NBER WORKING PAPER SERIES

DYNAMIC PRICE COMPETITION: THEORY AND EVIDENCE FROM AIRLINE MARKETS

Ali Hortaçsu
Aniko Oery
Kevin R. Williams

Working Paper 30347
http://www.nber.org/papers/w30347

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge, MA 02138
August 2022

We gratefully acknowledge financial and computational support from the Yale School of Management and the Becker Friedman Institute for Economics at the University of Chicago. We thank Jose Betancourt and Steve Li for their excellent research assistance. We thank seminar participants at NYU, Yale, and SICS 2022 for comments. The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research.

Dynamic Price Competition: Theory and Evidence from Airline Markets
Ali Hortaçsu, Aniko Oery, and Kevin R. Williams
NBER Working Paper No. 30347
August 2022
JEL No. C70,C73,D21,D22,D43,D60,L13,L93

ABSTRACT

We introduce a model of oligopoly dynamic pricing where firms with limited capacity face a sales deadline. We establish conditions under which the equilibrium is unique and converges to a system of differential equations. Using unique and comprehensive pricing and bookings data for competing U.S. airlines, we estimate our model and find that dynamic pricing results in higher output but lower welfare than under uniform pricing. Our theoretical and empirical findings run counter to standard results in single-firm settings due to the strategic role of competitor scarcity. Pricing heuristics commonly used by airlines increase welfare relative to estimated equilibrium predictions.

Ali Hortaçsu
Kenneth C. Griffin Department of Economics
University of Chicago
1126 East 59th Street
Chicago, IL 60637
and NBER
hortacsu@uchicago.edu

Aniko Oery
Yale School of Management
165 Whitney Ave
New Haven, CT 06520
aniko.oery@yale.edu

Kevin R. Williams
Yale School of Management
165 Whitney Avenue
New Haven, CT 06520
and NBER
kevin.williams@yale.edu

1 Introduction

Dynamic pricing is commonly used by firms selling fixed inventory by a set deadline. Examples range from seats on airlines and trains, tickets for entertainment events, reservations for cruises, to inventory in retailing. In these markets, prices adjust for several reasons. First, prices reflect changing opportunity costs—in the presence of scarcity, the cost of selling a unit of inventory today depends on the ability to sell it in the future. Second, demand may change over time. This can provide an incentive to hold inventory for certain customers. While in many of the aforementioned examples firms face competition, the prior empirical literature on dynamic pricing has studied it from a single firm's perspective and has consistently found that it yields higher welfare than under uniform pricing. With competition, today's prices not only affect today's demand, but also all firms' remaining inventories and hence, future equilibrium prices. These forces can alter price dynamics significantly. It is an open theoretical and empirical question how dynamic pricing affects market outcomes in an oligopoly.

In this paper we estimate the welfare effects of dynamic price competition in the airline industry using new theoretical insights and granular data on competing airlines. We introduce a general dynamic pricing game with multiple firms and products, and establish sufficient conditions for equilibrium existence and uniqueness, and for convergence to a system of differential equations. Our theoretical results show surprising departures from results established in single-firm settings. For example, a firm with excess inventory may charge high prices in order to induce a competitor to sell out early instead of offering low prices in order to reduce its excess inventory. Or, a firm with low inventory may charge low prices in order to raise future equilibrium prices. We empirically investigate the impact of these strategic incentives by estimating a model of air travel demand using unique and comprehensive pricing and bookings data of competing airlines in U.S. duopoly markets. Applying the equilibrium characterization to estimated demand, we find that dynamic pricing expands output, increases firm revenues, lowers consumer surplus, and decreases total welfare compared to uniform pricing. Our results contrast recent empirical studies, largely focused on a single firm, where dynamic pricing is found to increase total welfare (Hendel and Nevo, 2013; Castillo, 2020; Williams, 2022).[1]

[1]Hendel and Nevo (2013) consider markets for storable goods, where buyers are faced with an inventory prob-

We begin by extending earlier single-firm, dynamic pricing models (Gallego and Van Ryzin, 1994; Zhao and Zheng, 2000; Talluri and Van Ryzin, 2004) to oligopoly.[2] In the main text we focus on a duopoly where each firm offers a single product. In the appendix we extend our results to many firms, each offering an arbitrary number of products. Firms are exogenously endowed with limited capacity that must be sold by a deadline. Products are imperfect substitutes and satisfy general regularity conditions. Consumers arrive randomly according to time-varying Poisson rates and are allowed to have time-varying preferences. Each consumer decides whether to purchase an available product or select an outside option. Our demand assumptions are motivated by recent empirical evidence (Hortaçsu et al., 2021b) that support short-lived buyers. In every period, firms simultaneously choose prices after observing remaining inventories for all products. Demand is realized, inventory constraints are updated, and the process repeats until the perishability date or until all products are sold out. We call this game the *benchmark model*.

Our model produces a rich set of equilibrium strategies because prices affect both current demand and opportunity costs of remaining inventory for all firms. Firms internalize that their pricing decisions affect today's demand, which firm sells, and future equilibrium prices. The order of sales matters, creating novel incentives for how firms strategically set prices. We identify how the importance of own and competitor inventory govern price paths. Limited own inventory typically causes a firm to charge high prices (as in the single-firm setting), but it may result in relatively low prices if it raises future equilibrium prices. Such fire sales in order to reduce own inventory and soften future competition have also been identified by Dilme and Li (2019), where a single firm competes with its future self for forward-looking buyers.[3] In our model, this force occurs across firms. Limited competitor inventory typically increases future prices. The desire to create competitor scarcity is, however, particularly large for the firm with more inventory, as the rival firm with less inventory is closer to selling out. This can cause a firm to set especially

lem. As a result, competition among sellers does not affect the features of dynamic pricing. Castillo (2020) studies the matching of drivers and riders on the ride-hailing platform Uber, and Williams (2022) studies single-carrier airline markets with collected data.

[2]There exists a large literature on dynamic price competition in other settings, e.g., Maskin and Tirole (1988); Bergemann and Välimäki (2006) who do not consider limited capacity, and Sweeting et al. (2020) who study limit pricing. In related work, Dana (1999a) and Dana (1999b) allow firms to choose prices and quantities before demand uncertainty is resolved.

[3]Board and Skrzypacz (2016); Gershkov et al. (2018) consider forward-looking buyers when the firm can fully commit to a selling mechanism and hence, resist the temptation to fire-sale.

high prices in order to induce a rival with few units remaining to sell (out).[4] Although we do not endogenize the initial capacity choice (see Dana and Williams (2022) for a related example), we show that our model can produce strategies commonly observed in quantity-choice games, namely downward sloping (versus upward sloping) best-response curves.

We show that the marginal impact of own and competitor inventory on a firm's continuation payoff can be summarized by parameters in the stage game that we label "own-scarcity effects" and "competitor-scarcity effects." These scarcity effects are realized when a firm sells. The competitor scarcity effect enters a firm's payoff such that it is weighted by the competitor's demand. This makes the firms' stage game payoffs difficult to analyze as they are not (log) supermodular (Milgrom and Roberts, 1990), nor are they of the form considered in Caplin and Nalebuff (1991) and Nocke and Schutz (2018). We derive sufficient conditions for existence and uniqueness of equilibria of the stage game using a theorem in Kellogg (1976) and prove that close to the deadline, these conditions are satisfied for commonly used demand systems in empirical work, including (nested) logit demand. The conditions also ensure that the continuous-time limit of the unique discrete-time equilibrium price paths satisfy a system of differential equations.

The continuous-time characterization allows us to formalize the dynamic link between scarcity in the stage game and remaining inventory. We show that firms prefer to have asymmetric remaining capacities and that if the firm with the fewest number of units remaining sells, this will have the biggest impact on future prices. As a result, both firms prefer that the firm with less inventory sells first. In order to achieve this, the firm with more inventory tends to set higher prices, and the firm with less inventory tends to set a low price. Finally, we show that competition is fiercest when firms have the same number of units remaining, as any sale results in a large price increase.

We use our theoretical framework to quantify the welfare effects of dynamic price competition in the airline industry. This industry has been noted for significant price dispersion within and across routes (Borenstein and Rose, 1994; Stavins, 2001; Gerardi and Shapiro, 2009; Berry

[4]This force appears in Martínez-de Albéniz and Talluri (2011), where firms offer perfect substitutes. Shifting demand to rivals also appears in Dana and Williams (2022), where firms do not face uncertain demand. Similar incentives also arise in Edgeworth cycles (Dudey, 1992).

and Jia, 2010; Puller et al., 2012; Sengupta and Wiggins, 2014; Siegert and Ulbricht, 2020). We use new data sources that provide not only prices, but also all bookings (specifically, booking counts) for all competing carriers on a given route.[5] The booking counts include all tickets sold, e.g., directly with the airline or via an online travel agency. As a result, we observe the remaining inventory for every flight over time.

We estimate a Poisson demand model, where aggregate demand uncertainty is captured through Poisson arrivals, and preferences are modeled through discrete choice nested logit demand. Instead of fixing the market size, as commonly done in empirical work, we use search data for one airline to inform arrival process parameters that are then scaled up to account for unobserved searches, e.g., via online travel agencies or a competitor's website. We show that our results are robust to the choice in scaling parameter as well as the inclusion of unobserved preferences that are potentially correlated with price (beyond a rich set of fixed effects). In total, we estimate demand for 58 duopoly routes. We find significant variation in willingness to pay across routes and days from departure. In general, demand becomes more inelastic as the departure date approaches. Average own-price elasticities are -1.4.

With the demand estimates, we simulate equilibrium market outcomes using our differential equation characterization. This allows us to solve relatively large games (route-departure dates)—some feature over 131 million potential states. We recover the own/competitor-scarcity effects and firm strategies for all potential states. We verify equilibrium uniqueness and find that overwhelmingly (but not all) of the realized stage games are of strategic complements, i.e., best-response curves are downward sloping.

We compare market outcomes of dynamic pricing to uniform pricing, where each firm commits to a single price for each flight over time. We find the opposite welfare effect compared to recent work by Hendel and Nevo (2013) in retailing, Castillo (2020) in ride-share, and Williams (2022) for single-carrier airline markets.[6] Accounting for the competitive interactions, we find that dynamic pricing expands output but lowers total welfare compared to uniform pricing. This occurs because dynamic pricing softens price competition toward the departure date where

[5]The data were provided to us by a large U.S. airline that has elected to remain anonymous.

[6]Cho et al. (2018) and D'Haultfœuille et al. (2022) also investigate dynamic pricing in the single-firm setting. They report the revenue gains of dynamic pricing over uniform pricing, but do not consider total welfare.

demand is most inelastic, despite featuring lower prices on average. Early-arriving, price sensitive customers benefit from dynamic pricing, but late-arriving, price insensitive customers face higher prices. High prices at the end occur not only due to price targeting but also because the ability to react to competitors scarcity results in inefficiently low remaining inventory close to the deadline. Our estimates suggest that uniform pricing would increase total welfare by 2.2% but lower quantity sold by 6.4%.

We also investigate two pricing heuristics that mimic some industry pricing strategies.[7] The algorithms differ from recent work in economics that study reinforcement algorithms (Calvano et al., 2020; Asker et al., 2021; Hansen et al., 2021) in that the heuristics pursued by airlines do not incorporate additional information (learning) about demand or competitor strategies as the departure date approaches. The heuristics use demand estimates and potential fares competitors may charge, called *fare buckets* in the industry. We assume both firms use the same heuristic (see Brown and MacKay (2021) for work on algorithm choice). We find that heuristics lead to ambiguous effects on firm revenues but result in higher welfare than under the benchmark model. Our results show that the benchmark model with full information and dynamic pricing results in the lowest welfare among all counterfactuals.

2 Model of Dynamic Price Competition

We begin by detailing the demand assumptions that we use in our analysis in Section 2.1. Our exposition of demand is for an arbitrary number of products. In Section 2.2 we introduce supply-side notation by examining the single-firm case. We then present a duopoly pricing game with two products in Section 2.3 which we analyze in Section 3. In Appendix A, we generalize all formal results to a dynamic pricing game with arbitrary number of firms and products.

2.1 Demand Model

We consider firms selling a set of products denoted by $\mathscr{J} := \{1, \ldots, J\}$. Products are imperfect substitutes and must be scrapped with zero value at a deadline $T > 0$. We analyze a discrete-

[7] We observe how one firm has modeled competition based on internal documentation.

time environment with periods $t \in \{0, \Delta, \dots, T-\Delta\}$, $\Delta > 0$, and later study the continuous-time approximation as $\Delta \to 0$. In every period t, a single consumer arrives with probability $\Delta\lambda_t$, where λ_t is continuous in t. Therefore, each consumer can be indexed by her arrival time t.

If all products are available, then consumer t, facing a price vector $\mathbf{p} = (p_j)_{j\in\mathscr{J}}$, purchases product j with probability $s_j(\mathbf{p}; \boldsymbol{\theta}_t, \mathscr{J})$, where $\boldsymbol{\theta}_t \in \mathscr{T} \subset \mathbb{R}^n$ is a vector of $n \geq 1$ parameters that are smooth and deterministic in t. We impose the following regularity conditions on demand.

Assumption 1. *For all $\boldsymbol{\theta} \in \mathscr{T}$ and $\mathbf{p} \in \mathbb{R}^{\mathscr{J}}$, the following hold:*

i) Convergence for infinite prices: *For any j, $\lim_{p_j\to\infty} s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J})p_j = 0$. For any subset $\mathscr{A} \subset \mathscr{J}$ and $j \in \mathscr{A}$, the limit*[8]

$$s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) := \lim_{\substack{p_{j'}\to\infty \\ j'\notin\mathscr{A}}} s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J}) \in [0,1]$$

exists, where $p^{\mathscr{A}}_{j'} = p_{j'}$ for all $j' \in \mathscr{A}$, $\mathbf{p}^{\mathscr{A}} \in \mathbb{R}^{\mathscr{A}}$;

ii) Products are imperfect substitutes: *For all j, $s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J})$ is strictly decreasing in p_j and strictly increasing in $p_{j'}$ for $j' \neq j$;*

iii) The outside option is an imperfect substitute: *For any subset $\mathscr{A} \subset \mathscr{J}$ and $j \in \mathscr{A}$, $s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})$ is smooth in $\boldsymbol{\theta}$ and $\mathbf{p}^{\mathscr{A}}$, and for all $\underline{\mathbf{p}} \in \mathbb{R}^{\mathscr{A}}$ there exists a $C > 0$ such that for all $\mathbf{p}^{\mathscr{A}} \geq \underline{\mathbf{p}}$,*

$$C\, s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) < \frac{\partial s_0}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) \qquad \textit{for all } j, \tag{1}$$

where $s_0(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) := 1 - \sum_{j'\in\mathscr{A}} s_{j'}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})$.

Assumption 1-i) ensures that demand is well-defined when products sell out and Assumption 1-ii) simply states that all products are imperfect substitutes. Assumption 1-iii) can be viewed as a generalized concavity assumption as it makes sure that the profit maximizing prices of a static multi-product problem are interior and well-behaved. First, it implies $\frac{\partial s_0}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) > 0$,

[8] The limit takes all prices of products $j' \notin \mathscr{A}$ to infinity where the order does not matter.

i.e., the outside option is an imperfect substitute. Denoting the vector of choice probabilities by $\mathbf{s}(\cdot) := \big(s_j(\cdot)\big)_{j\in\mathscr{A}}$, it implies that the Jacobian matrix of demand $D_{\mathbf{p}^{\mathscr{A}}}\mathbf{s}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})$ is diagonally dominant since $\frac{\partial s_0}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) = \left|\frac{\partial s_j}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})\right| - \sum_{j'\in\mathscr{A}\setminus\{j\}} \frac{\partial s_{j'}}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) > 0$.[9] Then, $D_{\mathbf{p}}\mathbf{s}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})$ is non-singular by the Levy-Desplanques Theorem (see, e.g., Theorem 6.1.10. in Horn and Johnson (2012)). In addition, Assumption 1-iii) ensures that profit-maximizing prices are uniformly bounded from above. It is a relatively weak assumption that essentially asserts that $\frac{\partial s_0}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})/s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})$ remains bounded from zero when $p_j^{\mathscr{A}}$ is large.

Given Assumption 1, we can define for any $\boldsymbol{\theta}$, $\mathscr{A}$, and $\mathbf{p}\in\mathbb{R}^{\mathscr{A}}$ the vector of inverse quasi own-price elasticities of demand as

$$\hat{\boldsymbol{\epsilon}}(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) := \left(\left(D_{\mathbf{p}}\boldsymbol{s}(\mathbf{p};\boldsymbol{\theta},\mathscr{A})\right)^{\intercal}\right)^{-1}\boldsymbol{s}(\mathbf{p};\boldsymbol{\theta},\mathscr{A}).$$

Assumption 2 details the assumption that we place on demand elasticities.

Assumption 2. *The vector of inverse quasi own-price elasticities* $\hat{\boldsymbol{\epsilon}}(\mathbf{p};\boldsymbol{\theta},\mathscr{A})$ *satisfies*

$$\det\left(-D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}}(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) - I\right) \neq 0$$

for all $\mathbf{p}\in\mathbb{R}^{\mathscr{A}}$, $\boldsymbol{\theta}\in\mathscr{T}$, *and* $\mathscr{A}\subset\mathscr{J}$, *where* $I\in\mathbb{R}^{\mathscr{A}\times\mathscr{A}}$ *is the identity matrix.*

As discussed above, Assumption 1-iii) guarantees that for any marginal cost vector $\mathbf{c}\in\mathbb{R}^{\mathscr{A}}$, $\max_{\mathbf{p}\in\mathbb{R}^{\mathscr{A}}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta},\mathscr{A})^{\intercal}(\mathbf{p}-\mathbf{c})$ has an interior solution. Given Assumption 2, the system of first-order conditions (FOCs) has a unique solution. Together, these assumptions replace the commonly made assumption of quasi-concavity or log-concavity which is, for example, not satisfied for multinomial logit (see e.g., Hanson and Martin (1996)). We show in Appendix C that (nested) multinomial logit demand functions satisfy Assumptions 1 and 2.

We omit the conditioning arguments $\boldsymbol{\theta}$ and/or $\mathscr{A}$ in all expressions whenever the meaning is unambiguous. When the time index is relevant, we write $s_{j,t}(\mathbf{p}) := s_j(\mathbf{p};\boldsymbol{\theta}_t,\mathscr{A}_t)$ for $j\in\mathscr{A}_t\cup\{0\}$.

[9] Consistent with the common convention, the Jacobi matrix of a vector-valued function $f(\mathbf{x})\in\mathbb{R}^n$, $\mathbf{x}\in\mathbb{R}^n$ is $D_{\mathbf{x}}f(\mathbf{x}) := \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$, i denoting rows and j columns, and bold vectors $\mathbf{x}$ are column vectors.

We illustrate theoretical insights with a multinomial logit demand specification

$$s_{j,t}(\mathbf{p}) = \frac{\exp\left(\frac{\delta_j - \alpha_t p_j}{\rho}\right)}{1 + \sum\limits_{j' \in \mathscr{A}_t} \exp\left(\frac{\delta_{j'} - \alpha_t p_{j'}}{\rho}\right)}, \tag{2}$$

where α_t/ρ is the time-variant marginal utility to income, and $\rho > 0$ is a scaling factor. The parameter δ_j/ρ is the product-specific value of product j. Note that as $\rho \to 0$, competition collapses to standard Bertrand. As $\rho \to \infty$, products become perfectly differentiated. In our empirical analysis, we consider the more flexible nested logit demand model.

2.2 Single Firm Model

We discuss the single-firm, multi-product dynamic pricing model with two goals in mind. The first is to introduce supply-side notation that we carry over to the competitive model. The second is to showcase that the single-firm problem is well behaved and exhibits nice properties. These properties can fail in the oligopoly model.

A single firm M offers J products for sale with initial inventory $K_{j,0} \in \mathbb{N}$ of each product j. Let $\mathbf{K}_t = \left(K_{j,t}\right)_{j \in \mathscr{J}}$ denote the capacity vector at time t.[10] The firm's continuation payoff at time $t \leq T - \Delta$, given capacity vector $\mathbf{K}$, satisfies the dynamic program

$$\Pi_{M,t}(\mathbf{K};\Delta) = \\ \max_{\mathbf{p}} \Delta\lambda_t \sum_{j \in \mathscr{J}} \underbrace{s_{j,t}(\mathbf{p})\left(p_j + \Pi_{M,t+\Delta}(\mathbf{K} - \mathbf{e}_j;\Delta)\right)}_{\text{payoff from selling product } j} + \underbrace{\left(1 - \Delta\lambda_t \sum_{j \in \mathscr{J}} s_{j,t}(\mathbf{p})\right)}_{\text{probability of no purchase}} \Pi_{M,t+\Delta}(\mathbf{K};\Delta),$$

where $\mathbf{e}_j \in \mathbb{N}^{\mathscr{J}}$ is a vector of zeros with a one in the jth position. The firm faces three boundary conditions: (i) $\Pi_{M,T}(\mathbf{K};\Delta) = 0$ for all $\mathbf{K}$, (ii) $\Pi_{M,t}(\mathbf{0};\Delta) = 0$ for all t, where $\mathbf{0}$ is a vector of zeros, and (iii) $\Pi_{M,t}(\mathbf{K};\Delta) = -\infty$ if $K_j < 0$ for a $j \in \mathscr{J}$. The dynamic program captures that the continuation profits only depends on the capacity vector and time, where today's prices govern the probabilities of selling each product today. The boundary conditions ensure that remaining

[10]The capacity vector at time t captures at time t the remaining inventory.

inventory is scrapped with zero value after the deadline T and that the firm cannot oversell.

Since continuation values in period $t+\Delta$ are independent of past prices, the optimal price in each period solves a static maximization problem parameterized by $\boldsymbol{\omega} = (\omega_j)_{j\in\mathcal{J}}$, where $\omega_j = \Pi_{M,t}(\mathbf{K};\Delta) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j;\Delta)$ is commonly referred to as the *opportunity cost of selling product* j.[11] In particular, profit-maximizing prices are given by

$$\mathbf{p}^M(\boldsymbol{\omega},\boldsymbol{\theta}) := \arg\max_{\mathbf{p}} \sum_{j\in\mathcal{J}} s_j(\mathbf{p};\boldsymbol{\theta},\mathcal{J})\big(p_j - \omega_j\big).$$

By Kellogg (1976), Assumption 2 implies that there is a unique optimal price vector which is continuous in $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$. Then, by Lemma 4 in Appendix B, the continuous-time limit of this dynamic program exists, and solves the differential equation specified in the following lemma. The lemma formalizes that the loss in continuation profit if no sale occurs is given by the forgone expected flow payoff.

Lemma 1. *Let Assumptions 1 and 2 hold. Then,* $\Pi_{M,t}(\mathbf{K};\Delta)$ *converges uniformly to* $\Pi_{M,t}(\mathbf{K})$ *as* $\Delta \to 0$*, which satisfies*

$$\dot{\Pi}_{M,t}(\mathbf{K}) = -\lambda_t \max_{\mathbf{p}} \sum_{j\in\mathcal{J}} s_{j,t}(\mathbf{p})\Big(p_j - \Big(\Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)\Big)\Big) \tag{3}$$

with boundary conditions (i) $\Pi_{M,T}(\mathbf{K}) = 0$ *for all* $\mathbf{K}$*, (ii)* $\Pi_{M,t}(\mathbf{0}) = 0$ *for all* t*, and (iii)* $\Pi_{M,t}(\mathbf{K}) = -\infty$ *if* $K_j < 0$ *for a* $j \in \mathcal{J}$*.*

Given a capacity vector $\mathbf{K}$, corresponding available products $\mathcal{A} = \{j : K_j \neq 0\}$, and the vector of opportunity costs $\boldsymbol{\omega}_{M,t}(\mathbf{K}) := \Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)$ of products $j \in \mathcal{A}$, the FOC for profit-maximizing prices can be written in matrix form,

$$\mathbf{p} = \underbrace{\boldsymbol{\omega}_{M,t}(\mathbf{K})}_{\text{opportunity costs}} - \underbrace{\Big(\big(D_{\mathbf{p}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}_t,\mathcal{A})\big)^{\intercal}\Big)^{-1}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}_t,\mathcal{A})}_{\substack{=\hat{\epsilon}(\mathbf{p};\boldsymbol{\theta},\mathcal{A}) \\ \text{inverse quasi own-price elasticities}}}. \tag{4}$$

[11] Note that strictly speaking, the opportunity cost of selling product j is given by $\omega_j - \sum_{j'\neq j} \frac{s_j'(\mathbf{p})}{1-s_j(\mathbf{p})}\omega_{j'}$ as by selling product j, the firm forgoes the opportunity to sell any other product to the customer.

Hence, the optimal pricing policy $\mathbf{p}_t^M(\mathbf{K})$ is continuous in time.[12] The evolution of $\mathbf{p}_t^M(\mathbf{K}_t)$ is governed by the evolution of quasi own-price elasticities of demand and the evolution of the opportunity costs. The following proposition summarizes properties of the solution.

