Outside options, reputations, and the partial success of the Coase conjecture

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Abstract

A buyer and seller bargain over a good's price in continuous time, the buyer has a private value $v \in [\underline{v}, \overline{v}]$ and a positive outside option $w \in [\underline{w}, \overline{w}]$. Additionally, bargainers can either be rational or committed to some fixed price. When the sets of commitment types and buyer values are rich and the probability of commitment vanishes, outcomes are approximately equivalent to the seller choosing a take-it-or-leave-it offer below $\max\{\underline{w}, \underline{v}/2\}$. Prices are low and there is minimal delay, however, this needn't be efficient as the buyer sometimes chooses her outside option. Seller payoffs may increase in the buyer's outside option.

Keywords: Bargaining, reputation, coase conjecture

1 Introduction

What effect do outside options have on bargaining with incomplete information? The existing literature suggests a surprisingly dramatic impact. Most notably, consider an infinite horizon game where in every period the seller proposes a price for a good to a buyer with private information about her value $v \in [\underline{v}, \overline{v}]$ and there is a per period discount factor $\delta < 1$. If there are no outside options, then the Coase conjecture holds in equilibrium: if $\delta \approx 1$, as when offers are frequent, the seller must propose a price of approximately $\underline{v}$ immediately if there is "gap", $\underline{v} > 0$, or buyer strategies are stationary (Fudenberg and Tirole (1985), Gul et al. (1986)). The idea is that if today's offer $p > \underline{v}$ is rejected, the seller will update her beliefs and cut her price tomorrow, but in which case even a high value buyer would not accept today unless the price is already low $p \approx \underline{v}$. However, if the buyer can get a strictly positive outside option $w \in [\underline{w}, \overline{w}]$ by exiting the market at the end of the period, then Board and Pycia (2014) show the seller

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acts with complete commitment: she can choose any price in the first period, and the buyer either accepts it or exits the market. The logic is that if bargaining continues into period 2, the seller will never charge a price below the lowest net value buyer remaining, $u = v - w$, giving that buyer a continuation payoff below w, so she would prefer to exit in period 1 instead to avoid discounting.

How robust is Board and Pycia (2014)'s result? It relies heavily on only the seller being able to make offers, leaving the lowest net value type with no gains from trade. Paradoxically, if the buyer knew her type, and could take her outside option before the start of bargaining, she would never bargain (as the lowest net value who did would get a discounted continuation payoff below her outside option). If instead the buyer could also sometimes make offers, she might hope for a continuation payoff higher than her outside option when the game continued into period 2.[1] However, introducing offers by the informed party introduces signalling, which typically creates substantial indeterminacy in predictions. Off-path buyer offers can be "punished with beliefs", interpreted as coming from the weakest buyer with the highest net value $\overline{v} - \underline{w}$, which can support a wide variety of on path play.[2]

One way to potentially mitigate the power of belief punishments is to introduce a small probability of commitment types into the model, which always propose some fixed price and never back down (another form of incomplete information). Abreu and Gul (2000) first introduced these types on both sides of a bargaining problem with fixed surplus to divide (so there is only one type of rational bargainer); they showed discrete time outcomes converge to the unique equilibrium of a concession game when offers are frequent, independent of almost all details of the bargaining protocol.

When rational agents have private information, a small probability of commitment types can provide clear predictions despite the potential for signalling, because rational agents who imitate commitment types cannot be arbitrarily punished with beliefs as their behavior is on path. Abreu et al. (2015) consider a continuous time concession game with a fixed surplus, when one party's discount rate is private information (taking two values) and there are no outside options. They show that if the set of demands made by commitment types is sufficiently rich, then outcomes must be Coasean as the probability of commitment vanishes: there is immediate agreement on the same terms as would

[1] Certainly, if she instead made all offers, she could get the good for free.

[2] For similar reasons there are multiple equilibria in Board and Pycia (2014) if the buyer sometimes has an outside option of 0. In some the seller chooses any take-it-or-leave-it offer and the buyer is believed to have a high outside option if she remains in period 2, while others have a Coasean structure, with low and declining prices.

have been agreed if the informed party were known to be her most patient type.[3,4] By contrast, Rubinstein (1985) identifies a wide variety of equilibria in an alternating offers game without commitment types, which he selects between on the basis of axioms.

On the other hand, Compte and Jehiel (2002) suggest commitment types may have limited effects when agents have outside options. In an alternating offer protocol with a fixed surplus, and commitment types that aggressively offer an opponent less than her outside option, rational agents never imitate commitment behavior. However, this finding seems to rely heavily on more generous commitment types being unavailable.

In this paper, I introduce commitment types into a continuous time buyer-seller concession game, where the buyer has private information about her value and her positive outside option. The set of types is finite, and for each possible buyer value v there is some probability of the lowest outside option, $\underline{w} > 0$.

My main result shows that when the sets of buyer values and commitment types are rich, bargaining outcomes are partially Coasean as the probability of commitment vanishes: they are approximately those which would arise if the seller could choose any take-it-or-leave-it price below $p^* = \{\underline{v}/2, \underline{w}\}$, and the buyer could either immediately accept this, or exit. Loosely, the set of buyer values is rich if for any $d \in [\underline{v}, \overline{v}]$ there is a possible buyer value close to d, while the set of commitment types is rich if for any $d' \in [0, \overline{v}]$ there is some type which makes a demand close to d'.

The result suggests that the classical Coasean prediction (from Fudenberg and Tirole (1985), and Gul et al. (1986)) of low prices and minimal bargaining delays is robust to the presence of buyer outside options, in contrast to the potential for high prices in Board and Pycia (2014). Assuming $\underline{w} \leq \underline{v}$, as otherwise the type $(\underline{v}, \underline{w})$ would never bargain, we have $p^* \leq \underline{v}$ and so $\underline{v}$ remains an upper bound on prices, independent of the relative probabilities of different buyer types. Indeed, for small outside options $\underline{w} \leq \underline{v}/2$, we get the same upper bound on prices, $\underline{v}/2$, as with no outside options. This is similar to Binmore et al. (1989)'s finding of the irrelevance of small outside options under complete information.

However, the result also diverges from some features of the classical Coase conjecture in a similar direction to Board and Pycia (2014), most notably, the seller retains some

[3] This also matches the alternating offer division when the agent is known to be the patient type.

[4] On the other hand, if commitment types sometimes delay making their fixed demand, non-Coasean limit outcomes can occur as patient rational agents attempt to signal their type. More generally, we could consider commitment types whose demands vary over time in history contingent ways. These additional types have no effect on limit outcomes in the fixed surplus setting of Abreu and Pearce (2007), but do in Fanning (2016). Investigating their effect in the current setting is beyond this paper's scope and appears very challenging. Besides, fixed demand commitment types seem a more relevant behavioral assumption.

market power and the outcome may be inefficient with positive net value buyers choosing to exit. We have $\underline{v} - p^* < \underline{w}$ whenever $\underline{w} > \underline{v}/2$ and can have $p^* > \min_{(v,w)} v - w > 0$ more generally.

Moreover, the seller can benefit from larger buyer outside options since these allow higher prices, both when exogenously assuming $\underline{w} \leq \underline{v}$, and more generally treating the lower bound on values, $\underline{v}$ as endogenously determined by the buyer's choice to engage in bargaining. The seller may then sometimes welcome competitors, who provide more attractive outside options for buyers and allow her to partially escape the Coase conjecture. While a key message of Board and Pycia (2014) was similarly that the seller could benefit from positive buyer outside options, in that model the seller's payoff is always decreasing in the buyer's outside option (so long as it is positive).

A simple example highlights this possibility. Suppose the buyer's value is approximately uniformly distributed on $[1, 5]$ and her outside option is known to be w, so net values are uniformly distributed on $[1 - w, 5 - w]$. When $w \in [0.5, 1.25]$ she charges approximately $p^* = w$ for a payoff of $w(5 - 2w)/4$, which is increasing in w (her payoff is 0.5 when $w \leq 0.5$ and is decreasing in w for $w > 1.25$). This example allowed, but did not require, $w > 1$. When $w > 1$ the lower bound on buyer values which endogenously reflects those who choose to bargain is $\underline{v} \approx w$.

The result also highlights that private information about outside options is quite different from private information about values. For instance, if the buyer's outside option is known to be $w \approx 0$ and her value is approximately uniformly distributed on [0,1], then so is her net value, and the seller must give away the good almost for free, $p^* \approx 0$. On the other hand, if the buyer's value is known to be $v = 1$ and her outside option is approximately uniformly distributed on $[0, 1]$, then so are net values, but now the seller can get approximately her first-best commitment payoff of 0.25 by charging approximately $p^* = 0.5$. In Board and Pycia (2014), these two cases are equivalent given the identical distributions of net values; the seller charges 0.5 in both.

Despite the clear predictions with vanishing commitment and rich sets of types, more generally the model admits multiple equilibria. Each of these, however, has a similar structure to those of existing reputational models. At time 0, bargainers announce commitment demands. A rational buyer who finds the seller's price acceptable, $v - p_s > w$, randomizes over her initial demand choice, before eventually conceding. However, a rational buyer who finds the seller's price unacceptable, $v - p_s < w$, always imitates the lowest positive commitment price, $\underline{p}$, before eventually exiting; this choice is a key driver of the main results, however, its explanation is slightly subtle, and so is delayed.

After time 0, the seller concedes continuously to ensure a buyer with the highest remaining value is indifferent between conceding at one instant or the next (the skimming property holds) until she reaches a probability one reputation for commitment at time $T^* < \infty$. Similarly, the buyer's concession makes the seller indifferent to conceding on $(0, T^*)$, and she reaches a probability one reputation at T^*. Notably, however, the buyer must sometimes concede discontinuously after time 0, to compensate the seller for buyer types which exit.

The reason the seller cannot charge more than p^*, as commitment vanishes is similar to Abreu et al. (2015)'s Coasean result. If the seller asks for more $p_s > p^*$ with positive limit probability there is some rational buyer type (v', w') who finds that price acceptable, but makes a more generous counterdemand $p_b < v' - p_s$. The rate at which an agent concedes to make her opponent indifferent, is proportional to the generosity of her offer (the cost to her opponent of delaying his concession). By some finite time, the seller must believe any remaining buyer has a value below v', and thereafter concedes slower than the buyer, causing her reputation to grow at an exponentially slower rate. For both agents reach a probability one reputation at the same time, therefore, the seller must concede with probability approaching one at time 0.

Perhaps more surprising is the seller's ability to charge any price $p_s \leq p^*$, and retain some market power despite such prices being unacceptable to some buyer types. Such a price ensures that the seller is always more generous than any buyer who finds it acceptable, $v - p_s > p_b$, so that the seller concedes at a faster rate and builds her reputation more quickly; notice in particular that this is always true when $p_s \leq \underline{w}$ because acceptability implies $v - p_s \geq \underline{w}$. On the other hand, buyer types who find the seller's price unacceptable, $v - p_s < w$, always counterdemand $\underline{p}$, which is very ungenerous when the set of commitment types is rich, $\underline{p} \approx 0$, so only a small probability of buyer concession is needed to compensate for any buyer exit. That is not enough to counter the seller's faster rate of concession, and so for both bargainers to reach a probability one reputation at the same time after all demands, the buyer must either concede or exit with probability approaching one at time 0.

The set of buyer values must be rich for the Coasean prediction of low prices; with a sparse set of buyer values, equilibrium prices may be much higher than p^*. For instance, with binary values $v \in \{\underline{v}, \overline{v}\}$ where $\min\{\overline{v}/2, 2\underline{w}\} > \underline{v}$ the seller can charge $p_s = \overline{v}/2$, and the high value buyer will immediately accept in the limit. This is the same price she would get if the buyer was known to have value $\overline{v}$. This finding is similar to Ortner (2017) who shows that when the seller's costs change stochastically over time, the Coase conjecture holds with a (rich) continuum of buyer values, but need not with a discrete distribution. However, the forces driving the two results are quite different.

The main result also depends crucially on a rich set of commitment types, and in particular buyer commitment types that make ungenerous offers $\underline{p} \approx 0$. All types of rational buyer could benefit if they were constrained to make more generous offers, $\underline{p} >> 0$. With a more generous offer, the buyer's exit is more painful to the seller, and so the buyer must concede more to compensate for this, helping her build reputation faster.

Finally, although my analysis is focused on a continuous time concession game, the results extend to at least some discrete time protocols, when offers become frequent. In particular, for a slight adaption of Board and Pycia (2014)'s protocol where the buyer make offers at the start of a period, and the seller make counteroffers at the end, equilibrium outcomes must converge to those of the continuous time concession game.

The remainder of this section highlights additional literature, then section 2 outlines the model, which is analyzed in section 3, while section 4 describes some extensions.

1.1 Additional Literature

Hwang and Li (2017) show that the Coase conjecture may hold if a buyer's outside option arrives stochastically. The seller makes all offers, and the buyer's outside option arrives publicly at the end of each period with some probability (after the seller's offer). With frequent offers, the seller almost immediately offers the buyer a price that would make her lowest value type indifferent to waiting for the outside option. The logic driving the result is that buyers must immediately take an outside option when it arrives. Otherwise the lowest buyer type which did not, would receive a continuation payoff equal to that outside option in subsequent periods (as in Board and Pycia (2014)) and so she would prefer to avoid delay. If the stochastic arrival is not publicly observable multiple equilibria exist, some of which are Coasean and some not.

Nava and Schiraldi (2019) highlight what they call a robust Coase conjecture, when the seller can offer the buyer differentiated goods. The seller makes all offers and after purchasing one variety, the consumer receives no value from buying a second differentiated variety. When offers are frequent, the market clears instantaneously, with the buyer purchasing one of the varieties offered, however, the seller retains some market power and the outcome is not efficient. The seller offers a low price (possibly 0) for one variety, and a high price for the other and allows consumers to select between them. The low price for the first variety, effectively creates an outside option for the consumers, which by Board and Pycia (2014), effectively allows her to charge a monopolistic price for the second variety. The authors suggest that Board and Pycia (2014)'s result is similarly consistent with a properly understood Coase conjecture. However, seemingly many prices can clear the market with outside options, and the low prices identified in my analysis seem "more Coasean" than those of Board and Pycia (2014). Introducing a small probability of commitment types, and allowing buyers to make offers in the differentiated good model, seems likely to substantially reduce seller profits given its similarity to the outside option model.

In addition to outside options, the existing literature has identified many reasons the Coase conjecture might fail: a monopolist might rent rather than sell, or under-invest in capacity (Bulow (1982)), or use best-price provisions (Butz (1990)), or buyers may use non-stationary strategies when there is no gap between their values and the seller's (Ausubel and Deneckere (1989)). However, other factors that might be thought to interfere with the Coasean logic, merely see it confirmed in different guises. For instance, if a second buyer may arrive to compete with the first, the seller's profit is driven down to what she would get from waiting for that second buyer's arrival (Fuchs and Skrzypacz (2010)).

Peski (2021) also identifies a Coasean result when dividing two pies, one bargainer has private information about her relative value of the pies,and there is a vanishingly small probability that bargainers might be committed to a menu of divisions. Normalizing an agent's payoff from receiving all of both pies to equal one, in the limit the uninformed bargainer proposes a menu of all divisions that give her a payoff of 1/2 (a lower bound on her payoff facing any known type), and the informed party selects among these. The Coasean logic is similar to that in my paper and Abreu et al. (2015). If the uninformed bargainer demands more, there exists some informed type whose counterdemand causes her to concede (and thus build reputation) more quickly than the informed party.

Atakan and Ekmekci (2013) show that reputational bargaining with outside options endogenously determined by a search market can lead to inefficiency. Firms and workers flow into a search market at the same rate and are randomly matched to bargain. They exit after reaching an agreement that generates a unit of surplus or randomly dying. Bargainer i can be rational or a single commitment type that demands $\alpha_i > 1 - \alpha_{-i}$ of surplus, and returns to the search market if convinced her opponent is committed. If agents face no delay before rematching in the search market, steady state equilibria are inefficient with no immediate concession in bargaining by either party even if commitment is vanishingly unlikely. Immediate concession can't occur because it would give an opponent an outside option larger than her payoff from conceding. As in Compte and Jehiel (2002), the effect a richer set of commitment types is unclear.

Endogenous outside options are also central to Özyurt (2015), who shows that even vanishingly small reputational concerns allow a wide range of prices in Bertrand-competition like setting with two sellers and a single buyer. This occurs because if the buyer observes a seller undercut it's rival's posted price, she uses that as an outside option to obtain an even better price in bargaining with the high priced rival.

2 The model

In this section I outline a simple, continuous time concession game. However, in section 4 I describe a richer discrete time game, where offers are unrestricted, but whose outcomes must converge to those of the concession game as offers become frequent.

