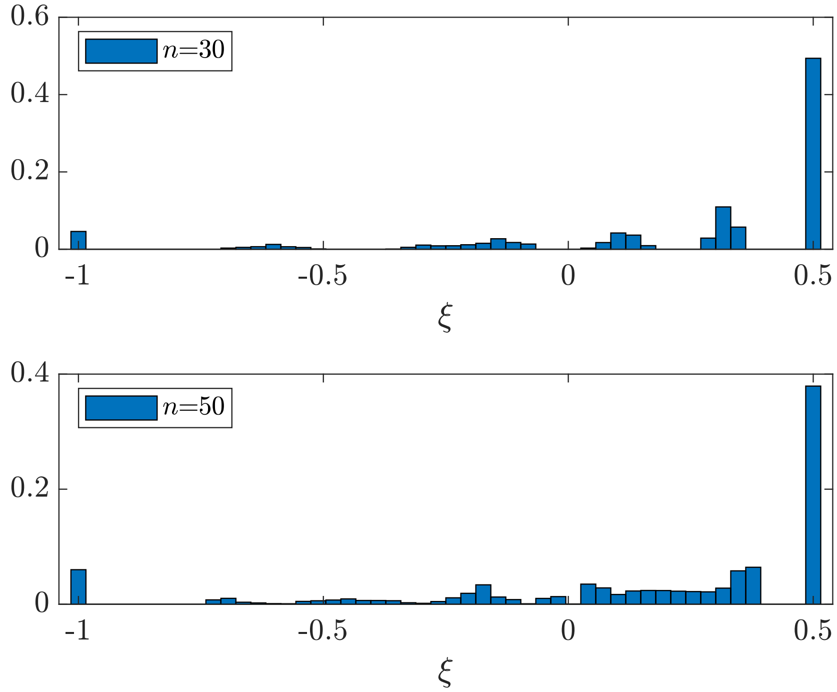

Figure 2: The weights Λ for $n = 30$ and 50. Based on 10,000 simulations.

3.4 Inference about the seller's expected revenue

We now consider the seller's expected revenue, that is, $\pi_K \equiv \mathbb{E}[V_{(2),j}]$. Employing Lemma 2.1, we have that

$$\frac{\pi_K - b_K}{a_K} \to \mathbb{E}\left[H_\xi(E_{1,j} + E_{2,j})\right] = \frac{\Gamma(2-\xi) - 1}{\xi} \equiv \pi^* \tag{3.11}$$

for $\xi \neq 0$. The case with $\xi = 0$ is obtained by taking the limit $\xi \to 0$. See Lemma A.4 in Appendix A.2 for details.

The derivation (3.11) yields that the seller's expected revenue is asymptotically equivalent to π^* after the scale and location normalization. To sidestep the issue of not knowing a_K and b_K, we impose the equivariance restriction such that our confidence interval U satisfies that $U(a\mathbf{P} + b) = aU(\mathbf{P}) + b$ for any $a \neq 0$ and b. Lemma 2.1 and the continuous mapping theorem yield that

$$\left(\frac{\pi_K - P_n}{P_{n-1} - P_n}, (P_1^*, \ldots, P_{n^*}^*)\right) \xrightarrow{d} \left(\frac{\pi^* - Z_n}{Z_{n-1} - Z_n}, (Z_1^*, \ldots, Z_{n^*}^*)\right) \equiv (Y_\pi^*, \mathbf{Z}^*).$$

Then the equivariance restriction implies that

$$\begin{aligned} \mathbb{P}(\pi_K^* \in U(\mathbf{P})) &= \mathbb{P}\left(\frac{\pi_K^* - P_n}{P_{n-1} - P_n} \in U\left(\frac{\mathbf{P} - P_n}{P_{n-1} - P_n}\right)\right) \\ &\to \mathbb{P}_\xi\left(Y_\pi^* \in U(\mathbf{Z}^*)\right). \end{aligned}$$

Similarly as (3.9), we can construct the confidence interval for π_K^* that minimizes the weighted average length and satisfies the uniform asymptotic coverage restriction, that is,

$$\begin{array}{l} \min_{U(\cdot)} \int_\Xi \mathbb{E}_\xi[\kappa_\xi(\mathbf{Z}^*)\text{lgth}(U(\mathbf{Z}^*))]dW(\xi) \\ \text{s.t. } \mathbb{P}_\xi\left(Y_\pi^* \in U(\mathbf{Z}^*)\right) \geq 1-\alpha \text{ for all } \xi \in \Xi. \end{array} \tag{3.12}$$

The solution of this problem is similarly written as

$$U(\mathbf{z}^*) = \left\{ y : \int_\Xi \kappa_\xi(\mathbf{z}^*) f_{\mathbf{Z}^*|\xi}(\mathbf{z}^*)dW(\xi) < \int_\Xi f_{Y_\pi^*,\mathbf{Z}^*|\xi}(y,\mathbf{z}^*)d\Lambda(\xi) \right\}, \tag{3.13}$$

where the density $f_{Y_\pi^*,\mathbf{Z}^*|\xi}$ is given in Appendix A.1.

We finish this section by discussing the optimal reserve price and its effect on the seller's expected revenue. In the classic setup where the seller's own value of the good, denoted as v_{0K}, is a constant, Riley and Samuelson (1981, Proposition 3) demonstrate that the optimal reserve price is also a constant regardless of the number of bidders. As the number of bidders increases, the transaction price will surpass the reserve price almost surely (cf. Wilson, 1977). As a consequence, the effect of the reserve price on the seller's revenue becomes degenerate asymptotically.

To construct the optimal reserve price that asymptotically affects the seller's revenue, we can assume that the seller's value of the good is of the same order of magnitude as the transaction price. To this end, we assume $(v_{0K} - b_K)/a_K \to v_0$ for some $v_0 \in \mathbb{R}$. Denote γ_K as the reserve price and then the seller's expected revenue becomes

$$\pi_K(\gamma_K) \equiv \mathbb{E}\left[\gamma_K \mathbf{1}\left[V_{(2),j} \leq \gamma_K \leq V_{(1),j}\right] + V_{(2),j}\mathbf{1}\left[\gamma_K \leq V_{(2),j}\right] + v_{0K}\mathbf{1}\left[V_{(1),j} \leq \gamma_K\right]\right].$$

Now as $K \to \infty$, the optimal γ_K should have the same order of magnitude as the transaction price $V_{(2),j}$. Writing the normalization $(\gamma_K - b_K)/a_K \to \gamma$, we employ Lemma 2.1 to derive the asymptotic expression of the revenue as

$$\begin{aligned} \frac{\pi_K(\gamma_K) - b_K}{a_K} &= \mathbb{E}\left[\frac{(r_K - b_K)}{a_K}\mathbf{1}\left[\frac{V_{(2),j} - b_K}{a_K} \leq \frac{\gamma_K - b_K}{a_K} \leq \frac{V_{(1),j} - b_K}{a_K}\right]\right] \\ &\quad + \mathbb{E}\left[\frac{\left(V_{(2),j} - b_K\right)}{a_K}\mathbf{1}\left[\frac{\gamma_K - b_K}{a_K} \leq \frac{V_{(2),j} - b_K}{a_K}\right]\right] \\ &\quad + \mathbb{E}\left[\frac{(V_{0K} - b_K)}{a_K}\mathbf{1}\left[\frac{V_{(1),j} - b_K}{a_K} \leq \frac{\gamma_K - b_K}{a_K}\right]\right] \\ &\to \gamma\mathbb{P}\left(H_\xi(E_{1,j} + E_{2,j}) \leq \gamma \leq H_\xi(E_{1,j})\right) + \mathbb{E}\left[H_\xi(E_{1,j} + E_{2,j})\mathbf{1}\left[\gamma \leq H_\xi(E_{1,j} + E_{2,j})\right]\right] \\ &\quad + v_0\mathbb{P}\left(H_\xi(E_{1,j}) \leq \gamma\right) \\ &\equiv \pi(\gamma), \end{aligned}$$

where $H_\xi(E_{1,j})$ and $H_\xi(E_{1,j}+E_{2,j})$ are jointly EV distributed as in (2.4). Their joint density is given by

$$f_{H_1,H_2|\xi}(x_1,x_2)=\begin{cases}(1+\xi x_1)^{-\frac{1+\xi}{\xi}}(1+\xi x_2)^{-\frac{1+\xi}{\xi}}\exp\left(-(1+\xi x_2)^{-1/\xi}\right) & \text{if } \xi\neq 0\\ \exp(-x_1)\exp(-x_2)\exp\left(-\exp(-x_2)\right) & \text{if } \xi=0\end{cases} \tag{3.14}$$

on $x_2 \le x_1$ and zero otherwise, where H_1 is short for $H_\xi(E_{1,j})$ and H_2 is short for $H_\xi(E_{1,j}+E_{2,j})$. The following lemma derives the asymptotic expression of the optimal reserve price. Given the knowledge of v_0, say zero, one can construct the confidence interval for the optimal reserve price γ_K^* similarly as in (3.13).