Proposition 1. *The solution to the continuous-time single-firm revenue maximization problem in Lemma 1 satisfies the following:*

i) $\Pi_{M,t}(\mathbf{K})$ *is decreasing in* t *for* $\mathbf{K} \neq \mathbf{0}$ *and increasing in* K_j*, for all* $j \in \mathscr{J}$ *and* $t < T$*;*

ii) $\omega_{j,t}(\mathbf{K})$ *is decreasing in* t *for* $\mathbf{K} \neq \mathbf{0}$ *and decreasing in* K_j*, for all* j *and* $t < T$*;*

iii) The stochastic process $\omega_{j,t\wedge\tau}(\mathbf{K}_t)$*,* $\tau := \inf\{t \geq 0 | K_{j,t} \leq 1\}$*, is a submartingale.*

Statements i) and ii) of Proposition 1 imply that more inventory and more time remaining increase continuation profits, every additional unit of inventory increases profits by less (concavity of profits in capacity), and opportunity costs are decreasing towards the deadline if $\mathbf{K}$ is held fixed. These properties have been established in the seminal paper by Gallego and Van Ryzin (1994) for the single-product setting. Statement (iii) says that, on average, opportunity costs are increasing. This formal result implies that given constant $\boldsymbol{\theta}_t \equiv \boldsymbol{\theta}$, or if α_t is non-decreasing in t in a (nested) multinomial logit specification, price paths are on average increasing in time by Equation 4. An important implication is that rational consumers should not wait to purchase because prices rise in expectation. As a result, long-lived buyers and short-lived buyers would behave the same upon arrival. Statement (iii) is not inconsistent with earlier work, e.g., McAfee and Te Velde (2006) and Williams (2022), where close to the deadline, observed average prices decline. The reason is that once a product sells out, its price is excluded from the average price, i.e., observed prices decline when there is one unit remaining.

2.3 Duopoly Model

We introduce a duopoly pricing game with two firms $f \in \{1,2\}$. Each firm controls exactly one product, i.e., $\mathscr{J} = \{1,2\}$. Therefore, we set $j = f$ and use the subscript f to denote both the firm

[12]Note that we abuse notation slightly by denoting the optimal price policy $\mathbf{p}_t^M(\mathbf{K})$, while also denoting the static optimal price parameterized by $(\boldsymbol{\omega}, \boldsymbol{\theta})$ by $\mathbf{p}^M(\boldsymbol{\omega}, \boldsymbol{\theta})$.

and product of interest. We generalize the results in this section to many firms with multiple products in Appendix A. Our exposition here focuses on the duopoly case with two products since this case is sufficient to highlight the key forces relevant for our analysis. Each firm f is initially endowed with $K_{f,0}$ units of its own product. In every period, firms simultaneously set prices $p_{f,t}$, and then a consumer arrives with probability $\Delta\lambda_t$. If a consumer arrives, she buys a product from firm f with probability $s_{f,t}(p_{1,t}, p_{2,t})$.

As in the single-firm case, the payoff-relevant state is given by the vector of inventories $\mathbf{K} := (K_1, K_2)$ at time t. We study Markov perfect equilibria in which each firm's strategy is measurable with respect to (K_1, K_2, t). We denote a Markov strategy of firm f by $p_{f,t}(\mathbf{K})$. Given equilibrium price vectors $\mathbf{p}_t^*(\mathbf{K}) := (p_{1,t}^*(\mathbf{K}), p_{2,t}^*(\mathbf{K}))$, firm f's value function satisfies[13]

$$\begin{aligned}\Pi_{f,t}(\mathbf{K};\Delta) = \Delta\lambda_t \Bigg(& \underbrace{s_f\left(\mathbf{p}_t^*(\mathbf{K})\right)\left(p_{f,t}^*(\mathbf{K}) + \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_f;\Delta)\right)}_{\text{payoff from own sale}} + \\ & \underbrace{s_{f'}\left(\mathbf{p}_t^*(\mathbf{K})\right)\Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_{f'};\Delta)}_{\text{payoff if } f' \text{ sells}}\Bigg) + \underbrace{\left(1-\Delta\lambda_t \sum_{h\in\{1,2\}} s_h\left(\mathbf{p}_t^*(\mathbf{K})\right)\right)}_{\text{probability of no purchase}} \cdot \Pi_{f,t+\Delta}(\mathbf{K};\Delta),\end{aligned} \tag{5}$$

where we denote the competitor by $f' \neq f$. The key difference to the single-firm's dynamic program is the additional term representing the payoff if the competitor sells. The boundary conditions are analogous to the single-firm case: (i) $\Pi_{f,T}(\mathbf{K};\Delta) = 0$ for all $\mathbf{K}$, (ii) $\Pi_{f,t}(\mathbf{K};\Delta) = 0$ if $K_f = 0$, (iii) $\Pi_{f,t}(\mathbf{K};\Delta) = -\infty$ if $K_f < 0$, (iv) $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_{f'};\Delta) = \Pi_{f,t}(\mathbf{K};\Delta)$ if $K_{f'} = 0$, $K_f \geq 0$.

Similar to the single-firm setup, given Markov pricing strategies, the continuation payoffs in period $t+\Delta$ are not impacted by past prices. Hence, $\mathbf{p}_t^*(\mathbf{K})$ is an equilibrium of a stage game in which firm f's payoff is given by $\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta)$. To describe this stage game, we denote for each firm $f \in \{1,2\}$ the change in continuation profit if product $j \in \{1,2\}$ is sold by

$$\omega_{j,t}^f(\mathbf{K};\Delta) := \Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta),$$

which we call the *scarcity effect* of product j on firm f. We refer to $\omega_{f,t}^f$ as the *own-scarcity*

[13]Formally, equilibrium prices are a function of Δ, which we omit in the main text for readability.

effect and $\omega^f_{f',t}$, $f' \neq f$, as the *competitor-scarcity effect*. We set $\omega^f_{f',t} := 0$ if $K_{f'} = 0$. Then, the stage game is parameterized by the matrix of scarcity effects

$$\Omega_t(\mathbf{K};\Delta) = \begin{pmatrix} \omega^1_{1,t}(\mathbf{K};\Delta) & \omega^1_{2,t}(\mathbf{K};\Delta) \\ \omega^2_{1,t}(\mathbf{K};\Delta) & \omega^2_{2,t}(\mathbf{K};\Delta) \end{pmatrix},$$

where by Equation 5, firm f's expected flow payoff is equal to[14]

$$\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta) = \Delta\lambda_t \left(s_{f,t}\left(\mathbf{p}^*_t(\mathbf{K})\right)\left(p^*_{f,t}(\mathbf{K}) - \omega^f_{f,t}(\mathbf{K})\right) - s_{f',t}\left(\mathbf{p}^*_t(\mathbf{K})\right)\omega^f_{f',t}(\mathbf{K})\right).$$

Hence, in the stage game, firms simultaneously choose prices to maximize payoffs

$$s_{f,t}(\mathbf{p}) \cdot \left(p_f - \omega^f_{f,t}(\mathbf{K})\right) - s_{f',t}(\mathbf{p}) \cdot \omega^f_{f',t}(\mathbf{K}), \qquad f' \neq f.$$

Intuitively, the firm incurs an opportunity cost of selling its own product as in the single-firm setting, but future prices are also affected by the future degree of competition. For example, firm f benefits from a sale of the competitor if $\omega^f_{f',t} < 0$. This provides the firm an incentive to shift demand to the competitor. The stage game can have different strategic properties depending on the size and sign of the scarcity effects (see Section 3.2).

For a stage game with $\omega^f_{f',t} \neq 0$, we cannot apply results from Caplin and Nalebuff (1991) or Nocke and Schutz (2018). Payoffs are neither super-modular nor log-supermodular (Milgrom and Roberts, 1990). The stage game is also not a potential game.

3 Analysis of the Duopoly Model

In this section, we derive theoretical properties of the dynamic pricing game. We start with an analysis of uniqueness and continuity of stage game equilibria, which allows us to generalize Lemma 1 to oligopoly. Uniqueness of equilibria is useful to make theoretical and empirical predictions, and the continuous-time approximation allows us to efficiently compute equilibria of the dynamic pricing game—which can reach a very large number of states $(\mathbf{K}, t)$ even with

[14]We omit the "Δ" in the scarcity effects and equilibrium prices for readability.

only two firms, two products, and a realistic number of units. Convergence to a system of differential equations is not guaranteed, even with well-behaved and commonly used demand models such as logit. Therefore, we provide conditions that can be checked to assure convergence. We discuss the unique economic forces of this dynamic pricing game in Section 3.2.

3.1 Equilibrium Existence, Uniqueness, and Continuity

3.1.1 Sufficient Condition for Equilibrium Uniqueness in the Stage Game

In this section, we consider the stage game for an arbitrary matrix of opportunity costs Ω. We drop the time index and capacity argument in all expressions temporarily. Our first result presents sufficient conditions for existence and uniqueness of an equilibrium of the stage game. We show in the proof of Lemma 2 that the boundedness and regularity conditions in Assumption 1 guarantee that firm best-responses are solutions to the FOCs of each firm's maximization problem. We can write the FOC of firm f's profit maximization problem as

$$g_f(\mathbf{p}) = p_f,$$

where

$$g_f(\mathbf{p}) := \underbrace{\omega_f^f + \frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})}\omega_{f'}^f}_{\substack{\text{net opportunity cost} \\ \text{of selling}}} - \underbrace{s_f(\mathbf{p})\left(\frac{\partial s_f(\mathbf{p})}{\partial p_f}\right)^{-1}}_{\substack{\text{inverse quasi} \\ \text{own-price elasticity}}}. \tag{6}$$

By Konovalov and Sándor (2010) (which is based on Kellogg (1976)), the following assumption guarantees that there is a unique solution to this system of equations.

Assumption 3. *Suppose the following two conditions hold:*

i) $\frac{\partial g_f}{\partial p_f}(\mathbf{p}) - 1 \neq 0$ *for all* $\mathbf{p}$ *and* $f = 1,2$*;*

ii) $\det\left(\left(D_{\mathbf{p}}\mathbf{g}(\mathbf{p})\right)^{\intercal} - I\right) \neq 0$ *for all* $\mathbf{p}$*, where* $\mathbf{g}(\mathbf{p}) := (g_1(\mathbf{p}), g_2(\mathbf{p}))^{\intercal}$.

Assumption 3-(i) is always satisfied for demand functions that are log-concave in each dimension. Further, note that if the competitor-scarcity effect is zero, one can see from Equation 6 that Assumption 2 implies Assumption 3. In the presence of competitor-scarcity effects, the net opportunity cost of selling depends on the ratio of derivatives of the demand of the two firms. Then, Assumption 3-(i) makes sure that the best response of each firm is always unique and 3-(ii) guarantees that $\mathbf{g}(\mathbf{p}) = \mathbf{p}$ has exactly one solution. Together, these two assumptions imply that the unique solution to the system of FOCs must be an equilibrium.

Lemma 2. *Let Assumptions 1, 2 and 3 hold. Then, the stage game admits a unique equilibrium.*

Even though Lemma 2 does not provide necessary conditions for uniqueness, it guides our construction of an Ω that violates Assumption 3 and yields multiple equilibria in Section 3.1.2. We will also demonstrate that, in general, there might not exist a well-behaved equilibrium price path if competitor-scarcity effects are strong. For example, any equilibrium could involve large price changes even if no product is being sold due to a switch in the equilibrium regime. However, we will later show in Proposition 2 that the conditions in Assumption 3 are satisfied in all stage games that are sufficiently close to the deadline. Furthermore, in our empirical analysis, the stage game always remains in a neighborhood of Ω where the equilibrium is unique.

Finally, note that Lemma 2 establishes uniqueness and existence simultaneously. Under the assumption of independence of irrelevant alternatives (see Section 3.3.1 for a formal definition), that is satisfied by a classic logit demand system and that is a commonly made assumption in theoretical analysis of oligopolies, we can establish equilibrium existence directly.

3.1.2 Continuity of Equilibrium Prices in Scarcity Effect Matrix Ω

Next, we study the stage game parameterized by scarcity effects Ω and demand parameters $\boldsymbol{\theta}$. To generalize the convergence result of Lemma 1 to an oligopoly, we need to show continuity of the equilibrium price $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$. In particular, we need to show that if Ω and $\boldsymbol{\theta}$ remain in a compact neighborhood in which the stage game admits a unique solution, then equilibrium prices denoted by $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ are continuous in Ω and $\boldsymbol{\theta}$. Consequently, a small change in the opportunity costs does not change prices substantially. In the dynamic game, this makes sure that as long as no sales occur, price paths do not make sudden significant changes over time.

Lemma 3. *Let Assumptions 1, 2 and 3 hold for a compact, path-connected set $\mathcal{O}$ of $(\Omega, \boldsymbol{\theta})$. Then, the unique equilibrium price vector $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ is continuous in $(\Omega, \boldsymbol{\theta})$ on $\mathcal{O}$.*

We require path-connectedness of the space of $(\Omega, \boldsymbol{\theta})$ since it allows us to use Browder's Theorem (see e.g., Solan and Solan (2021)) after parametrizing the multidimensional parameter space with a one-dimensional variable.

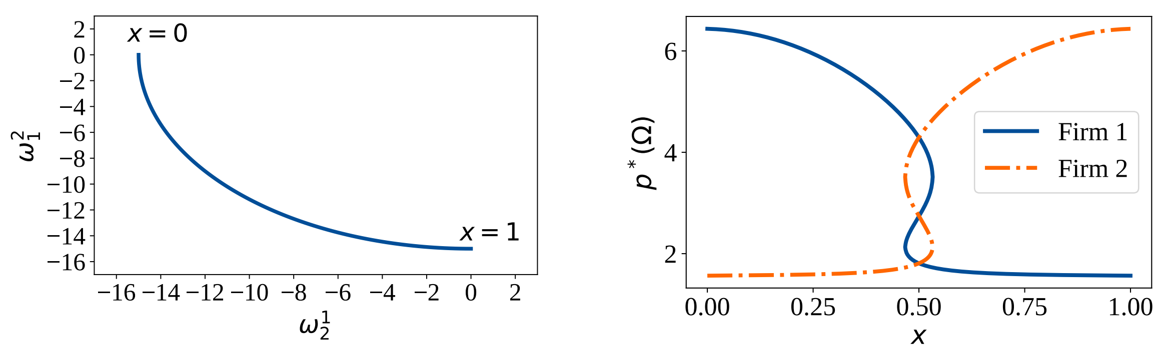
As discussed in Section 3.1, Assumption 3-ii) can be violated for non-zero values of scarcity effects. This can lead to multiplicities of equilibria and a failure of Lemma 5. In the dynamic pricing game, this can potentially translate to equilibrium price jumps that are not caused by a change in inventory in the dynamic game. We illustrate this point in the following example. Consider logit demand such that $\delta_1 = \delta_2 = 1$, and $\rho = 1$. Then, Assumption 3 is equivalent to

$$\Big(s_1(\mathbf{p}) + \alpha\omega_2^1 s_0(\mathbf{p})\Big)\Big(s_2(\mathbf{p}) + \alpha\omega_1^2 s_0(\mathbf{p})\Big) \neq 1 + \frac{1 - s_1(\mathbf{p}) - s_2(\mathbf{p})}{s_1(\mathbf{p})s_2(\mathbf{p})}.$$

Note that this condition does not depend on the firms' own-scarcity effects ω_1^1 and ω_2^2. Therefore, we set own-scarcity effects equal to zero and parameterize competitor-scarcity effects using a continuous function. We plot the parameterization of (ω_1^2, ω_2^1) in Figure 1-(a). The corresponding equilibrium prices for both firms are plotted in Figure 1-(b). It shows that (1) multiplicity of equilibria can occur and (2) any price path along x contains a price jump.

Figure 1: Multiplicities in stage-game equilibria

(a) Parametrization of $(\omega_2^1(x), \omega_1^2(x))$ (b) Multiplicity in equilibrium prices



Note: In these graphics we parameterize (ω_1^2, ω_2^1) with a curve $(\omega_2^1(x), \omega_1^2(x)) = \left(-15\cos\left(\frac{\pi}{2}x\right), -15\sin\left(\frac{\pi}{2}x\right)\right), x \in [0,1]$, where we set $(\omega_1^1, \omega_2^2) = (0,0)$, and assume logit demand with $\delta = (1,1)$, $\alpha_t = 1$ and scaling factor $\rho = 1$. Panel (a) depicts the parameterized curve and panel (b) equilibrium prices of both firms given (ω_1^2, ω_2^1) at varying values of x.

This example indicates that the dynamic pricing game might not converge to a system of differential equations and Lemma 4 cannot be immediately generalized to an oligopoly. However, given Assumption 2, Assumption 3-ii) is satisfied for any matrix of scarcity effects Ω in a neighborhood $\mathscr{O}$ that contains the zero matrix $\Omega = \mathbf{0}$ by continuity. This allows us to generalize Lemma 1 to an oligopoly as long as the time horizon is not too long.

3.1.3 Characterization of Continuous-time Limit

Using Lemma 5 and Lemma 4 in Appendix B, we can generalize Lemma 1 to an oligopoly as long as the time horizon is not too long. We state the result formally in the proposition below. The equilibrium characterization is useful because it allows us to simulate equilibrium outcomes in our empirical analysis for high-dimensional games.

Proposition 2 (Continuous-time Limit)**.** *Let Assumptions 1, 2, and 3 hold for a compact, path-connected set $\mathscr{O}$ containing $(\Omega, \boldsymbol{\theta}) = (\mathbf{0}, \boldsymbol{\theta}_T)$. For every $\mathbf{K}$, there exists a $T_0(\mathbf{K}) > 0$, non-increasing in $\mathbf{K}$, so that for any $T \leq T_0(\mathbf{K})$, there exists a unique equilibrium for any dynamic pricing game with sufficiently small Δ (holding all other parameters of the game fixed). The corresponding value function $\Pi_{f,t}(\mathbf{K};\Delta)$ converges to a limit $\Pi_{f,t}(\mathbf{K})$ as $\Delta \to 0$ that solves the differential equation*

$$\dot{\Pi}_{f,t}(\mathbf{K}) = -\lambda_t \Big(s_f(\mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t)) \big(p_f^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t) - (\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_j))\big) \\ - s_{f',t}(\mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t)) \big(\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_{f'})\big)\Big),$$

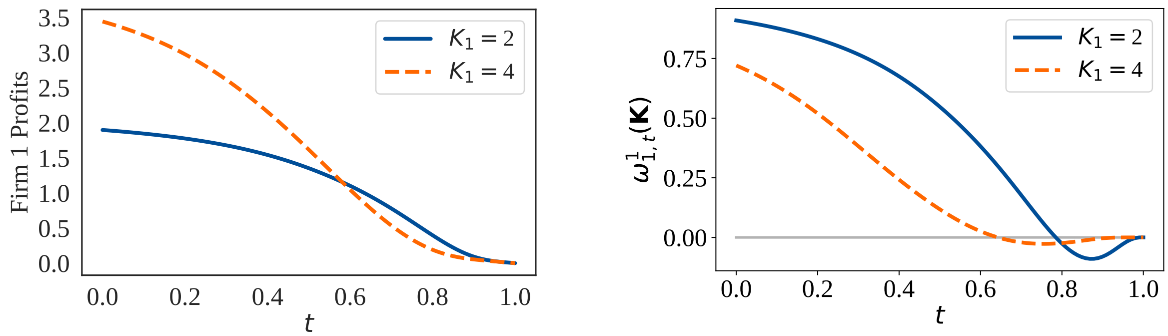
where $f' \neq f$, with boundary conditions (i) $\Pi_{f,T}(\mathbf{K}) = 0$ for all $\mathbf{K}$, (ii) $\Pi_{f,t}(\mathbf{K}) = 0$ if $K_f = 0$, (iii) $\Pi_{f,t}(\mathbf{K}) = -\infty$ if $K_f < 0$, and (iv) $\Pi_{f,t}(\mathbf{K} - \mathbf{e}_{f'}) = \Pi_{f,t}(\mathbf{K})$ if $K_{f'} = 0,\ K_f \geq 0$.

Using this equilibrium characterization, we can illustrate that the general insights from the single-firm setting (Proposition 1) do not hold in an oligopoly. In Figure 2, we consider a simulation using logit demand. We fix the capacity of firm 2 to be $K_2 = 3$ and vary the level of firm 1 capacity K_1 (either 2 or 4). In panel (a), we plot firm 1 profits over time for given capacities. The figure shows that firm 1 expects higher profits with $K_1 = 4$ than with $K_1 = 2$ far from the deadline, however, the firm expects higher profits with $K_1 = 2$ versus $K_1 = 4$ close to the deadline. That is, the value function is non-monotonic in own capacity. In panel (b), we plot the own-scarcity effect of firm 1. Contrary to the single-firm case, we see that the own-scarcity effect is also not monotonic in own capacity. In addition, note that the own-scarcity effect is actually negative close to the deadline but positive well before the deadline. We discuss the different forces in detail in Section 3.2. In Figure 10 in Appendix D, we show that *all* scarcity effects can be positive or negative (even within a single dynamic pricing game).

Figure 2: Simulated profits and own-scarcity effects when $K_2 = 3$ and K_1 varies

(a) Firm 1 equilibrium profit, $\Pi_t^1(K_1, 3)$

(b) Firm 1 own-scarcity effect, $\omega_{1,t}^1(K_1, 3)$



Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$. Panel (a) shows firm 1's profits over time, $t \in [0,1]$, for $\mathbf{K} = (2,3)$ and $\mathbf{K} = (4,3)$. Panel (b) shows firm 2's profits over time, $t \in [0,1]$, for the same states.

3.2 Economic Forces of the Game

3.3 Illustrative example

We start with an illustrative example that highlights how competitor-scarcity effects affect market outcomes before turning to formal results. Consider the dynamic game where only one firm is subject to scarcity, e.g., $K_1 = \infty$, $K_2 < \infty$. In that case, a sale of firm 1 does not create scarcity, i.e., $\omega_1^1 \equiv \omega_1^2 \equiv 0$, since the capacity vector remains unchanged even if firm 1 sells. A sale of firm 2 unambiguously creates scarcity and softens competition, so $\omega_2^1(\infty, K_2) = \Pi_{1,t}(\infty, K_2) - \Pi_{1,t}(\infty, K_2 - 1) < 0$, as once firm 2 sells out, firm 1 can charge $p^M := \arg\max_p s_1(p, \infty)p_1$. Typically $\omega_2^2(\infty, K_2) = \Pi_{2,t}(\infty, K_2) - \Pi_{2,t}(\infty, K_2 - 1) > 0$. Thus, the stage game payoffs are given by

$$\begin{cases} \text{Firm 1:} \quad s_1(\mathbf{p})p_1 - s_2(\mathbf{p})\omega_2^1, \\ \text{Firm 2:} \quad s_2(\mathbf{p})(p_2 - \omega_2^2). \end{cases}$$

Firm 1 has an incentive to shift demand to the competitor. It does so by charging a high price p_1 to increase the term $-s_2(\mathbf{p})\omega_2^1$. Thus, firm 1's competitor-scarcity effect ω_2^1 increases best-response prices (given any p_2 fixed). Firm 2 faces a single-firm optimization problem with residual demand $s_2(p_1,\cdot)$ that is increasing the competitor price. Thus, in equilibrium, the competitor-scarcity effect faced by firm 1 results in higher residual demand for firm 2. This indirect equilibrium effect further brings up overall equilibrium prices today.

This example highlights how competitor-scarcity effects directly affect best-response prices, but also indirectly lead to an increase in equilibrium prices. In general, when both firms have finite capacity the intuition is more complicated in two ways. First, a sale of a firm might strengthen future competition, leading to a positive competitor-scarcity effect $\omega_f^{f'}$. We can construct examples where this is the case, but typically firms set equilibrium prices that result in continuation payoffs that involve negative competitor-scarcity effects. We confirm this in our empirical application. Second, even if any sale softens future price competition, the indirect equilibrium effects are nuanced as long as both firms are subject to nonzero competitor-scarcity effects. This is because the equilibrium outcomes also depend on how the scarcity effects compare to each other and which firm's sale softens competition more.

In the following subsections, we discuss the implications of competitor-scarcity effects on the price level of the best response by fixing the competitor price. We then investigate equilibrium implications by studying how a firm responds to a competitor's price change strategically. We discuss how competitor prices can be strategic complements or strategic substitutes in the stage game. Finally, we show that close to the deadline, a sale of the product with minimum remaining inventory left softens competition the most, leading to the largest equilibrium price jump.

3.3.1 Scarcity Effects and Mark-ups

The effect of own- and competitor-scarcity are reflected in the first-order conditions of firms given by $g_f(\mathbf{p}) = p_f$, where g_f is defined in Equation 6. It shows that the net opportunity cost of selling given a price vector $\mathbf{p}$ is given by the sum of the own-scarcity effect ω_f^f and the competitor-scarcity effect $\omega^f_{f'}$ weighted by $\frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})} < 0$. It is immediate that the intuition of the illustrative example carries over: If competitor scarcity softens competition ($\omega^f_{f'} < 0$), then the best-response price of firm f is higher than if there was no competitor-scarcity effect.