A buyer and seller bargain in a continuous time concession game, where the seller has a single indivisible good. Time 0 is subdivided into 4 times, $0^1 < 0^2 < 0^3 < 0^4$, to allow for sequential decisions to be made with no discounting of payoffs between them. At time 0^1 the seller proposes a price $p_s \in P$ where $P \subset (0, \infty)$ is some finite set. At time 0^2, the buyer can observe the seller's price p_s. She can immediately concede to this price c, counterdemand $p_b \in (0, p_s) \cap P$, or exit the market e. At both the times 0^3 and 0^4, the seller can concede (accept her opponent's announced price), while the buyer can concede or exit the market. If there is no agreement before 0^4: the seller chooses a stopping time $t_s \in (0, \infty]$ to concede to the buyer's offer, while the buyer chooses a stopping time $t_b \in (0, \infty]$ and an action $a \in \{c, e\}$, where (t_b, c) denotes a decision to concede to the seller's offer at time t_b, while (t_b, e) denotes a decision to exit. Both concession and exit end the game and determine payoffs. If seller and buyer choose to concede at the same time, each price is agreed with equal probability. Similarly, if the seller chooses to concede at the same time the buyer chooses to exit, each outcome occurs with equal probability.

Both buyer and seller can either be rational or a commitment type. A rational seller has no value for the good and no outside option, while a rational buyer has a value for the good of $v > 0$, and an outside option of $w > 0$. If the good is traded at price p at time $t \geq 0$ then a rational seller gets a payoff $e^{-rt}p$, and a rational buyer gets the payoff $e^{-rt}(v - p)$, where r is a common discount rate (without loss of generality $r = 1$). If instead, the rational buyer exits the market at time t she receives her outside option for a payoff $e^{-rt}w$, while a rational seller gets a payoff of 0.

The distribution of the rational buyer's type (v, w) has finite support Θ with probability mass function g, so that $\sum_{(v,w)\in\Theta} g(v, w) = 1$. Let $V = \{v : (v, w) \in \Theta\}$ and $W = \{w :$

$(v, w) \in \Theta\}$. Let $\underline{v} = \min V$, $\overline{v} = \max V$ and $\underline{w} = \min W$, $\overline{w} = \max W$ respectively. I assume $v > w$ for all (v, w), as an agent with $w > v$ would never agree to a positive price for the good and would immediately exit the game; I explore how the buyer types present in bargaining may be endogenously determined in section 4. I also assume $g(v, \underline{w}) > 0$ for all $v \in V$, so there is always a chance of the minimum outside option; this is implicitly an assumption about the richness of the set of types.[5] I further assume that $v - w \neq p$ and $(v - p)/(p - p') \neq w'/(v' - p' - w')$ and $v - p \neq p'$ for all $(v, w), (v', w') \in \Theta$ and $p, p' \in P$ with $p > p'$; given the finiteness of Θ and P, this can be justified as an assumption of generic types.

The probability of player i being a commitment type is $z_i \in (0, 1)$. There is a finite set P_i of commitment types for agent i. Conditional of being committed, she is of type $p_i \in P_i \subset P$ with probability $\pi_i(p_i) \in (0, 1)$. Type p_i demands the price p_i in the bargaining game, concedes only if offered a better price (i.e. if $p_s < p_b$ for the buyer) and never exits the market.[6] Let $\underline{p} = \min\{p \in P\}$ and assume that $\underline{p} \in P_b$. While this may be a substantive assumption, it is also reasonable. Moreover, my results hold continue to hold in a discrete time model with a continuum of prices (where the assumption is not satisfied).

Let $\mu_s(p_s)$ be the probability that a rational seller proposes a price $p_s \in P$ at 0^1, and given p_s let $\mu_b^{p_s,v,w}(a)$ be the probability that a rational seller of type (v, w) chooses action $a \in P \cup \{e, c\}$ at 0^2. Hence, immediately after a seller's demand $p \in P_s$ and buyer's counterdemand $p' \in P_b$, the bargainers' reputations for commitment are:

$$\bar{z}_s^{p_s} = \frac{z_s \pi_s(p_s)}{z_s \pi_s(p_s) + (1 - z_s)\mu_s(p_s)}, \quad \bar{z}_b^{p_s,p_b} = \frac{z_b \pi_b(p_b)}{z_b \pi_b(p_b) + (1 - z_b) \sum_{(v,w)\in\Theta} \mu_b^{p_s,v,w}(p_b)}$$

and $\bar{z}_s(p_s) = 0$ if $p_s \notin P_s$ and $\bar{z}_b^{p_s}(p_b) = 0$ if $p_b \notin P_b$.[7] If $\mu_b^{p_s,v',w'}(p_b) > 0$ for some (v', w') then the probability that the buyer is of type (v, w) conditional on rationality is:

$$\bar{g}^{p_s,p_b}(v, w) = \frac{g(v, w)\mu_b^{p_s,v,w}(p_b)}{\sum_{(v',w')\in\Theta'} g(v', w')\mu_b^{p_s,v',w'}(p_b)}.$$

Given p_s, let $\Theta^{c,p_s} = \{(v, w) \in \Theta : v - w > p_s\}$ be the set of rational buyer types for whom

[5] While the precise specification of this assumption might seem non-generic, it is mainly for ease of exposition; without it the main result goes through under a more complicated definition of a rich set of buyer types.

[6] If some seller commitment types make low demands $p_s < \min_{(v,w)\in\Theta} v - w$, rational buyers would have a strict incentive to bargain with the seller even if they could take their outside option slightly beforehand.

[7] It is without loss of generality to assume this to be true even if respectively $\mu_s(p_s) = 0$ or $\mu_s^{p_s}(p_b) = 0$; commitment types cannot "deviate".

the seller's price is acceptable, $v - p_s > w$, and $\Theta^e = \{(v,w) \in \Theta : v - w < p_s\} = \Theta \setminus \Theta^{c,p_s}$ be the set of types which for whom the seller's price is not acceptable. Also let the probability that a buyer finds the seller's price acceptable, conditional on her rationality and her making demand p_b be

$$x^{p_s,p_b} = \sum_{(v,w)\in\Theta^{c,p_s}} \bar{g}^{p_s,p_b}(v,w).$$

I will abuse notation and refer to the continuation game at 0^3 as a subgame. Conditional on reaching such a subgame with demands p_s, p_b, let the probability that player i concedes by time $t \in \{0^3, 0^4\} \cup (0,\infty]$ be $F_i^{p_s,p_b}(t)$, and let the probability that buyer exits by time t in that subgame be $E_b^{p_s,p_b}(t)$. We can later back out the behavior of rational agents from these objects. A rational seller's utility in the subgame at 0^3 when she concedes at time t is then:

$$\begin{aligned} U_s^{p_s,p_b}(t) = &\int_{\tau<t} p_s e^{-r\tau} dF_b^{p_s,p_b}(\tau) + (1 - F_b^{p_s,p_b}(t) - E_b^{p_s,p_b}(t))p_b e^{-rt} \\ &+ \frac{1}{2}e^{-rt}\left((F_b^{p_s,p_b}(t) - F_b^{p_s,p_b}(t_-))(p_s+p_b) + (E_b^{p_s,p_b}(t) - E_b^{p_s,p_b}(t_-))p_b\right) \end{aligned}$$

where $G(t_-) = \sup_{\tau<t} G(\tau)$ with $G(0^3_-) = 0$ for $G : \{0^3, 0^4\} \cup (0,\infty] \to [0,1]$. The utility of a rational buyer with value v that concedes at time t is:

$$\begin{aligned} U_b^{p_s,p_b,v,c}(t) = &\int_{\tau<t} (v - p_b)e^{-r\tau} dF_s^{p_s,p_b}(\tau) + (1 - F_s^{p_s,p_b}(t))(v - p_s)e^{-rt} \\ &+ \frac{1}{2}e^{-rt}\left((F_s^{p_s,p_b}(t) - F_s^{p_s,p_b}(t_-))(2v - p_s - p_b)\right) \end{aligned}$$

The utility of a rational buyer with type (v,w) that exits at time t is:

$$\begin{aligned} U_b^{p_s,p_b,v,w,e}(t) = &\int_{\tau<t} (v - p_b)e^{-r\tau} dF_s^{p_s,p_b}(\tau) + (1 - F_s^{p_s,p_b}(t))we^{-rt} \\ &+ \frac{1}{2}e^{-rt}\left((F_s^{p_s,p_b}(t) - F_s^{p_s,p_b}(t_-))(w + v - p_b)\right). \end{aligned}$$

I will analyze the weak perfect Bayesian equilibria of this game, where at each information set (0^1, 0^2 and 0^3) agents' strategies are optimal given their beliefs, where agents' beliefs are consistent with Bayes' rule when possible, even off the equilibrium path, and an agent's actions, do not affect her belief about her opponent. However, my main result, providing tight bounds on equilibrium outcomes also holds for any Nash equilibrium.

3 Analysis

The analysis of the model in this section is broken down into three subsections. The first focusses on the continuation equilibrium in the subgame at 0^3, the second considers demand choices at 0^1 and 0^2, the third presents the main results as the probability of commitment vanishes.

3.1 Preliminary analysis

This subsection presents some prelimary results that help to identify the nature of a continuation equilibrium in the subgame at 0^3. Given attention to such a subgame, I drop the superscripts p_s and p_b on relevant variables.

Lemma 1 identifies some immediate properties of the continuation equilibrium, which are fairly standard in reputational concession games. First, if the seller but not the buyer reveals rationality ($p_s \notin P_s$ and $p_b \in P_b$) then the seller must immediately concede. Second, if both agents make commitment demands, then both agents reach a probability one reputation for commitment at the same finite time $T^* < \infty$; more generally, agent's reputations at time t are $\bar{z}_s(t) = \bar{z}_s/(1-F_s(t))$ and $\bar{z}_b(t) = \bar{z}_b/(1-E_b(t)-F_b(t))$. The lemma also partially characterizes the case where at least one agent hasn't made a commitment demand by defining $T_s = \min\{t \geq 0^4 : F_s(t) \geq 1 - \bar{z}_s\}$, $T_b = \min\{t \geq 0^4 : F_b(t) \geq (1 - \bar{z}_s)x\}$ and then $T^* = \min\{T_b, T_s\}$. Third, the seller concedes continuously after time 0. Fourth, if both agents make commitment demands, then the probability each has conceded, F_i, must be strictly increasing on $(0, T^*)$. Fifth, the skimming property holds, so that low value buyers concede later than high value buyers (if they concede at all).

However, there are many properties that are standard in most reputational concession games, which do not immediately hold. First, the lemma does not claim that if the buyer reveals rationality but not the seller, then the buyer must concede. This is not immediate because a rational buyer might not find the seller's demand acceptable $w > v - p_s$, and so will not concede. Second, it does not claim that the buyer's concession must be continuous after time 0. In fact, as I will argue shortly, the buyer's concession must sometimes be discontinuous after time 0 to compensate the seller for the utility she forgoes when the buyer exits.

Lemma 1. *In any equilibrium in the subgame at* 0^3 *after demands* p_s *and* p_b

(a) *It is without loss of generality to assume* $\bar{g}(v, w) = 0$ *if* $v - w < p_b$ *(henceforth, this is assumed throughout).*

(b) *If* $x = 0$*, as when* $\Theta^c = \emptyset$*, then without loss of generality* $F_s(0^3) = 1 - \bar{z}_s$*,* $E_b(0^3) = 0$*,* $E_b(0^4) = 1 - \bar{z}_b$*.*

(c) *If* $p_s \in P_s$ *then* $T^* < \infty$*.*

(d) *If* $p_s \notin P_s$*, but* $p_b \in P_b$*, then* $F_s(0^4) = 1$ *(and so clearly* $T^* < \infty$*).*

(e) *If* $p_s \in P_s$*,* $p_b \notin P_b$ *and* $x = 1$ *then* $F_b(0^4) = 1$*.*

(f) *If* $p_s \in P_s$*, then* $F_b(T^*) = (1 - \bar{z}_b)x$ *and* $E_b(T^*) = (1 - \bar{z}_b)(1 - x)$*. Similarly, if* $p_b \in P_b$ *then* $F_s(T^*) = (1 - \bar{z}_s)$*.*

(g) *If* F_s *jumps at* $t \geq 0^3$ *then* F_b *and* E_b *are constant on* $[t - \varepsilon, t]$ *for some* $\varepsilon > 0$*.*

(h) F_s *is continuous at* $t > 0^4$*.*

(i) *If* F_s *is continuous at* t *then so is* $U_b^{c,v,w}$ *and* $U_b^{e,v,w}$*. Likewise if* F_b *and* E_b *are continuous at* t *then so is* U_s*.*

(j) F_s *and* F_b *are strictly increasing on* $(0, T^*]$*.*

(k) *The skimming property holds: if a buyer with value* v *concedes at* t *then a buyer with value* $v' > v$ *will not concede after* $\max\{t, 0^4\}$*.*

Assume some rational buyer types find the seller's commitment price acceptable, $p_s \in P_s$ and $\Theta^c \neq \emptyset$ (Lemma 1 shows that otherwise the seller must immediately concede). Notice that if $(v, w) \in \Theta^c$ then certainly $(v, \underline{w}) \in \Theta^c$, so we can enumerate values $v > p_s + \underline{w}$ as follows. First let $v^1 = \min\{v \in V : v > \underline{w} + p_s\}$, then define $v^{k+1} = \min\{v \in V : v > v^k\}$ until, for some $K < \infty$ we have $v^K = \bar{v}$. If $v^1 > \underline{v}$, it is also useful to define $v^0 = \max\{v \in V : v < v^1\}$. Let $t^{K+1} = 0^4$, and for $k \in \{1, ..., K\}$ define

$$t^k = \min\{t \geq 0^4 : F_b(t) \geq (1 - \bar{z}_b) \sum_{(v,w) \in \Theta^c : v \geq v^k} \overline{g}(v, w)\}.$$

Given the skimming property (Lemma 1, part (k)), this is the first time by which all agents $(v^k, w) \in \Theta^c$ have conceded. Next define

$$\lambda_s^v := \frac{r(v - p_s)}{p_s - p_b}, \quad \underline{\lambda}^{v,w} := \frac{rw}{v - p_b - w}, \quad \lambda_b := \frac{rp_b}{p_s - p_b}.$$

Lemma 2 shows that the seller and buyer must respectively concede at rates $\lambda_s^{v^k}$ and λ_b on the interval (t^{k+1}, t^k), moreover, a buyer that finds the seller's price unacceptable, $(v, w) \in \Theta^e$ exits at time t^k if $\underline{\lambda}^{v,w} \in (\lambda_s^{v^k-1}, \lambda_s^{v^k})$. The logic behind this result is again fairly standard from the fact that each bargainer's concession probability increasing on $(0, T^*]$; for both bargainers to be indifferent to conceding over a dense set of times in (t^{k+1}, t^k), tightly pins down their concession behavior. Given the seller's concession behavior, the time that a buyer in Θ^e must exit can be immediately deduced.

Lemma 2. *Consider the subgame at 0^3 after demands $p_s \in P_s$ and $p_b \in P$ with $\theta^c \neq \emptyset$. On the interval (t^{k+1}, t^k) for $k \in \{1, ..., K\}$ the seller concedes at the rate $f_s(t)/(1-F_s(t)) = \lambda_s^{v^k}$ while the buyer concedes at rate $f_b(t)/(1 - E_b(t^{k+1}) - F_b(t)) = \lambda_b$. Furthermore, for $(v, w) \in \Theta^e$, if $\underline{\lambda}^{v,w} > \lambda_s^{v^k}$ then (v, w) exits before t^{k+1}, whereas if $\underline{\lambda}^{v,w} < \lambda_s^{v^k}$, then (v, w) exits after t^k.*

Lemma 3 then characterizes the probability that the buyer concedes at time t^k when the buyer exits the market with some probability. If $t^k \in (0, T^*)$ we must have that concession makes the seller exactly indifferent between conceding an instant before and an instant after t^k. However, if $t^k = T^*$ then we only get an upper bound on the probability of concession and if $t^k = 0^4$ we only get a lower bound on the probability of concession. The lemma also establishes that it is without loss to assume buyer never concedes or exits at 0^3 and the seller never concedes at 0^4.

Lemma 3. *Consider the subgame at 0^3 after demands $p_s \in P_s$ and $p_b \in P$ with $\theta^c \neq \emptyset$. It is without loss of generality to assume the buyer never concedes or exits at 0^3 and the seller never concedes at 0^4. Given this, suppose that at time $t > 0^3$ the buyer exits with conditional probability $\alpha = (E_b(t) - E_b(t_-))/(1 - F_b(t_-) - E_b(t_-))$ and concedes with conditional probability $\beta = (F_b(t) - F_b(t_-))/(1 - F_b(t_-) - E_b(t_-))$. If $\alpha p_b < \beta(p_s - p_b)$ then $t = 0^4$ and the seller cannot concede at 0^3. If $\alpha p_b > \beta(p_s - p_b)$ then $t = T^*$.*

3.2 Demand choice and equilibrium existence

This subsection considers bargainers demand choices. It shows that an equilibrium exists and further characterizes the buyer's demand choices, in particular showing that buyer types who find the seller's price unacceptable, $w > v - p_s$, always imitate the lowest commitment demand $\underline{p}$.