Lemma 3.2. *Under* (3.14), $\gamma^* \equiv \arg\max_\gamma \pi(\gamma)=(1+v_0)/(1-\xi)$.

4 First-price auctions

4.1 Auction format

We now consider first-price sealed-bid actions, where the highest bidder gets the object and pays the highest bid. By standard arguments, the symmetric equilibrium strategy is that a bidder with valuation v in an auction with K participants is to bid $\beta_K(v)\equiv \mathbb{E}[V_{(1),-j}|V_{(1),-j}<v]$, where $V_{(1),-j}$ denotes the highest bid among the remaining $(K-1)$ participants. See, e.g., Krishna (2009, Proposition 2.2) and Guerre et al. (2000). If this argument is applied to auction j (with K_j bidders), the equilibrium bid becomes

$$\beta_{K_j}(v) \;=\; \frac{K_j-1}{F_V(v)^{K_j-1}}\int_{-\infty}^{v} uF_V(u)^{K_j-2}f_V(u)du \;=\; v-\frac{\int_{-\infty}^{v}F_V(u)^{K_j-1}du}{F_V(v)^{K_j-1}}, \tag{4.1}$$

where the second equality holds by integration by parts. Since $\beta_{K_j}(v)$ is increasing in v, the auction j is won by the highest valuation bidder, who pays

$$P_j \;=\; V_{(1),j}-\frac{\int_{-\infty}^{V_{(1),j}}F_V(u)^{K_j-1}du}{F_V(V_{(1),j})^{K_j-1}}. \tag{4.2}$$

In Lemma A.3 in the Appendix, we use (4.2) to deduce that

$$\left\{\frac{P_j-b_K}{a_K}:j=1,\ldots,n\right\} \;\overset{d}{\to}\; \left\{X_j:j=1,\ldots,n\right\}, \tag{4.3}$$

where

$$X_j \;\equiv\; H_\xi\left(E_{1,j}\right)-\frac{\int_{-\infty}^{H_\xi\left(E_{1,j}\right)}G_\xi\left(h\right)dh}{G_\xi\left(H_\xi\left(E_{1,j}\right)\right)}, \tag{4.4}$$

$G_\xi(\cdot)$ is as in (2.3), and $\{(a_K,b_K)\in\mathbb{R}_{++}\times\mathbb{R}:K\in\mathbb{N}\}$, $\{E_{1,j}:\ j=1,\ldots,n\}$, and $H_\xi(\cdot)$ as in Lemma 2.1.

Once again, (4.3) cannot be directly used to conduct inference on the tail index ξ because the normalizing constants $\{(a_K,b_K)\in\mathbb{R}_{++}\times\mathbb{R}:K\in\mathbb{N}\}$ were unknown. To address this problem, we consider the self-normalized statistics in (3.3). In the case of a first-price auction, we have

$$\left\{P_j^* : j = 1, \ldots, n^*\right\} \stackrel{d}{\to} \left\{\frac{X_j - X_n}{X_{n-1} - X_n} : j = 1, \ldots, n^*\right\}. \tag{4.5}$$

Then (4.5) reveals that the asymptotic distribution of $\{P_j^* : j = 1, \ldots, n^*\}$, which further provides useful information about the tail index ξ and other tail-related features.

4.2 Inference about tail index, winner's utility, and seller's revenue

Given the asymptotic observation (4.4), the inference about ξ and other tail features follow analogously as in Sections 3.2 to 3.4. We only highlight the difference here.

We start with testing $\xi = -1$. In this case, $G_\xi(h)$ is simplified as $\exp(h-1)$ and then (4.4) becomes that $X_j = H_{-1}(E_{1,j}) - 1$. By construction, the normalized prices P_j^* is invariant to the constant shift by -1 and hence we can replace the joint density of $P_j^* : j = 1, \ldots, n^*$ with that of $Z_j^* : j = 1, \ldots, n^*$. See Appendix A.1 for the expression of $f_{\mathbf{Z}^*|\xi}$. Then the same test as in (3.6) is applicable and controls size asymptotically under the null hypothesis.

Now we consider inference about the winner's expected utility, which is given by $\mu_K = \mathbb{E}[V_{(1),j} - P_j]$. Under Lemma A.3 in the Appendix, we show that $\mu_K / a_K \to \Gamma(1-\xi)$. See also Gabaix, Laibson, and Li (2005, Proposition 8). Note that this expression is the same as in second-price auctions and coherent with the well-known revenue equivalence principle across auction formats (e.g., Krishna, 2009, Proposition 3.1).

To perform a similar analysis in Section 3.3, we need to derive the density of X as in (4.4), which does not have a closed-form expression. To see this, we again adopt the shorthand notation $H_1 = H_\xi(E_{1,j})$ and use (4.4) to write that

$$\begin{aligned} X &= H_1 - \frac{1}{G_\xi(H_1)} \int_{-\infty}^{H_1} G_\xi(h) dh \\ &= H_1 - \Gamma\left(-\xi, (1+\xi H_1)^{-1/\xi}\right) / \exp(-(1+\xi H_1)^{-1/\xi}), \end{aligned}$$

where we suppress the subscript j and denote $\Gamma(a,x) = \int_x^\infty t^{a-1} e^t dt$ as the incomplete gamma function. By the change of variables $H^* = (1+\xi H_1)^{-1/\xi}$ and the fact that $\Gamma(1+s,x) = s\Gamma(s,x) + x^s e^{-x}$, we have

$$\begin{aligned} \tilde{X} &\equiv 1 + \xi X \\ &= 1 + \xi H_1 - \xi \Gamma\left(-\xi, (1+\xi H_1)^{-1/\xi}\right) / \exp(-(1+\xi H_1)^{-1/\xi}) \\ &= \left[(H^*)^{-\xi} \exp(-H^*) - \xi\Gamma(-\xi, H^*)\right] \exp(H^*) \\ &= \Gamma(1-\xi, H^*) \exp(H^*) \end{aligned}$$

and hence

$$X = \frac{\Gamma(1-\xi, H^*) \exp(H^*) - 1}{\xi} \equiv e_\xi(H^*),$$

where the case with $\xi = 0$ is obtained by taking the limit.

Note that H^* has the standard exponential distribution and $e_\xi(x)$ is monotonically decreasing in x for any ξ. Letting the inverse function of e_ξ as e_ξ^{-1}, we then compute the density of X as

$$f_{X|\xi}(x) = -\frac{\partial e_\xi^{-1}(x)}{\partial x} \exp\left(-e_\xi^{-1}(x)\right). \tag{4.6}$$

Since $e_\xi(\cdot)$ does not a closed-form expression, evaluating this function and its inverse is challenging. To find a tractable alternative, we propose a numerical approximation based on Taylor expansion of the incomplete gamma function (e.g. Amore, 2005). Then we can construct approximations of the densities of $Y_\mu^* \equiv \Gamma(1-\xi)/(X_{n-1} - X_n)$ and the self-normalized X, which are given in Appendix A.1. The confidence intervals for the winner's utility can be constructed in the same way as in Sections 3.3. Moreover, the seller's expected revenue as in Section 3.4 can be applied with Z replaced by X given the equivalence revenue principle again.

5 Monte Carlo simulations

This section presents the finite sample performance of our proposed inference method and compares it with other existing ones based on the standard asymptotic framework with a diverging number of auctions. Section 5.1 considers second-price auctions and Section 5.2 considers first-price auctions.

5.1 Second-price auctions

We start with testing $\xi = -1$. Table 1 depicts the finite sample rejection probabilities of with the LR test (3.6). We generate i.i.d. values from four different distributions: the absolute value of standard Normal, the absolute value of Student-t(20), the Pareto distribution with exponent 0.25, and the standard uniform distribution over $[0, 3]$. The uniform distribution corresponds to the null hypothesis, and all other three distributions are the alternatives. We choose $W(\cdot)$ as the uniform weight and $\Xi = [-1, 0.5]$. The significance level is 5% and the results are based on 500 simulations. We set $n = 10$ or 100 and $K_j = K$ to be 10, 100, or K_j independently and uniformly generated from $\{80, 81, \ldots, 100\}$. The random K case examines the sensitivity of our method to the homogeneity in K.

The new test (3.6) performs well as long as the number of bidders is not too small. In particular, the new test overrejects when the number of bidders is only 10 since the EV convergence performs poorly. The new test quickly achieves the nominal size as the number of bidders increases. The testing power increases in the number of auctions.