We can describe the opportunity cost in the presence of competition more concretely for demand systems that satisfy the commonly made assumption of "Independence of Irrelevant Alternatives (IIA)." In that case, the weight in front of the competitor-scarcity effect has an intuitive interpretation and does not depend on the firm's own price. Logit demand, for example, satisfies IIA. We state the assumption formally below.

Assumption 4 (Independence of Irrelevant Alternatives (IIA))**.** *Suppose the following holds,*

$$\frac{\partial}{\partial p_1}\frac{s_2(\mathbf{p})}{s_0(\mathbf{p})} = \frac{\partial}{\partial p_2}\frac{s_1(\mathbf{p})}{s_0(\mathbf{p})} = 0.$$

Given Assumptions 1, 2 and 4, we establish the following proposition:[15]

Proposition 3 (Mark-up formula under IIA)**.** *Let Assumptions 1, 2 and 4 hold and* $-\frac{\partial}{\partial p_f}\frac{s_f(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}} \neq 1$ *for all* $\mathbf{p}$*. Then, there exists an equilibrium of the stage game for any scarcity matrix* Ω*. Any equilibrium price vector* $\mathbf{p}^*(\Omega,\boldsymbol{\theta})$ *solves*

$$p^*_f(\Omega,\boldsymbol{\theta}) \in \arg\max_{p_f} s_1(p_f, p^*_{f'}(\Omega,\boldsymbol{\theta}))(p_f - c_f(p_{f'};\Omega,\boldsymbol{\theta}))$$

for $f \in \{1,2\}$*,* $f' \neq f$*, where* $c_f(p_{f'};\Omega,\boldsymbol{\theta}) := \omega^f_f + \tilde{s}_{f'}(p_{f'})\omega^f_{f'}$*, and* $\tilde{s}_{f'}(p_{f'}) := \frac{s_{f'}(\mathbf{p})}{1-s_f(\mathbf{p})}$ *is the demand of firm* f' *conditional on firm* f *not selling.*

Proposition 3 implies that equilibrium prices $\mathbf{p}^*(\Omega,\boldsymbol{\theta})$ satisfy a markup formula

$$\frac{p^*_f(\Omega,\boldsymbol{\theta}) - c_f(p^*_{f'}(\Omega,\boldsymbol{\theta});\Omega,\boldsymbol{\theta})}{p^*_f(\Omega,\boldsymbol{\theta})} = -\frac{1}{\epsilon_f(\mathbf{p}^*_f(\Omega,\boldsymbol{\theta}))}, \tag{7}$$

where $\epsilon_f(\mathbf{p}) = \frac{\partial s_f(\mathbf{p})}{\partial p_f}\frac{p_f}{s_f(\mathbf{p})}$ is the elasticity of demand and the opportunity cost c_f does not depend on the own price. The weight in front of the competitor-scarcity effect has a natural interpreta-

[15] The general result in Appendix A additionally shows that with multiple products for each firm, the game can be transformed to a game in which each product is managed by its own firm given transformed payoff functions.

tion: it is the relative market share of the competitor relative to the outside option. Thus, if the outside option share is large and/or the competitor is small, a firm's decision is less affected by the competitor. If the competitor is large, the competitor-scarcity effect has a larger weight.

3.3.2 Prices as Strategic Substitutes vs Strategic Complements

Next, we study the stage game equilibrium implications of competition. In a static Bertrand game with imperfect substitutes, prices are strategic complements for commonly used demand specifications, including (nested) logit demand systems. Hence, competition unambiguously lowers prices. In the presence of competitor-scarcity effects, our model results in pricing games featuring strategic substitutes or strategic complements, even for simple demand systems.[16]

To see this, note that an increase in the competitor price increases firm f's best response price if $\frac{\partial g_f}{\partial p_{f'}} > 0$, i.e., the competitor's price is a strategic complement. On the other hand, if $\frac{\partial g_f}{\partial p_{f'}} < 0$, then an increase in the competitor price decreases firm f's best response price, i.e., the competitor's price is a strategic substitute. Hence, to determine whether prices are strategic complements versus substitutes, we calculate

$$\frac{\partial}{\partial p_{f'}} g_f(\mathbf{p}) = \frac{\partial}{\partial p_{f'}} \frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})} \omega^f_{f'} - \frac{\partial}{\partial p_{f'}} \Bigg(\underbrace{s_f(\mathbf{p}) \left(\frac{\partial s_f(\mathbf{p})}{\partial p_f} \right)^{-1}}_{\substack{\text{inverse quasi} \\ \text{own-price elasticity}}} \Bigg). \tag{8}$$

For logit demand, the weight in front of the competitor-scarcity effect in the best-response function corresponds to the relative market share of the competitor relative to the outside option $\frac{s_{f'}(\mathbf{p})}{1-s_f(\mathbf{p})} = -\frac{\exp(\delta_{f'} - \alpha p_{f'})}{1+\exp(\delta_{f'} - \alpha p_{f'})}$.By Assumption 1, $\frac{\exp(\delta_{f'} - \alpha p_{f'})}{1+\exp(\delta_{f'} - \alpha p_{f'})}$ is decreasing in $p_{f'}$, so

$$\frac{\partial}{\partial p_{f'}} \frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})} = -\frac{\partial}{\partial p_{f'}} \frac{\exp(\delta_{f'} - \alpha p_{f'})}{1+\exp(\delta_{f'} - \alpha p_{f'})} > 0.$$

Then, Equation 8 implies that if the competitor-scarcity effect $\omega^f_{f'}$ is positive, an increase in

[16] As noted in Vives (2018) and Nocke and Schutz (2018), static oligopoly games in multi-product environments are generally not games of strategic complements.

competitor price increases a firm's cost of selling a product. We retain strategic complementarity. In contrast, for negative $\omega^f_{f'}$, the cost is decreasing in the competitor's price. This is because if the competitor increases its price, it loses market shares, which decreases the upward pressure on the own price. When $\omega^f_{f'}$ is very negative, it might be that $\frac{\partial g_f}{\partial p_{f'}} < 0$. As a result, the competitor's price can become a strategic substitute to the firm's own price. Figure 11 in Appendix D plots best-response functions for an example using logit demand. In the left panel, the competitor-scarcity effects are positive, that is, price are strategic complements, and in the right panel, competitor scarcity effects are negative, so that prices are strategic substitutes.

All in all, positive own-scarcity effects ω^f_f and negative competitor-scarcity effects $\omega^f_{f'}$ shift best response functions upwards, but competitor-scarcity effects additionally change the slope of the best response functions (illustrated in Figure 12 in Appendix D).

3.3.3 The Influence of Remaining Inventory on Prices

Finally, we link remaining inventory to incentives to soften competition. We focus on demand that is constant over time ($\lambda_t \equiv \lambda$, $\boldsymbol{\theta}_t \equiv \boldsymbol{\theta}$) to single out the effects of remaining capacities. Similar forces occur with time-dependent demand, as we show in our empirical analysis.

Starting at the deadline, where all scarcity effects are equal to zero, equilibrium prices, $\mathbf{p}^*_T$, are equal to the stage game equilibrium prices. As we move away from the deadline, remaining inventory can influence pricing dynamics. We establish that the order of change of prices towards the deadline is determined by the product with the minimum remaining inventory in the market. The order of change is reduced by one only if a unit of the product with the minimum remaining inventory is sold. The proposition is formally stated below.

Proposition 4. *Let $\lambda_t \equiv \lambda$, $\boldsymbol{\theta}_t \equiv \boldsymbol{\theta}$. Then, for $\mathbf{K}$ with $\underline{K} := \min_f K_f$, the following holds:*

$$p_{f,t}(\mathbf{K}) = p^*_{f,T} + \mathcal{O}(|T-t|^{\underline{K}}),\ t \to T \ \ for\ f = 1,2,$$

i.e., price changes close to the deadline are at most of order $\underline{K}$. *If* $\lim_{t\to T} \frac{\partial^{\underline{K}}}{(\partial t)^{\underline{K}}} \Pi_{f,t}(\mathbf{K}-\mathbf{e}_h) \neq 0$ *for* *all f and h with $K_h = \underline{K}$, then*[17]

$$p_{f,t}(\mathbf{K}) = p^*_{f,T} + \Theta(|T-t|^{\underline{K}}),\ t \to T\ \textit{for}\ f = 1,2,$$

i.e., price changes are exactly of order $\underline{K}$.

This proposition implies that close to the deadline, price paths with capacity vectors with the same minimum remaining inventory $\underline{K}$ are close to each other, while after a sale of the firm with the lower remaining inventory price paths jump. This is because the prospect of softening future price competition is manifested in the scarcity effects of the product with minimum remaining inventory . If capacities are asymmetrically distributed, the competitor-scarcity effect of the firm with more remaining inventory is relatively large and negative because a sale of the competitor will increase future prices. In equilibrium, firms will set prices that induce the firm with the least inventory to sell with higher probability. If firms have the same capacity, then any sale leads to a price jump, regardless of which firm sells. Hence, a sale by either firm softens future price competition, leading to fierce price competition today as both firms want to leave the competitive state quickly. This can lead to both firms setting low prices, possibly even lower than the competitive price $\mathbf{p}^*_T$ absent scarcity effects.

We illustrate these price competition effects in Figure 13 in Appendix D. We consider firms with $\mathbf{K}$ =(5,4); (4,4); and (3,4) capacities. Note that (4,4) prices are the lowest (panel d). Own scarcity effects are higher (more positive) and competitor-scarcity effects are lower (more negative) with (3,4) versus (5,4) capacities.[18] Finally, Figure 14 in Appendix D illustrates both firm's price paths if starting from a capacity vector $\mathbf{K} = (3,5)$, firm 1 sells versus if firm 2 sells.

[17]Recall that $f(t) = \mathcal{O}(g(t))$ as $t \to T$ if $\exists \delta, C_1 > 0$ so that for all t with $0 < |T-t| < \delta$, $|f(t)| \leq C_1 g(t)$. $f(t) = \Theta(g(t))$ if additionally $\exists C_2 > 0$ so that $C_2 g(t) \leq |f(t)|$.

[18]The relationship between prices and competing firms' inventories has been explored in other contexts, e.g., see Israeli et al. (2022) on car dealership pricing.

4 Data and Descriptive Evidence

4.1 Data Description

Our empirical insights are derived from data provided to us through a research partnership with a large U.S. airline.[19] The core data set contains booking and pricing information covering competing airlines and was assembled by third parties that collect and combine contributed data. The data have strong parallels with other contributed data sets, such as the the Nielsen scanner data used to study retailing, in that we observe prices and quantities for competing firms. Our data cover the first nine months of departures in 2019.

The bookings data track flight-level sales counts over time. We use the tuple j, t, d to denote an airline-flight number, day before departure, departure date combination. The frequency of the data is daily. We observe separate booking counts for passengers flying between an origin-destination pair (OD) and consumers making connections. We call these consumers *local* and *flow* passengers, respectively. Our structural analysis focuses on local, nonstop traffic. We do not model the potential for consumers to connect while flying between an origin-destination pair. The data contain bookings for consumers who purchased directly with the airline and on other booking channels, e.g., online travel agencies. We label these bookings *direct* and *indirect*, respectively. Because we observe all booking counts, we can construct the load factor for each flight over time. We do not know the exact itinerary involved for each booking, e.g., a round-trip versus a one-way booking. Therefore, we assume that the price paid for each nonstop booking corresponds to the lowest available nonstop, one-way fare for that flight.

Our pricing data come from a separate third-party data provider that gathers and disseminates fare information for the airline industry. We observe daily prices at the flight level. We observe all fares, even when there are no bookings, including tickets of different qualities (cabins, fully refundable, etc.). Travelers overwhelmingly purchase the lowest available economy class fare offered (Hortaçsu et al., 2021b), which motivates our choice to concentrate our analysis on the lowest available economy class ticket. We do not model consumers choosing between cabins (economy vs. first class) nor the pricing decision for different versions of tickets.

[19]The airline has elected to remain anonymous.

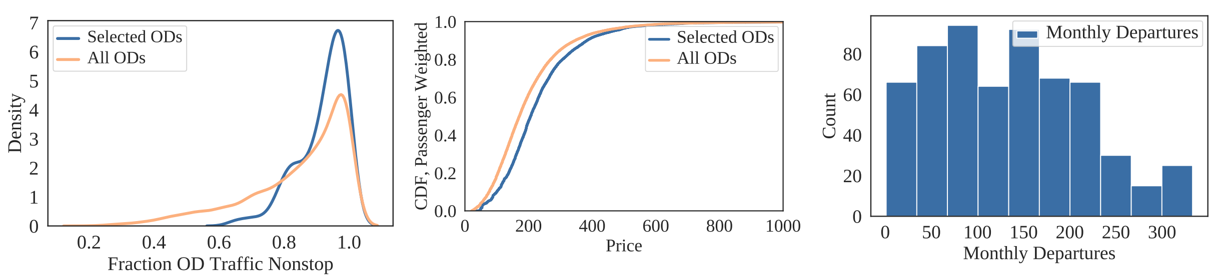
In order to gauge the dynamics of market sizes, we use clickstream search data provided to us by the air carrier. See Hortaçsu et al. (2021a) and Hortaçsu et al. (2021b) for more details. Observed searches understate true arrivals because some consumers may search and purchase through online travel agencies or directly with competitors. We extrapolate total arrivals by scaling up observed searches using hyperparameters that we describe below.

4.2 Route Selection

Our analysis concentrates on nonstop flight competition. We limit ourselves to routes where nonstop service is provided by exactly two airlines—by our data provider and one competitor. Our data contain more than one competitor airline, however, we will always refer to the competing airline of an OD pair as “the competitor.” We eliminate routes where the third-party data is incomplete, e.g., where a carrier provides direct bookings to the data provider but indirect bookings are missing. In addition to these criteria, we select routes in which most local traffic is traveling nonstop. These selection criteria allow us to avoid the additional complexity of modeling connecting traffic within an OD pair.

Figure 3: Summary Analysis from the DB1B Data

(a) PDF of Nonstop Traffic (b) CDF of Passenger-Weighted Fares (c) Monthly Departure Count



Note: Panel (a) records the PDF of nonstop traffic among local traffic in the DB1B data (orange) and for selected routes (blue). Panel (b) plots the CDF of prices for selected routes (blue) and all dual-carrier markets (orange). Panel (c) reports the number of aggregate monthly departures for the routes in our sample.

In Figure 3 we provide summary analysis of the 58 routes in our data using the publicly available DB1B data. These data contain 10% of bookings in the U.S. but lack information on the booking date, departure date, and flights involved. In panel (a), we show the distribution of local traffic flying nonstop. For the selected markets, most local traffic is traveling nonstop. In panel (b) we show that the distribution of fares in our markets is similar to the universe of dual-carrier markets. Finally, in panel (c) we use the publicly available T100 segment data to plot the total number of monthly departures for the routes in our sample. Over half of our sample contains routes in which there are less than five daily frequencies (across both airlines) between the origin and destination. Several routes feature twice daily service (one flight per airline). At the other extreme, one route in our data contains nearly 10 flights per day.

4.3 Descriptive Evidence

Table 1: Summary statistics

Data Series	Variable	Mean	Std. Dev.	Median	5th pctile	95th pctile
Fares						
	One-Way Fare ($)	233.7	111.4	218.6	92.1	390.7
	Num. Fare Changes	6.4	2.4	6.0	3.0	11.0
Bookings						
	Booking Rate-local	0.2	0.6	0.0	0.0	1.0
	Booking Rate-all	0.5	1.2	0.0	0.0	3.0
	Ending LF (%)	72.1	19.8	76.0	32.9	98.0

Note: One-Way fare is for the lowest economy class ticket available for purchase. Number of fare changes records the number of price adjustments observed for each flight. Booking rate-local excludes flow traffic. Booking rate-all includes both local and flow traffic. Ending load factor (LF) reports the percentage of seats occupied at departure time.

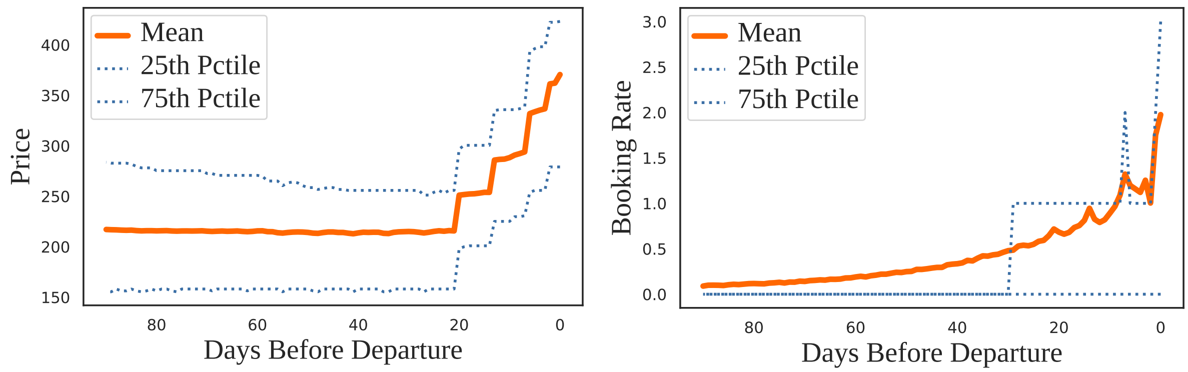
We provide a summary of the main data in Table 1. We focus on the last 90 days before departure due to sparsity in bookings beyond 90 days.[20] Average fares across airlines in our sample are $233. On average, each flight experiences about six price adjustments within 90 days. The average daily booking rate is less than one. Roughly 40% of observed bookings are for local traffic, the remaining are flow bookings. At the departure time, average load factors are 72%, which is lower than the industry average of about 80% for this time period. Roughly 3.5% of flights in our sample sell out.

In Figure 4 we plot average fares and booking rates by day before departure. The left panel (a) shows that average fares are fairly flat between 90 and 21 days before departure. The top end of the distribution is decreasing in this time window. There are noticeable "steps" in the

[20]The average load factor 90 days before departure is 10.0%.

Figure 4: Prices and Bookings by Day Before Departure

(a) Prices over Time (b) Bookings per Route-Departure Date over Time



Note: Panel (a) shows the average and interquartile range of flight prices over time. Panel (b) shows the average and interquartile range of flight booking rates per route-departure date over time. Greater than 30 days before departure, the 25th and 75th percentiles coincide.

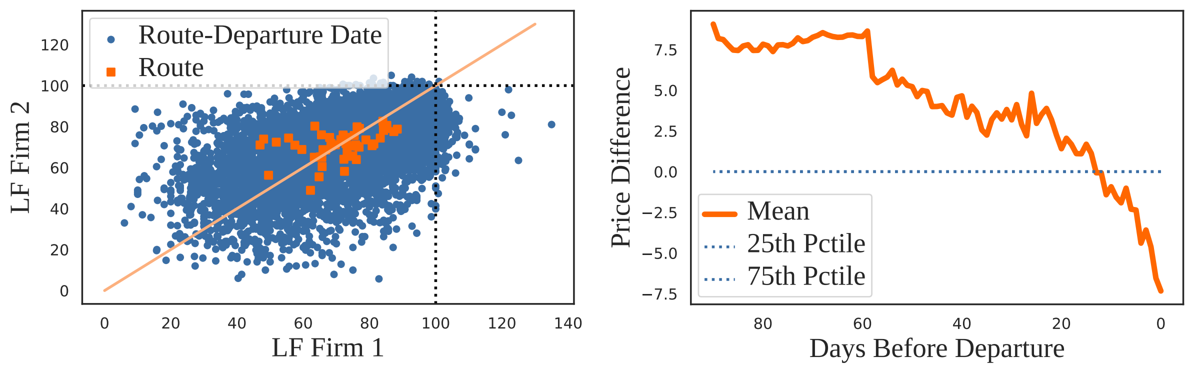
last 21 days before departure which highlights the use of advance purchase (AP) discounts in the industry. In the routes examined, we observe AP requirements at 21, 14, 7, and 3 days before departure. Note that fares increase by over 70% in three months. In the right panel (b) we highlight that bookings increase as the departure date approaches. This coincides with increasing prices and suggests that demand becomes more inelastic over time. The booking rate is greater than one per flight over the last month before departure.

In Figure 5 we compare outcomes across competitors. The left panel (a) provides a scatter plot of ending load factors at the route-departure date level for our data sample. The orange squares present route-level load factors. Note there exists a large mass of points both above and below the 45-degree line—no carrier consistently sells a larger fraction of capacity than the other carrier for all routes. The scatter plot also shows days where flights are sold out. In our analysis, we restrict firms to selling at most their capacity (recall, 3.5% of flights sell out). In the right panel (b) we plot the average fare difference across competitors (firm 1 minus firm 2) over time when exactly two flights are offered. Note that fares tend to be similar across competitors—the average difference is less than \$10. However, the gradient of the prices differs. One competitor has relatively higher prices well in advance of departure and relatively lower prices close to departure. Prices across airlines are nearly equal 50% of the time. This occurs because airlines have filed the same discrete fare(s), and the pricing heuristics have selected that fare to offer to consumers (Figure 16 in the Appendix presents an example fare menu for one carrier-route).

Figure 5: Load Factor and Price Differences across Carriers

(a) Load Factors

(b) Mean Price Difference



Note: Panel (a) shows the average load factor (across all flights) at the route-departure date level for both competitors in blue. The orange squares report average route-level load factors. The diagonal line is the 45-degree line. Panel (b) shows the average and the 25th and 75 percentiles of the difference in prices for markets in which each firm offers exactly one flight.

5 Demand Model and Estimates

5.1 Empirical Specification

We model nonstop air travel demand using a flexible nested logit demand model. Our approach differs from recent empirical work on airlines that uses a mixed-logit specification to model "business" and "leisure" travelers (Lazarev, 2013; Williams, 2022; Aryal et al., 2021; Hortaçsu et al., 2021b). Instead, we allow for time-varying elasticity as it better maps to our theoretical model and results in unique equilibrium price paths. We have found that mixed-logit models yield multiple equilibria in our setting, thus requiring an equilibrium selection mechanism.[21]

Define a market as an origin-destination (r), departure date (d), and day before departure (t) combination. Each flight j, leaving on date d, is modeled across time $t \in \{0, ..., T\}$. The first period of sale is $t = 0$, and the flight departs at T. Demand is modeled at the daily level over

[21]We have found up to four equilibria in a single stage game and nine fixed points of the system of first-order conditions using the mixed-logit model.

a 90-day horizon. Arriving consumers choose a flight that maximizes their individual utilities from the choice set $\mathscr{J}_{t,d,r}$, or select the outside option, $j=0$. Products are partitioned into two nests. The outside good belongs to its own nest, and all inside goods to the second nest.

We specify consumer arrivals to be

$$\lambda_{t,d,r}=\exp\left(\tau_r^{\text{OD}}+\tau_d^{\text{DD}}+\tau_{t,d}^{\text{SD}}+f(t)\right),$$

where the τs denote fixed effects for the route, departure date, and search date; $f(\cdot)$ is a polynomial series of degree three. We scale up these estimated arrival rates using hyperparameters to account for unobserved searches. Smoothness of $f(\cdot)$ allows us to use the differential equation equilibrium characterization.

Conditional on arrival, we specify consumer utilities as

$$u_{i,j,t,d,r}=\mathbf{x}_{j,t,d,r}\boldsymbol{\beta}-\alpha_t p_{j,t,d,r}+\zeta_{i,J}+(1-\sigma)\varepsilon_{i,j,t,d,r},$$

where $\zeta_{i,J}+(1-\sigma)\varepsilon_{i,j,t,d,r}$ follows a type-1 extreme value distribution, and $\zeta_{i,J}$ is an idiosyncratic preference for the inside goods. The parameter $\sigma\in[0,1]$ denotes correlation in preferences within the nests. We allow the price sensitivity parameter to vary over time (α_t) using three-day intervals of time; hence, we estimate 30 price sensitivity parameters.[22] We include a number of covariates in $\mathbf{x}$ where preferences are assumed to not vary across t: departure week of the year, departure day of the week, route, carrier, and departure time fixed effects. In our baseline model, we do not include an additional unobservable (ξ) that is potentially correlated with price. We discuss this extension in Section 5.4.

Arriving consumers solve their utility maximization problem such that consumer i chooses flight j if and only if

$$u_{i,j,t,d,r}\geq u_{i,j',d,t,r},\ \forall j'\in\mathscr{J}_{t,d,r}\cup\{0\}.$$

[22] We will later smooth these parameters in order to use the differential equation equilibrium characterization ($R^2=97.4\%$). Another approach would be to use constrained maximum likelihood.

Temporarily dropping the t, d, r subscripts, we define

$$D_{\mathscr{J}} := \sum_{j \in \mathscr{J}} \exp\left(\frac{\mathbf{x}_j \boldsymbol{\beta} - \alpha p_j}{1-\sigma}\right),$$

so that the probability that a consumer purchases j within the set of inside goods is equal to

$$s_{j|\mathscr{J}} := \frac{\exp\left(\frac{\mathbf{x}_j \boldsymbol{\beta} - \alpha p_j}{1-\sigma}\right)}{D_{\mathscr{J}}}.$$

It follows that the probability that a consumer purchases any inside good product is equal to

$$s_{\mathscr{J}} := \frac{D_{\mathscr{J}}^{1-\sigma}}{1 + D_{\mathscr{J}}^{1-\sigma}}.$$

Overall product shares are equal to $s_j = s_{j|\mathscr{J}} \cdot s_{\mathscr{J}}$, which are implicitly at the market level (t, d, r).