The preliminary analysis of the previous subsection shows that an equilibrium in the continuation game at 0^3 must look like something like Figure 1. Notice that in this example, both bargainers concede at time 0, with the seller conceding at 0^3 and the

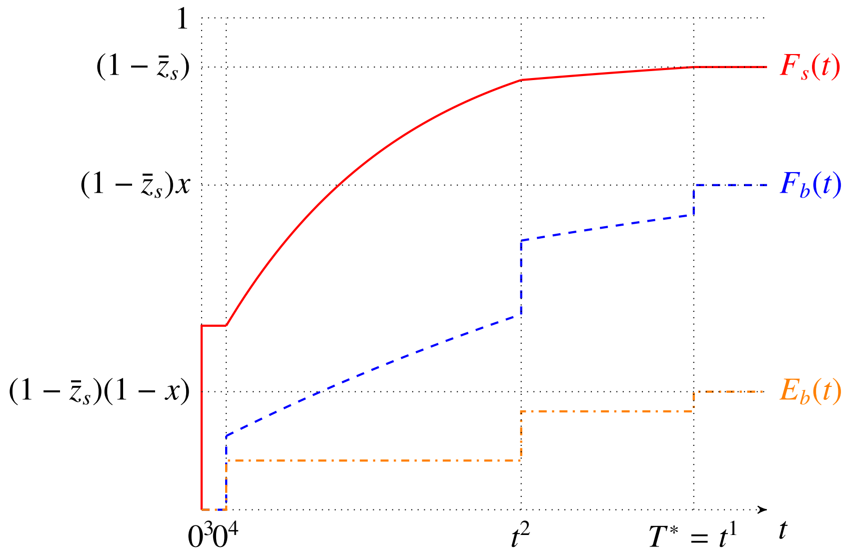

Figure 1. A straightforward equilibrium

buyer at 0^4. The fact that the buyer also exits at 0^4 makes the seller indifferent between conceding at 0^3 or later.

The equilibrium highlighted is not necessarily unique, however. The beliefs used in the figure, $(\bar{z}_s, \bar{z}_b, \bar{g})$, allow for many other equilibria. For instance, there is an equilibrium where the buyer does not concede with positive probability at time T^* (which is then different), but the equilbrium need not be unique even if there is no exit at T^* (depending on which types concede at each time t^k to compensate any buyer exit). For now, I am not interested in distinguishing between these equilibria, but simply want to show that one always exists.

I do this by first identifying a particular kind of continuation equilibrium in the subgame at 0^3, given demands $p_s \in P_s$ and $p_b \in P_s$, which I call a "straightforward" continuation equilibrium. Loosely it is one that looks slightly like Figure 1, so that buyer concession at T^* makes the seller indifferent to conceding an instant after T^*. It is defined by working backwards from time T^* when all agents have reached a probability one reputation. The particular details are not very important, beyond the fact that there is a unique straightforward continuation equilibrium, in which agents' payoffs are continuous in their beliefs $(\bar{z}_s, \bar{z}_b, \bar{g})$.

I then turn to demand choices for sellers and buyers at 0^1 and 0^2, and show that an equilibrium always exists. This is a simple identification of a fixed point using Kakutani, assuming there is always straightforward continuation play from time 0^3.

Proposition 1. *An equilibrium exists.*

The next Lemma is one of the most important in the paper and helps significantly clarify buyer's demand choice: it shows that rational buyers who find the seller's price unacceptable, Θ^e, always make the lowest commitment demand $\underline{p}$. The lemma reintroduces the superscripts p_b on variables where necessary. Combined with Lemma 1, this means that if the seller imitates a commitment demand and the buyer doesn't ($p_s \in P_s$ and $p_b \notin P_b$), then the buyer must immediately concede.

Lemma 4. *Consider the subgame $(p_s, \bar{z}_s, z_b, g, \Theta, P_b, P, \pi_s)$ at 0^2, after some demand $p_s \in P_s$. Let $p_b \in P_b$, and $p_b < p_b' \in P$ then: all types $(v, w) \in \Theta$ weakly prefer to demand p_b than p_b' and without loss of generality, $\sum_{(v,w)\in\Theta^e} \mu_b^{v,w}(p_b') = 0$. Furthermore, if $\sum_{(v^k,w)\in\Theta^c} \mu_b^{v^k,w}(p_b') > 0$, then $\sum_{(v^k,w)\in\Theta^c} \mu_b^{v^k,w}(p_b) > 0$ and $F_s^{p_b}(0^4)(p_s - p_b) = F_s^{p_b'}(0^4)(p_s - p_b')$.*

A driving force behind the lemma is that buyers who find the seller's price acceptable, Θ^c, must mix indifferent between some subset of demands, such that if they demand p_b' with positive probability, they must also choose the commitment demand $p_b < p_b'$ with positive probability. If this wasn't true, a rational seller would believe a buyer who demanded p_b is either committed or will eventually exit, and so the seller would immediately concede. However, in that case, all rational buyer types prefer the lower price, $p_b < p_b'$.

Since a value $\overline{v}$ buyer always concedes first (by the skimming property), she must be indifferent between demanding p_b and p_b' and then immediately conceding at $0^4 = t^{K+1,p_b}$, and so there must be less immediate seller concession after p_b than after p_b'; $F_s^{p_b'}(0^3) = F_s^{p_b}(0^3)(p_s - p_b)/(p_s - p_b') \geq F_s^{p_b}(0^3)$.

The next lowest value buyer v^{K-1} who finds the seller's price acceptable, must then either be indifferent between (i) demanding p_b and conceding at t^{K,p_b} and (ii) demanding p_b' and conceding at $t^{K,p_b'}$, or strictly prefer option (i). If she strictly preferred option (ii) then so would all lower value buyer types who find the seller's price acceptable, and so the seller would concede at rate $\lambda_s^{\overline{v},p_b}$ on $(0, T^{*,p_b'})$ and a rate of at most $\lambda_s^{\overline{v},p_b}$ on $(0, T^{*,p_b'})$. Other things equal, the seller concedes more slowly when facing a lower price, $\lambda_s^{\overline{v},p_b} < \lambda_s^{\overline{v},p_b'}$, and so $T^{*,p_b} = t^{K,p_b} > t^{K,p_b'}$. A buyer with value v^{K-1} prefers to receive concession at the rate $\lambda_s^{\overline{v},p_b}$ after demanding p_b than at the higher rate $\lambda_s^{\overline{v},p_b'}$ after demanding $p_b' > p_b$, $(v^{K-1} - p_b)\lambda_s^{\overline{v},p_b} > (v^{K-1} - p_b)\lambda_s^{\overline{v},p_b'}$, because delay is less costly for her than for $\overline{v}$ who is indifferent between them. Given that concession also lasts longer after p_b than p_b', $t^{K,p_b} > t^{K,p_b'}$, a value v^{K-1} buyer must therefore certainly prefer option (i), a contradiction. Extending this logic, for v^{K-1} to be indifferent between options (i) and (ii) we need $e^{-rt^{K,p_b'}}(1 - F_s^{p_b'}(t^{K,p_b'})) \leq e^{-rt^{K,p_b}}(1 - F_s^{p_b}(t^{K,p_b}))$, with a strict inequality when $F_s^{p'_b}(t^{K,p'_b}) > 0$; effectively there must again be less concession after the smaller price. This argument extends to $k < K - 1$.

Given this, a buyer who would choose to exit at time t^{k,p'_b} after the demand p'_b, must have a profitable deviation of instead exiting at time t^{k,p_b} after demanding p_b, whenever $F_s^{p'_b}(t^{k,p'_b}) > 0$; if $F_s^{p'_b}(t^{k,p'_b}) = 0$ then $t^{k,p'_b} = 0^4$ and without loss, this buyer could exit at 0^2 instead. This is because such a buyer benefits more from exit than concession, $w > v - p_s$, and must either be indifferent between (i) demanding p_b and conceding at t^{k,p_b} or (ii) demanding p'_b and conceding at t^{k,p'_b} or strictly prefer option (i). The gain that the buyer gets from exiting instead of conceding at t^{k,p_b} after the demand p_b is $e^{-rt^{k,p_b}}(1 - F_s^{p_b}(t^{k,p_b}))(w - v + p_s) > 0$; she only gains if the seller hasn't already conceded. As argued above, however, if she is indifferent between options (i) and (ii), we need $e^{-rt^{k,p'_b}}(1 - F_s^{p'_b}(t^{k,p'_b})) > e^{-rt^{k,p_b}}(1 - F_s^{p_b}(t^{k,p_b}))$ and so she gains more from exit at t^{k,p_b} after p_b than at t^{k,p'_b} after p'_b. The simplest case with $t^{k,p'_b} = 0^4$ and $F_s^{p'_b}(0^3) = F_s^{p_b}(0^3)(p_s - p_b)/(p_s - p'_b) > 0$ is illustrative: the buyer's payoff is $(F_s^{p'_b}(0^4) - F_s^{p_b}(0^4))(w - v + p_s) > 0$ higher when exiting at 0^4 after the demand p_b instead of p'_b. The case in which the buyer strictly prefers option (i) to (ii), can be similarly dealt with by finding some time $\hat{t}^{p_b} \in (t^{k+1,p_b}, t^{k,p_b})$ which leaves the buyer indifferent between option (ii) and demanding p_b before conceding at $\hat{t}^{k+1}$.

3.3 Vanishing commitment and main results

This section presents the paper's main result, that with a rich set of buyer values and commitment types, as commitment vanishes, bargaining outcomes converge to approximately those that would arise if the seller could make any take-it-or-leave-it offer below $p^* = \max\{\underline{w}, \underline{v}/2\}$. To reach that result, however, I first present two supporting lemmas which partially characterize outcomes as commitment vanishes in the subgame at 0^3.

The first lemma identifies conditions under which the seller must immediately concede with probability approaching one in the subgame at 0^3. Assume the probability of seller commitment vanishes weakly faster than buyer commitment, then these conditions are: (a) the buyer's commitment probability does not vanish, $\lim_n \bar{z}_b^n > 0$; (b) the seller expects to face a buyer type $(v, w) \in \Theta^c$ with positive limit probability, $\lim_n \bar{g}^n(v, w) > 0$, whose offer is more generous than hers $p_b > v - p_s$; (c) or $p_b = \underline{p}$ and there is some type that exits only at T^*, that is, $\underline{\lambda}^{v',w'} < \lambda_s^{v^1}$ for some $(v', w') \in \Theta^e$. Part (b), highlights the key Coasean force which ultimately drives the main results: any possibility of a low value buyer who makes a generous offer, $p_b > v - p_s$, means that seller must immediately back down.

The logic for (a) and (c) is straightforward. For (a), since the buyer's probability of commitment doesn't vanish, we have that T^* is bounded above; it satisfies $\bar{z}_b = 1 - E_b(T^*) - F_b(T^*) \leq e^{-\lambda_b T^*}$. For (c), since type (v', w') always demands $\underline{p}$ and then waits until T^* to exit we have that $\bar{g}(v', w') \geq g(v', w') > 0$ and so again T^* is bounded above; it satisfies $\bar{z}_b + (1 - \bar{z}_b)\bar{g}(v', w') = 1 - E_b(T^*) - F_b(T^*) \leq e^{-\lambda_b T^*}$. If T^* is bounded above and the seller didn't immediately concede with probability approaching one, however, she could not reach a probability one reputation by T^* as required, $(1 - F_s(0^4)) \leq \bar{z}_s e^{\lambda_s^{\bar{v}} T^*}$.

The logic for (b), is that by some finite time t' that is bounded above, satisfying $1 - E_b(t') - F_b(t') \leq (1 - \bar{z}_b)\bar{g}(v', w') = e^{-\lambda_b t'}$, the seller (she) must believe that any remaining buyer (he) who finds her price acceptable, is making a more generous offer than her own, $p_b > v - p_s$. To avoid the argument for (a) and (c), we must have $T^* \to \infty$. And so on the expanding interval (t', T^*), the seller must concede at a slower rate than the buyer $\lambda_s^v < \lambda_b$ which causes her reputation to grow at an exponentially slower rate, $d\bar{z}_i(t)/dt = \lambda_i \bar{z}_i(t)$. The seller must again, therefore, immediately concede with probability approaching one to ensure she reaches a probability one reputation at the same time as the buyer, T^*.

Lemma 5. *Consider some fixed demands $p_s \in P_s$ and $p_b \in P_b$ and rational buyer type distribution (g, Θ), and a sequence of bargaining subgames at 0^3 with $(\bar{z}_i^n, \bar{g}^n)$ which satisfy $\bar{z}_s^n \to 0$ and $\bar{z}_s^n / \bar{z}_b^n \leq L$ for some constant $L \geq 1$. Suppose that (for some subsequence if necessary) either:*

(a) or $\lim_n \bar{z}_b^n > 0$,

(b) $\lim_n \bar{g}^n(v', w') > 0$ for some $(v', w') \in \Theta^c$ with $v - p_s < p_b$,

(c) or $p_b = \underline{p}$ and $\underline{\lambda}^{v', w'} < \lambda_s^{v^1}$ for some $(v', w') \in \Theta^e$,

then, $\lim_n F_s^n(0) = 1$.

The next lemma identifies somewhat analogous conditions under which the buyer must immediately concede or exit with probability approaching one in the subgame at 0^3, assuming the probability of buyer commitment vanishes weakly faster than seller commitment. The first two conditions concern counterdemands $p_b > \underline{p}$, which are made only by buyers that eventually concede, $v - p_s > w$: (a) says that the seller's commitment probability does not vanish, $\lim_n \bar{z}_s^n > 0$; (b) says that any buyer's offer is less generous than the seller's, $p_b < v^1 - p_s$. The logic for these conditions is identical to that of conditions (a) and (b) in Lemma 5.

Condition (c) considers the counterdemand $p_b = \underline{p}$ and is made up of three subconditions: (i) the buyer's offer is less generous than the seller's, $p_b < v^1 - p_s$; (ii) it also sufficiently small that the seller would strictly prefer to concede an instant after time t than an instant before, if at t she expected all agents with types $(v^1, w) \in \Theta^c$ to concede and all agents in Θ^e to exit; and (iii) all types which find the seller's price unacceptable, exit before T^*, $\underline{\lambda}^{v',w'} > \lambda_s^{v^1}$ for all $(v', w') \in \Theta^e$. If the buyer did not immediately concede or exit when (c) is satisfied, then at time t^2 the only remaining rational buyers have $(v^1, w) \in \Theta^c$, and the probability the buyer has conceded is bounded away from x, so that her updated reputation for commitment $\bar{z}_b(t^2)$ vanishes at the same rate as $\bar{z}_b$. From that point on, however, the buyer concedes slower than the seller, and so her reputation could not reach a probability one reputation by T^* by the same logic as in (b).

Lemma 6. *Consider some fixed demands $p_s \in P_s$ and $p_b \in P_b$ and rational buyer type space Θ, and a sequence of bargaining subgames at 0^3 with $(\bar{z}_i^n, \bar{g}^n)$ which satisfy $\bar{z}_b^n \to 0$ and $\bar{z}_b^n / \bar{z}_s^n \leq L$ for some constant $L \geq 1$.*

(a) If $p_b > \underline{p}$ and $\lim_n \bar{z}_b^n > 0$, then $\lim_n F_b(0^4) = 1$.

(b) If $p_b > \underline{p}$ and $v^1 - p_b > p_s$, then $\lim_n F_b(0^4) = 1$.

(c) If $p_b = \underline{p} < v^1 - p_s$ and $p_b(1 - \lim_n x^n) < (p_s - p_b) \lim_n \sum_{(v^1,w) \in \Theta^c} \bar{g}^n(v^1, w)$ and $\underline{\lambda}^{v,w} > \lambda_s^{v^1}$ for all $(v', w') \in \Theta^e$, then $\lim_n F_b(0^4) = \lim_n x^n$ and $\lim_n E_b(0^4) = 1 - \lim_n x^n$

We are closing in on the main result, but must first more formally define what makes the sets of buyer values and commitment types rich. I say that the set of buyer values is $\varepsilon \geq 0$ rich if for any $d \in [\underline{v}, \overline{v}]$, there exists some $v \in V$ such that $|v - d| < \varepsilon$. This means that the difference between two consecutive buyer values must be less than 2ε. Given a rational buyer's type distribution, I say that the sets of agents commitment types is $\varepsilon' > 0$ rich if for any $d' \in [0, \overline{v} - \underline{w}]$, there exists some $p_i \in P_i$ such that $|p_i - d'| < \varepsilon'$ for $i = 1, 2$. Also define $H(p) = \sum_{(v,w):v-w<p} g(v, w)$ for any $p \in [0, \infty)$, as the fraction of rational buyers that find the price p unacceptable, $v - p < w$.

Proposition 2 bounds the seller's payoffs as bargainers' probability of commitment vanishes at the same rate. It says that for any $\delta > 0$ and any distribution of rational buyer types, where the set of values is ε rich, there exists $\varepsilon' > 0$ such that if the distribution of commitment types is ε' rich, the seller's limit payoff is a most δ better than what she would get if she could choose any take-it-or-leave-it price below $p^* + 2\varepsilon$ where $p^* = \max\{\underline{w}, \underline{v}/2\}$. However, that limit payoff cannot be more than 2ε worse than she would get if she could choose any take-it-or-leave-it price below p^*. Clearly if ε is small then this tightly pins down seller's payoff.