In Table 2, we examine the confidence intervals of the winner's expected utility by comparing the new method with three other existing ones: (i) the parametric approach assuming Normal value distribution, (ii) the infeasible approach that relies on observing both the largest and second largest bids, and (iii) the nonparametric kernel estimation. In particular, if both the first and the second largest bids were observed, then we observe $D_j \equiv V_{(1),j} - V_{(2),j}$ and simply construct the t-statistic $\overline{D}/\sqrt{n^{-1}\sum_{j=1}^{n}\left(D_j - \bar{D}\right)^2}$, where

# Bidders	10		20		100		U{80,...,100}	
# Auctions	10	100	10	100	10	100	10	100
Dist.				Rejection Prob.				
$\lvert N(0,1)\rvert$	0.18	1.00	0.19	1.00	0.19	0.99	0.16	1.00
$\lvert t(20)\rvert$	0.19	0.99	0.18	0.99	0.16	0.98	0.17	0.98
Pa(0.25)	0.08	0.49	0.09	0.59	0.09	0.68	0.08	0.70
U[0,3]	0.08	0.29	0.06	0.10	0.05	0.05	0.04	0.05

Table 1: Finite sample rejection prob. of the LR test for $\xi = -1$ for second-price auctions

$\overline{D} = n_{j=1}^{-1} \sum D_j$. Regarding the nonparametric kernel estimator, we suppose K is known. The value distribution F_V and its PDF f_V can be nonparametrically estimated by first estimating F_P and then inverting the functional that

$$\begin{aligned} F_P(x) &= F_{V_{(2)}}(x) \\ &= \sum_{r=K-1}^{K} \binom{K}{r} F_V(x)^r \left(1 - F_V(x)\right)^{K-r} \\ &= F_V(x)^K + K F_V(x)^{K-1} \left(1 - F_V(x)\right). \end{aligned} \tag{5.1}$$

Assume $V > 0$. Then the expected utility is such that

$$\begin{aligned} & \mathbb{E}\left[V_{(1),j} - V_{(2),j}\right] \\ = \; & K \int x F_V(x)^{K-1} f_V(x)\, dx - (K-1) K \int x \left(f_V(s)\left(1 - F_V(s)\right) F_V(x)^{K-2}\right) dx \\ = \; & \int_0^\infty \left(1 - F_V^K(x)\right) dx - K \int_0^\infty \left(1 - F_V^{K-1}(x)\right) dx + (K-1) \int_0^\infty \left(1 - F_V^K(x)\right) dx \\ = \; & K \left(\int_0^\infty \left(1 - F_V^K(x)\right) dx - \int_0^\infty \left(1 - F_V^{K-1}(x)\right) dx \right). \end{aligned}$$

This quantity can be estimated by plugging in $\hat{F}_V(\cdot)$ and the confidence intervals can be constructed by bootstrap. We use 500 simulation draws and 100 bootstrap samples. The significance level is 0.05.

Table 2 presents the coverage and length of these four intervals with the same choices of n and K as in Table 1. In the case with a random K_j, the true expected utility is still generated with $K_j = K = 100$, so that we allow for some misspecification.

We summarize the findings as follows. First, the new approach based on EV theory has excellent small sample coverage and length properties. Second, if data are generated from the Student-t distribution with 20 degrees of freedom, which is very close to the standard Normal distribution, the parametric approach based on Normal assumption still suffers severe undercoverage since the object of interest is in the tail. Unreported results show that only if the degree of freedom is larger than 40, approximating Student-t with Normal distribution leads to a satisfactory performance of the parametric approach. Third, comparing the length, the new approach even dominates the infeasible sample average approach. This finding implies that knowing the transaction price itself is good enough for learning the winner's expected utility. Finally, the nonparametric method works poorly because the sample size is not sufficiently large. The uniform

# Bidders	10				100				U{80,81,...,100}			
# Auctions	10		100		10		100		10		100	
Dist.	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth
New approach based on EV theory												
$\lvert N(0,1)\rvert$	0.94	1.46	0.94	0.24	0.95	1.17	0.98	0.18	0.95	1.22	0.94	0.18
$\lvert t(20)\rvert$	0.94	1.68	0.92	0.28	0.94	1.45	0.94	0.27	0.94	1.52	0.96	0.27
Pa(0.25)	0.98	1.47	0.94	0.54	0.94	2.63	0.95	0.96	0.94	2.53	0.95	0.93
U[0,3]	0.97	0.77	0.93	0.08	0.96	0.31	0.96	0.11	0.98	0.34	0.98	0.12
Parametric approach based on Normal assumption												
$\lvert N(0,1)\rvert$	0.93	0.21	0.75	0.07	0.92	0.15	0.93	0.05	0.92	0.15	0.84	0.05
$\lvert t(20)\rvert$	0.81	0.22	0.72	0.07	0.53	0.17	0.00	0.06	0.54	0.17	0.00	0.06
Pa(0.25)	0.01	0.34	0.00	0.11	0.04	0.22	0.00	0.09	0.00	0.22	0.00	0.07
U[0,3]	0.00	0.34	0.00	0.11	0.00	0.28	0.00	0.09	0.00	0.06	0.00	0.02
Infeasible sample average approach												
$\lvert N(0,1)\rvert$	0.89	0.47	0.94	0.17	0.90	0.45	0.95	0.16	0.89	0.35	0.96	0.12
$\lvert t(20)\rvert$	0.90	0.70	0.92	0.19	0.89	0.63	0.94	0.22	0.86	0.48	0.92	0.17
Pa(0.25)	0.79	0.75	0.90	0.29	0.80	1.30	0.92	0.52	0.77	1.32	0.87	0.49
U[0,3]	0.87	0.28	0.94	0.10	0.86	0.03	0.93	0.01	0.88	0.04	0.86	0.01
Nonparametric kernel approach												
$\lvert N(0,1)\rvert$	0.05	0.22	0.23	0.11	0.05	0.15	0.19	0.08	0.06	0.16	0.24	0.08
$\lvert t(20)\rvert$	0.02	0.24	0.16	0.13	0.03	0.19	0.14	0.11	0.04	0.20	0.14	0.11
Pa(0.25)	0.00	0.17	0.06	0.15	0.01	0.30	0.06	0.25	0.01	0.30	0.04	0.24
U[0,3]	0.65	0.22	0.92	0.08	0.67	0.03	0.96	0.01	0.79	0.03	0.73	0.01

Table 2: Finite sample performance of inference about winner's expected utility for second-price auctions. Based on 500 simulation draws. The significance level is 0.05.

distribution seems exceptional when n is 100. This is potentially because the winner's expected utility goes to zero given the bounded support.

5.2 First-price auctions

We now consider first-price auctions and start with testing for $\xi = -1$. We implement the test (3.6) with $W(\cdot)$ as the uniform weight and $\Xi = [-1, 0.5]$. We use the same DGPs as before. The simulation results are depicted in Table 3. Recall that only the uniform distribution corresponds to the null hypothesis and all the other three distributions correspond to the alternative hypothesis. Similarly as in the second-price auctions, the LR test has excellent size and power properties as long as the number of bidders is not too small.

In Table 4, we construct the 95% confidence intervals for the winner's expected utility. For easy implementation, we make the following miner modification. Regarding the parametric approach assuming Normal value distribution, we assume $V_{(1),j}$, the highest valuation is observed instead of the highest bid P_j (because

# Bidders	10		20		100		U{80,...,100}	
# Auctions	10	100	10	100	10	100	10	100
Dist.				Rejection Prob.				
$\lvert N(0,1)\rvert$	0.13	1.00	0.13	1.00	0.13	1.00	0.12	1.00
$\lvert t(20)\rvert$	0.14	1.00	0.18	1.00	0.18	1.00	0.18	1.00
Pa(0.25)	0.20	1.00	0.19	1.00	0.18	1.00	0.19	1.00
U[0,3]	0.03	0.10	0.05	0.07	0.05	0.05	0.07	0.17

Table 3: Finite sample rejection prob. of the LR test for $\xi = -1$ in first-price auctions. Based on 500 simulation draws. The significance level is 0.05.

the density of P_j cannot be analytically solved). Then using

$$\begin{aligned} \mathbb{E}\left[V_{(1),j} - P_j\right] &= \mathbb{E}\left[\frac{\int_0^{V_{(1)}} F_V^{K-1}(x)\,dx}{F_V^{K-1}(V_{(1)})}\right] \\ &= \int \frac{\int_0^{v} F_V^{K-1}(x)\,dx}{F_V^{K-1}(v)} d\left(F_V(v)\right)^K \\ &= K\int\left(\int_0^{v} F_V^{K-1}(x)\,dx\right) f_V(v)dv, \end{aligned} \tag{5.2}$$

we can estimate the expected utility by plugging in the pseudo maximum likelihood estimator. Since the above expression does not have an analytic form, we construct the confidence intervals with 100 bootstrap samples. Regarding the infeasible approach based on sample average, we assume both $V_{(1),j}$ and P_j are observed, so that the t-statistics based on $V_{(1),j} - P_j$ can be easily constructed. Regarding the nonparametric kernel estimator, suppose K is known and $V_{(1),j}$ instead of P_j is observed. Then the value distribution F_V can be nonparametrically estimated by first estimating $F_{V_{(1)}}$ and then using $F_V(\cdot) = F_{V_{(1)}}^{1/K}(\cdot)$. The expected utility is further estimated by plugging $\hat{F}_V(\cdot)$ into (5.2) and the intervals can be constructed by bootstrap. We use 500 simulation draws and 100 bootstrap samples.

Table 4 presents the coverage and length of the confidence intervals based on EV theory and the infeasible sample average ones. The same pattern can be found as in Table 2. The other two approaches are strictly dominated and hence not reported. In particular, the confidence intervals based on the Normal value distribution assumption deliver almost zero coverage in all scenarios except when data are generated from the Normal distribution. The kernel estimation performs poorly again given the small sample size.