Our assumptions imply that demand is distributed Poisson with a product purchase rate of $\min\{\lambda_{t,d,r} \cdot s_{j,t,d,r}, K_{j,t,d,r}\}$, where $K_{j,t,d,r}$ denotes the remaining inventory. Note as the length of a period decreases, at most one seat will be sold in any period.

5.2 Estimation Procedure

We estimate the model in two steps. In the first step, we estimate the arrival process parameters using Poisson regressions. We then estimate preferences of the Poisson demand model using maximum likelihood. We estimate standard errors using bootstrap.

We follow Hortaçsu et al. (2021b) in constructing arrivals using clickstream data for one airline. These data track all "clicks" or interactions on the firm's websites. We first sum the number of searches corresponding to each market (r, d, t), and then scale up estimated arrival rates to account for unobserved searches. This follows from the property of the Poisson distribution that the sum of Poisson variables is Poisson with added intensities, and from the assumption that consumers who search/purchase through alternative platforms (travel agents, other carriers' websites) have the same underlying preferences. We use the fraction of direct bookings by day before departure as weights when we scale up the estimated arrival rates. This adjusts arrivals for a single carrier. In our preferred specification, we then double these arrival rates to account for competitor indirect and direct searches, both of which are unobserved to us. Our demand estimates do not vary substantially under alternative scaling parameters (see Section 5.4).

5.3 Identification

In empirical work, it is customary to treat the market size as given. We use arrivals data to discipline our demand estimates and recover changes in willingness to pay over time. Without access to arrivals data, it is difficult to estimate preferences in models with demand uncertainty because a given booking could be observed due to many arrivals and price sensitive consumers, or few arrivals and price insensitive consumers. Consequently, researchers have resorted to supply-side optimality conditions in order to address this identification challenge (Williams, 2022; Aryal et al., 2021). Our arrivals data show that market participation increases over time in all routes studied, which informs how consumer preferences evolve. For example, if we assumed market sizes were constant over time, we would estimate early demand as being too elastic and late demand as being too inelastic.

Stochastic demand allows us to measure demand response to price changes. In the model, every booking changes the opportunity cost for the next unit and results in a discontinuous price jump. This is also reflected in our empirical application where a booking results in a "fare bucket" closing at a random time (see Hortaçsu et al. (2021b)). Our identification of demand uses a regression discontinuity type argument. When prices adjust within the three day interval of time where α_t is fixed, we can observe the resulting demand response, and estimate interval-specific price elasticities. A price change for one firm informs substitution patterns to other products versus the outside good (σ). The fixed effects are identified by booking rate differences across weeks of the year, route, times of the day, and airlines.

5.4 Demand Estimates

We summarize the demand estimates in Table 2. The nesting parameter is estimated to be 0.5 implying substantial substitution within inside goods. The price sensitivity parameters vary by nearly a factor of ten over time. We present a time series plot of α_t in Figure 6-(b). Almost all of our controls are significant, with day of the week and week of the year having the strongest influence on market shares. The competitor FEs are less important in driving variation in shares. We estimate the average own-price elasticity to be -1.44 (s.d. = 0.81), indicating slightly more elastic demand than in Hortaçsu et al. (2021b), which uses similar data for routes with a single carrier only.

Table 2: Demand Estimates Summary Table

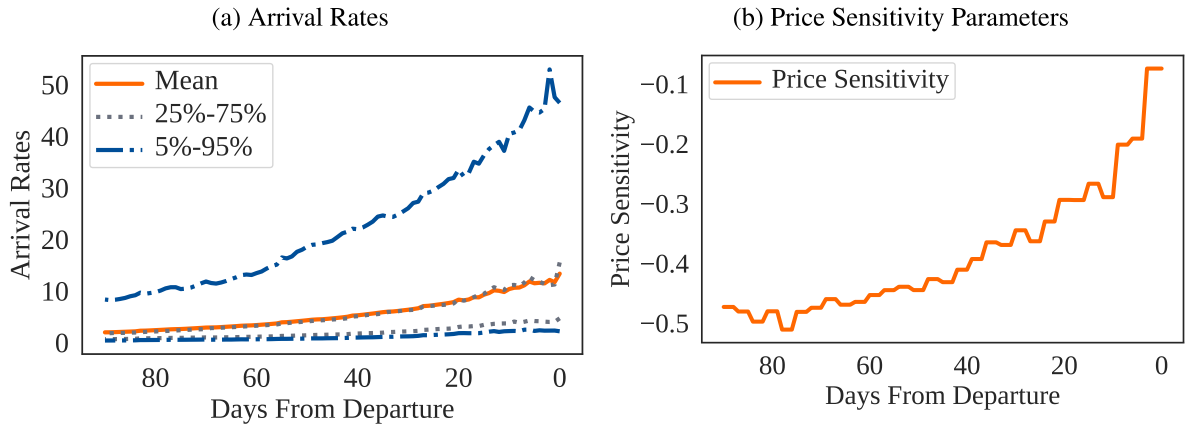
Variable	Symbol	Estimate	Std. Error.	Range	% Sig.
Nesting Parameter	σ	0.498	0.010	–	–
Price Sensitivity	α	–	–	[-0.511 ,-0.074]	100.0
Competitor FE	–	–	–	[0.000 ,0.071]	100.0
Day of Week FE	–	–	–	[-1.637 ,-0.961]	100.0
Departure Time FE	–	–	–	[-0.462 ,-0.050]	100.0
Route FE	–	–	–	[-0.177 ,0.226]	94.4
Week FE	–	–	–	[-0.953 ,0.699]	86.0
Sample Size	N		2,814,686		
Average Elasticity	e^D		-1.438		

Note: Demand estimates for the 58 routes in our sample.

In Figure 6-(a), we plot average adjusted arrival rates as well as different percentiles (5%, 25%, 75%, 95%) across markets. For each route, we estimate just a few arrivals 90 days before departure that rise to over 10 passengers per day close to departure. Recall that the average booking rate across flights is less than 2.0 (see Figure 4) so that market shares are low. An increase in interest in travel is a general finding across all of the routes in our sample. Note that while the 75th percentile closely followed the mean, the top part of the distribution is substantially higher, which corresponds to the routes with a larger number of departures.

Before turning to counterfactuals, we briefly discuss additional demand results. Our demand estimates are robust to the choice in scaling factor. Average demand elasticities with scaling

Figure 6: Arrival Rates and the Price Sensitivity Parameters



Note: Panel (a) shows fitted values of arrival rates over time adjusted for unobserved searches. The mean is the average arrival rate across all markets. The percentiles are also over markets. Panel (b) shows our estimates of the price sensitivity parameters in 3-day groupings.

parameters between 1.0 and 3.5 are between -1.40 and -1.46.[23] We have also estimated a demand model that incorporates an additional unobservable (ξ) that is potentially correlated with p. We use a 2-step estimation procedure where we first estimate the arrival process parameters and then use a control function to estimate the demand parameters using quasi-maximum likelihood estimation. Included in our set of instruments is a polynomial expansion of remaining inventory, indicators for AP fares, and the number of flights offered. With this approach, we estimate average demand elasticities to be -1.59, with a standard deviation of 0.87. These estimates are also robust to the choice in scaling parameter.

6 Counterfactual Analysis

With our demand estimates, we quantify the welfare effects of dynamic price competition by comparing equilibrium outcomes with dynamic pricing—the benchmark model—with the equilibrium outcomes with uniform pricing. Under uniform pricing, firms charge a single price for all seats on a given flight. We also discuss the use of pricing heuristics at the end of this section.

In the main text, we focus on 18 duopoly markets where each airline offers a single flight, i.e., each airline offers exactly a single flight on any given day. In Appendix D.5, we report

[23] With a scaling parameter equal to 0.5, average demand elasticities are −1.28.

results for all routes. The reason we separate the counterfactuals is that with more than two flights, solving for equilibria of the dynamic pricing game becomes computationally challenging. In order to run counterfactuals for all routes in our sample, we must reduce the number of flights studied. To do this, we adjust the choice set, utilities, and capacities for routes where an airline offers multiple flights a day. The appendix contains details of the procedure as well as the counterfactual results. We note that both the direction and magnitude of the welfare effects for the entire sample are consistent with the 18 routes reported here. Moreover, the direction of the welfare effect is the same for each route, but the magnitude differs.

Benchmark Model

We approximate the continuous-time model to solve for equilibrium prices for every route-departure date. We consider hourly decisions over 90 days. Both firms start with initial capacities K_f and $K_{f'}$. We solve for the equilibrium via backward induction, as outlined here.

In the last period, $t = T$, we have $\Pi_T(\mathbf{K}) = 0$. Therefore, in the last pricing period, $t = T - \Delta$, $\Omega_{T-\Delta}(\mathbf{K}) = \mathbf{0}$ and both firms solve static revenue maximization problems. We set the first-order conditions corresponding to the best-response functions equal to zero and solve for the fixed point. We denote the fixed-point price vector by $\mathbf{p}_{T-\Delta} = \mathbf{p}^*(\Omega_{T-\Delta}, \alpha_{T-\Delta})$, where $\Omega_{T-\Delta} = 0$. We denote the stage-game payoff in period t by $\pi_{f,t}(\mathbf{p}, \Omega)$. Then, using the differential equation, we can write $\dot{\Pi}_{f,T}(\mathbf{K}) = -\lambda_T \pi_{f,T}(p_T(\mathbf{K}), \mathbf{0})$, which allows us to calculate $\Pi_{f,T-\Delta}(\mathbf{K}) = \Pi_{f,T}(\mathbf{K}) - \Delta \cdot \dot{\Pi}_{f,T}(\mathbf{K})$ and $\omega^f_{f',T-2\Delta}(\mathbf{K}) = \Pi_{f,T-\Delta}(\mathbf{K}) - \Pi_{f,T-\Delta}(\mathbf{K} - \mathbf{e}_{f'})$. Given the updated own- and competitor-scarcity effect parameters, we again solve for equilibrium prices, $\mathbf{p}_{T-2\Delta} = \mathbf{p}^*(\Omega_{T-2\Delta}, \alpha_{T-2\Delta})$.[24] We continue the induction backwards in time until the first period using the recursion

$$\begin{cases} \Pi_{f,t-\Delta}(\mathbf{K}) = \Pi_{f,t}(\mathbf{K}) - \Delta \cdot \dot{\Pi}_{f,t-\Delta}(\mathbf{K}) & \forall f \\ \omega^f_{f',t-2\Delta}(\mathbf{K}) = \Pi_{f,t-\Delta}(\mathbf{K}) - \Pi_{f,t-\Delta}(\mathbf{K} - \mathbf{e}_{f'}) & \forall f, f' \end{cases}.$$

Due to the large number of state variables, we store Ω_t and $\mathbf{p}_t$ every 24 hours. This means prices are constant within a day, which maps well to our empirical setting as prices are adjusted

[24]We use a modified Powell method from MINPACK's hybrid routine to solve the system of first-order conditions corresponding to the best-response functions.

daily. We then appeal to modeling demand via multinomial distributions after drawing arrivals from a Poisson distributions in lieu of modeling each consumer's individual choice after drawing arrivals from Bernoulli distributions (as in the theoretical model). When demand exceeds remaining inventory and demand is censored, we assume random rationing in all counterfactuals.

Uniform Pricing

For the uniform pricing counterfactual, we assume that each firm sets a single price for each route-departure date independent of the timing of purchase. This is analogous to Williams (2022) who considers the single-firm setting. Holding the competitor price fixed, we simulate 10,000 flights to compute expected revenues for each route-departure date. We solve for the optimal price and iterate across best-response functions until convergence.

Implementation

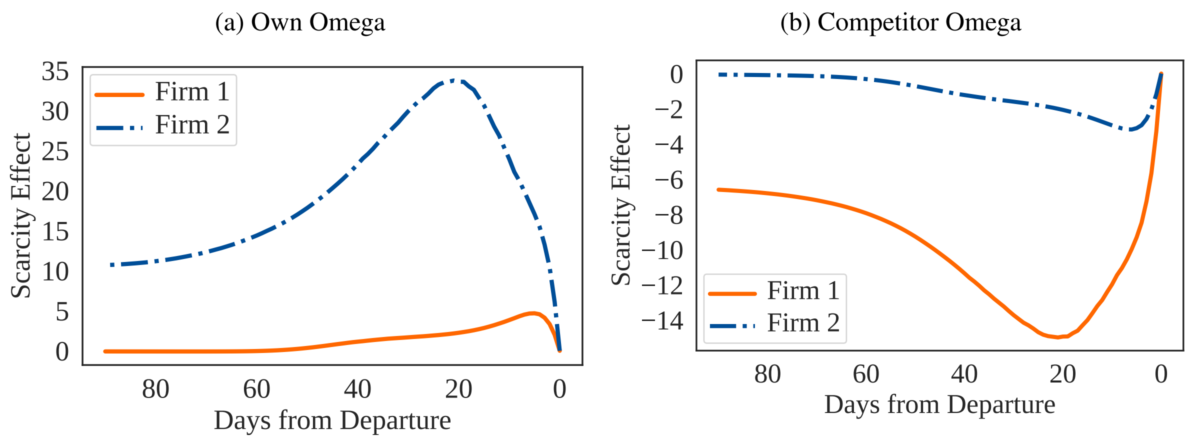
To implement all counterfactuals, we conduct 10,000 Monte Carlo experiments for every route, departure date combination. We smooth α_t using a polynomial regression in order to avoid discontinuities in the time derivatives which allows us to solve for equilibria ($R^2 = 0.974$).

We simulate all counterfactuals twice, once where flow traffic is subtracted from initial observed capacity in advance, and once where flow traffic is modeled through Poisson processes that make inventory units disappear independent of the price. We report the latter specification here as it approximates how our airline considers flow traffic. The appendix contains the former approach. Both the direction and magnitude of the welfare effects are similar across specifications.

6.1 Welfare Effects of Dynamic Price Competition

Before reporting our welfare numbers, we briefly describe the pricing forces of the benchmark model. We confirm in our empirical application that the sign of the scarcity effects can be positive or negative, and in fact, can vary within a particular game (route-departure date). Figure 15-(a) in the Appendix shows an example flight where the the own-scarcity effect changes from negative to positive to near zero over time. However, in our data, the scarcity effects tend to not change signs frequently, and we find less than 0.5% of states result in a game that is not one of strategic complements. Own-scarcity effects tend to remain positive. This can be seen in Figure 7. Average own-scarcity effects are largest close to the departure date. This is because selling a unit decreases a firm's continuation payoff the most when inventory is scarce. Average competitor-scarcity effects tend to be negative. This is because the sale of a competitor typically increases future prices. Therefore, both scarcity effects tend to raise average equilibrium prices in the stage game relative to a stage game that does not endogenize firm scarcity.

Figure 7: Benchmark Model Scarcity Effects



Note: Panel (a) reports the own-firm scarcity effect over time for both firms. Panel (b) reports the cross-firm competitor scarcity effect over time for both firms.

Scarcity effects are asymmetric across firms. Competitor-scarcity effects tend to be larger for firm 1 relative to firm 2. This asymmetry implies that the sale of firm 2 softens competition more than a sale of firm 1. At the same time, the own-scarcity effect is larger for firm 2. This is because firm 2 typically offers planes with lower capacities than firm 1.

Our other theoretical predictions are reflected in the counterfactuals and demonstrate how firms adjust prices over time. For example, Figure 15-(b) in the Appendix shows that close to the deadline, the predictions of Proposition 4 hold. We highlight in panel (c) that prices increase the most when the firm with minimum remaining inventory sells. In fact, the average price increase is over five times ($30) greater than the price increase ($5) if the firm with more seats remaining sells.

Table 3: Counterfactual Results for Single Product, Duopoly Routes

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	226.3	5571.5	5759.4	16698.2	28029.0	20.0	70.6	9.2
Uniform	250.8	4629.6	4925.7	19042.4	28597.6	19.2	69.7	7.9
% Diff.	10.8	-16.9	-14.5	14.0	2.0	-3.8	-0.9	-1.3

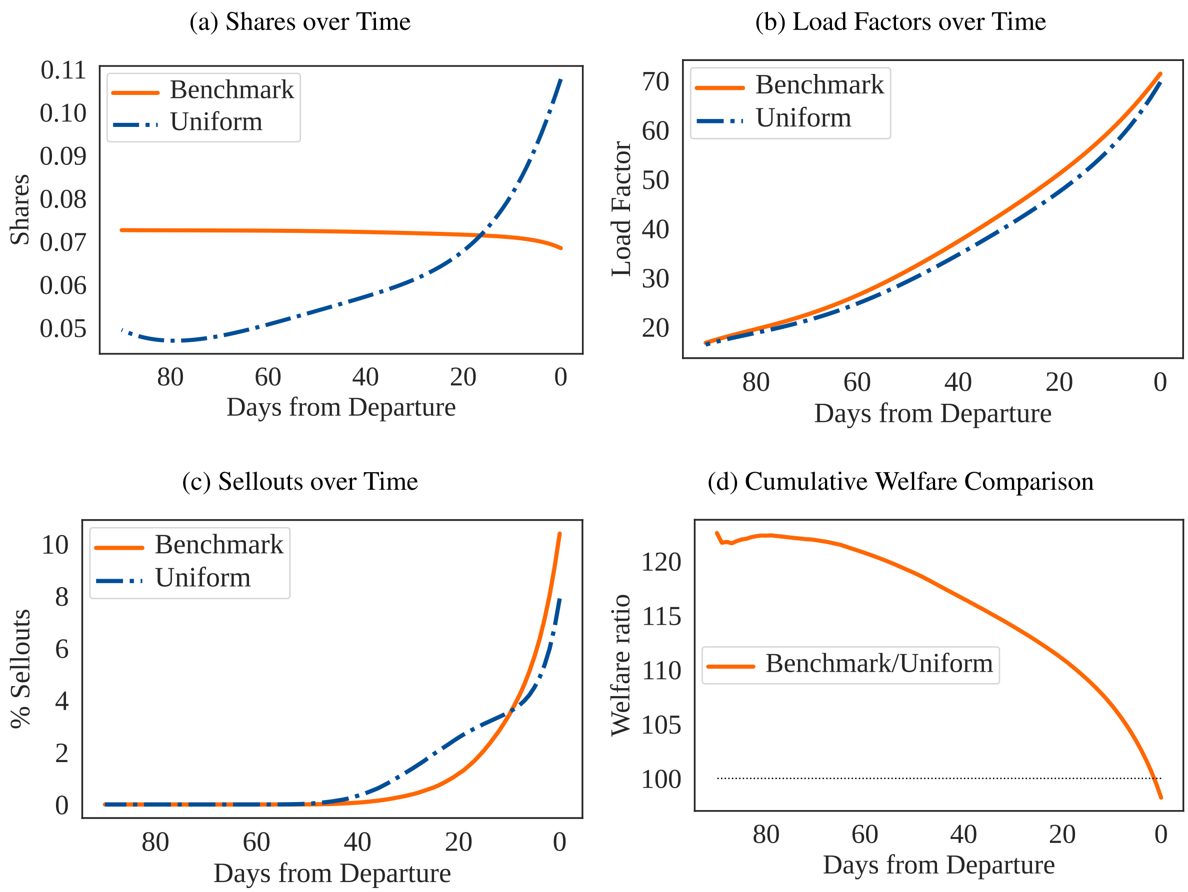
Note: Price is the average across routes (r) after computing the average across firms (f), departure dates (DD), days before departure (DFD) and simulation number (n) within a route. Firm revenues are similarly defined, except aggregated over DFD. CS is the expected consumer surplus, computed the same way as revenues. Welfare is the sum of revenues and CS. Q is the total number of seats sold. LF is the average fraction of seats sold (including flow traffic) at the departure time. Sellouts is the fraction of flights sold out.

We report market outcomes in Table 3. Average prices in our benchmark simulations (\$226.3) are close to the average observed prices (\$233.7). Prices are 10% higher under uniform pricing (\$250.8). Although uniform pricing features higher average prices, revenues are substantially lower for both firms (columns 2 and 3). Firm 1 benefits more from dynamic pricing because it has on average a larger (in size) competitor-scarcity effect. The revenue effects are driven by relatively higher fares for early-arriving, price sensitive customers and relatively lower for late-arriving, price insensitive customers under uniform pricing. Dynamic pricing expands output due to lower prices early on. This can be seen in panels (a) and (b) in Figure 8, which show purchase probabilities and cumulative load factors over time. Panel (c) shows that higher output also leads to more sell outs. Notably, the gap in load factors begins to close near the departure date when dynamic prices are high.

A key empirical finding of our analysis is that total output is higher, but total welfare is lower under dynamic pricing compared to uniform pricing. Hence, competition effects flip the results found in the single-firm case where dynamic pricing increases welfare (Hendel and Nevo, 2013; Castillo, 2020; Williams, 2022). Consumer surplus is 14% higher with uniform pricing, which is larger in magnitude than the associated revenue losses (between 14-17%) of not adjusting prices based on demand and scarcity. The welfare loss is driven by high fares close to the departure date, when most demand arrives. This can be seen in Figure 8-(d), which plots the ratio of welfare under dynamic pricing over uniform pricing.

Our welfare results are robust to how we handle local traffic. In Figure 18 and Table 5 in the Appendix, we show these welfare effects still hold if we first subtract off flow traffic in the first period. In Table 7 in the Appendix, we show that these welfare effects also hold for the entire sample under additional assumptions that make the equilibrium analysis tractable.

Figure 8: Counterfactual Summary Plots



Note: Panel (a) shows the average shares over time for the benchmark and uniform models. Panel (b) shows the average load factors over time for the same two models. Panel (c) shows the average sellouts over time for the same two models. Panel (d) shows the ratio of average cumulative welfare for the benchmark model with respect to the uniform one.

6.2 Analysis of Pricing Heuristics

We contrast our results with two pricing heuristics where firms do not internalize the scarcity of their competitor and do not explicitly account for the fact that their competitor is a strategic agent solving a dynamic pricing problem. These heuristics are based on documentation at one airline that outlines how it has considered incorporating competition into its pricing decisions.[25]

[25]Due to confidentiality reasons, we do not divulge which heuristic has been studied internally.

We assume both firms use the same heuristic.

For both heuristics, we consider discrete prices as they are used in actual airline pricing practices. Applied theory work, e.g., Asker et al. (2021), also consider discrete prices. We do not endogenize the pricing menu (set of discrete prices for all time periods), instead, we use the actual observed pricing menus for each carrier-route as an input. Typically, each carrier files between seven and fifteen "buckets" (prices) per route.[26] Oftentimes, a fare is restricted for a certain time period before departure, which is commonly referred to as an advance purchase discount. Figure 16-(a) in Appendix D shows an example fare menu for a given carrier-route in the data. Observed fares vary from less than $200 to over $3,000.

We label the heuristics "Lagged Model" and "Deterministic Model," respectively.[27] In the lagged model, each firm, having observed its competitor's last period price bucket, assumes this bucket will also be charged in the current and all future periods. Each firm then calculates its residual demand curves in all remaining periods and solves a single-firm dynamic programming problem. In the deterministic model, each firm simply assumes its competitor will price at the lowest possible bucket in all remaining periods.

Table 4: Heuristic Counterfactuals for Single Product, Duopoly Routes

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Lagged	104.6	104.1	105.3	103.3	103.9	100.0	100.1	101.0
Deterministic	98.0	99.4	100.8	108.2	104.9	103.9	101.4	109.2

Note: Price is the average across routes (r) after computing the average across firms (f), departure dates (DD), days before departure (DFD) and simulation number (n) within a route. Firm revenues are similarly defined, except aggregated over DFD. CS is the expected consumer surplus, computed the same way as revenues. Welfare is the sum of revenues and CS. Q is the total number of seats sold. LF is the average fraction of seats sold (including flow traffic) at the departure time. Sellouts is the fraction of flights sold out.

Counterfactual results appear in Table 6. Figure 9 plots market outcomes over time. We normalize market outcomes of the benchmark model to 100 and report percentage differences for the heuristics. We find that the use of heuristics can on average lead to higher or lower

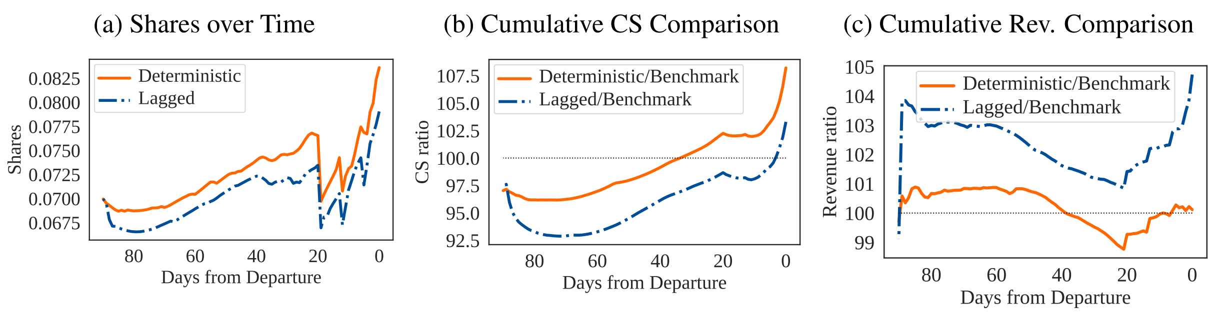
[26]Bucket prices can change by day before departure, which can result in an increase in the fare for a given bucket. However, the data suggests that a more consequential change in buckets over time is their availability.