The key ideas behind the upper bound on seller payoffs are as follows: All rational buyer types that find the seller's price unacceptable, counterdemand $\underline{p}$, implying an upper bound of $(1 - H(p_s))p_s + \underline{p}$ for seller payoffs after demanding p_s, where $\underline{p} < \delta$ if the set of commitment types is $\varepsilon' \approx 0$ rich. If the seller's payoff were to exceed the upper bound in the Proposition, therefore, she must demand $p_s \in (p^* + 2\varepsilon, \overline{v} - \underline{v})$ with positive limit probability. Moreover, we must have that p_s ensures that a rational never buyer exits at $T^{*,p_s,\underline{p}}$, that is $\lambda^{v^{1,p_s},p_s,\underline{p}} < \underline{\lambda}^{v,w,p_s,\underline{p}}$ for $(v,w) \in \Theta^{e,p_s}$, otherwise a rational buyer would always counterdemand $\underline{p} \approx 0$ and the seller would immediately concede (by Lemma 5, part (c))

Given $p_s \in (p^* + 2\varepsilon, \overline{v} - \underline{v})$ and a $\varepsilon' \approx 0$ rich set of commitment types, however, there is a demand $p_b < p^* + 2\varepsilon$ such that $v^{1,p_s} - p_s < p_b \in P_b$. If $(v^{1,p_s}, \underline{w})$ demands $p'_b \geq p_b$ with positive limit probability then bargainers updated reputations, $\overline{z}_i^{p'_b,p_s}$, converge to 0 at the same rate, and the buyer eventually concedes at a faster rate than the seller, so that the seller must immediately concede with probability approaching one (Lemma 5, part (b)). A similar outcome arises if $\lim_n \overline{z}_b^{p_b,p_s} > 0$ (by Lemma 5, part (a)). To avoid immediate seller concession to p_b, therefore, we need $\lim_n g^{p_b}(v^{1,p_s}, \underline{w}) = \lim_n \overline{z}_b^{p_b,p_s} = 0$. If $(v^{1,p_s}, \underline{w})$ demanded $p'_b > p_b$ with positive limit probability, her limit payoff would be $v^{1,p_s} - p'_b$, which is the same limit payoff she could get by demanding p_b and conceding at 0^4, as the seller must then immediately concedes to p_b with probability $(p_s - p'_b)/(p_s - p_b) < 1$. However, if $(v^{1,p_s}, \underline{w})$ demanded p'_b and then conceded at $t^{2,p_s,p_b} \to \infty$ she would do strictly better, as she strictly prefers to wait on $(0, t^{2,p_s,p_b})$ given that the seller concedes at a weakly faster rate than $\lambda_s^{v^{2,p_s},p_s,p_b}$. On the other hand if $(v^{1,p_s}, \underline{w})$ imitated demands $p'_b \in (\underline{p}, v^{1,p_s} - p_s)$ with positive limit probability, she would need to subsequently immediately concede by Lemma 6, part (b) for a payoff of $v^{1,p_s} - p_s$. That would also be true if she demanded $\underline{p}$ with probability one in the limit by Lemma 6, part (c) given $0 \approx \underline{p} < (p_s - \underline{p})g(v^{1,p_s}, \underline{w})$. In either case, however, $(v^{1,p_s}, \underline{w})$ would then benefit by instead demanding p_b and conceding at $t^{2,p_s,p_b} \to \infty$.

The key ideas behind the lower bound on payoffs are similar: Given any $p \leq p^*$, with a $\varepsilon' \approx 0$ rich set of commitment types there exists a commitment demand $p_s \in [p - 2\varepsilon, p)$ such that no rational buyers exit at $T^{*,p_s,\underline{p}}$, $\lambda^{v^{1,p_s},p_s,\underline{p}} < \underline{\lambda}^{v,w,p_s,\underline{p}}$ for $(v,w) \in \Theta^{e,p_s}$. Moreover, since $p_s \leq p^*$, for any counterdemand $p_b < p_s$, the seller always concedes at a faster rate than the buyer, $v^{1,p_s} - p_s > p_b$. If $(v^{1,p_s}, \underline{w})$ demands $p_b > \underline{p}$ with positive limit probability, therefore, she would subsequently need to immediately concede with probability approaching one for a limit payoff of $v^{1,p_s} - p_s$. However, that conclusion also holds if she instead demanded $\underline{p}$ with probability one in the limit, given Lemma 6, part (c) and $0 \approx \underline{p} < (p_s - \underline{p})g(v^{1,p_s}, \underline{w})$.

Proposition 2. *For any $\delta > 0$, and any $\varepsilon > 0$ rich distribution of rational buyer types (g, Θ), there exists some $\varepsilon' > 0$ such that for any sequence of bargaining games $(z_i^n, \pi_i, P_i, g, \Theta, P)_{i\in s,b}$ with a ε' rich distribution of commitment types, $z_i^n \to 0$ and $z_s^n/z_b^n \in [1/L, L]$ for some $L \geq 1$, the seller's payoffs V_s^n satisfy:*

$$\max_{p\in[0,p^*]}(1 - H(p))p - 2\varepsilon \leq \liminf_n V^n \leq \limsup_n V^n \leq \max_{p\in[0,p^*+2\varepsilon]}(1 - H(p))p + \delta.$$

While Proposition 2 characterizes seller limit payoffs, Proposition 3 characterizes limit outcomes (in particular buyer payoffs) more tightly under regularity conditions on the seller's profit function from take-it-or-leave-it offers, $(1 - H(p))p$. These are similar in spirit to requiring that $\arg\max_{p\leq p^*}(1 - H(p))p$ is unique, but are slightly stronger to account for the richness of the sets of buyer values and commitment types. In particular, define $\check{p}(p) = \mathbb{1}_{p\leq \underline{v}-\underline{w}}p + \mathbb{1}_{p>\underline{v}-\underline{w}} \max\{v - \underline{w} \leq p : v \in V\}$, so that $\check{p}(p) \in (p - 2\varepsilon, p]$ when the set of buyer values is ε-rich. If $\hat{p}_s \leq p^*$ is the seller's optimal take-it-or-leave-it price below $p^* + 2\varepsilon$ and the take-it-or-leave-it price $\check{p}(\hat{p}_s)$ is better than a price strictly below $\check{p}(\hat{p}_s)$ or in $(\hat{p}_s, p^* + 2\varepsilon]$ then the seller always proposes a price in $p_s \in [\hat{p}_s - 2\varepsilon, \hat{p}_s)$ in the limit, and the buyer either immediately accepts or exits.

The buyer must either immediately concede or exit following a demand just below $\check{p}(\hat{p}_s)$ by the same logic which gave the lower bound on seller profits in Proposition 2. The regularity condition ensures, that the seller's payoff from this demand is better than she could get from proposing any lower take-it-or-leave-it price, or any price in $(\hat{p}_s, p^*+2\varepsilon]$, which are in turn always better than demanding $p_s > p^* + 2\varepsilon$, which would result in the seller having to immediately concede to a counterdemand strictly below $p^* + 2\varepsilon$.

Proposition 3. *For any $\varepsilon > 0$ rich distribution of rational buyer types (g, P), there exists some $\varepsilon' > 0$ such that for any sequence of bargaining games $(z_i^n, \pi_i, P_i, g, \Theta, P)_{i\in s,b}$ with a ε' rich distribution of commitment types, $z_i^n \to 0$ and $z_s/z_b \in [1/L, L]$ for some $L \geq 1$, payoffs $V_s^{v,w,n}$ for buyer $(v, w) \in \Theta$ satisfy:*

$$\liminf_n V_s^{v,w,n} \geq \max\{w, v - p^* + 2\varepsilon\}.$$

Further suppose that $\hat{p}_s \in \arg\max_{p\in[0,p^*+2\varepsilon]}(1-H(p))p = \arg\max_{p\in[0,p^*]}(1-H(p))p$ and $\check{p}(\hat{p}_s)(1-H(\check{p}(\hat{p}_s))) > \max_{p\in[0,\check{p}(\hat{p}_s))\cup(\hat{p}_s,p^*+2\varepsilon]} p(1-H(p)$ then in the limit with probability approaching one, the seller proposes a price $p_s \in [\hat{p}_s-2\varepsilon, \hat{p}_s)$, all buyers with $v-w > p_s$ immediately accept and all buyers with $v - w < p_s$ immediately exit. Hence, buyer *payoffs satisfy*

$$\max\{w, v - \hat{p}_s\} \le \liminf_n V_s^{v,w,n} \le \limsup_n V_s^{v,w,n} \le \max\{w, v - \hat{p}_s - 2\varepsilon\}.$$

4 Extensions and discussion

This section discusses some of the implications of my model, and presents extensions of it. In particular, I show how results can be extended to a discrete time bargaining protocol, allow the choice to bargain to be endogenous, highlight the importance to buyers of being able to make counteroffers, and illustrate how the main results depend of the richness of the sets of agents' types.

4.1 Discrete time

Below I outline a discrete time bargaining protocol, which is a minimal modification of Board and Pycia (2014)'s protocol, where outcomes must converge to those of the concession game as offers become frequent. I also discuss the difficulties extending the results to still other protocols.

In period 1 the seller can propose a price $p_s \in [0, \infty)$. The buyer observes this and can then accept, reject, or exit (taking her outside option). If still bargaining in period $n \ge 2$, the buyer can propose a price $p_b \in [0, \infty)$. The seller observes this and can accept, or make a counterdemand $p_s \in [0, \infty)$. If the seller makes a counterdemand the buyer observes this and can accept, reject, or exit. If one of the players accepts, or exits, the game ends. If the price p is agreed in period n, a rational seller gets $\delta^{n-1}p$ and a rational buyer $\delta^{n-1}(v - p)$ where $\delta = e^{-r\Delta}$ for some period length $\Delta > 0$. If the buyer exits in period n, rational payoffs are 0 and $\delta^{n-1}w$ respectively. The description of types is unchange except now assume $\underline{p} = \min P_b < \min_{(v,w)\in\Theta} v - w$.

In this model, in any equilibrium the buyer never reveals rationality before the game ends. This is an immediate consequence of Lemma 1 from Board and Pycia (2014). The reason is that if a rational buyer did reveal rationality, a rational seller will never propose or accept a price strictly below that of the lowest net value type she still considers feasible. Given that, the lowest net value type's continuation payoff will be (weakly) less than her outside option w. Of course, if such a type faces a committed seller, her continuation payoff is below $\max\{v - p_s, w\}$. In either case, however, the buyer could have obtained the payoff $\max\{v - p_s, w\}$ in the previous period without it being discounted, hence, she would never wait.

On the other hand, for any $\varepsilon > 0$, there exists $\overline{\Delta} > 0$ such that if the seller has revealed rationality but the buyer has not at history h, then the buyer's continuation payoff is at least $\max\{v-p_b, w\}-\varepsilon$ and the seller's is at most $p_b+\varepsilon$. This follows almost immediately from Abreu and Gul (2000)'s Lemma 1, a reputational Coase conjecture argument. If the buyer doesn't modify her demand, eventually the seller must be convinced that she faces a commitment type at some time T^*, and must then concede to p_b, but in which case the buyer should not accept a price much greater p_b to start with. And so, as offers become frequent ($\Delta \to 0$) the game must converge to a concession game. Abreu and Gul (2000)'s Proposition 4, provides a complete proof of such a convergence result in a fixed surplus setting, e.g. one rational buyer type with $v > w = 0$.

What happens with more general bargaining protocols (Abreu and Gul (2000)'s result with a fixed surplus applies to almost any protocol)? Typically, there will still always be an equilibria which converges to a concession game. Assume the seller always believes she faces a buyer compatible with her commitment demand $p_s < \overline{v} - \underline{w}$ if the buyer reveals rationality, then such a buyer must concede almost immediately when offers are frequent. However, it is unclear whether one can rule out other equilibria: if the buyer reveals rationality but the seller has not, it could still be that the seller's commitment demand is incompatible with the rational buyer, which means that the buyer cannot always immediately concede with frequent offers. The ability to subsequently punish such a buyer with beliefs may then allow for outcomes that incentivizes the buyer to reveal her rationality without her then immediately conceding or exiting. In the protocol considered above, this was not a problem because the seller had all the bargaining power (making offers at the end of each period), so that revealing rationality was never worthwhile, preventing any role for off path beliefs punishments.

4.2 Endogenous entry

In this subsection, I explore the effect of the making participation in bargaining endogenous, first when there is negligible and then non-negligable delay between when the buyer can first take her outside option and when bargaining occurs. I find that the possibility highlighted in the introduction, of the seller's payoff increasing in the buyer's outside option, remains present in both cases. Bargaining outcomes with non-negligable delay, however, alway appear to be Coasean, in that there is always immediate agreement (as commitment vanishes) at a price of $\check{v}/2$ where $\check{v}$ is the lowest value buyer that chooses to participate. A hold-up problem, however, may mean that the outcome is very inefficient even when the delay before bargaining is small; the buyer may never choose to bargain despite large potential gains from trade.

In the baseline model, I assumed that a rational buyer type's value always exceeded her outside option, $v > w$, as otherwise such she would never accept any positive price for the good, and would immediately exit. This was helpful for the model exposition, and had no effect on the results since Lemma 1 showed that it was without loss of generality to assume type (v, w) always exits at time 0^2 if $v - w < \underline{p}$, which is certainly true if $v < w$. However, to fully appreciate the implications of outside options, it is useful to now consider more general type distributions that allow $v < w$, and let the choice to bargain be endogenous. Let this larger set of buyer types be $\hat{\Theta}$.

Suppose that buyers could take their outside option very slightly before the start of bargaining, the "negligible" delay case. If there is a rich set of commitment types, all positive net value buyer types will choose to accept this negligible delay because they receive payoffs strictly above their outside options when a seller commitment type demands $\min P_s < \min_{(v,w)\in\Theta} v - w$. This willingness to always (ever) accept negligible delay is in contrast to Board and Pycia (2014). In this case, $\underline{v} = \min\{v > \underline{w} : (v, \underline{w}) \in \hat{\Theta}\}$ is the lowest buyer value with a positive net value for the good that will choose to bargain, as in the baseline model.[8]

As highlighted in the introduction, such an endogenously determined lowest value $\underline{v}$ allows the seller's payoff (as commitment vanishes) to increase in the buyer's outside option. For instance if buyer values are approximately uniform on $[0, 1]$ and the outside option $w > 0$ is known, then net values are approximately uniformly distributed on $[-w, 1 - w]$ with $\underline{v} \approx w$. For $w \leq 0.33$ the seller will charge approximately $p^* = w$ for a payoff of $(1 - 2w)w$, which is increasing in w for $w \leq 0.25$. For $w \in [0.33, 1]$ she charges $(1 - w)/2$ for a payoff of $(1 - w)^2/4$. With no (or minimal) outside option, the seller would need to give away the good almost for free, $p^* \approx 0$.

What happens if the delay required before bargaining is non-negligable? The benefit to all buyer types from bargaining outlined above (occasional low prices $\min P_s$ from committed sellers) becomes vanishingly small as commitment vanishes, and on its own will not justify non-negligible delay in taking an outside option. Suppose, therefore, that the buyer (with a rich set of values) can either immediately take her outside option w, or wait to bargain with any bargaining payoffs then discounted by $\delta < 1$ (i.e. she has to wait $-ln(\delta)/r$ before getting to bargain). Assume a unique limit price $\check{p}$ is proposed and immediately accepted in equilibrium as commitment vanishes (arguments can be

[8] I continue to regard types with $v = w$ as non-generic.

extended to allow for mixing in the limit). The only buyer types which wait to bargain, therefore, are those with $w \leq \delta(v - \check{p})$, who actually end up purchasing the good; there's no point waiting to bargain only to end up taking the outside option. Let $\Theta^{\check{p}} = \{(v, w) \in \hat{\Theta} : w \leq \delta(v - \check{p})\}$ and $\underline{v}^{\check{p}} = \min\{v : (v, \underline{w}) \in \Theta^{\check{p}}\}$.