6 Empirical application

We illustrate the new inference method with the car license plate auctions in Hong Kong. The data are available from Ng, Chong, and Du (2010). The auction is implemented as the standard oral ascending auction, which is equivalent to the second-price auction under the symmetric IPV setup. The number of bidders is large and approximately the same across auctions. Plates are categorized into special and non-special ones. The former can be individually designed and will not be tradable after the auction. The non-special ones have random patterns and can be traded after their initial sales.

# Bidders	10				100				U{80,81,...,100}			
# Auctions	10		100		10		100		10		100	
Dist.	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth	Cov	Lgth
	New approach based on EV theory											
$\lvert N(0,1)\rvert$	0.94	1.05	0.98	0.38	0.96	1.17	0.99	0.29	0.97	1.12	1.00	0.46
$\lvert t(20)\rvert$	0.94	1.08	0.96	0.42	0.97	1.09	0.99	0.38	0.96	1.10	1.00	0.62
Pa(0.25)	0.96	1.22	0.85	0.51	0.96	1.73	0.95	0.89	0.97	1.82	1.00	1.43
U[0,3]	0.89	0.98	0.99	0.24	0.93	0.20	0.98	0.13	0.95	0.23	0.86	0.17
	Infeasible sample average approach											
$\lvert N(0,1)\rvert$	0.86	0.31	0.94	0.11	0.86	0.24	0.93	0.09	0.86	0.25	0.95	0.09
$\lvert t(20)\rvert$	0.88	0.40	0.93	0.14	0.83	0.37	0.92	0.13	0.83	0.36	0.92	0.13
Pa(0.25)	0.81	0.69	0.88	0.28	0.76	1.22	0.91	0.47	0.74	1.19	0.87	0.46
U[0,3]	0.89	0.03	0.95	0.01	0.85	0.00	0.95	0.00	0.00	0.00	0.00	0.00

Table 4: Finite sample performance of inference about winner's expected utility for first-price auctions. Based on 500 simulation draws. The significance level is 0.05.

In each auction, we observe the plate number, auction date, whether or not the plate was successfully sold, and if so, the transaction price. We do not observe the reserve prices of the special plates, the number of bidders, the bidder's identities, or the bid increments. The reserve price for non-special plates is 1000 Hong Kong dollars (HKDs).

One may concern that some plate patterns are preferable to others. To control for this observed heterogeneity, we focus on the non-special plates that satisfy the following criteria: letters are not HK, letters are not the same (e.g., not AA. BB, CC, etc.), the numbers on the plate are not in order (e.g. not 1369), or in reverse order (e.g., not 9631), the number part of the plate has 4 digits, and none of these digits is either an 8 or a 4 (indicating good or bad fortune in pronunciation), and the sold price is larger than the reserve price.

Table 6 presents the confidence intervals for the expected utility of the winner and the p-values of the test (3.6) for $\xi = -1$ using the data from each year. The significance level is 0.05. We can make several interesting findings. First, the winner's expected utility is very large in the economic sense since the middle values of the confidence intervals range from 1000 to 6800 HKDs. Second, there is a sharp difference in winner's utility before and after the year of 2006, which reflects the structural change that special plates became available for auction in that year. Third, the hypothesis that $\xi = -1$ is satisfied for most of the years. But it is rejected at 10% level in the years 1997 and 2003, suggesting that existing inference methods relying on the bounded support and positive density conditions might be unappropriate.

7 Concluding remarks

This paper studies inference in first-price or second-price sealed-bid auctions with a large number of symmetric independent private values. In this context, we propose an asymptotic framework where the number of bidders diverges, while the number of auctions remains fixed. Under this new framework, we propose new

year	n	95% CI		p-value	year	n	95% CI		p-value
1997	26	2.24	9.20	0.06	2003	46	0.82	2.98	0.00
1998	27	1.81	6.96	>0.10	2004	12	0.74	4.94	>0.10
1999	32	1.28	5.07	>0.10	2005	22	0.61	3.80	>0.10
2000	29	1.72	6.18	>0.10	2006	17	0.13	1.61	>0.10
2001	34	1.29	4.52	>0.10	2007	31	0.81	2.71	>0.10
2002	18	1.13	5.56	>0.10	2008	24	0.59	3.47	>0.10

Table 5: 95% winner's expected utility (measured in 1000 HKDs) and the p-values for testing $\xi = -1$ using (3.6) in Hong Kong car license plate auctions. See the main text for more details.

inference methods for the winner's expected utility, the seller's expected revenue, and the tail properties of the valuation distribution. We conclude with several remarks.

First, the new framework provides the asymptotically exact inference as the number of bidders diverges, but is valid even if the number of observed auctions is small. This is in contrast with the existing methods whose asymptotic derivations rely on a diverging number of auctions. If the number of auctions is also diverging, we could improve our method by consistently estimating the tail index ξ of the valuation distribution and other features that can be expressed as functions of ξ. Second, the new framework requires observing only the transaction price in each auction. This is in contrast with the existing methods that require multiple bids in each auction. If multiple bids are available, we could use the joint asymptotic distributions of these largest order statistics to improve the power of our test and shorten the length of our confidence intervals. Third, the new framework does not requires observing the number of bidders, which can be difficult to obtain in practice. This could happen when some bidders do not submit their bids as they are already below the standing ask price. Our method sidesteps this issue as long as the true number of bidders is large and the transaction price is above the reserve price.

Finally, our analysis has focused on the symmetric IPV setup but allows for extension to the cases with asymmetric bidders and non-independent valuations provided additional conditions are imposed. One possibility is the conditional IPV setup (e.g. Li, Perrigne, and Vuong, 2000), where in each auction the ith bidder's valuation V_i consists of a common component v plus an independent signal σ_i. The effect of v can be eliminated by imposing the invariance restriction, and then the transaction price can be employed to learn the tail features of the distribution of σ_i. Such extensions are investigated in other ongoing research projects.

A Appendix

Appendix A.1 provides computational details omitted from the main text. Appendix A.2 presents all mathematical proofs.

A.1 Computational details

The following expressions are used in Section 3 for the second-price auctions:

$$f_{\mathbf{Z}^*|\xi}\left(\mathbf{z}^*\right) = n!\Gamma\left(2n\right)\int_0^{b(\xi)} s^{n-2}\exp\left(\begin{array}{c} -2n\log\left(\sum_{j=1}^{n^*}(1+z_j^*\xi s)^{-1/\xi} + (1+\xi s)^{-1/\xi}\right) \\ -(1+\frac{2}{\xi})\left(\sum_{j=1}^{n^*}\log\left(1+z_j^*\xi s\right) + \log\left(1+\xi s\right)\right)\end{array}\right) ds,$$

$$\kappa_\xi\left(\mathbf{z}^*\right) f_{\mathbf{Z}^*|\xi}\left(\mathbf{z}^*\right) = n!\Gamma\left(2n-\xi\right)\int_0^{b(\xi)} s^{n-1}\exp\left(\begin{array}{c} (\xi-2n)\log\left(\sum_{j=1}^{n^*}(1+z_j^*\xi s)^{-1/\xi} + (1+\xi s)^{-1/\xi}\right) \\ -(1+\frac{2}{\xi})\left(\sum_{j=1}^{n^*}\log\left(1+z_j^*\xi s\right) + \log\left(1+\xi s\right)\right)\end{array}\right) ds,$$

where $b\left(\xi\right) = -1/\xi$ for $\xi < 0$, and $b\left(\xi\right) = \infty$ otherwise,

$$f_{Y_\mu^*,\mathbf{Z}^*|\xi}(y,\mathbf{z}^*) = n!\Gamma\left(1-\xi\right)^{n-1} y^{-n}\int_{a(\xi)}^{c(\xi)}\exp\left(\begin{array}{c} -\sum_{j=1}^{n^*+2}(1+\xi s+\xi\frac{z_j^*}{y}\Gamma)^{-1/\xi} \\ -(1+\frac{2}{\xi})\sum_{j=1}^{n^*+2}\log\left(1+\xi s+\xi\frac{z_j^*}{y}\Gamma\right)\end{array}\right) ds$$

where Γ denotes $\Gamma\left(1-\xi\right)$, $z_{n-1}^* = 1$, $z_n^* = 0$, and $a\left(\xi\right)$ and $c\left(\xi\right)$ are such that for all $s \in \left(a\left(\xi\right), c\left(\xi\right)\right)$, $1+\xi s+\xi\Gamma y^{-1} > 0$ and $1+\xi s > 0$, and

$$f_{Y_\pi^*,\mathbf{Z}^*|\xi}(y,\mathbf{z}^*) = n!y^{-n}\int_{a_1(\xi)}^{c_1(\xi)}\left|\pi^*-s\right|^{n-1}\exp\left(\begin{array}{c} -\sum_{j=1}^{n^*+2}(1+\xi(s+z_j^*(\pi^*-s)/y)^{-1/\xi} \\ -(1+\frac{2}{\xi})\sum_{j=1}^{n^*+2}\log\left(1+\xi(s+z_j^*(\pi^*-s)/y\right)\end{array}\right) ds,$$

where $a_1\left(\xi\right)$ and $c_1\left(\xi\right)$ are such that for all $s \in \left(a_1\left(\xi\right), c_1\left(\xi\right)\right)$, $1+\xi(s+(\pi^*-s)y^{-1} > 0$, $1+\xi s > 0$ and $(\pi^*-s)/y > 0$.