[27]These heuristics can also produce ambiguous welfare effects. Figure 17 in Appendix D provides simulations where price levels with heuristics can be higher or lower than in the benchmark model depending on demand parameters.

prices compared to the benchmark model. As a result, revenues can also be higher or lower compared to the benchmark model. Both models result in significant price matching, at the same frequency observed in the data (between 45-47%).

With the deterministic model, firms start out with low prices and sell too many units to price elastic early customers, instead of late-arriving price-insensitive customers. This results in more frequent sellouts. However, prices remain low close to the deadline, leaving late-arriving customers with more surplus. Instead, the lagged model allows firms to soften stage-game competition. As a result, more seats are saved for late-arriving, price-insensitive customers. However, firms extract less surplus close to the deadline because the algorithm bounds how quickly firms can increase prices. Thus, we conclude that the use of heuristics can lead to ambiguous revenue effects but both produce higher welfare relative to the benchmark model.[28]

Figure 9: Heuristic Counterfactuals Results over Time



Note: Panel (a) shows the average shares over time for the two heuristic models. Panel (b) shows the ratios of cumulative consumer surplus for the two models with respect to the benchmark. Panel (c) shows the ratios of cumulative revenue for the two models with respect to the benchmark.

7 Conclusion

In this paper we estimate the welfare effects of dynamic pricing for oligopolies in an important industry—airline markets featuring nonstop flight competition. We develop new theoretical insights on dynamic pricing in an oligopoly when firms are endowed with limited initial capacity and compete in prices toward a sales deadline. The stage game differs from commonly studied

[28]These findings are also robust to how we handle flow traffic. See Table 6 and Figure 19.

oligopoly pricing games such as Caplin and Nalebuff (1991) and captures dynamic incentives through scarcity effects. We establish conditions for equilibrium existence and uniqueness, and for continuity of equilibrium prices in the continuous-time limit. We show that little intuition from the single-firm case carries over to an oligopoly; for example, firm payoffs are not monotonic in own remaining inventory. We then use unique and comprehensive booking and pricing data for competing airlines to estimate a model of air travel demand. With demand estimated and the equilibrium characterization, we find that dynamic pricing expands output, but lowers total welfare compared to uniform pricing. This contrasts with recent empirical work examining the single-firm case where dynamic pricing has been found to increase welfare. Finally, we show that pricing heuristics used by airlines result in ambiguous revenue effects but higher welfare than the benchmark dynamic pricing model.

References

Aryal, Gaurab, Charles Murry, and Jonathan W Williams, "Price discrimination in international airline markets," *arXiv preprint arXiv:2102.05751*, 2021.

Asker, John, Chaim Fershtman, and Ariel Pakes, "Artificial Intelligence and Pricing: The Impact of Algorithm Design," Technical Report, National Bureau of Economic Research 2021.

Bergemann, Dirk and Juuso Välimäki, "Dynamic price competition," *Journal of Economic Theory*, 2006, *127* (1), 232–263.

Berry, Steven and Panle Jia, "Tracing the woes: An empirical analysis of the airline industry," *American Economic Journal: Microeconomics*, 2010, *2* (3), 1–43.

Board, Simon and Andrzej Skrzypacz, "Revenue management with forward-looking buyers," *Journal of Political Economy*, 2016, *124* (4), 1046–1087.

Borenstein, Severin and Nancy L Rose, "Competition and price dispersion in the US airline industry," *Journal of Political Economy*, 1994, *102* (4), 653–683.

Brown, Zach Y and Alexander MacKay, "Competition in pricing algorithms," Technical Report, National Bureau of Economic Research 2021.

Burgess, CE, "Continuous functions and connected graphs," *The American Mathematical Monthly*, 1990, *97* (4), 337–339.

Calvano, Emilio, Giacomo Calzolari, Vincenzo Denicolo, and Sergio Pastorello, "Artificial intelligence, algorithmic pricing, and collusion," *American Economic Review*, 2020, *110* (10), 3267–97.

Caplin, Andrew and Barry Nalebuff, "Aggregation and imperfect competition: On the existence of equilibrium," *Econometrica: Journal of the Econometric Society*, 1991, pp. 25–59.

Castillo, Juan Camilo, "Who Benefits from Surge Pricing?," *Available at SSRN 3245533*, 2020.

Cho, Sungjin, Gong Lee, John Rust, and Mengkai Yu, "Optimal dynamic hotel pricing," in "2018 Meeting Papers," Vol. 179 2018.

Dana, James D, "Equilibrium price dispersion under demand uncertainty: the roles of costly capacity and market structure," *The RAND Journal of Economics*, 1999, pp. 632–660.

_ , "Using yield management to shift demand when the peak time is unknown," *The Rand Journal of Economics*, 1999, pp. 456–474.

_ **and Kevin R Williams**, "Intertemporal price discrimination in sequential quantity-price games," *Marketing Science*, 2022.

de Albéniz, Victor Martínez and Kalyan Talluri, "Dynamic price competition with fixed capacities," *Management Science*, 2011, *57* (6), 1078–1093.

D'Haultfœuille, Xavier, Ao Wang, Philippe Février, and Lionel Wilner, "Estimating the Gains (and Losses) of Revenue Management," *arXiv preprint arXiv:2206.04424*, 2022.

Dilme, Francesc and Fei Li, "Revenue management without commitment: Dynamic pricing and periodic flash sales," *The Review of Economic Studies*, 2019, *86* (5), 1999–2034.

Dontchev, Asen L and R Tyrrell Rockafellar, *Implicit functions and solution mappings*, Vol. 543, Springer, 2009.

Dudey, Marc, "Dynamic edgeworth-bertrand competition," *The Quarterly Journal of Economics*, 1992, *107* (4), 1461–1477.

Fleming, Wendell H and Halil Mete Soner, *Controlled Markov processes and viscosity solutions*, Vol. 25, Springer Science & Business Media, 2006.

Gallego, Guillermo and Garrett Van Ryzin, "Optimal dynamic pricing of inventories with stochastic demand over finite horizons," *Management science*, 1994, *40* (8), 999–1020.

Gerardi, Kristopher S and Adam Hale Shapiro, "Does competition reduce price dispersion? New evidence from the airline industry," *Journal of Political Economy*, 2009, *117* (1), 1–37.

Gershkov, Alex, Benny Moldovanu, and Philipp Strack, "Revenue-maximizing mechanisms with strategic customers and unknown, markovian demand," *Management Science*, 2018, *64* (5), 2031–2046.

Hansen, Karsten T, Kanishka Misra, and Mallesh M Pai, "Frontiers: Algorithmic collusion: Supra-competitive prices via independent algorithms," *Marketing Science*, 2021, *40* (1), 1–12.

Hanson, Ward and Kipp Martin, "Optimizing multinomial logit profit functions," *Management Science*, 1996, *42* (7), 992–1003.

Hendel, Igal and Aviv Nevo, "Intertemporal price discrimination in storable goods markets," *American Economic Review*, 2013, *103* (7), 2722–51.

Horn, Roger A and Charles R Johnson, *Matrix analysis*, Cambridge university press, 2012.

Hortaçsu, Ali, Olivia R Natan, Hayden Parsley, Timothy Schwieg, and Kevin R Williams, "Incorporating Search and Sales Information in Demand Estimation," Technical Report, National Bureau of Economic Research 2021.

_ , _ , _ , _ , and _ , “Organizational structure and pricing: Evidence from a large us airline,” Technical Report, National Bureau of Economic Research 2021.

Israeli, Ayelet, Fiona Scott-Morton, Jorge Silva-Risso, and Florian Zettelmeyer, “How Market Power Affects Dynamic Pricing: Evidence from Inventory Fluctuations at Car Dealerships,” *Management Science*, 2022, *68* (2), 895–916.

Kellogg, RB, “Uniqueness in the Schauder fixed point theorem,” *Proceedings of the American Mathematical Society*, 1976, *60* (1), 207–210.

Konovalov, Alexander and Zsolt Sándor, “On price equilibrium with multi-product firms,” *Economic Theory*, 2010, *44* (2), 271–292.

Lazarev, John, “The welfare effects of intertemporal price discrimination: an empirical analysis of airline pricing in US monopoly markets,” 2013.

Maskin, Eric and Jean Tirole, “A theory of dynamic oligopoly, II: Price competition, kinked demand curves, and Edgeworth cycles,” *Econometrica: Journal of the Econometric Society*, 1988, pp. 571–599.

McAfee, R Preston and Vera Te Velde, “Dynamic pricing in the airline industry,” *Handbook on economics and information systems*, 2006, *1*, 527–67.

Milgrom, Paul and John Roberts, “Rationalizability, learning, and equilibrium in games with strategic complementarities,” *Econometrica: Journal of the Econometric Society*, 1990, pp. 1255–1277.

Nocke, Volker and Nicolas Schutz, “Multiproduct-firm oligopoly: An aggregative games approach,” *Econometrica*, 2018, *86* (2), 523–557.

Puller, Steven L, Anirban Sengupta, and Steven N Wiggins, “Does scarcity drive intra-route price dispersion in airlines?,” 2012.

Sengupta, Anirban and Steven N Wiggins, “Airline pricing, price dispersion, and ticket characteristics on and off the internet,” *American Economic Journal: Economic Policy*, 2014, *6* (1), 272–307.

Siegert, Caspar and Robert Ulbricht, “Dynamic oligopoly pricing: Evidence from the airline industry,” *International Journal of Industrial Organization*, 2020, *71*, 102639.

Solan, Eilon and Omri N Solan, “Browder’s Theorem through Brouwer’s Fixed Point Theorem,” *arXiv preprint arXiv:2107.02428*, 2021.

Stavins, Joanna, “Price discrimination in the airline market: The effect of market concentration,” *Review of Economics and Statistics*, 2001, *83* (1), 200–202.

Sweeting, Andrew, James W Roberts, and Chris Gedge, “A model of dynamic limit pricing with an application to the airline industry,” *Journal of Political Economy*, 2020, *128* (3), 1148–1193.

Talluri, Kalyan and Garrett Van Ryzin, “Revenue management under a general discrete choice model of consumer behavior,” *Management Science*, 2004, *50* (1), 15–33.

Vives, Xavier, “Strategic complementarities in oligopoly,” in “Handbook of Game Theory and Industrial Organization, Volume I,” Edward Elgar Publishing, 2018.

Williams, Kevin R., “The Welfare Effects of Dynamic Pricing: Evidence From Airline Markets,” *Econometrica*, 2022, *90* (2), 831–858.

Zhao, Wen and Yu-Sheng Zheng, “Optimal dynamic pricing for perishable assets with nonhomogeneous demand,” *Management science*, 2000, *46* (3), 375–388.

A General Model with Many Firms and Many Products

In Appendix A, we formulate the generalized results stated in Section 3 for the duopoly case. We directly prove those general statements in Appendix B.

A.1 General Model of Dynamic Price Competition

Consider a market with $F \geq 1$ firms and $J \geq F$ products, denoting the set of firms by $\mathscr{F} := \{1, \dots, F\}$ and the set of products by $\mathscr{J} := \{1, \dots, J\}$. Each firm f sells products in $\mathscr{J}_f$, where $\left(\mathscr{J}_f\right)_{f \in \mathscr{F}}$ is a partition of $\mathscr{J}$; that is, $\mathscr{J} = \bigcup_{f \in \mathscr{F}} \mathscr{J}_f$ and $\mathscr{J}_f \cap \mathscr{J}_{f'} = \emptyset$ for $f \neq f'$, i.e., each product is sold by exactly one firm. Each firm f is equipped with an initial inventory $K_{j,0} \in \mathbb{N}$ of each of its products $j \in \mathscr{J}_f$. Demand is as specified in Section 2.1, satisfying Assumptions 1 and 2.

The dynamic pricing game is the canonical generalization of the duopoly game introduced in Section 2.3. In every period t, each firm f simultaneously sets prices $p_{j,t}$ for its products $j \in \mathscr{J}_f$, and then a consumer arrives with probability $\Delta\lambda_t$. If a consumer arrives, she buys product j with probability $s_{j,t}(\mathbf{p}_t)$. Let $\mathbf{s}_f(\mathbf{p}) = \left(s_j(\mathbf{p})\right)_{j \in \mathscr{J}_f}$ be the demand of firm f.

Like for the duopoly, the payoff-relevant state is given by the vector of inventories $\mathbf{K} := (K_j)_{j \in \mathscr{J}}$ and time t. We study Markov perfect equilibria in which each firm's strategy is measurable with respect to $(\mathbf{K}, t)$. We denote a Markov pricing strategy of firm f by $\mathbf{p}_{f,t}(\mathbf{K}) = \left(p_{j,t}(\mathbf{K})\right)_{j \in \mathscr{J}_f}$.

Given equilibrium price vectors $\mathbf{p}_t^*(\mathbf{K}) := \left(p_{j,t}^*(\mathbf{K})\right)_{j \in \mathscr{J}}$, firm f's value function satisfies[29]

$$
\begin{aligned}
\Pi_{f,t}(\mathbf{K};\Delta) = \; & \Delta\lambda_t \Bigg(\underbrace{\sum_{j \in \mathscr{J}_f} s_{j,t}(\mathbf{p}_t^*(\mathbf{K}))\,(p_{j,t}^*(\mathbf{K}) + \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta))}_{\text{payoff from own sale}} + \\
& \underbrace{\sum_{j' \neq \mathscr{J} \setminus \mathscr{J}_f} s_{j',t}(\mathbf{p}_t^*(\mathbf{K}))\,\Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_{j'};\Delta)}_{\text{payoff if } j' \text{ sells}} \Bigg) + \underbrace{\left(1 - \Delta\lambda_t \sum_{j' \in \mathscr{J}} s_{j'}(\mathbf{p}_t^*(\mathbf{K}))\right)}_{\text{probability of no purchase}} \Pi_{f,t+\Delta}(\mathbf{K};\Delta),
\end{aligned}
$$

with boundary conditions (i) $\Pi_{f,T}(\mathbf{K};\Delta) \equiv 0$ for all $\mathbf{K}$, (ii) $\Pi_{f,t}(\mathbf{K};\Delta) \equiv 0$ if $K_j = 0$ for all $j \in \mathscr{J}_f$

[29]Formally, equilibrium prices are a function of Δ, which we omit here for readability.

and (iii) $\Pi_{f,t}(\mathbf{K};\Delta) = -\infty$ if $K_j < 0$ for a $j \in \mathscr{J}_f$, (iv) $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_j;\Delta) = \Pi_{f,t}(\mathbf{K};\Delta)$ if $K_j = 0$ for a $j \notin \mathscr{J}_f$, $K_j \geq 0$ for all $j \in \mathscr{J}_f$. The *scarcity effect* of product j on firm f in state $(\mathbf{K}, t)$ is then

$$\omega^f_{j,t}(\mathbf{K}) := \Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta).$$

Then, the stage game is parameterized by the matrix of scarcity effects

$$\Omega_t(\mathbf{K}) = \left(\omega^f_{j,t}(\mathbf{K})\right)_{f,j} \in \mathbb{R}^{\mathscr{F}\times\mathscr{J}},$$

where firm f's flow payoff $\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta)$ is equal to

$$\Delta\lambda_t \left(\sum_{j\in\mathscr{J}_f} s_{j,t}\left(\mathbf{p}^*_t(\mathbf{K})\right)\left(p^*_{j,t}(\mathbf{K}) - \omega^f_{j,t}(\mathbf{K})\right) - \sum_{j'\notin\mathscr{J}_f} s_{j',t}\left(\mathbf{p}^*_t(\mathbf{K})\right)\omega^f_{j',t}(\mathbf{K}) \right).$$

Hence, the equilibrium prices in period t correspond to equilibria of the stage game where each firm f simultaneously chooses prices to maximize payoffs

$$\sum_{j\in\mathscr{J}_f} s_{j,t}(\mathbf{p})\left(p_j - \omega^f_{j,t}(\mathbf{K})\right) - \sum_{j'\notin\mathscr{J}_f} s_{j',t}(\mathbf{p})\omega^f_{j',t}(\mathbf{K}).$$

This payoff function can be written as $\mathbf{s}_f(\mathbf{p})^\intercal \mathbf{p}_f - \mathbf{s}(\mathbf{p})^\intercal \boldsymbol{\omega}^f$ in matrix form, where we adopt the convention that bold vectors are column vectors.

A.2 Analysis of General Oligopoly Market

We follow closely the structure of Section 3 and state the generalized results here.

A.2.1 Equilibrium Existence, Uniqueness, and Continuity

Analogously to Equation 6, we define

$$\mathbf{g}_f(\mathbf{p}) := \underbrace{\left(\left(D_{\mathbf{p}_f}\mathbf{s}_f(\mathbf{p})\right)^\intercal\right)^{-1} D_{\mathbf{p}_f}\left(\mathbf{s}(\mathbf{p})^\intercal\boldsymbol{\omega}^f\right)^\intercal}_{\substack{\text{net opportunity costs} \\ \text{of selling}}} - \underbrace{\left(\left(D_{\mathbf{p}_f}\mathbf{s}_f(\mathbf{p})\right)^\intercal\right)^{-1}\mathbf{s}_f(\mathbf{p})}_{\substack{\text{inverse quasi} \\ \text{own-price elasticities}}} \in \mathbb{R}^{\mathscr{J}_f} \tag{9}$$

Then, we can generalize Assumption 3 and Lemma 2 as follows.

General Assumption 3. *The following two conditions hold,*

i) $\det\left(D_{\mathbf{p}_f}\mathbf{g}_f(\mathbf{p})-I_{\mathscr{J}_f}\right)\neq 0$ *for all* $\mathbf{p}$ *and* f*;*

ii) $\det\left(D_{\mathbf{p}}\big(\mathbf{g}(\mathbf{p})\big)-I_{\mathscr{J}}\right)\neq 0$ *for all* $\mathbf{p}$*, where* $\mathbf{g}(\mathbf{p}):=\left(\mathbf{g}_f(\mathbf{p}):f\in\mathscr{F}\right)\in\mathbb{R}^{\mathscr{J}}$.

General Lemma 2. *Let Assumptions 1, 2, and General Assumption 3 hold. Then, the stage game admits a unique equilibrium. The equilibrium price vector is finite for all available products.*

A.2.2 Continuity of Equilibrium Prices in Scarcity Effect Matrix Ω

General Lemma 3. *Let Assumptions 1, 2, and General Assumption 3 hold for a compact, path-connected set* $\mathscr{O}$ *of* $(\Omega,\boldsymbol{\theta})$*. Then the unique equilibrium price vector* $\mathbf{p}^*(\Omega,\boldsymbol{\theta})$ *is continuous in* $(\Omega,\boldsymbol{\theta})$ *on* $\mathscr{O}$*.*

A.2.3 Characterization of Continuous-time Limit

General Proposition 2 (Continuous-time Limit)**.** *Let Assumptions 1, 2, and General Assumption 3 hold for a compact, path-connected set* $\mathscr{O}$ *containing* $(\Omega,\boldsymbol{\theta})=(\mathbf{0},\boldsymbol{\theta}_T)$*. For every* $\mathbf{K}$*, there exists a* $T_0(\mathbf{K})>0$*, non-increasing in* $\mathbf{K}$*, so that for any* $T\leq T_0(\mathbf{K})$ *there exists a unique equilibrium of the dynamic pricing game for sufficiently small* Δ*. The corresponding value function* $\Pi_{f,t}(\mathbf{K};\Delta)$ *converges to a limit* $\Pi_{f,t}(\mathbf{K})$ *as* $\Delta\to 0$ *that solves the differential equation*

$$\begin{aligned}\dot{\Pi}_{f,t}(\mathbf{K})=-\lambda_t\quad&\Bigg(\sum_{j\in\mathscr{J}_f}s_j(\mathbf{p}^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}_t))\left(p_j^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}_t)-(\Pi_{f,t}(\mathbf{K})-\Pi_{f,t}(\mathbf{K}-\mathbf{e}_j))\right)\\&-\sum_{j'\notin\mathscr{J}_f}s_{j'}(\mathbf{p}^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}_t))\left(\Pi_{f,t}(\mathbf{K})-\Pi_{f,t}(\mathbf{K}-\mathbf{e}_{j'})\right)\Bigg)\end{aligned}$$

where $f'\neq f$*, with boundary conditions (i)* $\Pi_{f,T}(\mathbf{K})=0$ *for all* $\mathbf{K}$*, (ii)* $\Pi_{f,t}(\mathbf{K})=0$ *if* $K_j=0$ *for all* $j\in\mathscr{J}_f$*, (iii)* $\Pi_{f,t}(\mathbf{K})=-\infty$ *if* $K_j<0$ *for a* $j\in\mathscr{J}_f$*, and (iv)* $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_{j'})=\Pi_{f,t}(\mathbf{K})$ *if* $K_{j'}=0$ *for a* $j'\notin\mathscr{J}_f$*,* $K_j\geq 0$ *for all* $j\in\mathscr{J}_f$*.*

A.2.4 Economic Forces of Dynamic Price Competition

General Assumption 4 (Independence of Irrelevant Alternatives (IIA))**.** $\frac{\partial}{\partial p_j}\frac{s_{j_1}(\mathbf{p})}{s_{j_2}(\mathbf{p})} = 0$ *for* $j \neq j_1, j_2 \in \mathscr{J} \cup \{0\}$.

Given Assumptions 1, 2 and General Assumption 4, we can show that the game with multi-product firms can be transformed into a game of single-product firms.

General Proposition 3 (Mark-up formula under IIA)**.** *Let Assumptions 1, 2 and General Assumption 4 hold and* $-\frac{\partial}{\partial p_j}\frac{s_j(\mathbf{p})}{\frac{\partial s_j}{\partial p_j}} \neq 1$ *for all* $\mathbf{p}$. *Then, there exists an equilibrium of the stage game for any scarcity matrix* Ω. *All equilibrium prices* $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ *coincide with the equilibrium prices of a game with a set* $\mathscr{J}$ *of players who each simultaneously choose a price* p_j *maximizing*

$$s_j(\mathbf{p})\big(p_j - c_j\left(\mathbf{p}_{-j};\Omega,\boldsymbol{\theta}\right)\big)$$

with a cost function

$$c_j\left(\mathbf{p}_{-j};\Omega,\boldsymbol{\theta}\right) := \omega_j^f - \sum_{j'\in\mathscr{J}_f\setminus\{j\}} \tilde{s}_{j,j'}(\mathbf{p}_{-j})(p_{j'} - \omega_j^f) + \sum_{j'\notin\mathscr{J}_f} \tilde{s}_{j,j'}(\mathbf{p}_{-j})\omega_{j'}^f \tag{10}$$

and $\tilde{s}_{j,j'}(\mathbf{p}_{-j}) := \frac{s_{j'}(\mathbf{p})}{1-s_j(\mathbf{p})}$.

Proposition 3 implies that even with multiple firms and products, the first-order conditions (FOCs) that implicitly define the best response functions of the firms, can be written in a markup formulation for each product, with $\epsilon_j(\mathbf{p}) = \frac{\partial s_j(\mathbf{p})}{\partial p_j}\frac{p_j}{s_j(\mathbf{p})}$ being the elasticity of demand, as

$$\frac{p_j^*(\Omega,\boldsymbol{\theta}) - c_j(\mathbf{p}_{-j};\Omega,\boldsymbol{\theta})}{p_j^*(\Omega,\boldsymbol{\theta})} = -\frac{1}{\epsilon_j(\mathbf{p}^*(\Omega,\boldsymbol{\theta}))}. \tag{11}$$

General Proposition 4. *Let* $\lambda_t \equiv \lambda$, $\boldsymbol{\theta}_t \equiv \boldsymbol{\theta}$. *Then, for* $\mathbf{K}$ *with* $\underline{K} := \min_j K_j$, *the following holds:*

$$p_{j,t}(\mathbf{K}) = p_{j,T}^* + \mathscr{O}(|T-t|^{\underline{K}}),\ t \to T \text{ for all } j,$$

i.e., price changes close to the deadline are at most of order $\underline{K}$. *If* $\lim_{t\to T}(\Pi_{f,t})^{(\underline{K})}(\mathbf{K} - \mathbf{e}_{j'}) \neq 0$ *for* *all f and j' with $K_{j'} = \underline{K}$, then*

$$p_{j,t}(\mathbf{K}) = p^*_{j,T} + \Theta(|T-t|^{\underline{K}}),\ t \to T \text{ for all } j,$$

i.e., price changes are exactly of order $\underline{K}$.

B Proofs

B.1 Technical results

B.1.1 Continuous time limit

We use the following result for the proofs of Lemma 1 and General Proposition 2.