The limit bargaining outcomes must then be approximately those where the seller chooses any take-it-or-leave-it-price p_s below $p^{*,\check{p}} = \max\{\underline{v}^{\check{p}}/2, \underline{w}\}$ where for consistency we need $p_s = \check{p}$. Suppose that $p_s < p^*$ and $p_s < \underline{u}^{\check{p}} = \min\{v - w : (v, w) \in \Theta^{\check{p}}\}$ then the seller could profitably increase her demand. Hence, if $p_s < p^{*,\check{p}}$ we need $p_s = \underline{u}^{\check{p}}$, but we can never have $p_s = \underline{u}^{\check{p}}$ because then $w > \delta(v - p_s)$ for some $(v, w) \in \Theta^{\check{p}}$, a contradiction. And so we must have $p_s = p^{*,\check{p}} < \underline{u}^{\check{p}}$. If $p_s = p^{*,\check{p}} = \underline{w} \geq \underline{v}^{\check{p}}/2$ then $\underline{v}^{\check{p}} - p_s \leq \underline{v}^{\check{p}}/2 \leq \underline{w} < \underline{w}/\delta$, a contradiction to $p_s < \underline{u}^{\check{p}}$. Hence, we must have $\check{p} = p^{*,\check{p}} = \underline{v}^{\check{p}}/2 > \underline{w}$. And so, $\underline{u}^{\check{p}} > \check{p} = \check{v}/2$ where $\check{v} = \min\{v \geq 2\underline{w}/\delta : (v, w) \in \hat{\Theta}\} = \underline{v}^{\check{p}}$ is the lowest value buyer that chooses to bargain.

Fascinatingly, this selection of buyers suggests that bargaining outcomes may appear entirely Coasean, with immediate agreement by bargainers on the same price $\check{v}/2$ that would have been agreed if the bargainer was known to be her "toughest" type $(\check{v}, \underline{w})$, when in reality they are not, with substantial inefficiency due to many types that could have benefited from bargaining, with $\delta v > w$, failing to show up. This effect is partially due to a holdup problem: the delay incurred by the lowest value buyer is a sunk cost, and the seller can then hold her up for half of that value. Moreover, although the buyer's "small" outside option may seem irrelevant to bargaining (as in Binmore et al. (1989) under complete information), as there is always immediate agreement on a price that provides bargainers with strictly more than that outside option, $\check{v}/2 > \underline{w}$, in reality outside options may have a substantial effect on outcomes, by endogenously determining $\check{v}$.

To understand some of the implications of this, let's return to the example above, where buyer values are approximately uniform on $[0, 1]$ and the outside option $w \in (0, \delta/2)$ is known. We then have $\check{v} = 2w/\delta$ and the seller charges w/δ for expected profits of $(1 - 2w/\delta)w/\delta$. This payoff is increasing in w for $w \leq \delta/4$. If $w < 0.33$ then as $\delta \to 1$, the seller's payoff converges to her payoff without any delay, of $(1 - 2w)w$. Her payoff is decreasing in δ when $\delta \geq 1/(2 - 4w)$, so that for $w < 0.25$ the seller benefits from some delay, $\delta < 1$; this is because the greater exclusion of low value buyers allows her to charge a higher price, $w/\delta > w$, which outweighs the cost of a reduced probability of bargaining. For $w > 0.33$, even as $\delta \to 1$ the seller's payoff remains bounded below her no delay ($\delta = 1$) payoff of $(1 - w)^2/4$. This is due to the hold up problem: even with minimal delay, the seller cannot commit to the low price of $(1 - w)/2 < w < w/\delta$ she would charge without any delay. For $w \geq \delta/2$ this hold up problem is so severe that no bargaining ever takes place. This hold up problem,is not inherently due to incomplete information, however, with a similar prediction of no trade when the rational buyer is known to have $v = 1$ and $w > \delta/2$

4.3 No buyer offers

I now return to the model with exogenous participation in bargaining, and ask what happens if the buyer might be one of a rich set of commitment types, but cannot make offers (and must merely choose when to accept the seller's offers)? In this case, Board and Pycia (2014)'s prediction that the seller can choose any take-it-or-leave-it price should hold as commitment becomes vanishingly unlikely. This shows the clear benefits to buyers of being able to make (generous) offers to the seller. It might also suggest that while we should expect Coasean outcomes in bargaining when both sides can make offers, we might not when buyers must simply react to a seller's posted price.

In a discrete time game with no buyer offers and frequent seller offers, Abreu and Gul (2000)'s reputational Coase conjecture implies that if the seller reveals rationality she must almost immediately propose a price of $\underline{p} \approx 0 < \min_{(v,w)\in\Theta} v - w$. Hence, outcomes must effectively converge to a continuous time concession game where the seller can choose any commitment price from a rich set but the buyer can only choose $\underline{p}$ (the seller screens through commitment types that accept larger prices arbitrarily quickly). Given that, for any $d \in [\underline{v}, \overline{v}] - \underline{w}$, the seller can choose a price arbitrarily close to d which is more generous than $\underline{p} < v^1 - p_s$ for buyers who find it acceptable. And so, as the probability of commitment vanishes, the buyer must either immediately accept this price or exit, an outcome is consistent with Board and Pycia (2014).

Unlike my results when the buyer can make offers, the conclusion above depends on buyer outside options being strictly positive. If instead $\underline{w} = 0$ the seller would propose a price below $\underline{v}$ as commitment vanished (to ensure no buyer exited at T^* or waited forever). That alternative prediction is broadly consistent those of Inderst (2005), who assumes the buyer is always rational and cannot make offers (and so accepts $\underline{v}$), while the seller might be a commitment type.

4.4 Rich type space requirements

In this subsection I illustrate the need for the sets of buyer values and commitment types to be rich for my results to be meaningful.

I first highlight that if the set of buyer values is not sufficiently rich, prices may be much higher than the Coase conjecture would lead us to expect. My main result shows that as commitment vanishes rational seller's will never demand more than $p^* + 2\varepsilon$ as commitment vanishes when the set of buyer values is ε-rich (and there is a rich set of commitment types). For the set of values to be ε rich we need $\varepsilon > \max_{d\in[\underline{v},\overline{v}]} \min_{v\in V} |v - d|$. If the set of values is sparse, however, ε may be large.

For instance, binary values $v \in \{\underline{v}, \overline{v}\}$ are only ε rich if $\varepsilon > \overline{v} - \underline{v}$. As highlighted in the introduction, if we additionally assume $\min\{\overline{v}/2, 2\underline{w}\} > \underline{v}$ the seller can in fact charge $p_s = \overline{v}/2 \in (p^*, p^* + 2\varepsilon)$. This equals the highest price she could charge if the buyer was known to have value $\overline{v}$, even when $\overline{v}$ is very large. As commitment vanishes, any high value buyer immediately concedes to that demand, while any low value buyer immediately exits. This is because that demand is more generous than any counterdemand $p_b \in (\underline{p}, p_s)$ of the high value seller, $\overline{v} - p_s > p_b$, so the seller concedes and builds reputation faster. Moreover, after the ungenerous counterdemand $\underline{p} \approx 0$, a low value buyer never waits as the seller concedes at rate $\lambda_s^{\overline{v},\underline{p},p_s} = r(\overline{v} - p_s)/(p_s - \underline{p}) \approx r$ since $\lambda^{\underline{v},\underline{w}} = r\underline{w}/(\underline{v} - \underline{p} - \underline{w}) < r$. Introducing a third buyer value v' that is slightly higher than $\overline{v}/2 + \underline{w}$ rules out such a high price, because such a buyer could counterdemand slightly more than $\underline{w}$ while being more generous than the seller, $v' - p_s < p_b$, so that she builds reputation more quickly, meaning the seller would need to immediately concede.

The requirement of a rich set of buyer values is only needed because $\underline{w} > 0$. It is straightforward to extend my results to show that the buyer would choose her best take-it-or-leave-it offer below $p^* = \underline{v}/2$ if $\underline{w} = 0$ regardless of the richness of buyer values. In this case, if the seller made a demand $p_s > \underline{v}$ with positive limit probability, then buyers with type $(\underline{v}, 0)$ would demand $\underline{p}$ and then wait until at least T^* to exit. In that case, the seller would immediately have to concede with probability approaching one, by the same logic as Lemma 5 part (c). If $p_s \in (\underline{v}/2, \underline{v})$, however, then $(\underline{v}, 0)$ can counterdemand $p_b \approx \underline{v} - p_s < \underline{v}/2$, to which the seller would again have to immediately concede by Lemma 5 part (b).

The assumption that there is a positive probability of the lowest buyer outside option is also implicitly an assumption that the set of types is rich, without which the main result need not hold. For instance, suppose that when the buyer's value is v, her outside option is known to be $w = \alpha v$ for some $\alpha \in (0.5, 1)$. Further assume that the buyer's value is approximately uniformly distributed on $[1, \beta]$ for some $\beta > 2\alpha/(1 - \alpha)$. This example does *not* satisfy our assumption as a buyer with $v > 1$ can never have the lowest outside option $\underline{w} = \alpha$. In this case, as commitment vanishes outcomes are approximately those where the seller makes her first best take-it-or-leave-it offer of $p_s = \beta(1 - \alpha)/2 > p^* =$

$\underline{w} = \alpha$. The reason why prices can be greater than p^* is that lowest value buyer which finds this price acceptable, $v \approx \beta/2$, receives $v - p_s \approx \alpha\beta/2$ when she accepts, which is greater than $p_s \approx \beta(1-\alpha)/2$ and so also greater than any counterdemand $p_b < p_s$. And so, the buyer's reputation will grow slower than the seller's and she must immediately concede or exit at time 0. Of course, in we instead assume that with probability $\varepsilon \approx 0$ the buyer's outside option is α, and with probability $1 - \varepsilon$ it is αv then the main result again applies, and the seller would instead charge approximately α. In that case, if the buyer tried to charge $p_s \approx \beta(1-\alpha)/2$ then with positive probability, the buyer of type $(v', \underline{w})$ with $v' - p_s \approx \underline{w}$ finds this price acceptable and will make a more generous counterdemand, $v' - p_s < p_b \approx \underline{w}$ that ensures the seller must immediately concede.

The result also depends on a rich set of commitment types, and in particular buyer commitment types that make ungenerous offers $\underline{p} \approx 0$. All types of rational buyer could benefit if they were constrained to make more generous offers, $\underline{p} >> 0$. This point is complimentary to the one made in the previous subsection, that the buyer does worse if unable to make (generous) offers.

Consider the following example highlighting the potential benefit to the buyer of being unable to make low offers. The buyer has value 5, or 6 each with probability 0.24, value 13 with probability 0.48, and values $\{7, 8, ..., 12\}$ each with probability 1/150, and a known outside option of $w = 3 = p^*$. With a rich set of commitment types we must have an upper bound on seller prices is 4. However, the seller will in fact charge just less than $p^* = 3$ for a limit payoff of 2.28 (the buyer with value 5 immediately exits).

On the other hand, if the buyer can only imitate commitment prices greater than $\underline{p} = 1.5$, there are multiple equilibrium limits, which include one where the seller always proposes a price just below 2 (as well as one similar to the limit when $\underline{p} \approx 0$ where the seller makes take-it-or-leave-it price just below 3). To see how the low price equilibrium hangs together, notice that a rational seller certainly prefers to charge just below 2 instead of $p_s \in (2, 2.63)$ as those prices risk much higher disagreement with little benefit. After demands $p_s \in (2.63, 3)$, that are never made only in the limit, the buyer immediately concedes with probability approximately $(2 - \underline{p})/(p_s - \underline{p}) < 0.44$ to make the seller indifferent between this demand and demanding just below 2. With the residual limit probability the buyer demands $\underline{p}$. Buyer and seller then concede continuously at rates $\lambda_s^{p_s, \underline{p}, \bar{v}}$ and $\lambda_b^{p_s, \underline{p}}$ until time T^*, at which the buyer exits with probability 0.24 (in the limit, when she has value $\underline{v} = 5$) and concedes with probability $0.24\underline{p}/(p_s - \underline{p})$. This concession is much larger when $\underline{p} >> 0$, which helps the buyer build her reputation. It is also important for this construction (and can be easily verified) that the buyer with value $\underline{v}$ prefers to wait to concede at time T^* given the high initial rate of concession

$$\lambda_s^{p_s,\underline{p},\bar{v}} > \underline{\lambda}_b^{p_s,\underline{p},\underline{v},w}.$$

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A Proofs of results

Proof of Lemma 1. For (a), suppose that $\mu_b^{p_s,v,w}(p_b) > 0$ for some $v - w < p_b$, then such an agent would certainly always exit before 0^4 as her best payoff in the continuation game is less than w, and if the buyer ever conceded to her with positive probability she would have a strictly profitably deviation of conceding at 0^2 instead.

For (b), notice that since buyer can never concede in equilibrium, without loss of generality, $F_b(t) = 0$. Suppose that $F_s(t) < 1 - \bar{z}_s$ for $t > 0$ then deviating to concede at 0^3 is always a profitable deviation. This deviation would also be profitable for the seller if she conceded at 0^4 and $E_b(0^4) > 0$, while if $E_b(0^4) = 0$ then it is still weakly better for the seller to concede at 0^3 than 0^4, hence in all cases we may assume $F_s(0^3) = 1 - \bar{z}_s$. If $F_s(0^3) > 0$, then clearly $E_b(0^3) = 0$ (as exit at 0^4 would be a profitable deviation for the buyer given (a)). If $\bar{z}_s < 1$, then we must, however, have $E_b(0^4) = 1 - \bar{z}_b$ given $w > 0$.

The logic for (c) is standard. If $p_s \in P_s$, then if a rational buyer does not concede, she must believe her opponent will concede shortly afterwards, and so increase her belief that the buyer is committed if there is no concession. Repeating this argument, the buyer must eventually become convinced of her opponent's commitment by some time $T^* < \infty$ and will then concede or exit.

The reasoning for (d) is similar: given $p_b \in P_b$ if the seller does not immediately concede, she must eventually become convinced of the buyer's commitment by some $T^* < \infty$ and will then concede. Given that, however, no rational buyer will concede to her on $[T^* - \varepsilon, T^*]$ for sufficiently small $\varepsilon > 0$ (strictly preferring to wait for the buyer's concession), implying that she must have conceded by $T^* - \varepsilon$, a contradiction. The logic for (e) is analogous.

For the first part of (f), notice that at time T^*, the buyer knows that the seller is committed to her demand and so will immediately either concede or quit. For the second part, the seller likewise knows that the buyer is committed to her demand at T^*, and so will immediately concede.

For (g), we can assume that $v - p_b > w$ for all buyers by (a). Clearly we also have $v - p_b > v - p_s$ given $p_b < p_s$. Hence, given the seller's positive concession at t, the buyer would strictly prefer to concede, or respectively exit, an instant after time t than on $[t - \varepsilon, t]$ for $\varepsilon > 0$ small. Given (g), if F_s jumped at $t > 0^4$ then F_b is constant on $[t - \varepsilon, t]$, and hence the seller would prefer to concede strictly before t, implying (h). Notice that (i) is immediate from the definitions.

Suppose that (j) did not hold, so that $F_i(t) = F_i(t')$ for some $0 < t < t' \leq T^*$ and i. Let $t_i^* = \sup\{\tau : F_i(\tau) = F_i(t)\}$. Clearly, agent j will not concede at $\tau \in (t, t_i^*)$ as conceding slightly beforehand would strictly improve her payoff, and hence $t_s^* = t_b^*$. As F_s and hence $U_b^{c,v,w}$ is continuous after t, conceding at or slightly after t_b^* delivers a strictly lower buyer payoff than conceding at some point on the interval (t, t_i^*). Hence, t_b^* cannot be the supremum, a contradiction.

For (k), given that conceding at t is optimal for type (v, w) we can assume $t \geq 0^4$ and that F_s is continuous at t (if the seller conceded with positive probability at 0^3 or 0^4 then the buyer wouldn't), and at t' by (g) and (h). So let $D(v) = U_b^{v,w,c}(t) - U_b^{v,w,c}(t')$ for $t' > t$:

$$D(v) = -\int_{\tau \in (t,t')} (v - p_b) e^{-r\tau} dF_s(\tau) + (v - p_s)\left((1 - F_s(t))e^{-rt} - (1 - F_s(t'))e^{-rt'}\right) \geq 0$$

Notice that

$$\begin{aligned} dD(v)/dv &= -\int_{\tau \in (t,t')} e^{-r\tau} dF_s(\tau) + (1 - F_s(t))e^{-rt} - (1 - F_s(t'))e^{-rt'} \\ &\geq \left(1 - \frac{(v - p_s)}{(v - p_b)}\right)\left((1 - F_s(t))e^{-rt} - (1 - F_s(t'))e^{-rt'}\right) > 0 \end{aligned}$$

where the first inequality uses $D(v) \geq 0$, and the second uses $(1 - F_s(t))e^{-rt} > (1 - F_s(t'))e^{-rt'}$ and $p_s > p_b$. Hence, $D(v') > 0$. □

Proof of Lemma 2. By Lemma 1 part (j), F_i is strictly increasing on $(0, T^*)$. This implies that if $T^* > 0$, the set of times O_i^c at which it is optimal for some type of agent i to concede, must be dense in $(0, T^*) \cap (t^{k+1}, t^k)$. By the skimming property only types $(v^k, w) \in \Theta^c$ concede on (t^{k+1}, t^k). By Lemma 1 parts (h) and (i), we also have that F_s and so $U_b^{c,v,w}$ are continuous at $t > 0$. Combined with the density of O_b^c in (t^{k+1}, t^k) therefore, we must have that $U_b^{c,v^k,w}$ is differentiable on that interval, with a derivative equal to zero, $dU_b^{c,v^k,w}(t)/dt = 0$. This immediately implies that the seller must concede at rate $\lambda_s^{v^k}$ on that interval.