For first-price auctions, we first employ the following numerical approximation for the function $e_\xi\left(\cdot\right)$ in Section 4.2. Consider the asymptotic series (e.g., Amore, 2005) that

$$\Gamma\left(a,x\right) \approx x^{a-1}\exp\left(-x\right)\left[1+\frac{a-1}{x}+\frac{\left(a-1\right)\left(a-2\right)}{x^2}+...\right],$$

Then taking the first order approximation $\Gamma\left(a,x\right) \approx x^{a-1}\exp\left(-x\right)$ yields that $e_\xi\left(x\right) \approx (x^{-\xi}-1)/\xi$.

The above approximation is more precise for large values of x. To obtain a better performance, we can approximate $\Gamma\left(1-\xi,x\right)\exp\left(x\right)$ by some polynomial function of x as motivated by the expressions in second-auction prices. In particular, we approximate $\log\left(\Gamma\left(1-\xi,x\right)\exp\left(x\right)\right)$ by $-r_1\log x + r_2$, where r_1 and r_2 are some constants depending on ξ. To this end, we obtain r_1 and r_2 for each ξ by first generating an equally-spaced grid of 50000 x_i's within the 10^{-6} and $1-10^{-6}$ quantiles of a standard exponentially distributed random variable and then regressing $\log(r\left(1-\xi,x_i\right)\exp(x_i))$ on $\log(x_i)$ and a constant. The regression coefficients serve as a good candidate for $(-r_1, r_2)$.

Given this approximation, we have that

$$\begin{aligned} e_\xi(x) &\approx \frac{\exp\left(-r_1\log(x)+r_2\right)-1}{\xi} = \frac{x^{-r_1}e^{r_2}-1}{\xi} \\ e_\xi^{-1}(x) &\approx (1+\xi x)^{-1/r_1}\underbrace{\left(e^{-r_2}\right)^{-1/r_1}}_{\equiv r_3} = r_3(1+\xi x)^{-1/r_1} \\ \frac{\partial e_\xi^{-1}(x)}{\partial x} &\approx -\frac{r_3\xi}{r_1}(1+\xi x)^{-1/r_1-1}. \end{aligned}$$

Then using $f_{X|\xi}(x) = -\frac{\partial e_\xi^{-1}(x)}{\partial x}\exp\left(-e_\xi^{-1}(x)\right)$, we approximate the joint distribution of $\mathbf{X}^*$ as

$$f_{\mathbf{X}^*|\xi}(x^*) = \frac{n!\Gamma(n)|r_1|}{|\xi|}\int_0^{b(\xi)} s^{n-2}\exp\left(\begin{array}{c} -n\log\left(r_3\sum_{j=1}^n(1+\xi x_j^* s)^{-1/r_1}\right) \\ -(1+\frac{1}{r_1})\sum_{j=1}^n\log\left(1+\xi x_j^* s\right)+n\log\left(\left|\frac{r_3\xi}{r_1}\right|\right)\end{array}\right)ds,$$

where $x_{n-1}^*=1$, $x_n^*=0$, and $n=n^*+2$.

Similarly, we have

$$\kappa_\xi\left(\mathbf{x}^*\right)f_{\mathbf{X}^*|\xi}\left(\mathbf{x}^*\right) = n!|r_1|\Gamma(n-r_1)|\xi|^{-1}\int_0^{b(\xi)} s^{n-1}\exp\left(\begin{array}{c}(r_1-n)\log\left(r_3\sum_{j=1}^n(1+\xi x_j^* s)^{-1/r_1}\right) \\ -\sum_{j=1}^n(1+\frac{1}{r_1})\log\left(1+\xi x_j^* s)\right)+n\log\left(\left|\frac{r_3\xi}{r_1}\right|\right)\end{array}\right)ds,$$

and

$$f_{Y_\mu^*,\mathbf{X}^*|\xi}(y,\mathbf{x}^*) = n!\Gamma(1-\xi)^{n-1}y^{-n}\int_{a(\xi)}^{c(\xi)}\exp\left(\begin{array}{c}-\sum_{j=1}^n(1+\frac{1}{r_1})\log\left(1+\xi s+\xi\frac{z_j^*}{y}\Gamma(1-\xi)\right) \\ +n\log\left(\frac{r_3\xi}{r_1}\right)-r_3\sum_{j=1}^n(1+\xi s+\xi\frac{z_j^*}{y}\Gamma(1-\xi))^{-1/r_1}\end{array}\right)ds,$$

where $a(\xi)$ and $c(\xi)$ are such that for all $s\in\left(a(\xi),c(\xi)\right)$, $1+\xi s+\xi\Gamma y^{-1}>0$ and $1+\xi s>0$.

All the above integrals are numerically approximated by Gaussian quadrature for implementation.

To determine the Lagrange multipliers Λ, we use the algorithm developed by Elliott et al. (2015). See also Müller and Wang (2017). In particular, we implement the following steps for inference about the winner's expected utility. The program for the optimal reserve price is similar.

Step 1 Discretize the space Ξ into a fine grid $\Xi_M\equiv\{\xi_1,\ldots,\xi_M\}$ and generate random draws $\mathbf{Z}$ from the EV distribution with each ξ_j in this grid.

Step 2 Start with an arbitrary initial vector $\lambda^{(0)}$, say $\{1/M,\ldots,1/M\}$. Solve problem (3.9) numerically and evaluate the coverage $\mathbb{P}_{\xi_j}\left(\frac{\Gamma(1-\xi_j)}{Z_{n-1}-Z_n}\in U\left(\mathbf{Z}^*\right)\right)$ numerically. Denote these estimated coverage as the vector $\hat{\mathbb{P}}$.

Step 3 Update the vector of λ by setting $\lambda^{(s+1)}=\lambda^{(s)}+\varepsilon\left(\hat{\mathbb{P}}-(1-\alpha)\right)$ where ε is some small step length, say 0.01, and $\alpha=0.05$ is the significance level. The idea is to increase the weight on ξ_j if the constructed interval has overcoverage when ξ_j is the parameter generating the data and symmetrically to decrease the weight if the interval has undercoverage.

Step 4 Iterate steps 2 and 3 for S times, say 2000, and record $\lambda^{(S)}$ and numerically compute the W-weighted average length (3.8). Denote it as $V_{\lambda^{(S)}}$.

Step 5 Numerically determine the constant c^* such that $V_{c^*\lambda^{(S)}}=(1+\varepsilon)V_{\lambda^{(S)}}$ for $\varepsilon=0.01$. The output $c^*\lambda^{(S)}$ is a suitable candidates for the Lagrangian multipliers.

Step 6 Generate a finer grid $\Xi_{\tilde{M}}$ with $\tilde{M} > M$ and estimate $\hat{\mathbb{P}}$ with $\lambda^{(S)}$. If all components of $\hat{\mathbb{P}}$ are larger than or equal to $1-\alpha$, stop and use $\lambda^{(S)}$. Otherwise, use $\Xi_{\tilde{M}}$ as the initial grid and repeat the above steps. Due to continuity, the uniform size control can be nearly achieved over Ξ if the grid is fine enough. This is easily checked numerically.

For any given n, the Lagrange multipliers only need to be determined once. The tables of the Lagrange multipliers and the corresponding MATLAB code are provided on our website: https://sites.google.com/site/yulongwanghome/.

A.2 Proofs

Proof of Lemma 2.1. Since the n auctions are i.i.d., it suffices to show the marginal convergence result for any auction $j = 1, \ldots n$, i.e.,

$$\left(\frac{V_{(1),j} - b_K}{a_K}, \ldots, \frac{V_{(d),j} - b_{K_j}}{a_K}\right) \overset{d}{\to} \left(H_\xi(E_{1,j}), \ldots, H_\xi\left(\sum\nolimits_{s=1}^{d} E_{s,j}\right)\right) \text{ as } K \to \infty, \tag{A.1}$$

where $\{E_{s,j} : s = 1, \ldots, d\}$ are i.i.d. standard exponential random variables. We fix $j = 1, \ldots n$ for the remainder of the proof.