Lemma 4. *Consider a continuous price function $(\Omega, \boldsymbol{\theta}) \mapsto \mathbf{p}^*(\Omega, \boldsymbol{\theta}) = (p^*_j(\Omega, \boldsymbol{\theta}))_j$ on a compact set $\mathcal{O}$, and a bounded and continuous function $\mathbf{A} : \mathbb{R}^{\mathcal{J}} \times \mathbb{R}^{\mathcal{F}\times\mathcal{J}} \times \mathcal{T} \to \mathbb{R}^{\mathcal{F}}$. Let $\Pi_{f,t}(\mathbf{K};\Delta)$, $f \in \mathcal{F}$, be a solution to the difference equations*

$$\left(\frac{\Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t}(\mathbf{K};\Delta)}{\Delta}\right)_f = -\lambda_t \mathbf{A}\Big(\mathbf{p}^*\big(\Omega(\mathbf{K};\Delta)), \boldsymbol{\theta}_t\big),\ \Omega(\mathbf{K};\Delta),\ \boldsymbol{\theta}_t\Big)$$

where $\Omega(\mathbf{K};\Delta) = (\omega^f_{j,t}(\mathbf{K};\Delta))_{f,j}$, $\omega^f_{j,t}(\mathbf{K};\Delta) := \Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta)$, with boundary conditions (i) $\Pi_{f,T}(\mathbf{K};\Delta) = 0$, (ii) $\Pi_{f,t}(\mathbf{K};\Delta) = 0$ if $K_j = 0$ for all $j \in \mathcal{J}_f$, (iii) $\Pi_{f,t}(\mathbf{K};\Delta) = -\infty$ if $K_j < 0$ for a $j \in \mathcal{J}_f$, and (iv) $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_j;\Delta) = \Pi_{f,t}(\mathbf{K};\Delta)$ if $K_j = 0$ for a $j \notin \mathcal{J}_f$, $K_{j'} \geq 0$ for all $j' \in \mathcal{J}_f$. Then, $(\Pi_{f,t}(\mathbf{K};\Delta))_f$ converges and any limit $(\Pi_{f,t}(\mathbf{K}))_f$ satisfies

$$\big(\dot{\Pi}_{f,t}(\mathbf{K})\big)_f = -\lambda_t \mathbf{A}\Big(\mathbf{p}^*\big(\Omega(\mathbf{K}), \boldsymbol{\theta}_t\big),\ \Omega(\mathbf{K}),\ \boldsymbol{\theta}_t\Big),$$

where $\Omega(\mathbf{K}) = (\omega^f_{j,t}(\mathbf{K}))_{f,j}$, $\omega^f_{j,t}(\mathbf{K}) := \Pi_{f,t}(\mathbf{K}) - \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_j)$, with boundary conditions (i) $\Pi_{f,T}(\mathbf{K}) = 0$, (ii) $\Pi_{f,t}(\mathbf{K}) = 0$ if $K_j = 0$ for all $j \in \mathcal{J}_f$, (iii) $\Pi_{f,t}(\mathbf{K};) = -\infty$ if $K_j < 0$ for a $j \in \mathcal{J}_f$, and (iv) $\Pi_{f,t}(\mathbf{K}-\mathbf{e}_{j'}) = \Pi_{f,t}(\mathbf{K})$ if $K_{j'} = 0$ for a $j' \notin \mathcal{J}_f$, $K_j \geq 0$ for all $j \in \mathcal{J}_f$.

Proof. Since $\mathbf{A}$ is bounded, the difference equations show that $(\Pi_f(\mathbf{K};\Delta))_{f\in\mathcal{F},\mathbf{K}\leq\mathbf{K}_0}$ is equicontinuous and equibounded in t as $\Delta \to 0$. Hence, by the Arzela-Ascoli Theorem, there exist limit points $(\Pi_f(\mathbf{K}))_{f\in\mathscr{F},\mathbf{K}\le\mathbf{K}_0}$. We claim that

$$\left(\Pi_{f,t}(\mathbf{K})\right)_f = \int_t^T \lambda_u \mathbf{A}\Big(\mathbf{p}^*\big(\Omega_u(\mathbf{K}),\boldsymbol{\theta}_u\big),\, \Omega_u(\mathbf{K}),\, \boldsymbol{\theta}_u\Big) du. \tag{12}$$

To this end, we note that if we let $\lceil u\rceil_\Delta$ to be the smallest number that is divisible by Δ and larger or equal than u

$$\left(\Pi_{f,t}(\mathbf{K};\Delta)\right)_f = \int_t^T \lambda_{\lceil u\rceil_\Delta} \mathbf{A}\Big(\mathbf{p}^*\big(\Omega_{\lceil u\rceil_\Delta}(\mathbf{K};\Delta),\boldsymbol{\theta}_{\lceil u\rceil_\Delta}\big),\, \Omega_{\lceil u\rceil_\Delta}(\mathbf{K};\Delta),\, \boldsymbol{\theta}_{\lceil u\rceil_\Delta}\Big) du. \tag{13}$$

We take the limit $\Delta \to 0$ on both sides. The left-hand side of (13) converges to the left-hand side of (12). On the right-hand side, $\Omega_{\lceil u\rceil_\Delta}(\mathbf{K};\Delta)$ converges to $\Omega_u(\mathbf{K})$. Hence, by continuity of $\mathbf{p}^*$ and $\mathbf{A}$ the integrand in (13) converges to the integrand in (12). By the dominated convergence theorem the right-hand side of (13) converges to the right-hand side of (12). Thus, any limiting value function exists and must satisfy (12). ■

B.1.2 Continuity of stage game prices

Lemma 5. *Let $\mathscr{P} \subset \mathbb{R}^{\mathscr{J}}$ be compact and convex and $\mathscr{O}$ a path-connected set of $(\Omega,\boldsymbol{\theta})$. Further, let $g : \mathscr{P}\times\mathscr{O}\to\mathscr{P}, (\mathbf{q};\Omega,\boldsymbol{\theta})\mapsto\mathbf{p}$ be (i) continuously differentiable in $\mathbf{q}$, (ii) continuous in Ω and $\boldsymbol{\theta}$, (iii) such that it implicitly defines a unique $\mathbf{p}^*(\Omega,\boldsymbol{\theta})$ satisfying $g(\mathbf{p}^*(\Omega,\boldsymbol{\theta});\Omega,\boldsymbol{\theta}) = \mathbf{p}^*(\Omega,\boldsymbol{\theta})$ for all $(\Omega,\boldsymbol{\theta})\in\mathscr{O}$, (iv) where $\mathbf{p}^*(\Omega,\boldsymbol{\theta})$ is uniformly bounded on $\mathscr{O}$. Then, $\mathbf{p}^*(\Omega,\boldsymbol{\theta})$ depends continuously on Ω and $\boldsymbol{\theta}$.*

Proof. To show continuity, we consider a sequence $(\Omega_n,\boldsymbol{\theta}_n)_{n\ge1}$ converging to some $(\Omega_\infty,\boldsymbol{\theta}_\infty)$. Thanks to path-connectedness of $\mathscr{O}$ there exists a continuous path $\mathbf{r}:[0,1]\to\mathscr{O}$ and a sequence $a_n\uparrow 1$ such that $\mathbf{r}(a_n) = (\Omega_n,\boldsymbol{\theta}_n)$ and $\mathbf{r}(1) = (\Omega_\infty,\boldsymbol{\theta}_\infty)$. By Browder's Theorem (Theorem 1.1 in Solan and Solan (2021)), the set $\mathscr{G} := \{(\mathbf{p}^*(\mathbf{r}(a));a) : a\in[0,1]\}\subset\mathscr{P}\times[0,1]$ is connected. Further, note that $\mathscr{G}$ is the pre-image of $\{\mathbf{0}\}\times[0,1]$ under the continuous function $(\mathbf{q},a)\mapsto(g(\mathbf{q},r(a))-\mathbf{q},a)$, so it is closed. Since $\mathbf{p}^*(\mathbf{r}(a))$ is uniformly bounded for all a, $\mathscr{G}$ is compact. By the main theorem of connectedness, each set $\mathscr{G}_j := \{(p_j^*(\mathbf{r}(a));a) : a\in[0,1]\}\subset\mathbb{R}\times[0,1]$

is connected, for all j because $\mathbf{p}^*(\Omega,\boldsymbol{\theta})$ is the unique solution to $g(\mathbf{p}^*(\Omega,\boldsymbol{\theta});\Omega,\boldsymbol{\theta}) = \mathbf{p}^*(\Omega,\boldsymbol{\theta})$. Furthermore, as projections of a compact set, $\mathscr{G}_j$ are compact for all j. Then, by Burgess (1990), the function $a \mapsto p_j^*(\mathbf{r}(a))$ is continuous, so $p_j^*(\Omega_n,\boldsymbol{\theta}_n) = p_j^*(\mathbf{r}(a_n)) \to p_j^*(\mathbf{r}(1)) = p_j^*(\Omega_\infty,\boldsymbol{\theta}_\infty)$. Hence $\mathbf{p}^*(\Omega,\boldsymbol{\theta})$ is continuous in $\Omega,\boldsymbol{\theta}$ on $\mathscr{O}$. ■

B.2 Proofs of Single Firm Model

B.2.1 Proof of Lemma 1

In the following we omit the conditioning argumet $\mathscr{A}$.

Step 1: All profit-maximizing prices $\mathbf{p}^M$ are interior. First, we show that given $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$,

$$\mathbf{p}^M \in \arg\max_{\mathbf{q}} \sum_{j\in\mathscr{J}} s_j(\mathbf{q};\boldsymbol{\theta})(q_j - \omega_j)$$

is bounded from below by a vector $\underline{\mathbf{p}} = (\underline{p}+\omega_1,\dots,\underline{p}+\omega_J)$, $\underline{p}\in\mathbb{R}$. We proceed with a proof by contradiction. Suppose such a $\underline{\mathbf{p}}$ did not exist. Then, for any $\underline{p}\in\mathbb{R}$ there exists an optimal price vector $\mathbf{p}^M$ and a j such that $p_j^M - \omega_j = \min_{j'}(p_{j'}^M - \omega_{j'}) < \underline{p}$. At this optimal price $\mathbf{p}^M$ (which could include (minus) infinite prices), the derivative of the stage game profit with respect to any price dimension has to be smaller than or equal to zero by optimality. The derivative with respect to p_j at $\mathbf{p}^M$ (or as we converge to $\mathbf{p}^M$ if it includes (minus) infinite prices) is

$$\begin{aligned}
&\lim_{\mathbf{p}\to\mathbf{p}^M} \sum_{k\neq j} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_k-\omega_k) + s_j(\mathbf{p};\boldsymbol{\theta}) + \frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_j-\omega_j) && \geq \\
&\lim_{\mathbf{p}\to\mathbf{p}^M} -(p_j-\omega_j)\underbrace{\left(\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right| - \sum_{k\neq j}\frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right)}_{=\frac{\partial s_0}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})>0 \text{ by Assumption 1-iii)}} + s_j(\mathbf{p};\boldsymbol{\theta}) && \geq \\
&\lim_{\mathbf{p}\to\mathbf{p}^M} -\underline{p}\frac{\partial s_0}{\partial p_j}(\mathbf{p};\boldsymbol{\theta}) + s_j(\mathbf{p};\boldsymbol{\theta}) \xrightarrow[\underline{p}\to-\infty]{} \infty && \text{by Assumption 1-iii).}
\end{aligned}$$

Thus, for sufficiently small $\underline{p}$, this yields a contradiction, i.e. any optimal price vector $\mathbf{p}^M$ is bounded by a vector $\underline{\mathbf{p}}$ from below.

Next, we show that given $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$, any profit maximizing price vector $\mathbf{p}^M$ is bounded by a vector $\bar{\mathbf{p}} = (\bar{p}+\omega_1,\dots,\bar{p}+\omega_J)$, $\bar{p} \in \mathbb{R}$. We again proceed with a proof by contradiction. Suppose such a $\bar{\mathbf{p}}$ did not exist. Then, for any $\bar{p} \in \mathbb{R}$, there exists an optimal price vector $\mathbf{p}^M$ and a j such that $p_j^M - \omega_j = \max_{j'}\left(p_{j'}^M - \omega_{j'}\right) > \bar{p}$. At the optimal price $\mathbf{p}^M$ (which could include (minus) infinite prices), the derivative of the stage game profit with respect to any price dimension has to be greater than or equal to zero by optimality. There exists a constant $C > 0$ satisfying Assumption 1-iii) as we have established a lower bound $\underline{p}$ for $\mathbf{p}^M$. The derivative with respect to p_j at $\mathbf{p}^M$ (or as we converge to $\mathbf{p}^M$ if it includes (minus) infinite prices) is

$$\begin{aligned}
&\lim_{\mathbf{p}\to\mathbf{p}^M} \underbrace{\sum_{k\neq j} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_k-\omega_k)}_{\geq 0} + s_j(\mathbf{p};\boldsymbol{\theta}) + \frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_j-\omega_j) && \leq \\
&\lim_{\mathbf{p}\to\mathbf{p}^M} \sum_{k\neq j} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_j-\omega_j) + C^{-1}\frac{\partial s_0}{\partial p_j}(\mathbf{p};\boldsymbol{\theta}) + \frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_j-\omega_j) && = \\
&\lim_{\mathbf{p}\to\mathbf{p}^M} \underbrace{\frac{\partial s_0}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})}_{>0}(C^{-1}-(p_j-\omega_j)) \leq \lim_{\mathbf{p}\to\mathbf{p}^M} \frac{\partial s_0}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(C^{-1}-\overline{p}) \xrightarrow[\overline{p}\to\infty]{} -\infty && .
\end{aligned}$$

by Assumption 1-iii). Thus, for sufficiently large $\bar{p}$, this yields a contradiction. Hence, any optimal price vector $\mathbf{p}^M$ is bounded by a vector $\bar{\mathbf{p}}$ from above.

Step 2: Uniqueness of profit-maximizing price $\mathbf{p}^M$. It follows from Step 1 that any profit-maximizing price $\mathbf{p}^M$ of the stage game must satisfy the FOCs of the firm. Assumption 1 ensures that the Jacobian matrix $D_{\mathbf{p}}s(\mathbf{p};\boldsymbol{\theta})$ is non-singular by the Levy-Desplanques Theorem (see e.g. Theorem 6.1.10. in Horn and Johnson (2012)). Hence, the FOCs can be written as Equation 4. Because of Assumption 2 there is a unique solution to this system of equations by Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010).

Step 3: Convergence. We can apply the Implicit Function Theorem to Equation 4 by Assumption 2 and it follows that the unique optimal price $\mathbf{p}^M(\Omega,\boldsymbol{\theta})$ is continuous in Ω and $\boldsymbol{\theta}$. Convergence to Equation 3 follows by Lemma 4.

B.2.2 Proof of Proposition 1

Proof. i) To see that $\Pi_{M,t}(\mathbf{K})$ is decreasing in t, note that in Equation 3, setting $p_j > (\Pi_{M,t}(\mathbf{K}-\Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)))$ results in a positive stage-game payoff, so $\dot{\Pi}_{M,t}(\mathbf{K}) < 0$.

Next, we show that $\Pi_{M,t}(\mathbf{K}) > \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)$ for all j by induction in $\sum_j K_j$.

Induction start: It is immediate that $\Pi_{M,t}(\mathbf{e}_j) \geq \Pi_{M,t}(\mathbf{0}) = 0$ for all j and $t \leq T$.

Induction hypothesis: Assume that $\Pi_{M,t}(\mathbf{K}) > \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)$ for all $\mathbf{K}$ with $\sum_j K_j = \bar{K}$ and $j \in \mathscr{J}$.

Induction step: Now, consider a capacity vector $\mathbf{K}$ with $\sum_j K_j = \bar{K}+1$. The solution of the differential equation for the profits is

$$\Pi_{M,t}(\mathbf{K}) = \int_t^T \lambda_z \sum_j s_j(\mathbf{p}_z^M(\mathbf{K}))\,(p_{j,z}^M(\mathbf{K}) + \Pi_{M,z}(\mathbf{K}-\mathbf{e}_j)) \cdot e^{-\int_t^z \lambda_u \sum_{j'} s_{j'}(\mathbf{p}_u^M(\mathbf{K}))du}\, dz.$$

By sub-optimality of the prices $\mathbf{p}_t^M(\mathbf{K}-\mathbf{e}_k)$ given capacity vector $\mathbf{K}$, we have for all k

$$\begin{aligned}
&\Pi_{M,t}\big(\mathbf{K}\big) \geq \\
&\int_t^T \lambda_z \sum_j s_j\big(\mathbf{p}_z^M(\mathbf{K}-\mathbf{e}_k)\big)\big(p_{j,z}^M(\mathbf{K}-\mathbf{e}_k) + \underbrace{\Pi_{M,z}(\mathbf{K}-\mathbf{e}_j)}_{\substack{>\Pi_{M,z}\left(\mathbf{K}-\mathbf{e}_k-\mathbf{e}_j\right) \\ \text{by induction hypothesis}}}\big) \cdot e^{-\int_t^z \lambda_u \sum_{j'} s_{j'}(\mathbf{p}_u^M(\mathbf{K}-\mathbf{e}_k))du}\, dz \\
&> \Pi_{M,t}\big(\mathbf{K}-\mathbf{e}_k\big).
\end{aligned}$$

ii) Next, we show that $\Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j) \leq \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j) - \Pi_{M,t}(\mathbf{K}-2\mathbf{e}_j)$ for all j. To this end, let

$$H(\mathbf{x};\boldsymbol{\theta}) = -\max_{\mathbf{p}} \sum_j s_j(\mathbf{p};\boldsymbol{\theta})(p_j - x_j).$$

Note that H is concave as a minimum of affine functions, strictly increasing in $\mathbf{x}$. Since H is concave and continuous, by the Fenchel-Moreau Theorem, it admits the representation

$$H(\mathbf{x};\boldsymbol{\theta}) = \inf_{\mathbf{s}} (\mathbf{s}\cdot\mathbf{x} - H^*(\mathbf{s};\boldsymbol{\theta}))$$

where $H^*(\mathbf{s};\boldsymbol{\theta}) = \inf_{\mathbf{x}}(\mathbf{x}\cdot\mathbf{s} - H(\mathbf{x};\boldsymbol{\theta}))$ is the concave conjugate of H. Moreover,

$$\dot{\Pi}_{M,t}(\mathbf{K}) = \lambda_t H(\nabla\Pi_t(\mathbf{K});\boldsymbol{\theta}_t)$$

where $\nabla\Pi_{M,t}(\mathbf{K}) = \left(\Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)\right)_j$. Thus, $\Pi_{M,t}(\mathbf{K})$ is the value function for the optimal control problem

$$\Pi_{M,t}(\mathbf{K}) = \sup_{\mathbf{s}\in\mathscr{A}} \mathbb{E}\left[\int_t^T \lambda_u H^*(\mathbf{s}_u;\boldsymbol{\theta}_u)du \,\middle|\, \mathbf{X}_t^{\mathbf{s}} = \mathbf{K}\right] =: \sup_{\mathbf{s}} J_t(\mathbf{K},\mathbf{s})$$

where $\mathbf{X}_t^{\mathbf{a}}$ is the process which jumps by $-\mathbf{e}_j$ at rate $\lambda_t s_{j,t}$ and $\mathbf{s}\in\mathscr{A}$ are processes adapted with respect to the filtration on the probability space supporting $\mathbf{X}^{\mathbf{s}}$, with the property $s_{j,t} = 0$ if $X_{j,t}^{\mathbf{s}} = 0$ (Theorem 8.1 in Fleming and Soner (2006)). Let $\mathbf{s}_{\mathbf{K}}^*$ be the optimal control in the previous equation and $\mathbf{s}_{K-2}^*$ be the optimal control when $\mathbf{K}$ is replaced by $\mathbf{K}-2\mathbf{e}_j$. Then, note that since $\mathbf{s}_{\mathbf{K}}^*, \mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^* \in \mathscr{A}$, $\frac{\mathbf{s}_{\mathbf{K}}^*+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^*}{2} \in \mathscr{A}$ because the process $\left(\mathbf{X}_s^{\frac{\mathbf{s}_{\mathbf{K}}^*+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_f}^*}{2}}\right)_s$ can be chosen as $\left(\frac{\mathbf{X}_s^{\mathbf{s}_{\mathbf{K}}^*}+\mathbf{X}_s^{\mathbf{s}_{\mathbf{K}-2\mathbf{e}_f}^*}}{2}\right)_s$ ("coupling argument"). Hence,

$$\begin{aligned}
&\Pi_{M,t}(\mathbf{K}) + \Pi_{M,t}(\mathbf{K}-2\mathbf{e}_j) - 2\Pi_{M,t}\left(\mathbf{K}-\mathbf{e}_j\right) && \leq \\
&J_t(\mathbf{K}, s_{\mathbf{K}}^*) + J_t(\mathbf{K}-2\mathbf{e}_j, s_{\mathbf{K}-2\mathbf{e}_j}^*) - 2\, J_t\left(\mathbf{K}-\mathbf{e}_j, \frac{\mathbf{s}_{\mathbf{K}}^* + \mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^*}{2}\right) && \leq \\
&\mathbb{E}\left[\int_t^T \lambda_u\left(H^*(\mathbf{s}_{\mathbf{K},u}^*) + H^*(\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j,u}^*) - 2H^*\left(\frac{\mathbf{s}_{\mathbf{K},u}^* + \mathbf{s}_{\mathbf{K}-2\mathbf{e}_f,u}^*}{2}\right)\right)du \,\middle|\, \mathbf{X}_t^{\mathbf{s}_{\mathbf{K}}^*} = \mathbf{K}, \mathbf{X}_t^{\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^*} = \mathbf{K}-2\mathbf{e}_j,\right] && \leq 0.
\end{aligned}$$

iii) To show that $\omega_{j,t\wedge\tau}^M(\mathbf{K}_t)$ is a submartingale, we show that for any capacity vector $\bar{\mathbf{K}}$ with $\bar{K}_j \geq 2$:

$$\lim_{\Delta\to 0} \frac{\mathbb{E}_0\left[\omega_{j,t+\Delta}^M(\mathbf{K}_{t+\Delta}) - \omega_{j,t}^M(\mathbf{K}_t)\middle|\mathbf{K}_t = \bar{\mathbf{K}}\right]}{\Delta} \geq 0.$$

To this end, first, note that $\mathbf{K}_t$ is right-continuous in t. Consider $\bar{\mathbf{K}}$ with $\bar{K}_j \geq 2$. Then, we have that

$$\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}=$$
$$\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_{t+\Delta})|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}+\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}=$$
$$\dot{\omega}^M_{j,t}(\bar{\mathbf{K}})+\lambda_t\sum_{j'}s_{j',t}(\mathbf{p}^M_t(\bar{\mathbf{K}}))\left(\omega^M_{j,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})-\omega^M_{j,t}(\bar{\mathbf{K}})\right)$$

by right-continuity of the process $\mathbf{K}_t$. By (3), we can write

$$\dot{\omega}^M_{j,t}(\bar{\mathbf{K}}) = -\lambda_t\Big[\sum_{j'}s_{j',t}(\mathbf{p}^M_t(\bar{\mathbf{K}}))\left(p^M_{j',t}(\bar{\mathbf{K}})-\omega^M_{j',t}(\bar{\mathbf{K}})\right)-s_{j',t}(p^M_t(\bar{\mathbf{K}}-\mathbf{e}_j))\left(p^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\right)\Big].$$

and we know that

$$\begin{aligned}-\omega^M_{j',t}(\bar{\mathbf{K}})+\omega^M_{j,t}(\bar{\mathbf{K}})-\omega^M_{j,t}(\bar{\mathbf{K}}-\mathbf{e}_{j'}) &= \Pi^M(\bar{\mathbf{K}}-\mathbf{e}_{j'})-\Pi^M(\bar{\mathbf{K}}-\mathbf{e}_j)-\Pi^M(\bar{\mathbf{K}}-\mathbf{e}_{j'})+\Pi^M(\bar{\mathbf{K}}-\mathbf{e}_{j'}-\mathbf{e}_j)\\ &= \omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\end{aligned}$$

Hence, $\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}$ is equal to

$$-\lambda_t\Big[\sum_{j'}s_{j',t}\left(\mathbf{p}^M_t(\bar{\mathbf{K}})\right)\left(p^M_{j',t}(\bar{\mathbf{K}})-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\right)-s_{j',t}\left(\mathbf{p}^M_t(\bar{\mathbf{K}}-\mathbf{e}_j)\right)\left(p^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\right)\Big]$$

Then, note that by optimality of $\mathbf{p}^M_t(\bar{\mathbf{K}}-\mathbf{e}_j)$,

$$\sum_{j'}s_{j',t}\left(\mathbf{p}^M_t(\bar{\mathbf{K}})\right)\left(p^M_{j',t}(\bar{\mathbf{K}})-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j)\right)\le\sum_{j'}s_{j',t}\left(\mathbf{p}^M_t(\bar{\mathbf{K}}-\mathbf{e}_j)\right)\left(p^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_j))-\omega^M_{j',t}(\bar{\mathbf{K}}-\mathbf{e}_{j'})\right).$$

Hence, $\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}\ge 0$. ■

B.3 Proofs of General Oligopoly Model

B.3.1 Proof of General Lemma 2

Step 1: All equilibrium prices $\mathbf{p}^*$ are interior. First, we show that for fixed Ω and $\boldsymbol{\theta}$, any equilibrium price vector $\mathbf{p}^*$ is bounded from below by a vector $\bar{\mathbf{p}} = ((\bar{p}+\omega_j^f)_{j\in\mathscr{J}_f} : f \in \mathscr{F})$, $\bar{p} \in \mathbb{R}$. We proceed with a proof by contradiction. Suppose such a $\underline{\mathbf{p}}$ did not exist. Then, for any $\underline{p}$ there exists an equilibrium price vector $\mathbf{p}^*$ and a j such that $p_j^* - \omega_j^f = \min_{f'} \min_{k\in\mathscr{J}_{f'}} p_k^* - \omega_k^{f'} < \underline{p}$. Additionally, let $k^* = \operatorname{argmax}_{k\notin\mathscr{J}_f} \omega_k^f$. At this equilibrium price vector $\mathbf{p}^*$ (which could include (minus) infinite prices), the derivative of firm f's stage game profit with respect to all firm f's prices has to be smaller or equal to zero by optimality. The derivative with respect to p_j at $\mathbf{p}^*$ (or as we converge to $\mathbf{p}^*$ if it includes (minus) infinite prices) is

$$\lim_{\mathbf{p}\to\mathbf{p}^*} \frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_j-\omega_j^f) + \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_k-\omega_k^f) - \sum_{k\notin\mathscr{J}_f} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\omega_k^f + s_j(\mathbf{p};\boldsymbol{\theta}) \geq$$

$$\lim_{\mathbf{p}\to\mathbf{p}^*} -\left(\underbrace{\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right| - \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})}_{\geq 0 \text{ by Assumption 1-iii)}}\right)\left(p_j - \omega_j^f + \underbrace{\frac{\sum_{k\notin\mathscr{J}_f} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})}{\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right| - \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})}}_{\in(0,1) \text{ by Assumption 1-iii)}} |\omega_{k^*}^f|\right)$$

$$+ s_j(\mathbf{p};\boldsymbol{\theta}) \geq$$

$$\lim_{\mathbf{p}\to\mathbf{p}^*} -\left(\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right| - \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right)\left(\underline{p} + |\omega_{k^*}^f|\right) + s_j(\mathbf{p};\boldsymbol{\theta}) \xrightarrow[\underline{p}\to-\infty]{} \infty$$

by Assumption 1-iii). Thus, for sufficiently small $\underline{p}$, this yields a contradiction, i.e. any equilibrium price vector $\mathbf{p}^*$ is bounded by a vector $\underline{\mathbf{p}}$ from below.