A buyer of type $(v, w) \in \Theta^e$, with $\underline{\lambda}^{v,w} > \lambda_s^{v^k}$ prefers to exit earlier on (t^{k+1}, t^k) than later, as the inequality implies $dU_b^{e,v^k,w}(t)/dt < 0$ on that interval. Moreover, given the skimming property and the continuity of $U_b^{e,v^k,w}$ at $t > 0$ (parts (h) and (i) of Lemma 1), such a buyer would prefer to concede at some point in (t^{k+1}, t^k) than at any later time (as any buyer concession after t^k is at an even slower rate). Likewise, if $\underline{\lambda}^{v,w} < \lambda_s^{v^k}$ for $(v, w) \in \Theta^e$, then such a buyer prefers to concede later on (t^{k+1}, t^k) than earlier as $dU_b^{e,v^k,w}(t)/dt > 0$. Given the skimming property, therefore, she will certainly not concede before t^k (as any buyer concession before t^{k+1} is even faster).

I next claim that F_b is continuous on (t^{k+1}, t^k). If F_b jumped at some $t \in (t^{k+1}, t^k)$, then F_s would necessarily be constant on $[t - \varepsilon, t]$, for some small $\varepsilon \in (0, t - t^{k+1}]$, because we have established that the buyer will not exit on (t^{k+1}, t^k), while the seller prefers that the buyer concedes to her, rather than that she concedes. This, however, would contradict the required seller concession rate of $\lambda_s^{v^k}$ on that interval.

Given the continuity of F_b and E_b on (t^{k+1}, t^k), U^s is also continuous, by Lemma 1 part (i). Combined with the fact that O_s^c is dense in (t^{k+1}, t^k), we must then have $dU_s(t)/dt = 0$ and so the buyer must concede at rate λ_b. □

Proof. 3 First, suppose that a seller conceded with positive probability at time 0^4, then certainly a rational buyer cannot concede or exit at 0^3 or 0^4 (or the buyer would strictly prefer to concede or exit an instant after 0^4). Hence, outcomes are not affected by switching such seller concessions to time 0^3. Likewise, if the buyer conceded or exited at 0^3, then certainly the seller cannot concede at 0^3 or 0^4, or the buyer's decision would not be optimal. Hence, outcomes are not affected by moving any buyer concession or exit to time 0^4.

Now suppose $\alpha p_b < \beta(p_s - p_b)$ at $t \in (0, T^*]$, then by Lemma 2, we must certainly have $t \in \{t^{K+1}, ..., t^1\}$, and since F_b has at most finitely many jumps at times $t^K, ..., t^1$, there exists some $\varepsilon > 0$ such that the seller would prefer to concede an instant after t than on $(t - \varepsilon, t]$ to receive that concession. This would contradict that F_s is strictly increasing on $(0, T^*)$, Lemma 1 part (j), and hence if $\alpha p_b < \beta(p_s - p_b)$ then $t = 0^4$ (we have already argued that it is without loss to assume no buyer concession or exit at 0^3). Clearly, in this case seller cannot find it optimal to concede at 0^3 (or 0^4).

Now suppose instead $\alpha p_b > \beta(p_s - p_b)$, then the seller would prefer to concede at t compared to conceding on $(t, t + \varepsilon]$ for some sufficiently small $\varepsilon > 0$. By Lemma 1 part (j), F_s must be strictly increasing on $(0, T^*)$, and so we must have $T^* = t$. If $t = 0^4$, a rational seller would strictly prefer to concede at 0^3 rather than at 0^4 or $(0, \varepsilon)$, where we can assume without loss that sellers never concede at 0^4 and buyers never concede or exit at 0^3, and hence any rational buyers must have always conceded by 0^3, while $T^* = 0^4$.

□

However, I will first define some preliminary objects, which will be useful in the analysis.

Proof of Proposition **??**. I prove this result by taking advantage of the following Lemma (proved subsequently), that a unique "straightforward" continuation equilibrium exists at time 0^3, in which rational agents payoffs are continuous in their beliefs. The precise nature of such an equilibrium is not important beyond these facts, and its precise definition is left until the proof.

Lemma 7. *Consider the subgame at 0^3 after demands $p_s \in P_s$ and $p_b \in P_b$ with fixed Θ. A unique "straightforward" continuation equilibria exists, for which agents' continuation payoffs are continuous at the beliefs $(\bar{z}_s, \bar{z}_b, \bar{g})$ where $\bar{z}_i \geq z_i \pi_i(p_i) > 0$.*

Now consider the parameters of a problem $(z_i, \pi_i, g, \Theta)_{i=s,b}$. Let $\Delta_s = \Delta(P_s)$ be the set of seller demand choice distributions at 0^1. Let $\Delta_s^{p_s} \subset \Delta(P_b \cup \{e\})$ be the set of rational buyer demand choice distributions at 0^2 after seller demand p_s such that $\mu_b^{v,w,p_s}(e) = \mathbb{1}_{v-w>\underline{p}}$ and $\mu_b^{v,w,p_s}(p_b) = 0$ for $p_b \geq p_s$. Then $\Delta_b = \prod_{p_s \in P_s} \Delta_b^{p_s}$.

Let $U_b^{v,w,p_b,p_s}(\mu_s, \mu_b)$ be the expected payoff of rational buyer (v, w) at 0^3 given demands $p_i \in P_i$, the demand choice distributions, $\mu_s \in \Delta_s(P_s)$ and $\mu_b \in \Delta_b$ combined with straightforward equilibrium continuation play. Also let $U_s^{p_s}(\mu_s, \mu_b)$ be the expected payoff of the seller at 0^2 given the demand $p_s \in P_s$, the demand choice distributions $\mu_s \in \Delta_s(P_s)$ and $\mu_b^{p_s} \in \Delta_b$ with straightforward equilibrium continuation play at 0^3. We then define:

$$B(\mu_s, \mu_b) = \{(\hat{\mu}_s, \hat{\mu}_b) \in \Delta_s \times \Delta_b : \hat{\mu}_s(p_s) > 0 \Rightarrow U_s^{p_s}(\mu_s, \mu_b) \geq U_s^{p'_s}(\mu_s, \mu_b), \forall p'_s \in P_s,$$
$$\hat{\mu}_b^{p_s}(p_b) > 0 \Rightarrow U_b^{v,w,p_b,p_s}(\mu_b, \mu_s) \geq U_b^{v,w,p'_b}(\mu_s, \mu_b), \forall p'_b \in P_b\}.$$

It is clear that this self-correspondence is non-empty and convex-valued and has a closed graph given that $U_b^{v,w,p_b,p_s}(\mu_s, \mu_b)$ and $U_s^{p_s}(\mu_s, \mu_b)$ are continuous in (μ_b, μ_s) by Lemma 7. Hence, by Kakutani, it admits a (non-empty) fixed-point. This fixed point describes equilibrium demand choices and implies beliefs for $p_i \in P_i$. After the demand $p_b \notin P_b$, the seller always believes the rational buyer has a type $(\bar{v}, \underline{w})$. The buyer then immediately concedes if $p_s \leq \bar{v} - \underline{w}$ and the seller immediately concedes otherwise. □

Proof of Lemma 7. I first define some prelimary objects that will help describe a straightforward equilibrium, given some arbitrary beliefs $d = (\bar{z}_s, \bar{z}_b, \bar{g})$ with $\bar{z}_i > 0$. For $y \in [0, (1 - \bar{z})x]$ let

$$\bar{k}(y) = \max\{k \leq K + 1 : \sum_{(v^m, w) \in \Theta^c : m \geq k} \bar{g}(v, w)(1 - \bar{z}_b) \geq y\},$$

Clearly, $\bar{k}(0) = K + 1$, and $\bar{k}((1 - \bar{z})x) = 1$ if $\bar{g}(v^2, w) > 0$ for $(v^2, w) \in \Theta^c$. This is decreasing and lower semi continuous in y. Also define $\underline{k}(y) = \bar{k}(y)$ if $y < (1 - \bar{z})x$ and $\underline{k}((1 - \bar{z})x) = 0$. Loosely, if fraction y of buyers have conceded by time t then $t \in (t^{\bar{k}(y)+1}, t^{\bar{k}(y)}]$.

For $k \in \{1, ..., K\}$ let

$$\bar{G}^e(k) = \sum_{(v,w) \in \Theta^e : \underline{\lambda}^{v,w} > \lambda_s^{v^k}} \bar{g}(v, w)$$

while $\bar{G}^e(K + 1) = 0$ and $\bar{G}^e(0)$. Notice that $\bar{G}^e(\bar{k}(y))$ is increasing and lower semi continuous.in y.

Next define

$$\pi(y, \hat{y}) = (p_s - p_b)(\hat{y} - y) - p_b(1 - \bar{z}_b)(G^e(\underline{k}(\hat{y})) - G^e(\bar{k}(y))).$$

Loosely, this the difference between the present value payoff of p_b a seller gets by conceding an instant before time t, and the payoff she would receive conceding conceding an instant after t, if at time t a fraction $(\hat{y} - y)$ of buyers concede and $(1 - \bar{z})(G^e(\underline{k}(y)) - G^e(\bar{k}(y)))$ exit. And then let:

$$\tilde{y}(\hat{y}) = \min\{y \geq 0 : \pi(y, \hat{y}) \leq 0\}.$$

Loosely, $\hat{y} - \tilde{y}(\hat{y})$ is the maximum probability of concession at exactly time t, if a fraction $\hat{y}$ of buyers must have conceded by time t. after time t compared to conceding an instant before time t, such that the buyer prefers to concede an instant before t compared to an instant after.

It is useful to outline equilibrium strategies starting at time $T^* = t^1$, which I relabel as "time" $0 = \tau^1$, and more generally will define equilibrium objects in terms of $\tau = T^* - t \in [0, \infty)$. Define $\hat{F}^1_s(\tau_1) = (1 - \bar{z}_s)$, $\hat{F}^1_b(\tau_1) = (1 - \bar{z}_b)x$, $\hat{E}^1_b(\tau_1) = (1 - \bar{z}_b)G^e(1)$, and then by induction for $k \in \{1, ..., K\}$ and $\tau \geq \tau^k$, let $1 - \hat{F}^k_s(\tau) = (1 - \hat{F}^k_s(\tau^k))e^{\lambda^{v^k}_s(\tau - \tau_1)}$, $\hat{E}^k_b(\tau) = \hat{E}^k_b(\tau^k)$, $1 - \hat{E}^k_b(\tau^k) - \hat{F}^k_b(\tau) = (1 - \hat{E}^k_b(\tau^k) - \hat{F}^k_s(\tau^k))e^{\lambda_b(\tau - \tau^k)}$. Effectively, $\hat{F}^k_s$ (respectively $\hat{F}^k_b$) correspond to the concession probability of the seller (buyer) assuming she concedes at rate $\lambda^{v^k}_s$ (λ_b) on $(t, t^k) = (T^* - \tau, T^* - \tau^k)$ if $F_i(t^k_-) = \hat{F}^k_i(\tau^k)$ and $E_b(t^k_-) = \hat{E}^k_b(\tau^k)$. Then for $k \leq K$ (where recall that $v^K = \overline{v}$) define

$$\tau^{k+1} = \min\{\tau \geq \tau^k : \tilde{y}(\hat{F}^k_b(\tau)) < \hat{F}^k_b(\tau) \text{ or } \overline{k}(\hat{F}^k_b(\tau)) > k\}$$

with $\hat{F}^{k+1}_s(\tau^{k+1}) = \hat{F}^k_s(\tau^{k+1}))$, $\hat{F}^{k+1}_b(\tau^{k+1}) = \tilde{y}(\hat{F}^k_b(\tau))$ and $\hat{E}^{k+1}_b(\tau^{k+1}) = (1 - \bar{z}_b)G^e(k + 1)$. Notice that we can have $\tau^{k+1} = \tau^k$. In fact, define $\ell^k = \min\{\ell \geq k : \tau^{\ell+1} > \tau^k\}$ so that $\tau^{\ell^k} = \tau^k$.

Next define $\hat{F}_s(0) = (1 - \bar{z}_s)$, $\hat{F}_b(0) = (1 - \bar{z}_b)x$, $\hat{E}_b(0) = (1 - \bar{z}_b)(1 - x)$, $\hat{F}_s(\tau) = \hat{F}^k_s(\tau)$, $\hat{F}_b(\tau) = \hat{F}^k_b(\tau)$, $\hat{E}_b(\tau) = \hat{E}^k_b(\tau) = (1 - \bar{z}_b)G^e(k)$ if $\tau \in (\tau^k, \tau^{k+1}]$. Let $\tau_b = \tau^{K+1}$ where recall that. Let $\hat{F}_s(\tau) = \hat{F}^K_s(\tau)$ for $\tau \geq \tau^{K+1}$ and then define $\tau_s = \min\{\tau : \hat{F}_s(\tau) \geq 0\}$ and $\tau^* = T^* = \min\{\tau_b, \tau_s\}$. Finally, let $F_s(0^3) = \hat{F}_s(\tau^*)$, $E_b(0^3) = F_b(0^3) = 0$, then for $t \in [0^4, T^*]$ let $F_s(t) = \hat{F}_s(\tau^* - t)$, $F_b(t) = \hat{F}_b(\tau^* - t)$ and $E_b(t) = \hat{E}_b(\tau^* - t)$.

By construction, for $k \in \{1, ..., K\}$, we have $t^k = \tau^* - \tau^k$ if $\tau^k < \tau^*$ and $t^k = 0^4$ otherwise. Suitable rational agent concession and exit strategies can clearly be backed out of these functions by skimming property and Lemma 2. By construction, no agent has a profitable deviation. In particular, concession on (t^{k+1}, t^k) is at rates λ_b and $\lambda^{v^k}_s$ respectively to make a rational seller or buyer with value v^k indifferent between conceding on that interval. If $\tau^* = 0$ then $F_s(0^3) = (1 - \bar{z}_s)$. Otherwise, buyer concession at $t^k \geq 0^4$ is calibrated to always leave a rational seller indifferent between conceding an instant before or after t^k (given the probability of exit at t^k).

Claim: For an arbitrary sequence of distributions $d^n \to d$ suppose that $\lim_n \tau^{k,n} = \tau^k$ as well as $\lim_n \hat{F}^{k,n}_b(\tau^{k,n}) = F^k_b(\tau^k)$ and $\lim_n \hat{E}^{k,n}_b(\tau^{k,n}) = E^k_b(\tau^k)$. I claim that: $\lim_n \tau^{\ell,n} = \tau^\ell$ for all $\ell \in \{k + 1, ..., \ell^{k+1}\}$ and $\lim_n \hat{F}^{\ell^{k+1},n}_b(\tau^{\ell^{k+1},n}) = F^{\ell^{k+1}}_b(\tau^{\ell^{k+1}})$ and $\lim_n \hat{E}^{\ell^{k+1},n}_b(\tau^{\ell^{k+1},n}) = E^{\ell^{k+1}}_b(\tau^{\ell^{k+1}})$.

Step 1. For any $\tau > \tau^k$ we must have $\tau > \tau^{k,n}$ for large n and $1 - \hat{F}^{k,n}_b(\tau) = (1 - \hat{F}^{k,n}_b(\tau^{k,n}))e^{\lambda^{v^k}_s(\tau - \tau^{k,n})} \to 1 - \hat{F}^k_b(\tau)$. Since $\overline{k}$ is lower semi continuous, $\lim_n \overline{k}^n(\hat{F}^{k,n}_b(\tau)) \leq \overline{k}(\hat{F}^k_b(\tau))$. More precisely, if $\sum_{(v^m,w)\in\Theta^c:m\geq k'} \bar{g}^n(v, w)(1 - \bar{z}^n_b) \geq \hat{F}^{k,n}_b(\tau)$ for all n, then the inequality also holds in the limit.

Step 2. On the other hand, for any $\varepsilon > 0$, if $\overline{k}(1 - \hat{F}^k_b(\tau)) = k' > k$ then $\lim_n \overline{k}^n(\hat{F}^{k,n}_b(\tau + \varepsilon) \geq k'$ since $\sum_{(v^m,w)\in\Theta^c:m\geq k'} \bar{g}^n(v, w)(1 - \bar{z}^n_b) > \hat{F}^k_b(\tau + \varepsilon/2)) \geq \hat{F}^{k,n}_b(\tau + \varepsilon))$ for all large n. Hence, if $\overline{k}(\hat{F}^k_b(\tau^{k+1})) > k$ then $\lim_n \tau^{k+1,n} = \tau^{k+1}$.

Step 3. Suppose, next that $\overline{k}(\hat{F}^k_b(\tau^{k+1})) = k$ and so for all sufficiently large n, $\overline{k}^n(\hat{F}^{k,n}_b(\tau^{k+1})) = k$.