Let x be any continuity point of G_ξ. Since $K_j \to \infty$ as $K \to \infty$, (2.2) implies that there is a sequence of constants $\{(a_{K_j}, b_{K_j}) \in \mathbb{R}_{++} \times \mathbb{R} : K_j \in \mathbb{N}\}$ s.t.

$$F(a_{K_j} x + b_{K_j})^{K_j} \to G_\xi(x) \text{ as } K \to \infty. \tag{A.2}$$

Under (A.2), de Haan and Ferreira (2006, Theorem 2.1.1), implies that

$$\left(\frac{V_{(1),j} - b_{K_j}}{a_{K_j}}, \ldots, \frac{V_{(d),j} - b_{K_j}}{a_{K_j}}\right) \overset{d}{\to} \left(H_\xi(E_{1,j}), \ldots, H_\xi\left(\sum\nolimits_{s=1}^{d} E_{s,j}\right)\right) \text{ as } K \to \infty. \tag{A.3}$$

Note that

$$\left(\frac{V_{(1),j} - b_K}{a_K}, \ldots, \frac{V_{(d),j} - b_K}{a_K}\right) = \left(\frac{V_{(1),j} - b_{K_j}}{a_{K_j}}, \ldots, \frac{V_{(d),j} - b_{K_j}}{a_{K_j}}\right) \frac{a_{K_j}}{a_K} + \left(\frac{b_K - b_{K_j}}{a_K}, \ldots, \frac{b_K - b_{K_j}}{a_K}\right). \tag{A.4}$$

under (A.3) and (A.4), (A.1) follows from showing that

$$\left(\frac{a_{K_j}}{a_K}, \frac{b_{K_j} - b_K}{a_K}\right) \to (1, 0) \text{ as } K \to \infty. \tag{A.5}$$

We devote the remainder of this proof to establish (A.5).

Let x be any continuity point of G_ξ. By (A.2) and $K_j/K \to 1$ as $K \to \infty$,

$$F(a_{K_j} x + b_{K_j})^K \to G_\xi(x) \text{ as } K \to \infty. \tag{A.6}$$

Under (2.2) and (A.6), de Haan (1976, Lemma 1) implies that for all $y \in \mathbb{R}$,

$$G_\xi\left(\lim_{K\to\infty}\left(\frac{a_{K_j}}{a_K} y + \frac{b_{K_j} - b_K}{a_K}\right)\right) = G_\xi(y). \tag{A.7}$$

By de Haan and Ferreira (2006, Theorem 1.1.3), there are three possible specifications for G_ξ. If $\xi = 0$, $G_\xi(y) =$

$\exp(-\exp(-y))$, which is strictly increasing. If $\xi > 0$, $G_\xi(y) = \exp(-(1+\xi y)^{-1/\xi})1[1+\xi y > 0]$, which is strictly increasing for $y > -1/\xi$. Finally, if $\xi < 0$, $G_\xi(y) = \exp(-(1+\xi y)^{-1/\xi})1[1+\xi y > 0] + 1[1+\xi y \leq 0]$, which is strictly increasing for $y < -1/\xi$. Thus, in all three cases, there is a continuum of values of y, which we can denote by S_ξ, s.t. G_ξ is is strictly increasing and, therefore, invertible. Then, (A.7) implies that for all $y \in S_\xi$,

$$\frac{a_{K_j}}{a_K} y + \frac{b_{K_j} - b_K}{a_K} \ \to \ y \ \text{ as } K \to \infty. \tag{A.8}$$

Since S_ξ includes a continuum of values, (A.8) implies (A.5), as desired. ■

Lemma A.1. *Under* (2.2),

$$\sup_{x \in \mathbb{R}} |F_V(a_K x + b_K)^K - G_\xi(x)| \to 0 \quad \text{as} \quad K \to \infty.$$

Proof. This proof is analogous to that of van der Vaart (1998, Lemma 2.11). ■

Lemma A.2. *For any sequence* $\{x_K : K \in \mathbb{N}\}$ *with* $x_K \to x \in S_\xi \equiv \{s : G_\xi(s) > 0\}$,

$$L_K(x_K) \to L(x),$$

where the sequence of functions $\{L_K : K \in \mathbb{N}\}$ *and the function* L *are defined as follows*

$$\begin{aligned} L_K(x) &\equiv x - \frac{\int_{-\infty}^x F_V(ha_K + b_K)^K dh}{F_V(xa_K + b_K)^K} \\ L(x) &\equiv x - \frac{\int_{-\infty}^x G_\xi(h)\, dh}{G_\xi(x)}. \end{aligned}$$

Proof. As a preliminary step, we now show that for any $x \in \mathbb{R}$,

$$\int_{-\infty}^x \left(F_V(ha_K + b_K)^K dh - G_\xi(h)\right) dh \to 0 \text{ as } K \to \infty. \tag{A.9}$$

We begin by showing the result for $x = 0$, i.e.,

$$\int_{-\infty}^0 \left(F_V(ha_K + b_K)^K dh - G_\xi(h)\right) dh \to 0 \text{ as } K \to \infty. \tag{A.10}$$

For a fixed K, consider the following argument:

$$\int_{-\infty}^0 \left(F_V(ha_K + b_K)^K - G_\xi(h)\right) dh \overset{(1)}{=} E\left[\min\{Z_\xi, 0\}\right] - E\left[\min\left\{\frac{V_{(1),K} - b_K}{a_K}, 0\right\}\right] \tag{A.11}$$

where Z_ξ denotes a RV with CDF G_ξ, and (1) holds by integration by parts, $\lim_{x\to-\infty} xF_V(xa_K + b_K)^K = \lim_{x\to-\infty} xG_\xi(x) = 0$. Since $\xi < 1$, we have that $E|Z_\xi| < \infty$. Under $E|Z_\xi| < \infty$ and the fact that valuations are non-negative, Pickands (1968, Theorem 2.1) shows that

$$\lim_{K\to\infty} E\left[\min\left\{\frac{V_{(1),K} - b_K}{a_K}, 0\right\}\right] = E\left[\min\{Z_\xi, 0\}\right]. \tag{A.12}$$

By combining (A.11) and (A.12), (A.10) holds. Next, consider the following argument for $x \neq 0$:

$$\left|\int_{-\infty}^x \left(F_V(ha_K + b_K)^K - G_\xi(h)\right) dh\right| \leq \left\{ \begin{array}{c} \left|\int_{-\infty}^0 \left(F_V(ha_K + b_K)^K - G_\xi(h)\right) dh\right| \\ +|x| \sup_y \left|F_V(ya_K + b_K)^K - G_\xi(y)\right| \end{array} \right\} \overset{(1)}{\to} 0 \text{ as } K \to \infty.$$

as desired, here (1) holds by (A.10) and Lemma A.1. Between both cases, we conclude that (A.9) holds, as desired.

As a second preliminary step, we now show that

$$\int_{-\infty}^{x} G_\xi(h)\,dh < \infty. \tag{A.13}$$

To show this, consider the following argument:

$$\int_{-\infty}^{x} G_\xi(h)\,dh =_{(1)} xG_\xi(x) - \int_{-\infty}^{x} hg_\xi(h)\,dh \leq xG_\xi(x) + E\left[|Z_\xi|\right] \overset{(2)}{<} \infty,$$

as desired, where Z_ξ denotes a RV with CDF G_ξ, (1) holds by integration by parts and $\lim_{x\to-\infty} xG_\xi(x) = 0$, and (2) holds by $\xi < 1$.

We are now ready to show the desired results. For any $x \in S_\xi$, consider the following argument.

$$\begin{aligned}
\left|L_K(x_K) - L(x)\right| &= \left|\frac{\int_{-\infty}^{x_K} F_V(ha_K+b_K)^K dh}{F_V(x_Ka_K+b_K)^K} - \frac{\int_{-\infty}^{x} G_\xi(h)\,dh}{G_\xi(x)}\right| \\
&= \frac{\left|\begin{array}{c} G_\xi(x)\left(\int_{-\infty}^{x_K} F_V(ha_K+b_K)^K dh - \int_{-\infty}^{x} F_V(ha_K+b_K)^K dh\right) \\ G_\xi(x)\left(\int_{-\infty}^{x} F_V(ha_K+b_K)^K dh - \int_{-\infty}^{x} G_\xi(h)\,dh\right) \\ -\left(\int_{-\infty}^{x} G_\xi(h)\,dh\right)\left(F_V(x_Ka_K+b_K)^K - G_\xi(x_K)\right) \\ -\left(\int_{-\infty}^{x} G_\xi(h)\,dh\right)\left(G_\xi(x_K) - G_\xi(x)\right)\end{array}\right|}{G_\xi(x)\left[\left(F_V(x_Ka_K+b_K)^K - G_\xi(x_K)\right) + \left(G_\xi(x_K) - G_\xi(x)\right) + G_\xi(x)\right]} \\
&\overset{(1)}{\leq} \frac{\left|\begin{array}{c} |x_K - x| + \left|\int_{-\infty}^{x}\left(F_V(ha_K+b_K)^K dh - G_\xi(h)\right)dh\right| \\ +\left(\int_{-\infty}^{x} G_\xi(h)\,dh\right)\sup_{y\in\mathbb{R}}\left|F_V(ya_K+b_K)^K - G_\xi(y)\right| \\ +\left(\int_{-\infty}^{x} G_\xi(h)\,dh\right)\left|G_\xi(x_K) - G_\xi(x)\right|\end{array}\right|}{G_\xi(x)\left[G_\xi(x) - \sup_{y\in\mathbb{R}}\left|F_V(ya_K+b_K)^K - G_\xi(y)\right| - \left|G_\xi(x_K) - G_\xi(x)\right|\right]},
\end{aligned} \tag{A.14}$$

where (1) holds by $G_\xi(x) \leq 1$ and $\sup_{y\in\mathbb{R}} F_V(ya_K+b_K)^K \leq 1$. As $K \to \infty$, we can deduce that the numerator and the denominator of the RHS of (A.14) converge to zero and $G_\xi(x)^2 > 0$, respectively. This conclusion relies on $x \in S_\xi$ (and so $G_\xi(x)^2 > 0$), $x_K \to x \in S$ as $K \to \infty$, the continuity of G_ξ, (A.9), (A.13), and Lemma A.1. From this conclusion and (A.14), the desired result follows. ■