Next, we show that for fixed Ω and $\boldsymbol{\theta}$, any equilibrium price vector $\mathbf{p}^*$ is bounded from above by a vector $\bar{\mathbf{p}} = ((\bar{p}+\omega_j^f)_{j\in\mathscr{J}_f} : f \in \mathscr{F})$, $\bar{p} \in \mathbb{R}$, by contradiction. Suppose such a $\bar{\mathbf{p}}$ did not exist. Then, for any $\bar{p}$, there exists an equilibrium price vector $\mathbf{p}^*$ and a j such that $p_j^* - \omega_j^f = \max_{f'} \max_{k\in\mathscr{J}_{f'}} p_k^* - \omega_k^{f'} > \bar{p},\ j \in \mathscr{J}_f$. At the equilibrium price $\mathbf{p}^*$ (which could include (minus) infinite prices), the derivative of firm f's stage game profit with respect to all firm f's prices has to be greater or equal to zero by optimality. There exists a constant $C > 0$ satisfying Assumption 1-iii) as we have established a lower bound $\underline{p}$ for $\mathbf{p}^*$. Additionally, let $k^* = \text{argmax}_{k \notin \mathscr{J}_f} |C^{-1} + \omega_k^f|$. The derivative of firm f's payoff with respect to p_j at $\mathbf{p}^*$ (or as we converge to $\mathbf{p}^M$ if it includes (minus) infinite prices) is

$$\begin{aligned}
&\lim_{\mathbf{p}\to\mathbf{p}^*} \frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_j - \omega_j^f) + \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})(p_k - \omega_k^f) - \sum_{k\notin\mathscr{J}_f} \frac{\partial s_k}{\partial p_j}(\mathbf{p})\omega_k^f + s_j(\mathbf{p}) && \leq \\
&\lim_{\mathbf{p}\to\mathbf{p}^*} \left(\frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta}) + \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta}) \right)(p_j - \omega_j^f) + C^{-1}\left(\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right| - \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta}) \right) \\
&+ \sum_{k\notin\mathscr{J}_f} \frac{\partial s_k}{\partial p_j}(\mathbf{p})\left| -C^{-1} - \omega_k^f \right| && \leq \\
&\lim_{\mathbf{p}\to\mathbf{p}} \left(\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right| - \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta}) \right) \\
&\left(C^{-1} - \bar{p} + \underbrace{\frac{\sum_{k\notin\mathscr{J}_f} \frac{\partial s_k}{\partial p_j}(\mathbf{p})}{\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})\right| - \sum_{k\in\mathscr{J}_f\setminus\{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p};\boldsymbol{\theta})}}_{\in(0,1)} \left| C^{-1} + \omega_{k^*}^f \right| \right) \xrightarrow[\bar{p}\to\infty]{} -\infty.
\end{aligned}$$

Thus, for sufficiently large $\bar{p}$, this yields a contradiction. Hence, any equilibrium price vector $\mathbf{p}^*$ is bounded by a vector $\bar{\mathbf{p}} = ((\bar{p} + \omega_j^f)_{j\in\mathscr{J}_f} : f \in \mathscr{F})$ from above.

All in all, it follows that the best response of each firm must be within a box with extreme points $\bar{\mathbf{p}}$ and $\underline{\mathbf{p}}$.

Step 2: Uniqueness of equilibrium price $\mathbf{p}^*$. It follows from Step 1 that any equilibrium price $\mathbf{p}^*$ of the stage game is a solution to the system of FOCs. Assumption 2 ensures that the Jacobi matrix $D_{\mathbf{p}^f} s(\mathbf{p}^f;\boldsymbol{\theta})$ non-singular by the Levy-Desplanques Theorem (see e.g. Theorem 6.1.10. in Horn and Johnson (2012)). Hence, the FOCs can be written as $\mathbf{g}(\mathbf{p}) = \mathbf{p}$ where $\mathbf{g}$ is as defined in General Assumption 3. By General Assumption 3-ii), there is a unique solution to this system of equations by Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010). Further, by General Assumption 3-i) and Kellogg (1976), there is a unique solution of the first order condition of each firm's optimization problem, given by $\mathbf{g_f}(\mathbf{p}) = \mathbf{p}^f$. Thus, for any competitor prices, there exists a unique best response of each firm f, which solves $\mathbf{g}_f(\mathbf{p}) = \mathbf{p}^f$ and the unique solution to $\mathbf{g}(\mathbf{p}) = \mathbf{p}$ must be an equilibrium.

B.3.2 Proof of General Lemma 3

Let Assumptions 1, 2, and General Assumption 3 hold for a compact, path-connected set $\mathscr{O}$ of $(\Omega, \boldsymbol{\theta})$. Then, by General Lemma 2, all stage games with parameters $(\Omega, \boldsymbol{\theta}) \in \mathscr{O}$ admit a unique and finite equilibrium that are unifomly bounded on $\mathscr{O}$. Hence, we can apply Lemma 5.

B.3.3 Proof of General Proposition 2

Let Assumptions 1, 2, and General Assumption 3 hold for a compact, path-connected set $\mathscr{O}$ containing $(\Omega, \boldsymbol{\theta}) = (\mathbf{0}, \boldsymbol{\theta}_T)$. By General Lemma 2 and General Lemma 3, the stage games for $(\Omega, \boldsymbol{\theta}) \in \mathscr{O}$ have a unique solution $p^*(\Omega, \boldsymbol{\theta})$ that is continuous in $(\Omega, \boldsymbol{\theta})$. Then, convergence follows by Lemma 4.

B.3.4 Proof of General Proposition 4

Let $\lambda_t = \lambda$, $\boldsymbol{\theta}_t = \boldsymbol{\theta}$. So, we will drop the parameter $\boldsymbol{\theta}$ in the notation in this proof. For t close to T, we know from General Lemma 2 that the equilibirum of the stage game is unique and the price vectors $\mathbf{p}_t^*(\mathbf{K}) = \mathbf{p}^*(\Omega_t(\mathbf{K}))$ are implicitly defined by a system of equations given by

$$\left(D_{\mathbf{p}_f}\mathbf{s}_f\left(\mathbf{p}_t^*(\mathbf{K})\right)\right)^{\mathsf{T}}\mathbf{p}_t^*(\mathbf{K}) - \left(D_{\mathbf{p}_f}\mathbf{s}(\mathbf{p}_t^*(\mathbf{K}))\right)^{\mathsf{T}}\boldsymbol{\omega}_t^f(\mathbf{K}) + \mathbf{s}_f\left(\mathbf{p}_t^*(\mathbf{K})\right) = 0 \qquad \forall f.$$

The only time-dependent variables are then $\Omega_t(\mathbf{K}) = (\boldsymbol{\omega}_t^f(\mathbf{K}))_{f \in \mathscr{F}}$. Hence, $\mathbf{p}_t^*$ and Ω_t are continuous in t. Due to the ODE, Ω_t is continuously differentiable, so $\mathbf{p}_t^*$ is continuously differentiable. Inductively it follows that as we take derivatives of the ordinal differential equation, if Ω_t is n times continuously differentiable, then $\mathbf{p}_t^*$ is n times continuously differentiable. The n-th time derivative $(p_t^*)^{(n)}(\mathbf{K})$ depends on the time derivatives $\Omega_t(\mathbf{K}), \ldots, \Omega_t^{(n)}(\mathbf{K})$ and is well defined because the implicit function is smooth in $\mathbf{p}$ and Ω. We are interested in the limit as $t \to T$. We show by induction in n that if $K_j > n$ for all j, then as $t \to T$, $(\omega_{j,t}^f)^{(n)}(\mathbf{K}) = 0$ for all f, j which implies the claim by Taylor's theorem.

Induction start: First, $\lim_{t\to T}\Omega_t = 0$. Furthermore, we can write for all f and j:

$$\begin{aligned}\dot{\omega}^f_{j,t}(\mathbf{K}) =& \dot{\Pi}_{f,t}(\mathbf{K}) - \dot{\Pi}_{f,t}(\mathbf{K}-\mathbf{e}_j)\\ =& -\lambda\Big[\underbrace{\mathbf{s}_f(\mathbf{p}^*(\Omega_t(\mathbf{K})))^\intercal \mathbf{p}^*_f(\Omega_t(\mathbf{K})) - \mathbf{s}(\mathbf{p}^*(\Omega_t(\mathbf{K})))^\intercal \boldsymbol{\omega}^f_t(\mathbf{K})}_{=:G^1_f(\Omega_t(\mathbf{K}))}\\ & - \underbrace{\left(\mathbf{s}_f\left(\mathbf{p}^*\left(\Omega_t(\mathbf{K}-\mathbf{e}_j)\right)\right)^\intercal \mathbf{p}^*_f\left(\Omega_t(\mathbf{K}-\mathbf{e}_j)\right) - \mathbf{s}_f\left(\mathbf{p}^*\left(\Omega_t(\mathbf{K}-\mathbf{e}_j)\right)\right)^\intercal \boldsymbol{\omega}^f_t(\mathbf{K}-\mathbf{e}_j)\right)}_{=:G^1_f(\Omega_t(\mathbf{K}-\mathbf{e}_j))}\Big]\end{aligned}$$

Thus, as $t \to T$, $\dot{\omega}^f_{j,t}(\mathbf{K}) = 0$ if $K_j > 1$. If $j \in \mathscr{J}_f$ and $K_j = 1$, then $\dot{\omega}^f_{j,t}(\mathbf{K}) < 0$. If $j \notin \mathscr{J}^f$ and $K_j = 1$, then by the competition effect $\dot{\omega}^f_{j,t}(\mathbf{K}) > 0$. This implies that $\dot{p}^*_{j,T}(\mathbf{K}) < 0$ if $K_j = 1$ and $\dot{p}^*_{j,T}(\mathbf{K}) = 0$ otherwise.

Induction assumption: Letting for $\Omega^{(m)}_t(\mathbf{K})$ be that matrix of m-th derivatives of $\omega^f_j(\mathbf{K})$, we can write for all f and j

$$(\omega^f_{j,t})^{(n-1)}(\mathbf{K}) = -\lambda\left[G^{n-1}_f\left(\left(\Omega^{(m)}_t(\mathbf{K})\right)^{n-2}_{m=0}\right) - G^{n-1}_f\left(\left(\Omega^{(m)}_t(\mathbf{K}-\mathbf{e}_j)\right)^{n-2}_{m=0}\right)\right]$$

where $G^{n-1}_f((\Omega^{(m)}_t(\mathbf{K}-\mathbf{e}_j))^{n-2}_{m=0}) = \frac{\partial^{n-2}}{(\partial t)^{n-2}} G^1_f(\Omega_t(\mathbf{K}))$. If $K_j > n-1$ for all j, then as $t \to T$, $(\omega^f_{j,t})^{(n-1)}(\mathbf{K}) = 0$ for all f, j.

Induction step: Given the induction assumption, we can also calculate the next order derivative recursively

$$(\omega^f_{j,t})^{(n)}(\mathbf{K}) = -\lambda\big[G^n((\Omega^{(m)}_t(\mathbf{K}))^{n-1}_{m=0}) - G^n(\Omega^{(m)}_t(\mathbf{K}-\mathbf{e}_j))^{n-1}_{m=0}))\big].$$

Then, note if $\min_i K_i > n$, then $(\omega^f_{j,t})^{(n)}(\mathbf{K}) = 0$ by the Induction Assumption. If $\min_i K_i = n$,

$$(\omega^f_{j,t})^{(n)}(\mathbf{K}) = -\lambda\big[-G^n(\Omega^{(m)}_t(\mathbf{K}-\mathbf{e}_j))^{n-1}_{m=0}))\big] = -\lambda\frac{\partial^{n-1}}{(\partial t)^{n-1}} G^1_f((\mathbf{K}-\mathbf{e}_j)).$$

B.3.5 Proof of General Proposition 3

Let Assumptions 1, 2, and General Assumption 4 hold. First, note that General Assumption 4 implies that for $j_1, j_2 \neq k$

$$\frac{s_{j_1}(\mathbf{p})}{s_{j_2}(\mathbf{p})} = \frac{\frac{\partial s_{j_1}}{\partial p_k}(\mathbf{p})}{\frac{\partial s_{j_2}}{\partial p_k}(\mathbf{p})}.$$

By Step 1 in the proof of General Lemma 2 and by Assumption 2, any equilibrium price vector of the stage game $\mathbf{p}^*(\Omega;\boldsymbol{\theta})$ must satisfy for all $j \in \mathscr{J}_f$ the FOCs of firm f's payoff given by:

$$p_j - \omega_j^f + \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \frac{\frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}(p_{j'} - \omega_{j'}^f) - \sum_{j' \notin \mathscr{J}_f} \frac{\frac{s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} \omega_{j'}^f = -\frac{s_j(\mathbf{p})}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}.$$

Since $\frac{\partial s_j}{\partial p_j}(\mathbf{p}) = -\sum_{k \in \mathscr{J} \setminus \{j\}} \frac{\partial s_k}{\partial p_j}(\mathbf{p}) - \frac{\partial s_0}{\partial p_j}$, this can be rewritten as

$$p_j - \omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \frac{1}{\sum_{k \in \mathscr{J} \setminus \{j\}} \frac{s_k(\mathbf{p})}{s_{j'}(\mathbf{p})} + \frac{s_0(\mathbf{p})}{s_{j'}(\mathbf{p})}}(p_{j'} - \omega_{j'}^f) + \sum_{j' \notin \mathscr{J}_f} \frac{1}{\sum_{k \in \mathscr{J} \setminus \{j\}} \frac{s_k(\mathbf{p})}{s_{j'}(\mathbf{p})} + \frac{s_0(\mathbf{p})}{s_{j'}(\mathbf{p})}} \omega_{j'}^f = -\frac{s_j(\mathbf{p})}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}$$

$$\Leftrightarrow p_j - \omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \frac{s_{j'}(\mathbf{p})}{1 - s_j(\mathbf{p})}(p_{j'} - \omega_{j'}^f) + \sum_{j' \notin \mathscr{J}_f} \frac{s_{j'}(\mathbf{p})}{1 - s_j(\mathbf{p})} \omega_{j'}^f = -\frac{s_j(\mathbf{p})}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}.$$

By Assumption 4, for $j' \neq j$, $\frac{\partial}{\partial p_j} \frac{s_{j'}(\mathbf{p})}{1-s_j(\mathbf{p})} = 0$, we can define $\tilde{s}_{j,j'}((p_{j'})_{j' \neq j}) := \frac{s_{j'}(\mathbf{p})}{1-s_j(\mathbf{p})}$ and

$$c((p_{j'})_{j' \neq j}; \Omega) := \omega_j^f + \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \tilde{s}_{j,j'}((p_{j'})_{j' \neq j})(p_{j'} - \omega_j^f) - \sum_{j' \notin \mathscr{J}_f} \tilde{s}_{j,j'}((p_{j'})_{j' \neq j}) \omega_{j'}^f.$$

Thus, the FOCs of the stage game are equivalent to the first order conditions of a game with $\mathscr{J}$ players where each player j's payoff is given by

$$s_j(\mathbf{p})\big(p_j - c((p_{j'})_{j' \neq j}; \Omega)\big).$$

We call this game the "auxiliary game with J players." Note that the derivative of player j's payoff is greater or equal than zero if and only if

$$\frac{\partial s_j(\mathbf{p})}{\partial p_j}\left(p_j - c\left((p_{j'})_{j'\neq j};\Omega\right)\right) + s_j(\mathbf{p}) \geq 0.$$

Hence any equilibrium of the stage game is an equilibrium of a game with J players with the above payoffs and vice versa.

In order to show existence of equilibria of the stage game, it is sufficient to show existence of equilibria of the auxiliary game with J players and the above payoffs. First, recall that by Step 1 in the proof of General Lemma 2, all best response prices are interior and hence, if an equilibrium exists, it must satisfy the FOCs. Further, since we assume $-\frac{\partial}{\partial p_j}\frac{s_j(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}} \neq 1$ for all $\mathbf{p}$, the first-order condition has a unique solution which must be a maximizer of player j's payoff function. All in all, the best response function of player j, $\mathscr{R}_j$, maps a compact set of prices $\mathbf{q}$ into a compact set of prices $\mathbf{p}$. For $\epsilon > 0$, consider the mapping

$$\Phi : (\mathbf{p},\mathbf{q}) \mapsto \left(p_j - \epsilon\Big(p_j - c_j(\mathbf{q}_{-j};\Omega,\boldsymbol{\theta}) + \frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}\Big)\right)_{j\in\mathscr{J}}$$

Then $D_{\mathbf{p}}\Phi$ is a diagonal matrix with diagonal entries

$$\phi_j := 1 - \epsilon\Bigg(1 + \underbrace{\frac{\partial}{\partial p_j}\frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}}_{\geq 0}\Bigg)$$

Let $\epsilon > 0$ be so that $\phi_j > 0$ for all j. Then all diagonal entries are in $(0,1-\epsilon)$ and Φ is Lipschitz continuous with Lipschitz constant $\max_j \phi_j$. Further $D_{\mathbf{q}}\Phi$ is bounded because it is continuous. Then, the implicit function theorem in the form of Theorem 1.A.4 in Dontchev and Rockafellar (2009) implies continuity of $\mathscr{R} = ((\mathscr{R}_j)_j)$. Hence, by Brouwer's fixed-point theorem $\mathscr{R} = ((\mathscr{R}_j)_j)$ has a fixed point.

C Nested Logit Calculations

Since our empirical application uses a nested logit specification, we verify in the following that all assumptions made in the model are satisfied for a nested logit demand model given by

$$s_j(\mathbf{p}) = \underbrace{\frac{e^{\frac{\delta_j - \alpha p_j}{1-\sigma}}}{\sum_{j\in\mathscr{J}} e^{\frac{\delta_j - \alpha p_j}{1-\sigma}}}}_{=:s_{j|\mathscr{J}}(\mathbf{p})} \frac{\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i - \alpha p_i}{1-\sigma}}\right)^{1-\sigma}}{1+\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i - \alpha p_i}{1-\sigma}}\right)^{1-\sigma}} \qquad s_0(\mathbf{p}) = \frac{1}{1+\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i - \alpha p_i}{1-\sigma}}\right)^{1-\sigma}}.$$

Note that the same properties follow for regular logit by setting $\sigma = 0$ and replacing α with $\frac{\alpha}{\rho}$. To simplify notation, let $D_J := \sum_{i\in\mathscr{J}} e^{\frac{\delta_i - \alpha p_i}{1-\sigma}}$. Then,

$$\frac{\partial s_j}{\partial p_j} = -\frac{\alpha}{1-\sigma} s_j \left(1 - \left(\sigma s_{j|\mathscr{J}} + (1-\sigma) s_j\right)\right) \qquad \frac{\partial s_j}{\partial p_{j'}} = \frac{\alpha}{1-\sigma} s_{j'} \left(\sigma s_{j|\mathscr{J}} + (1-\sigma) s_j\right).$$

It is easy to check that Assumptions 1-i) and ii) are satisfied. We show that Assumption 1-iii) is satisfied. Letting $\underline{s}_0 \equiv s_0\left(\underline{\mathbf{p}}\right)$, the constant in Equation 1 is given by $C = \alpha \underline{s}_0 > 0$ since then

$$\frac{\partial s_0}{\partial p_j} = \alpha s_j s_0 > C s_j.$$

$$(D_{\mathbf{p}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1} = -\frac{1-\sigma}{\alpha s_0} \cdot \begin{pmatrix} \frac{\sigma}{1-\sigma}\frac{1+D_J^{1-\sigma}}{D_J^{1-\sigma}} + \frac{s_0}{s_1} + 1 & \frac{\sigma}{1-\sigma}\frac{1+D_J^{1-\sigma}}{D_J^{1-\sigma}} + 1 & \dots & \frac{\sigma}{1-\sigma}\frac{1+D_J^{1-\sigma}}{D_J^{1-\sigma}} + 1 \\ \frac{\sigma}{1-\sigma}\frac{1+D_J^{1-\sigma}}{D_J^{1-\sigma}} + 1 & \ddots & & \frac{\sigma}{1-\sigma}\frac{1+D_J^{1-\sigma}}{D_J^{1-\sigma}} + 1 \\ \vdots & & \ddots & \vdots \\ \frac{\sigma}{1-\sigma}\frac{1+D_J^{1-\sigma}}{D_J^{1-\sigma}} + 1 & \dots & \dots & \frac{\sigma}{1-\sigma}\frac{1+D_J^{1-\sigma}}{D_J^{1-\sigma}} + \frac{s_0}{s_J} + 1 \end{pmatrix}$$

Hence, $\hat{\boldsymbol{\epsilon}} = ((D_p \mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^\intercal)^{-1}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}) = -\frac{1}{\alpha s_0}\mathbf{1}$ and noting that $\frac{\partial}{\partial p_j}\left(\frac{1}{s_0}\right) = -\alpha \frac{s_j}{s_0}$,

$$D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}} = \begin{pmatrix} \frac{s_1}{s_0} & \cdots & \frac{s_J}{s_0} \\ & \ddots & \\ \frac{s_1}{s_0} & \cdots & \frac{s_J}{s_0} \end{pmatrix}.$$

It follows that Assumption 2 is satisfied:

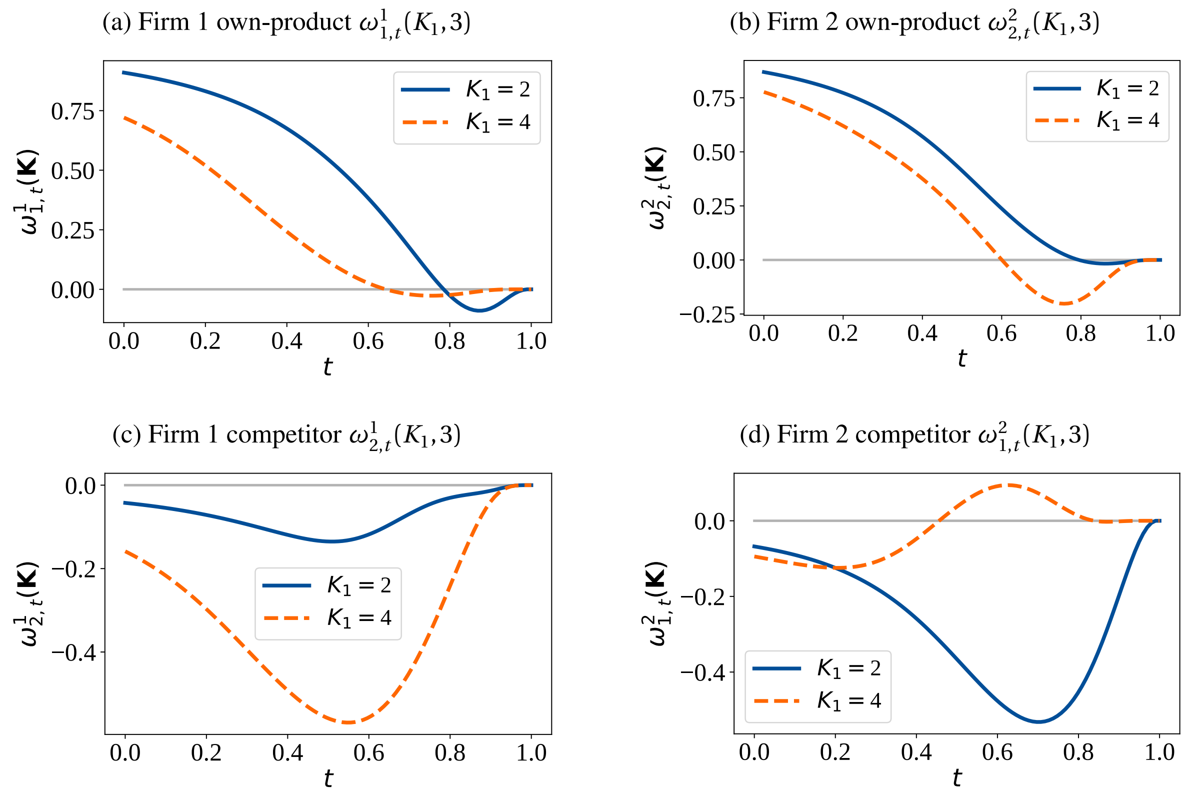
$$\det\left(-D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}} - I\right) = (-1)^J \frac{1}{s_0} \neq 0.$$

It follows immediately hat the properties are satisfied for all subsets $\mathscr{A} \subset \mathscr{J}$.