We must then have $\tilde{y}(\hat{F}_b^k(\tau^{k+1})) < \hat{F}_b^k(\tau^{k+1})$ by the definition of τ^{k+1}. Suppose that for some subsequence $\lim_n \tau^{k+1,n} < \tau^{k+1}$ (where clearly this is impossible if $\tau^{k+1} = \tau^k$) then for some $y^n \leq \hat{F}_b^{k,n}(\tau^{k+1})$ (since $\overline{k}^n(\hat{F}_b^{k,n}(\tau^{k+1})) = k$), we must have

$$\pi^n(y^n, \hat{F}_b^{k,n}(\tau^{k+1,n})) = (p_s - p_b)(\hat{F}_b^{k,n}(\tau^{k+1,n}) - y^n) - p_b(1-\bar{z})(G^{e,n}(k) - G^{e,n}(\overline{k}^n(y^n))) \leq 0$$

and these inequalities are preserved in the limit, $\pi(\lim_n y^n, \hat{F}_b^k(\lim_n \tau^{k+1,n}) \leq 0$ for $\lim_n y^n \leq \hat{F}_b^k(\tau^{k+1})$. However, this contradicts the definition of τ^{k+1}.

Step 4. On the other hand, suppose that $\lim_n \tau^{k+1,n} > \tau^{k+1} + \varepsilon$ for some $\varepsilon > 0$ and so $\overline{k}^n(\hat{F}_b^{k,n}(\tau^{k+1} + \varepsilon) = k$ for large n. For small enough $\varepsilon' > 0$, we must have $\pi(y, \hat{F}_b^k(\tau^{k+1}))$ is continuous and strictly decreasing for $y \in [-\varepsilon', 0] + \tilde{y}(\hat{F}_b^k(\tau^{k+1})$. For small enough $\varepsilon' > 0$, we must then have

$$\begin{aligned}&\lim_n (p_s - p_b)(\hat{F}_b^{k,n}(\tau^{k+1} + \varepsilon) - \tilde{y}(\hat{F}_b^k(\tau^{k+1}) + \varepsilon') - p_b(1-\bar{z})(G^{e,n}(k) - G^{e,n}(\overline{k}((\tau^{k+1}) + \varepsilon')))) \\ =&(p_s - p_b)(\hat{F}_b^k(\tau^{k+1} + \varepsilon) - \tilde{y}(\hat{F}_b^k(\tau^{k+1}) + \varepsilon') - p_b(1-\bar{z})(G^e(k) - G^e(\overline{k}((\tau^{k+1}) + \varepsilon')))) < 0.\end{aligned}$$

And so for all sufficiently large n we must have $\pi^n(\tilde{y}(\hat{F}_b^k(\tau^{k+1}) - \varepsilon', \hat{F}_b^{k,n}(\tau^{k+1} + \varepsilon) < 0$, which contradicts $\lim_n \tau^{k+1,n} > \tau^{k+1} + \varepsilon$. This establishes $\lim_n \tau^{k+1,n} = \tau^{k+1}$.

Notice that $\tau^{\ell^{k+1}} = \tau^{k+1} = \lim_n \tau^{k+1,n} \leq \lim_n \tau^{\ell^{k+1},n}$. However, if $\lim_n \tau^{\ell^{k+1},n} > \tau^{k+1} + \varepsilon$ for some $\varepsilon > 0$, we can repeat the steps above with minimal adaptions, to get a contradiction. Hence, $\tau^{k+1} = \lim_n \tau^{\ell^{k+1},n}$. Moreover, by definition $F_b^{\ell^{k+1}}(\tau^{k+1}) = \tilde{y}(F_b^k(\tau^{k+1}))$ and $\ell^{k+1} = \overline{k}(F_b^{\ell^{k+1}}(\tau^{k+1}))$. Hence if $F_b^{\ell^{k+1}}(\tau^{k+1}) > \lim_n F_b^{\ell^{k+1},n}(\tau^{k+1,n})$, minor adaptions to the argument in Step 3, imply $\tilde{y}(F_b^k(\tau^{k+1})) \leq \lim_n F_b^{\ell^{k+1},n}(\tau^{k+1,n})$, a contradiction. Similarly, if $F_b^{\ell^{k+1}}(\tau^{k+1}) < \lim_n F_b^{\ell^{k+1},n}(\tau^{k+1,n})$, then minor adaptions to the argument in Step 4, generate a contradiction. $E_b^{\ell^{k+1}}(\tau^{k+1}) = \lim_n E_b^{\ell^{k+1},n}(\tau^{k+1,n})$ then follows as a corollary. This establishes the Claim. Notice that treating F_i as a distribution function (we can let $F_i(\infty) = 1$), we clearly have that $F_i^n \to_w F_i$ weakly in distribution.

Question: more detail needed here?

Given the Claim, it is clear that $\tau^{k,n} \to \tau^k$, $\tau_b^n \to \tau_b$, $\tau_s^n \to \tau_s$, as well as $F_s^n(0^3) \to F_s(0^3)$. The payoff of a rational buyer with value v who concedes at t^k is

$$U_b^{v,c}(t^k) = (v - p_b)F_s(0^3) + (v - p_b)\int_{t \in (0,t^k)} e^{-rt} dF_s(t) + (v - p_s)e^{-rt^k}(1 - F_s(t^k))$$

Given that $F_s^n \to_w F_s$ where F_s is continuous at t^k, it is clear that $U_b^{v,n}(t^{k,n}) \to U_b^v(t^k)$. Similarly, the payoff of a rational buyer who exits at time t^k is $U_b^{v,w,e}(t^k) = U_b^{v,c}(t^k) + (w - v + p_s)e^{-rt^k}(1 - F_s(t^k))$ so that $U_b^{v,w,e,n}(t^{k,n}) \to U_b^{v,w,e}(t^k)$.

We now turn to the rational seller, who's payoff can be expressed as $V_s = \max\{p_b, U_s(T_+^*)\}$ where $U_s(T_+^*) = \int_{s \leq T^*} p_s e^{-rs} dF_b(s) + e^{-rT^*}(1 - \bar{z}_s)p_b$ is the payoff from conceding an instant after T^*. Given that $\lim_n F_b^n(T^{*,n}) = F_b(T^*) = 1 - \bar{z}_b$, $\lim_n T^{*,n} = T^*$ and $F_i^n \to_w F_i$ it is immediate that $U_s^n(T_+^{*,n}) \to U_s(T_+^*)$ and so $V_s^n \to V_s$. This completes the proof.

□

Proof of Lemma 4. Lemma 8, listed below is useful for proving this result.

Lemma 8. *Consider the subgame $(p_s, \bar{z}_s, z_b, g, \Theta, P_b, P, \pi_s)$ at 0^2, after some demand $p_s \in P_s$. If $F_s^{p_b}(0^4) > 0$ for some $p_b \in P$ then $F_s^{p_b'}(0^4) > 0$ for all $p_b' \in P_b$ with $p_b' < p_s$ and $\mu_b^{v,w}(c) = \mu_b^{v,w}(e) = 0$ for all $(v, w) \in \Theta$ where $v - w > \underline{p}$. Whenever $F_s^{p_b}(0^4) = 0$ for all $p_b \in P$, it is without loss of generality to assume $E_b^{p_b'}(0^4) = F_b^{p_b'}(0^4) = 0$, with any time 0 buyer concession and exit occurring at 0^2. Given this assumption, if $\mu_b^{v^k,w}(c) < 1$ for some $(v^k, w) \in \Theta^c$, then $\mu_b^{v^m,w'}(c) = 0$ for any $(v^m, w') \in \Theta^c$ with $m < k$.*

Assume (as is shown to be without loss in Lemma 8) that if $F_s^{p_b}(0^4) = 0$ for all $p_b \in P$ then $F_b^{p_b}(0^4) = 0$. Let $\tilde{v}^{p_b} = \max\{v : \sum_{(v,w)\in\Theta^c} \mu_s^{p_s}(p_b) > 0\}$. The payoff to $(\tilde{v}^{p_b}, w) \in \Theta^c$ to demanding arbitrary p_b is $F_s^{p_b}(0^4)(\tilde{v}^{p_b} - p_b) + (1 - F_s^{p_b}(0^4))(\tilde{v}^{p_b} - p_s)$, while her payoff from demanding p_b' is at least $F_s^{p_b'}(0^4)(\tilde{v}^{p_b} - p_b) + (1 - F_s^{p_b'}(0^4))(\tilde{v}^{p_b} - p_s)$, by the skimming property (Lemma 1, part (k)). Clearly if $F_s^{p_b'}(0^4)(p_s - p_b') > F_s^{p_b}(0^4)(p_s - p_b)$ then type $\tilde{v}^{p_b}$ will not imitate type p_b (a contradiction), whereas if $F_s^{p_b'}(0^4)(p_s - p_b') < F_s^{p_b}(0^4)(p_s - p_b)$ then type $\tilde{v}^{p_b'}$ will not imitate type p_b'. Hence if p_b and p_b' are both imitated with positive probability then $F_s^{p_b'}(0^4)(p_s - p_b') = F_s^{p_b}(0^4)(p_s - p_b)$; if $p_b < p_b'$ therefore $F_s^{p_b'}(0^4) \geq F_s^{p_b}(0^4)$. Certainly, if $p_b' > p_b \in P_b$, and p_b' is demanded with positive probability by some rational agent, then p_b is certainly imitated with positive probability as otherwise a rational seller will always concede when facing p_b and $F_s^{p_b'}(0^4)(p_s - p_b') < F_s^{p_b}(0^4)(p_s - p_b)$, a contradiction.

Let $\check{v}^{p_b}$ be the maximum value buyer such that some $(\check{v}^{p_b}, w) \in \Theta^c$ demands p_b with positive probability, but has not always conceded by time 0^4. Suppose that $p_b' > p_b \in P_b$ is demanded with positive probability but $\check{v}^{p_b'}$ is not well defined because all rational buyers concede or quit by 0^4. A rational seller must have conceded at 0^3 with strictly positive probability (else we could move the buyer's concession and exit to 0^2). Given $F_s^{p_b'}(0^4)(p_s - p_b') = F_s^{p_b}(0^4)(p_s - p_b) > 0$, a buyer of type $(v, w) \in \Theta^e$ would strictly prefer to demand p_b and then exit at 0^4 than to demand p_b' since $F_s^{p_b'}(0^4)(v - p_b - w) > F_s^{p_b'}(0^4)(v - p_b' - w) = F_s^{p_b}(0^4)(p_s - p_b)(v - p_b' - w)/(p_s - p_b')$ as $(v - p_b - w)/(p_s - p_b)$ is decreasing in p_b since $v - p_s < w$. However, in which case $x^{p_b'} = 1$ and so the seller strictly prefer to concede an instant after 0^4 than at 0^3, a contradiction. Because $F_s^{p_b}(0^4) < 1 - \bar{z}_s = F_s^{p_b}(T^{*,p_b})$ given $p_b \in P_b$, we must also have $\check{v}^{p_b}$ well defined. The argument above also shows more generally that without loss of generality, a buyer with value $(v, w) \in \Theta^e$ will never demand $p_b' > p_b \in P_b$ and then exit at time 0.

I next claim that $\check{v}^{p_b} = \check{v}^{p_b'}$. Suppose not, and that $\check{v}^{p_b} < \check{v}^{p_b'}$. The payoff to $(\check{v}^{p_b}, w) \in \Theta^c$ from demanding p_b is $F_s^{p_b}(0^4)(\check{v}^{p_b} - p_b) + (1 - F_s^{p_b}(0^4))(\check{v}^{p_b} - p_s)$, which we have established above is also her payoff from demanding p_b' and then conceding an instant after 0^4. However, since

$\lambda_s^{\check{v}^{p'_b},p'_b} > \lambda_s^{\check{v}^{p_b},p'_b}$, the payoff to type $(\check{v}^{p_b}, w)$ from demanding p'_b and waiting to concede after the strictly positive interval on which she receives that concession rate (by Lemma 2) must give a strictly larger than from demanding p_b, a contradiction. Hence, we must have $\check{v}^{p'_b} = \check{v}^{p_b}$.

Recall that an agent with value v^m with $m \geq 1$ is indifferent between conceding at any point in the interval $[t^{m+1,p_b}, t^{m,p_b}]$ after demanding p_b. Now assume that (i) a buyer with value v and one with value v^m were both indifferent between demanding p_b and conceding at t^{m+1,p_b} or demanding $p'_b > p_b$ and conceding at t^{m+1,p'_b} and (ii) that $F_s^{p'_b}(t^{m+1,p'_b}) \geq F_s^{p_b}(t^{m+1,p_b})$ and $t^{m+1,p'_b} \geq t^{m+1,p_b}$ so in particular $e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) \geq e^{-rt^{m+1,p'_b}}(1 - F_s^{p'_b}(t^{m+1,p'_b}))$. Let the difference in payoffs for a buyer with value v between demanding $p_b \in P_b$ and conceding at t^{m,p_b} or demanding $p'_b > p_b$ and conceding at time t^{m,p_b} for an agent with value v be $D^m(v) = U^{c,p_b,v}(t^{m,p_b}) - U^{c,p'_b,v}(t^{m,p'_b})$. Given (i) we must have $D^m(v) = D^m(v) - D^{m+1}(v)$, and so:

$$\begin{aligned} D^m(v) = &\int_{t^{m+1,p_b} < \tau < t^{m,p_b}} e^{-r\tau}(v - p_b) dF_s^{p_b}(\tau) - \int_{t^{m+1,p'_b} < \tau < t^{m,p'_b}} e^{-r\tau}(v - p_b) dF_s^{p'_b}(\tau) \\ &- (e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) - e^{-rt^{m,p_b}}(1 - F_s^{p_b}(t^{m,p_b})))(v - p_b) \\ &+ (e^{-rt^{m+1,p'_b}}(1 - F_s^{p'_b}(t^{m+1,p'_b})) - e^{-rt^{m,p'_b}}(1 - F_s^{p_b}(t^{m,p'_b})))(v - p'_b) \end{aligned}$$

This implies that

$$\begin{aligned} \frac{dD^m(v)}{dv} &= \int_{t^{m+1,p_b} < \tau < t^{m,p_b}} e^{-r\tau} dF_s^{p_b}(\tau) - e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) + e^{-rt^{m,p_b}}(1 - F_s^{p_b}(t^{m,p_b})) \\ &\quad - \int_{t^{m+1,p'_b} < \tau < t^{m,p'_b}} e^{-r\tau} dF_s^{p'_b}(\tau) + e^{-rt^{m+1,p'_b}}(1 - F_s^{p'_b}(t^{m+1,p'_b})) - e^{-rt^{m,p'_b}}(1 - F_s^{p_b}(t^{m,p'_b})) \\ &= -\frac{p_s - p_b}{v^m - p_b} e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b}))(1 - e^{-r(t^{m,p_b} - t^{m+1,p_b})} \frac{1 - F_s^{p_b}(t^{m,p_b})}{1 - F_s^{p_b}(t^{m+1,p_b})})) \\ &\quad + \frac{p_s - p'_b}{v^m - p'_b} e^{-rt^{m+1,p'_b}}(1 - F_s^{p'_b}(t^{m+1,p'_b}))(1 - e^{-r(t^{m,p'_b} - t^{m+1,p'_b}} \frac{1 - F_s^{p'_b}(t^{m,p'_b})}{1 - F_s^{p'_b}(t^{m+1,p'_b})})) \end{aligned}$$

where the second line imposes that type v^m is indifferent between conceding at t^{m+1,p_b} or t^{m,p_b} for any demand p_b (as required by Lemma 2), that is:

$$\int_{t^{m+1,p_b} < \tau < t^{m,p_b}} e^{-r\tau} dF_s^{p_b}(\tau) = (e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) - e^{-rt^{m,p_b}}(1 - F_s^{p_b}(t^{m,p_b}))) \frac{v^m - p_s}{v^m - p_b}$$

and also $(v^m - p_s)/(v^m - p_b) - 1 = -(p_s - p_b)/(v^m - p_b)$.

Other things equal it is clear that $dD^m(v)/dv$ is strictly decreasing in t^{m,p_b} and strictly increasing in t^{m,p'_b} and equals 0 when $t^{m,p'_b} = t^{m+1,p'_b}$ and $t^{m,p_b} = t^{m+1,p_b}$.