Lemma A.3. *Under* (A.2),

$$\left\{\frac{P_j - b_K}{a_K} : j = 1,\ldots,n\right\} \overset{d}{\to} \left\{\left(H_\xi(E_{1,j}) - \frac{\int_{-\infty}^{H_\xi(E_{1,j})} G_\xi(h)\,dh}{G_\xi\left(H_\xi(E_{1,j})\right)}\right) : j = 1,\ldots,n\right\},$$

and $\{(a_K, b_K) \in \mathbb{R}_{++}\times\mathbb{R} : K \in \mathbb{N}\}$, $\{E_{1,j} :\ j = 1,\ldots,n\}$, *and* H_ξ *as in Lemma 2.1.*

Proof. Since the n auctions are i.i.d., it suffices to show the marginal convergence result for any auction j, i.e.,

$$\frac{P_j - b_K}{a_K} \overset{d}{\to} H_\xi(E_{1,j}) - \frac{\int_{-\infty}^{H_\xi(E_{1,j})} G_\xi(h)\,dh}{G_\xi\left(H_\xi(E_{1,j})\right)}, \tag{A.15}$$

where $E_{1,j}$ is a standard exponential random variable. We fix $j = 1,\ldots n$ for the remainder of the proof.

Consider the following argument.

$$\begin{aligned}
\frac{P_j - b_K}{a_K} &= \frac{P_j - b_{K_j-1}}{a_{K_j-1}}\frac{a_{K_j-1}}{a_K} + \frac{b_{K_j-1} - b_K}{a_K} \\
&= \left(\frac{V_{(1),j} - b_{K_j-1}}{a_{K_j-1}} - \frac{\left(\int_{-\infty}^{V_{(1),j}} F_V(u)^{K_j-1} du\right)/a_{K_j-1}}{F_V(V_{(1),j})^{K_j-1}}\right)\frac{a_{K_j-1}}{a_K} + \frac{b_{K_j-1} - b_K}{a_K} \\
&\overset{(1)}{=} \left(L_{K_j-1}\left(\frac{V_{(1),j} - b_{K_j-1}}{a_{K_j-1}}\right)\right)\frac{a_{K_j-1}}{a_K} + \frac{b_{K_j-1} - b_K}{a_K},
\end{aligned} \tag{A.16}$$

where (1) holds by the change of variable $u = ha_{K_j-1} + b_{K_j-1}$, and by defining $\{L_K : K \in \mathbb{N}\}$ as in Lemma A.2.

Since $(K_j - 1)/K \to 1$ as $K \to \infty$, the same argument as Lemma 2.1 shows that

$$\left(\frac{a_{K_j-1}}{a_K}, \frac{b_{K_j-1} - b_K}{a_K}\right) \to (1, 0) \quad \text{as } K \to \infty. \tag{A.17}$$

Next, notice that Lemma 2.1 and (A.17) imply that

$$\frac{V_{(1),j} - b_{K_j-1}}{a_{K_j-1}} = \frac{V_{(1),j} - b_K}{a_K}\frac{a_K}{a_{K_j-1}} + \frac{b_K - b_{K_j-1}}{a_K} \overset{d}{\to} H_\xi(E_1) \quad \text{as } K \to \infty. \tag{A.18}$$

In addition, $H_\xi(E_1) \in S_\xi \equiv \{x \in \mathbb{R} : G_\xi(x) > 0\}$. Then, by (A.18) and the extended CMT (e.g. van der Vaart, 1998, Theorem 1.11.1),

$$L_{K_j-1}\left(\frac{V_{(1),j} - b_{K_j-1}}{a_{K_j-1}}\right) \overset{d}{\to} L\left(H_\xi(E_1)\right) = O_p(1) \quad \text{as } K \to \infty, \tag{A.19}$$

where L is defined as in Lemma A.2. Then, (A.15) follows from combining (A.16), (A.17), and (A.19). ■

Proof of Lemma 3.1. By de Haan and Ferreira (2006, Theorem 1.2.1), it suffices to establish that

$$\lim_{t\downarrow 0}\frac{1 - F_V(v^* - tx)}{1 - F_V(v^* - t)} = x$$

for all $x > 0$. Condition (iii) allows us to use L'Hospital rule to obtain that

$$\lim_{t\downarrow 0}\frac{1 - F_V(v^* - tx)}{1 - F_V(v^* - t)} = \lim_{t\downarrow 0}\frac{f_V(\dot{v} - tx)}{f_V(\ddot{v} - t)}x$$

where $\dot{v}$ and $\ddot{v}$ are both in the ε-neighborhood of v^* if t is small enough. Then by the continuous mapping theorem and Conditions (ii) and (iii), we have $\lim_{t\downarrow 0}\frac{f_V(\dot{v})}{f_V(\ddot{v})} = 1$, which yields the result. ■

Lemma A.4. *$\pi = (\Gamma(2 - \xi) - 1)/\xi$ if $\xi \neq 0$, and $-1 + \mathring{\gamma}$ otherwise, where $\mathring{\gamma} \approx 0.57721$ is the Euler's constant.*

Proof of Lemma A.4. We introduce the shorthand notation $\tilde{\pi} = 1 + \xi\pi$ and $\tilde{H}_2 = H_\xi(E_1 + E_2)$ where we suppress the subscript j given the i.i.d.ness. Note that for $1 + \xi x \geq 0$,

$$f_{\tilde{H}_2|\xi}(x) = \begin{cases} x^{-2/\xi-1}\exp\left(-x^{-1/\xi}\right)/|\xi| & \text{if } \xi \neq 0 \\ \\ \exp(-2x)\exp\left(-\exp(-x)\right) & \text{otherwise} \end{cases}$$

When $\xi \neq 0$, we have that $\tilde{\pi} = 1 + \xi\pi(\gamma) = |\xi|^{-1} \int_0^\infty x^{-2/\xi} \exp\left(-x^{-1/\xi}\right) dx = \Gamma(2-\xi)$, implying that $\pi = \left(\Gamma(2-\xi) - 1\right)/\xi$.

When $\xi = 0$, we have that $\pi = \int_{-\infty}^{\infty} x \exp(-2x) \exp\left(-\exp(-x)\right) dx = -1 + \mathring{\gamma}$, which is equivalent to $\lim_{\xi \to 0} \left(\Gamma(2-\xi) - 1\right)/\xi$. ■

Proof of Lemma 3.2. Recall that we impose $\xi < 1$. We divide the rest of the argument depending on the sign of ξ. We introduce the short-hand notation that $H_1 = H_\xi(E_{1,j})$ and $H_2 = H_\xi(E_{1,j} + E_{2,j})$.

Case 1: $\xi \in (0,1)$. In that case,

$$\begin{aligned}
1 + \xi\pi(\gamma) &\overset{(1)}{=} (1+\xi\gamma)\mathbb{P}(H_2 \leq \gamma \leq H_1) + \mathbb{E}[1 + \xi H_2 | \gamma \leq H_2]\mathbb{P}(\gamma \leq H_2) + (1+\xi v_0)\mathbb{P}(H_1 < \gamma) \\
&\overset{(2)}{=} \tilde{\gamma} P(\tilde{H}_2 \leq \tilde{\gamma} \leq \tilde{H}_1) + \mathbb{E}[\tilde{H}_2 | \tilde{\gamma} \leq \tilde{H}_2] P(\tilde{\gamma} \leq \tilde{H}_2) + \tilde{v}_0 \mathbb{P}(\tilde{H}_1 < \tilde{\gamma}) \\
&\overset{(3)}{=} \exp(-\tilde{\gamma}^{-1/\xi})\tilde{\gamma}^{1-1/\xi} + \int_0^{\tilde{\gamma}^{-1/\xi}} t^{1-\xi} \exp(-t) dt + \tilde{v}_0 \exp(-\tilde{\gamma}^{-1/\xi}) \\
&\overset{(4)}{=} \exp(-(1+\xi\gamma)^{-1/\xi})(1+\xi\gamma)^{1-1/\xi} + \int_0^{(1+\xi\gamma)^{-1/\xi}} t^{1-\xi} \exp(-t) dt + (1+\xi v_0) \exp(-(1+\xi\gamma)^{-1/\xi}),
\end{aligned} \tag{A.20}$$

where (1) holds by definition of $\pi(\gamma)$, (2) holds by using $\tilde{\gamma} = 1 + \xi\gamma$, $\tilde{v}_0 = (1 + \xi v_0)$ and $\tilde{H}_j = 1 + \xi H_j$ for $j = 1, 2$, (3) holds by computation from (3.14), and (4) holds by replacing back $\tilde{\gamma} = 1 + \xi\gamma$. The computation that delivers (3) is greatly simplified by the transformation of the random variables $\tilde{H}_j = 1 + \xi H_j$ for $j = 1, 2$, whose joint and marginal PDFs are

$$\begin{aligned}
f_{\tilde{H}_1, \tilde{H}_2}(x_1, x_2) &= x_1^{-1/\xi - 1} x_2^{-1/\xi - 1} \exp(-x_2^{-1/\xi})\xi^{-2} 1[x_1 \geq x_2 \geq 0], \\
f_{\tilde{H}_j}(x) &= x^{-j/\xi - 1} \exp(-x^{-1/\xi})\xi^{-1} 1[x \geq 0] \qquad \text{for } j = 1, 2.
\end{aligned}$$

Since $\xi > 0$, maximizing $\pi(\gamma)$ is equivalent to maximizing $1 + \xi\pi(\gamma)$. The corresponding first and second order conditions imply Lemma 3.2.