D Additional Tables and Figures

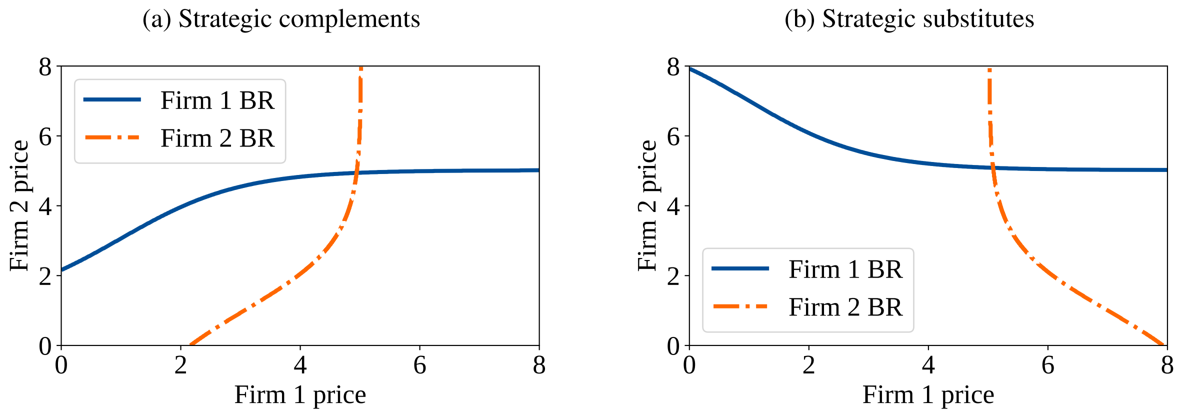
D.1 Simulations

Figure 10: Simulated scarcity effects for $K_2 = 3$, K_1 varying



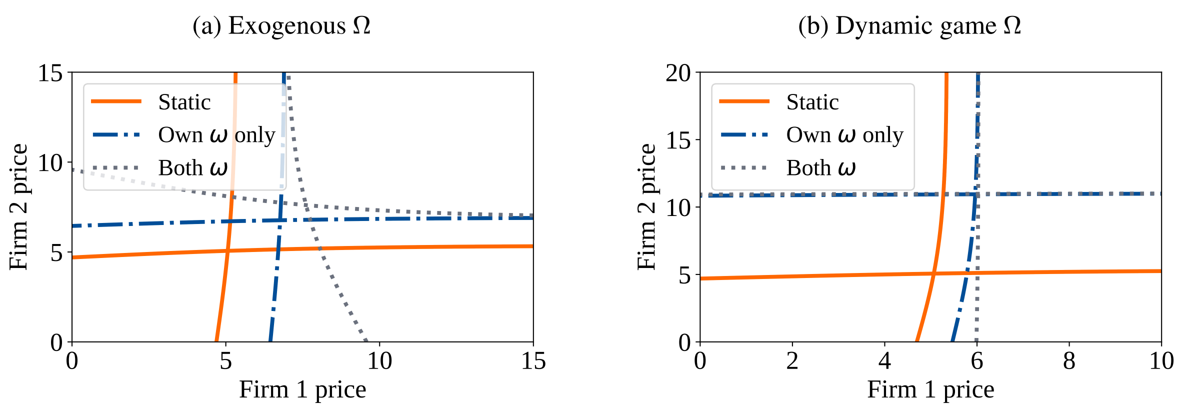
Notes: The simulations assume $\delta = (1, 1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$.

Figure 11: Strategic complements and substitutes in the stage game



Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 1$, as well as $\omega_1^1 = \omega_2^2 = 4$. Panel (a) shows both firms' best response functions for $\omega_2^1 = \omega_1^2 = 4$. Panel (b) shows both firms' best response functions for $\omega_2^1 = \omega_1^2 = -4$.

Figure 12: Effects of own and competitor scarcity on prices

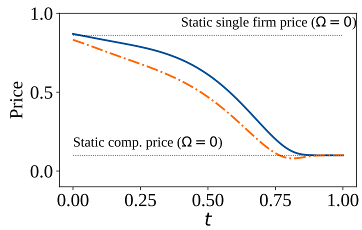


Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 4$. Panel (a) shows both firms' best response functions for $\omega_1^1 = \omega_1^2 = 2$ and $\omega_2^1 = \omega_2^2 = -6$ when no ωs are considered in the profits (orange), when only the own ωs are considered (blue), and when both ωs are considered (grey). Panel (b) shows an analogous figure for the Ω matrix obtained at $t = 0$ in the dynamic duopoly game with $T = 2$ and $\lambda_t \equiv 10$ at the state $\mathbf{K} = (20,1)$.

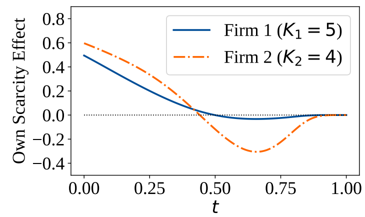
Figure 13: Simulated prices and scarcity effects

$\mathbf{K} = (5,4)$

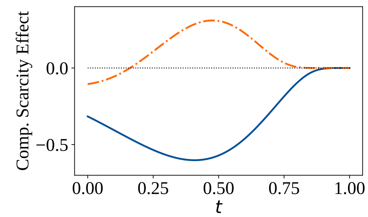
(a) Price paths over time



(b) Own ω over time

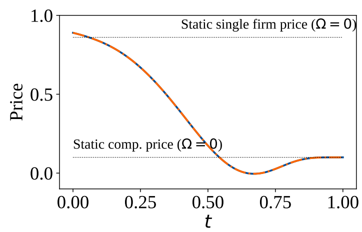


(c) Competitor ω over time

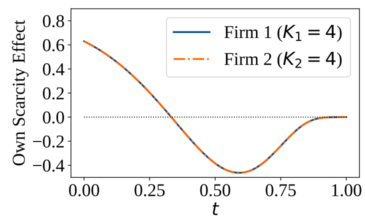


$\mathbf{K} = (4,4)$

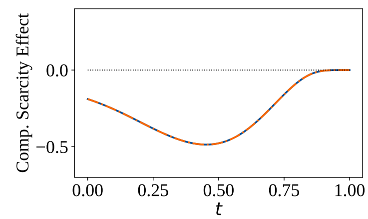
(d) Price paths over time



(e) Own ω over time

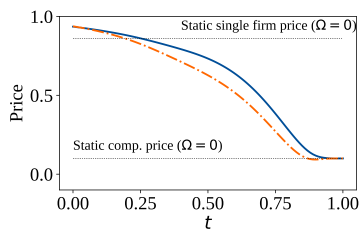


(f) Competitor ω over time

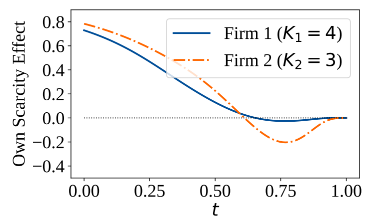


$\mathbf{K} = (3,4)$

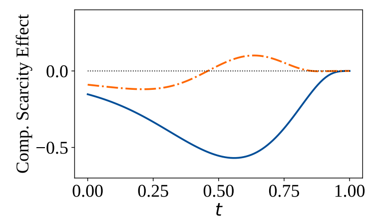
(g) Price paths over time



(h) Own ω over time



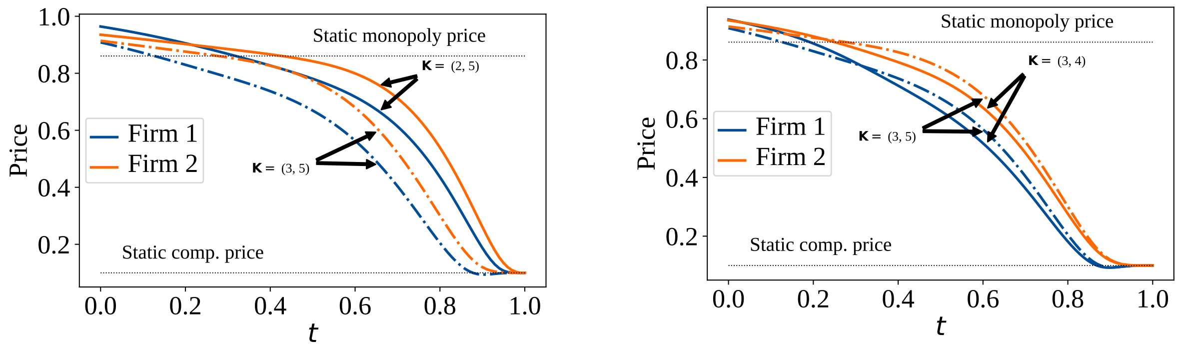
(i) Competitor ω over time



Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$.

Figure 14: Price paths for varying levels of capacity

(a) Sale of a product with minimum inventory (b) Sale of a product without minimum inventory

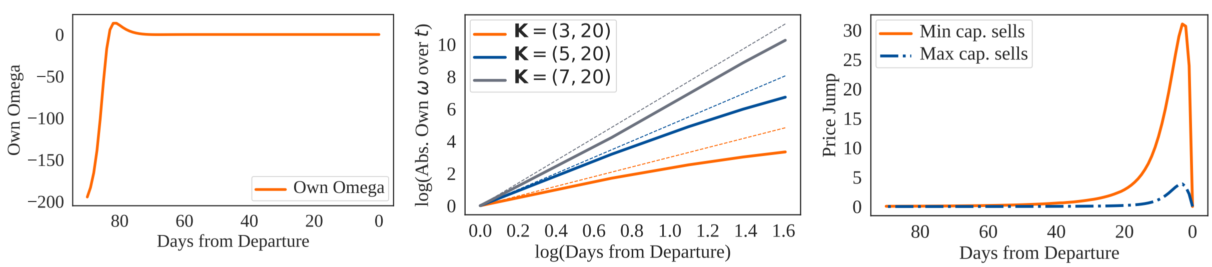


Notes: These simulations correspond to logit demand with parameter values $\delta_j = 1$, $\alpha = 1$, $\lambda = 10$ and scale factor $\rho = 0.05$. Panel (a) shows both firm's price paths for $\mathbf{K} = (3,5)$ and $\mathbf{K} = (2,5)$. Panel (b) shows both firm's price paths for $\mathbf{K} = (3,5)$ and $\mathbf{K} = (3,4)$.

D.2 Empirical Evidence of Dynamic Pricing Forces

Figure 15: Example of a negative own Opportunity Costs

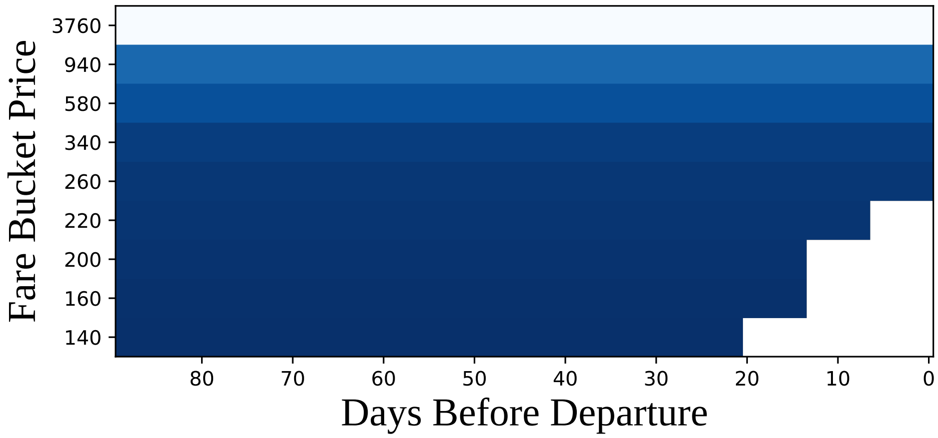
(a) Example own Omega over Time (b) Log. Abs. Own ω over Time (c) Average Price Jumps over Time



Note: Panel (a) shows the own ω over time for a given state in one of our benchmark solutions. Panel (b) shows the log of the absolute value of the own ω over time for three states in one of our Benchmark solutions. The dotted lines represent the behavior these curves would follow if the omegas were proportional to $|T-t|^{\min(\mathbf{K})}$. Panel (c) shows the price change if the firm with the minimum and maximum capacities sell a unit.

D.3 Pricing Heuristics

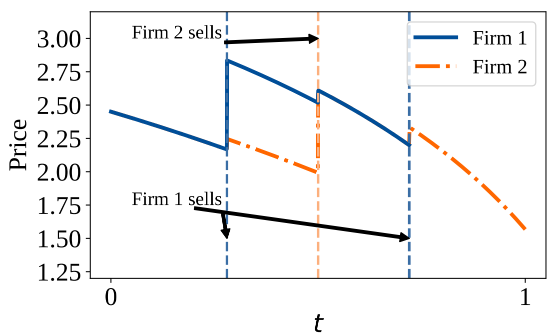
Figure 16: Fare Menu Example



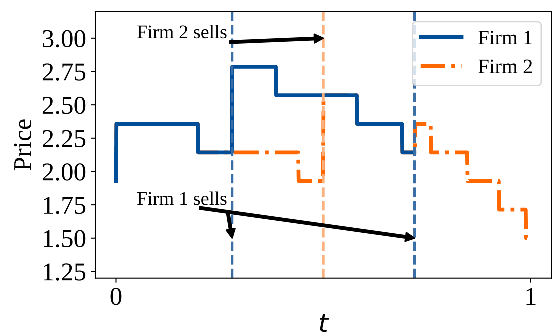
Note: Example pricing menu over time. Prices rounded to nearest $20.

Figure 17: Price Path Realizations comparing Benchmark model to Heuristics

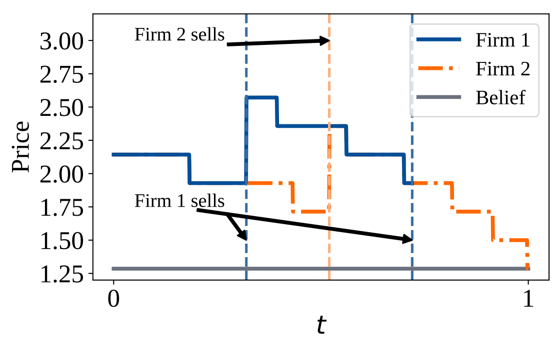
(a) Price paths for the benchmark model



(b) Price paths for the lagged algorithm



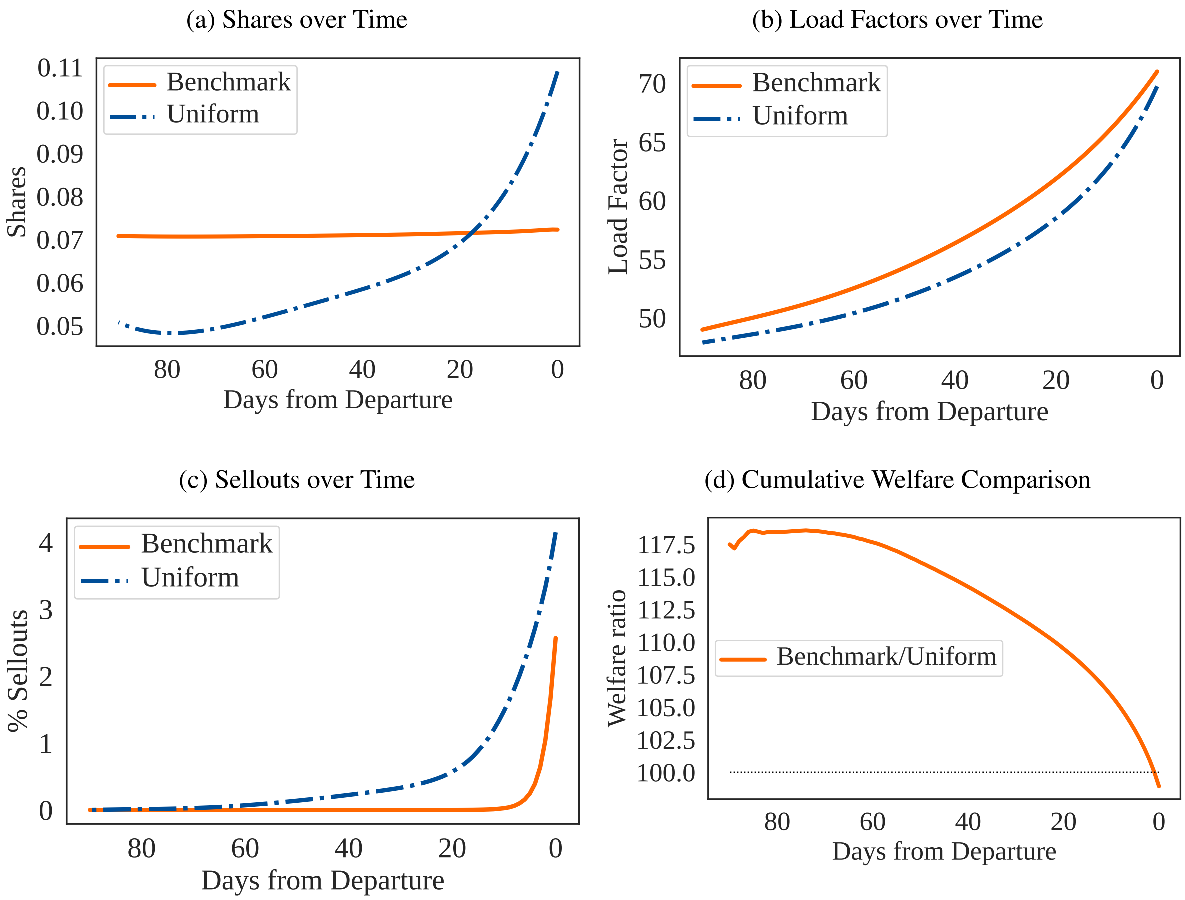
(c) Price paths for the deterministic algorithm



Notes: We assume demand follows a logit specification with an initial capacity vector of $\mathbf{K}_0 = (2,2)$. Time is continuous for $t \in [0,1]$. There are three panels: panel (a) depicts the equilibrium price path for the benchmark model, panel (b) considers prices if firms use the lagged model, and panel (c) considers prices if firms use the deterministic model. The vertical lines mark realized sales times; the color denotes the firm that received the sale. These simulations correspond to the parameter values $\delta_j = 1$, $\alpha = 1$, $\rho = 1$, $\lambda = 10$ and $\mathbf{K}_0 = [2,2]$. In the heuristic model, firms assume that the competitor prices at the level given by the grey line.

D.4 Welfare Calculations with Restricted Capacities

Figure 18: Counterfactual Summary Plots, Restricted Capacities



Note: Panel (a) shows the average shares over time for the benchmark and uniform models. Panel (b) shows the average load factors over time for the same two models. Panel (c) shows the average sellouts over time for the same two models. Panel (d) shows the ratio of average cumulative welfare for the benchmark model with respect to the uniform one.

Table 5: Counterfactual Results for Single Product, Duopoly Routes, Restricted Capacities

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	228.0	5407.2	5731.4	17186.4	28325.1	19.6	70.1	2.1
Uniform	244.4	4521.7	4693.6	19513.1	28728.4	19.2	69.7	4.1
% Diff.	7.2	-16.4	-18.1	13.5	1.4	-2.0	-0.4	2.0

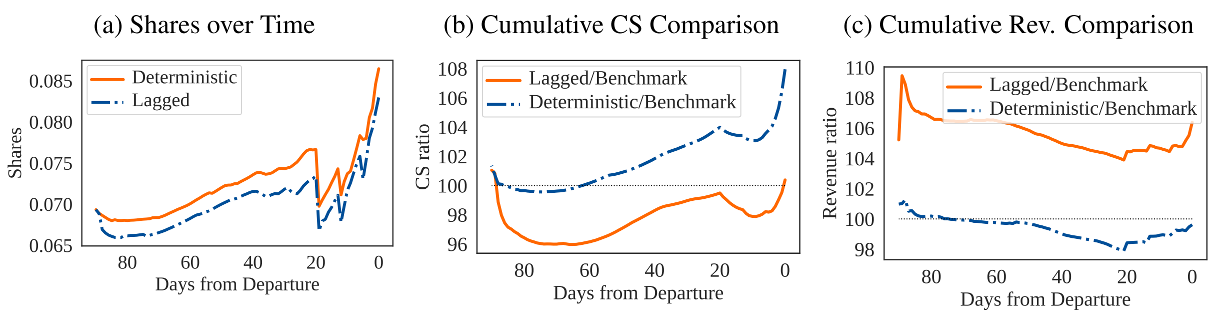
Note: Price is the average across routes (r) after computing the average across firms (f), departure dates (DD), days before departure (DFD) and simulation number (n) within a route. Firm revenues are similarly defined, except aggregated over DFD. CS is the expected consumer surplus, computed the same way as revenues. Welfare is the sum of revenues and CS. Q is the total number of seats sold. LF is the average fraction of seats sold (including flow traffic) at the departure time. Sellouts is the fraction of flights sold out.

Table 6: Heuristic Counterfactuals for Single Product, Duopoly Routes, Restricted Capacities

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
Lagged	103.2	102.7	104.8	103.6	103.7	99.6	100.0	98.9
Deterministic	97.3	99.3	100.5	107.5	104.5	102.9	101.2	125.8

Note: Price is the average across routes (r) after computing the average across firms (f), departure dates (DD), days before departure (DFD) and simulation number (n) within a route. Firm revenues are similarly defined, except aggregated over DFD. CS is the expected consumer surplus, computed the same way as revenues. Welfare is the sum of revenues and CS. Q is the total number of seats sold. LF is the average fraction of seats sold (including flow traffic) at the departure time. Sellouts is the fraction of flights sold out.

Figure 19: Heuristic Counterfactuals Results over Time, Restricted Capacities



Note: Panel (a) shows the average shares over time for the two heuristic models. Panel (b) shows the ratios of cumulative consumer surplus for the two models with respect to the benchmark. Panel (c) shows the ratios of cumulative revenue for the two models with respect to the benchmark.

D.5 Welfare Calculations for Entire Sample

i) In our counterfactuals we consider only two products. In order to include routes that have more than one flight per carrier per day, we adjust the choice set, utilities, and capacities for all routes.

ii) We take the mean utilities (δ) across observed flights for each route-carrier-departure date.

iii) We use the maximum observed capacity for each route-carrier-departure date. Although it may be natural to sum the capacities when restricting the choice set, we have found that large capacities presents a significant computational burden.

iv) We use the observed arrival process for each route-departure date. We do not adjust the estimated arrival processes as the inside good shares tend to be small. That is, because most consumers choose the outside good, we do not scale down arrival rates to account for smaller choice sets.

Table 7: Counterfactual Results for Entire Sample

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	220.4	5566.9	6007.6	16742.7	28317.2	20.4	80.3	20.7
Uniform	262.3	4797.5	5266.0	18979.5	29043.0	19.1	78.8	15.1
% Diff.	19.0	-13.8	-12.3	13.4	2.6	-6.4	-1.5	-5.6

Note: Price is the average across routes (r) after computing the average across firms (f), departure dates (DD), days before departure (DFD) and simulation number (n) within a route. Firm revenues are similarly defined, except aggregated over DFD. CS is the expected consumer surplus, computed the same way as revenues. Welfare is the sum of revenues and CS. Q is the total number of seats sold. LF is the average fraction of seats sold (including flow traffic) at the departure time. Sellouts is the fraction of flights sold out.