Case 2: $\xi < 0$. Despite the change in sign, an analogous derivation shows that (A.20) also holds. Since $\xi < 0$, maximizing $\pi(\gamma)$ is equivalent to minimizing $1 + \xi\pi(\gamma)$. The corresponding first and second order conditions then imply Lemma 3.2.

Case 3: $\xi = 0$. In that case,

$$\begin{aligned}
\pi(\gamma) &\overset{(1)}{=} \gamma \mathbb{P}(H_2 \leq \gamma \leq H_1) + \mathbb{E}[H_2 | \gamma \leq H_2]\mathbb{P}(\gamma \leq H_2) \\
&\overset{(2)}{=} \gamma \exp(-\gamma) \exp(-\exp(-\gamma)) + \int_\gamma^\infty t \exp(-2t) \exp(-\exp(-t)) dt,
\end{aligned}$$

where (1) holds by definition of $\pi(\gamma)$ and (2) holds by computation from (3.14). From here, the first and second order conditions imply Lemma 3.2. ■

References

Adams, C. P. (2007): "Estimating demand from eBay prices," *International Journal of Industrial Organization*, 25, 1213–1232.

Amore, P. (2005): "Asymptotic and exact series representations for the incomplete Gamma function," *Europhysics Letters*, 71, 1–7.

An, Y., Y. Hu, and M. Shum (2010): "Estimating first-price auctions with an unknown number of bidders: a misclassification approach." *Journal of Econometrics*, 157, 328–341.

Andrews, D. W. K. and W. Ploberger (1994): "Optimal tests when a nuisance parameter is present only under the alternative," *Econometrica*, 62, 1383–1414.

Aradillas-Lopez, A., A. Gandhi, and D. Quint (2013): "Identification and inference in ascending auctions with correlated private values," *Econometrica*, 81, 489–534.

Athey, S. C. and P. A. Haile (2002): "Identification of standard auction models," *Econometrica*, 70, 2107–2140.

——— (2007): "Nonparametric approaches to auctions," *Handbook of Econometrics*, 6, 3847–3965.

Brendstrup, B. and H. J. Paarsch (2006): "Identification and estimation in sequential, asymmetric, English auctions," *Journal of Econometrics*, 134, 69–94.

——— (2007): "Semiparametric identification and estimation in multi-object, English auctions," *Journal of Econometrics*, 141, 84–108.

Chesher, A. and A. Rosen (2017): "Incomplete English auction models with heterogeneity," *cemmap working paper*, CWP27/17.

David, H. A. and H. N. Nagaraja (2004): *Order Statistics*, John Wiley & Sons.

de Haan, L. (1976): "Sample extreme: an elementary introduction," *Statistica Neerlandica*, 30, 161–172.

de Haan, L. and A. Ferreira (2006): *Extreme value theory: an introduction*, New York: Springer.

Di Tillio, A., M. Ottaviani, and P. N. Sørensen (2021): "Strategic sample selection," *Econometrica*, 89, 911–953.

Elliott, G., U. K. Müller, and M. W. Watson (2015): "Nearly optimal tests when a nuisance parameter is present under the null hypothesis," *Econometrica*, 83, 771–811.

Embrechts, P., C. Klüpperberg, and T. Mikosch (1997): *Modelling extremal events for insurance and finance*, Springer, New York.

Freyberger, J. and J. Larsen, Bradley (2020): "Identification in ascending auctions, with an application to digital rights management," *Quantitative Economics*, forthcoming.

Gabaix, X., D. Laibson, and H. Li (2005): "Extreme value theory and the effects of competition on profits," *Massachusetts Institute of Technology, Working Paper.*

Gentry, M. L., T. P. Hubbard, D. Nekipelov, and H. J. Paarsch (2018): "Structural econometrics of auctions: a review," *Foundations and Trends in Econometrics*, 9, 79–302.

Gentry, M. L. and T. Li (2014): “Identification in Auctions With Selective Entry,” *Econometrica*, 82, 315–344.

Guerre, E. and Y. Luo (2022): “Nonparametric Identification of First-Price Auction with Unobserved Competition: A Density Discontinuity Framework,” *Working Paper*.

Guerre, E., I. Perrigne, and Q. Vuong (2000): “Optimal nonparametric estimation of first-price auctions,” *Econometrica*, 68, 525–574.

Haile, P. A. and E. Tamer (2003): “Inference with an incomplete model of English auctions,” *Journal of Political Economy*, 111, 1–51.

Hickman, B. R., P. Hubbard, Timothy, and Y. Sağlam (2012): “Structural econometric methods in auctions: a guide to the literature,” *Journal of Econometric Methods*, 1, 67–106.

Hickman, B. R., T. P. Hubbard, and H. J. Paarsch (2017): “Identification and estimation of a bidding model for electronic auctions,” *Quantitative Economics*, 8, 505–551.

Hong, H., H. J. Paarsch, and P. Xu (2014): “On the asymptotic distribution of the transaction price in a clock model of a multi-unit, oral, ascending-price auction within the common-value paradigm,” *Rand Journal of Economics*, 44, 664–685.

Hong, H. and M. Shum (2004): “Rates of information aggregation in common value auctions,” *Journal of Economic Theory*, 116, 1–40.

Kim, K. i. and J. Lee (2014): “Nonparametric Estimation and Testing of the Symmetric IPV Framework with Unknown Number of Bidders,” *Working Paper*.

Komarova, T. (2013): “A new approach to identifying generalized competing risksmodels with application to second-price auctions,” *Quantitative Economics*, 4, 269–328.

Krishna, V. (2009): *Auction theory*, Academic press.

Lehmann, E. L. and J. P. Romano (2005): *Testing Statistical Hypothesis*, New York: Springer.

Li, T. (2005): “Econometrics of first-price auctions with entry and binding reservation prices,” *Journal of Econometrics*, 126, 173–200.

Li, T., I. Perrigne, and Q. Vuong (2000): “Conditionally independent private information in OCS wildcat auctions,” *Journal of Econometrics*, 98, 129–161.

Li, T. and X. Zheng (2009): “Entry and Competition Effects in First-Price Auctions: Theory and Evidence from Procurement Auctions,” *Review of Economic Studies*, 76, 1397–1429.

Luo, Y. and R. Xiao (2022): “Identification of Auction Models Using Order Statistics,” *Working Paper*.

Maskin, E. and J. G. Riley (1984): “Optimal auction with risk averse buyers,” *Econometrica*, 52, 1473–1518.

Mbakop, E. (2017): “Identification of auctions with incomplete bid data in the presence of unobserved heterogeneity,” *Working Paper*.

Menzel, K. and P. Morganti (2013): "Large sample properties for estimators based on the order statistics approach in auctions," *Quantitative Economics*, 4, 329–375.

Müller, U. K. (2011): "Efficient tests under a weak convergence assumption," *Econometrica*, 79, 395–435.

Müller, U. K. and Y. Wang (2017): "Fixed-k asymptotic inference about tail properties," *Journal of the American Statistical Association*, 112, 1134–1143.

Ng, T., T. Chong, and X. Du (2010): "The Value of Superstitions," *Journal of Economic Psychology*, 31, 293–309.

Pickands, III, J. (1968): "Moment convergence of sample extremes," *Annals of Mathematical Statistics*, 881–889.

Riley, J. G. and W. F. Samuelson (1981): "Optimal Auctions," *American Economic Review*, 71, 381–192.

Shneyerov, A. and A. C. L. Wong (2011): "Identification in first-price and Dutch auctions when the number of potential bidders is unobservable," *Games and Economic Behavior*, 72, 574–582.

Song, U. (2015): "Identification of auction models with an unknown number of bidders," *working paper*.

van der Vaart, A. W. (1998): *Asymptotic statistics*, Cambridge, UK: Cambridge University Press.

Virág, G. (2013): "First-price auctions with resale: the case of many bidders," *Economic Theory*, 52, 129–163.

William, Steven, R. and K. E. Zachariadis (2021): "The Asymptotics of Price and Strategy in the Buyer's Bid Double Auction," *Working Paper*.

Wilson, R. (1977): "A bidding model of perfect competition," *Review of Economic Studies*, 44, 511–518.