Given some equilibrium t^{m,p'_b}, t^{m+1,p'_b} and t^{m+1,p_b} we must have $T^{*,p_b} \geq t^{m+1,p_b} + t^{m,p'_b} - t^{m+1,p'_b}$. Suppose not, then let $t^{m,p'_b} - t^{m+1,p'_b} = q > T^{*,p_b} - t^{m+1,p_b}$. Since $\lambda_s^{v,p_b} \leq \lambda_s^{v^m,p_b} < \lambda_s^{v^m,p'_b}$ for all $v \leq v^m$ (since $p_b < p'_b$) we have $(1 - F_s^{p_b}(T^{*,p_b}))/(1 - F_s^{p_b}(t^{m+1,p_b})) > e^{-\lambda_s^{v^m,p_b} q} \geq e^{-\lambda_s^{v^m,p'_b} q} =$

$(1 - F_s^{p_b'}(t^{m,p_b'}))/(1 - F_s^{p_b'}(t^{m+1,p_b'}))$, and so given $F_s^{p_b'}(t^{m+1,p_b'}) \geq F_s^{p_b}(t^{m+1,p_b})$ by (ii) we would then have $(1 - F_s^{p_b}(T^{*,p_b})) > 1 - F_s^{p_b}(t^{m,p_b'})) \geq \bar{z}_s$, a contradiction since $p_b \in P_b$. Suppose next that $q = t^{m,p_b'} - t^{m+1,p_b'} = t^{m,p_b} - t^{m+1,p_b}$ and in this case let $\hat{D}^v(q)$ be $dD^m(v)/dv$ defined as a function of q. We then have:

$$\frac{d\hat{D}^v(q)}{dq} = -re^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b}))e^{-(r+\lambda_s^{p_b})q} + re^{-rt^{m+1,p_b'}}(1 - F_s^{p_b'}(t^{m+1,p_b'}))e^{-(r+\lambda_s^{p_b'})q}$$

where I use the fact that $r + \lambda_s^{p_b} = r(v^m - p_b)/(p_s - p_b)$. Notice that $e^{(r+\lambda_s^{p_b'})q}d\hat{D}^v(q)/dq$ is strictly decreasing in q and so since $e^{-rt^{m+1,p_b}}(1 - F_s^{p_b}(t^{m+1,p_b})) \geq e^{-rt^{m+1,p_b'}}(1 - F_s^{p_b'}(t^{m+1,p_b'}))$ by (ii) we must have $d\hat{D}^v(0)/dq \leq 0$, and so $d\hat{D}^v(q)/dq < 0$ for all $q > 0$. Since $\hat{D}^v(0) = 0$ we must have $\hat{D}^v(q) < 0$ for all $q > 0$. Since $dD^m(v)/dv$ is strictly decreasing in t^{m,p_b}, if $t^{m,p_b} - t^{m+1,p_b} \geq t^{m,p_b'} - t^{m+1,p_b'}$ then $dD^m(v)/dv \leq 0$ with $dD^m(v)/dv < 0$ if $t^{m,p_b'} > t^{m+1,p_b'}$. On the flip side, if $dD^m(v)/dv \geq 0$ then we must certainly have $t^{m,p_b} - t^{m+1,p_b} \leq t^{m,p_b'} - t^{m+1,p_b'}$.

Given this we must in fact always have $dD^m(v)/dv \leq 0$. Suppose not so that $dD^m(v)/dv > 0$, then since $D^m(v^m) = 0$ we would have $D(v) < 0$ for all $v < v^m$. Hence, all such buyers would strictly prefer to demand p_b' and concede at $t^{m,p_b'}$ than demand p_b and concede at t^{m,p_b}. This would then imply that $T^{*,p_b} = t^{m,p_b}$. However, we observed earlier that $T^{*,p_b} \geq t^{m+1,p_b} + t^{m,p_b'} - t^{m+1,p_b'}$, and know that $t^{m,p_b} - t^{m+1,p_b} \geq t^{m,p_b'} - t^{m+1,p_b'}$ implies $d\hat{D}^v(q)/dq \leq 0$ a contradiction.

Hence either (a) $dD^m(v)/dv < 0$ and no agent with $v < v^m$ imitates p_b' only to concede, or (b) $dD^m(v)/dv = 0$ in which case, we know that $t^{m,p_b} - t^{m+1,p_b} \leq t^{m,p_b'} - t^{m+1,p_b'}$ and hence $t^{m,p_b'} \geq t^{m,p_b}$ and $F_s^{p_b'}(t^{m,p_b'} \geq F_s^{p_b}(t^{m,p_b})$, both strictly if $t^{m,p_b'} > 0$, given that $F_s^{p_b'}(t^{m+1,p_b'}) \geq F_s^{p_b}(t^{m+1,p_b})$ and $t^{m+1,p_b'} \geq t^{m+1,p_b}$ by (ii).

Given that $m \leq k$ is arbitrary, induction establishes that all $(v, w) \in \Theta^c$ weakly prefer $p_b \in P_b$ over $p_b' > p_b$. In either case (a) or (b), notice that if $D^m(v^m) = 0$, there is always some $\hat{t}^{m,p_b} \in [t^{m+1,p_b}, t^{m,p_b}]$ such that all buyer types are indifferent between demanding p_b' and conceding at $t^{m,p_b'}$ or demanding p_b and conceding at $\hat{t}^{m,p_b}$ where we must have $\hat{t}^{m,p_b} \leq t^{m,p_b'}$ and $F_s^{p_b'}(t^{m,p_b'}) \geq F_s^{p_b}(t^{m,p_b})$, where these inequalities are strict if $t^{m,p_b'} > 0$ (notice that $\hat{t}^{m,p_b} = t^{m,p_b}$ if $dD^m(v)/dv = 0$).

Next, consider the incentives of an agent $(v, w) \in \Theta^e$. We saw earlier that without loss of generality, such a buyer would never demand p_b' only to exit at 0^4. Hence, suppose that conditional on demanding p_b' it was optimal for such an agent to concede at $t^{m,p_b'} > 0$. We can assume that $t^{m,p_b'} > t^{m+1,p_b'}$, as otherwise it is optimal to concede at $t^{m+1,p_b'}$. We know that such a buyer is indifferent between demanding p_b and conceding at time $\hat{t}^{m,p_b}$ or demanding p_b' and conceding at time $t^{m,p_b'} > 0$. However, in that case such a buyer must then strictly prefer to demand p_b and exit at time $\hat{t}^{m,p_b}$, than demand p_b' and exit at $t^{m,p_b'}$ since $w > v - p_s$. To see this, let $\hat{D}^m(v, w) = U^{e,v,w,p_b}(\hat{t}^{m,p_b}) - U^{e,v,w,p_b'}(t^{m,p_b'})$ be the difference in payoffs between these two strategies and consider the notationally simplest case in which $\hat{t}^{m,p_b} = t^{m,p_b}$, then:

$$\hat{D}^m(v,w) = D^m(v) + (e^{-rt^{m,p_b}}(1 - F_s^{p_b}(t^{m,p_b})) - e^{-rt^{m,p'_b}}(1 - F_s^{p_b}(t^{m,p'_b})))(w - v + p_s) > 0$$

where the strict inequality holds since $D^m(v) = 0$ for all v, but we have $e^{-rt^{m,p_b}}(1 - F_s^{p_b}(t^{m,p_b})) > e^{-rt^{m,p'_b}}(1 - F_s^{p_b}(t^{m,p'_b}))$ and $w > v - p_s$. Hence, demanding p'_b is never optimal for $(v,w) \in \Theta^e$.

Now suppose that $\sum_{(v^k,w)\in\Theta^c} \mu_b^{v^k,w,n}(p'_b) > 0$. Since we know that no agent with $(v,w) \in \Theta^e$ demands p'_b we must have $v^k \leq \check{v}^{p'_b}$. We wish to argue that $\sum_{(v^k,w)\in\Theta^c} \mu_b^{v^k,w,n}(p_b) > 0$. Suppose by way of contradiction, that $v^k < \check{v}^{p'_b} = \check{v}^{p_b}$ is the maximal value such that $\sum_{(v^k,w)\in\Theta^c} \mu_b^{v^k,w,n}(p'_b) > 0$ but $\sum_{(v^k,w)\in\Theta^c} \mu_b^{v^k,w,n}(p_b) = 0$. Since we know that no agent with $(v,w) \in \Theta^e$ demands p'_b we must have $t^{k,p'_b} > t^{k+1,p'_b}$ (buyer concession must be continuous after time 0), but combined with $t^{k,p_b} = t^{k+1,p_b}$ we would then have $dD^k(v)/dv > 0$, which contradicts our earlier finding that we always need $dD^k(v)/dv \leq 0$. □

Proof of Lemma 8. Certainly, if $F_s^{p_b}(0^4) > 0$, then a rational buyer $(v,w) \in \Theta^c$ gets a payoff of at least $F_s^{p_b}(0^4)(v - p_b) + (1 - F_s^{p_b}(0^4))(v - p_s) > (v - p_s)$ from imitating p_b and so will certainly not concede at 0^2. Suppose then that $F_s^{p'_b}(0^4) = 0$ for some $p'_b \in P_b$, and suppose that $\mu_b^{v,w}(p'_b) > 0$ for some $(v,w) \in \Theta^c$ and let $\tilde{v} = \max\{v : (v,w) \in \Theta^c, \mu_b^{v,w}(p'_b) > 0\}$. The payoff of type $(\tilde{v},w) \in \Theta^c$ from demanding p'_b is then exactly $(\tilde{v} - p_s)$, a contradiction. However, if $\mu_b^{v,w}(p'_b) = 0$ for all $(v,w) \in \Theta^c$ then a rational seller will certainly concede before 0^4, implying $F_s^{p'_b}(0^4) \geq F_s^{p_b}(0^4) > 0$, a contradiction. Given this, if $\underline{p} \in P_s$ satisfies $\underline{p} < v - w$ for some buyer $(v,w) \in \Theta^e$, then demanding $\underline{p}$ gives her a payoff of at least $F_s^{\underline{p}}(0^4)(v - \underline{p}) + (1 - F_s^{p_b}(0^4))w > w$, and so she will certainly not exit at 0^2.

Next observe that if $F_s^{p_b}(0^4) = 0$ for all $p_b \in P$, any exit and concession by a rational buyer that occurs by 0^4 can instead be assumed to occur at 0^2 without affecting outcomes and so $F_b^{p_b}(0^4) = 0$. Given this, if $F_s^{p_b}(0^4) = 0$ for all $p_b \in P$ and $\mu_b^{v^k,w}(c) < 1$, then for some $p_b \in P$ agent $\mu_b^{v^k,w}(p_b) > 0$ and in the subgame at 0^3 with that counterdemand, the seller concedes at a rate greater than $\lambda_s^{v^k}$ on an interval $(0,\varepsilon)$ with $\varepsilon > 0$ (by Lemma 2). Imitating p_b and conceding at ε would then give a buyer of type $(v^m, w') \in \Theta^c$ with $m < k$ a payoff strictly greater than $v^m - p_s$, and so she will certainly not concede at 0^2. Of course if $F_s^{p_b}(0^4) > 0$ for some $p_b \in P$, then she will not concede at 0^2 either.

□

Proof of Lemma 5. Suppose not, then since $\bar{z}_s = 1 - F_s(T^*) \geq (1 - F_s(0^4))e^{-\lambda_s^{\bar{v}} T^*}$ by Lemma 2, we must have $T^* \to \infty$.

For (a), define $t^* = -ln(\lim_n \bar{z}_b^n - \varepsilon)/\lambda_b < \infty$ for some $\varepsilon \in (0, \lim_n \bar{z}_b^n)$. However, for all large enough n we must have $T^* \leq t^*$ since $\lim_n \bar{z}_b^n - \varepsilon < \bar{z}_b = 1 - E_b(T^*) - F_b(T^*) \leq e^{-\lambda_b T^*}$ by Lemma 2. Given this, for the remaining claims, assume that $\lim_n \bar{z}_s^n = 0$

For (b), notice that $1 - E_b(t) - F_b(t) \leq e^{-\lambda_b t}$ by Lemma 2. Hence by the skimming property (Lemma 1 part (k)), for any $\varepsilon \in (0, \lim_n \bar{g}^n(v,w))$ for large n, at time $t^* = -ln(\lim_n \bar{g}^n(v',w') - \varepsilon)/\lambda_b < \infty$ all remaining rational buyers with $(v,w) \in \Theta^c$ must have $v \leq v' < p_b + p_s$ and hence $\lambda_b > \lambda_s^v$. But since $\bar{z}_b = 1 - E_b(T^*) - F_b(T^*) \leq e^{-\lambda_b T^*}$ and $\bar{z}_s = 1 - F_s(T^*) \geq (1 - F_s(0^4))e^{-\lambda_s^{\bar{v}} t^* - \lambda_s^v (T^* - t^*)}$ (by Lemma 2) we have

$$1 - F_s(0^4) \leq \frac{\bar{z}_s^n}{\bar{z}_b^n} e^{(\lambda_s^v - \lambda_b)(T^* - t^*) + (\lambda_s^{\bar{v}} - \lambda_b)t^*} \leq L e^{(\lambda_s^{v'} - \lambda_b)(T^* - t^*) + (\lambda_s^{\bar{v}} - \lambda_b)t^*}$$

where the right hand side converges to 0 as $T^* \to \infty$ since $\lambda_s^{v'} - \lambda_b < 0$. This clearly contradicts $\lim_n F_s^n(0) \neq 1$.

For (c), notice that type $(v', w') \in \Theta^e$ always demands $\underline{p}$ so that $\lim_n \bar{g}^n(v', w') \geq g(v', w') > 0$, and will not exit until after any type $(v^1, w) \in \Theta^c$, by Lemma 2. For any $\varepsilon \in (0, \lim_n \bar{g}^n(v', w'))$ let $t^* = -ln(\lim_n \bar{g}^n(v,w) - \varepsilon)/\lambda_b < \infty$. Since $1 - E_b(t) - F_b(t) \leq e^{-\lambda_b t}$, for large n, by time t^* all $(v^1, w) \in \Theta^c$ must have conceded, and so $t^* \geq T^*$, which contradicts $T^* \to \infty$.

□

Proof of Lemma 6. The logic for (a) and (b) are almost identical to that of Lemma 6, parts (a) and (b). Given $p_b > \underline{p}$ we have always have $x = 1$ so that $\bar{z}_b = 1 - F_b(T^*) = (1 - F_b(0^4))e^{-\lambda_b T^*}$ by Lemma 2. Hence, if $\lim_n F_b(0^4) \neq 1$ then we must have $T^* \to \infty$.

For (a) notice that $\bar{z}_s = 1 - F_s(T^*) \geq e^{-\lambda_s^{\bar{v}} T^*}$ and so $\lim_n T^*$ is bounded above by $-ln(\lim_n \bar{z}_s)/\lambda_s^{\bar{v}}$ a contradiction.

For (c), by assumption $v^1 - p_s > p_b$ and so $\lambda_s^{v^1} > \lambda_b$. We need $\bar{z}_s = 1 - F_s(T^*) \leq e^{-\lambda_s^{v^1} T^*}$ and so

$$(1 - F_b(0^4)) \leq \frac{\bar{z}_b}{\bar{z}_s} e^{(\lambda_b - \lambda_s^{v^1})T^*} \leq L e^{(\lambda_b - \lambda_s^{v^1})T^*}$$

where the right-hand side clearly converges to 0 as $T^* \to \infty$, implying $\lim_n F_b(0^4) = 1$.

For (b) let $\bar{t}^n = \min\{t \geq 0^4 : F_b^n(t) \geq \sum_{(v,w) \in \Theta^c : v > v_1} \bar{g}^n(v^1, w)\}$. By time $\bar{t}^n$ there are only rational buyers with type $(v^1, w) \in \Theta^c$ remaining. Clearly, we must have $F_b(\underline{t}^n) \leq \sum_{(v,w) \in \Theta^c : v > v_1} \bar{g}^n(v, w)$. First consider some subsequence for which $\bar{t}^n > 0$ for all n. Then by Lemma 3 we must have $(F_b(\bar{t}^n) - F_b(\underline{t}^n))(p_s - p_b)/p_b \leq E_b(\bar{t}^n) - E_b(\underline{t}^n)$ where the right hand side is certainly less than $1 - x^n$ and so for small enough $\varepsilon > 0$, for all sufficiently large n

$$F_b(\bar{t}^n) \leq (1 - x^n)p_b/(p_s - p_b) + \sum_{(v,w) \in \Theta^c : v > v_1} \bar{g}^n(v, w) < \lim_n \sum_{(v,w) \in \Theta^c} \bar{g}^n(v, w) - \varepsilon.$$

We must then have $1 - F_b(\bar{t}^n) - E_b(\bar{t}^n) \geq \varepsilon$ for all sufficiently large n. Similarly, suppose along some subsequence we always have $\bar{t}^n = 0^4$, and so certainly $E_b(0^4) = 1 - x^n$. For this subsequence, if $\lim_n F_b(0^4) \neq \lim_n x^n$ we must then again have $1 - F_b(\bar{t}^n) - E_b(\bar{t}^n) \geq \varepsilon$ for some $\varepsilon > 0$. For any subsequence with $\bar{t}^n = 0^4$ or $\bar{t}^n > 0^4$, therefore, we must have

$1 - F_b(\bar{t}^n) - E_b(\bar{t}^n) \geq \varepsilon$ for some $\varepsilon > 0$ for all large n. In that case, we must have $\bar{z}_b = 1 - F_b(T^*) - E_b(T^*) = (1 - F_b(\bar{t}^n) - E_b(\bar{t}^n))e^{-\lambda_b(T^*-t^n)}$ and so clearly $(T^* - \bar{t}^n) \to \infty$. Combined with $\bar{z}_s \leq 1 - F_s(T^*) \leq e^{-\lambda_s^{v1}T^*}$ we get

$$(1 - F_b(\bar{t}^n) - E_b(\bar{t}^n)) \leq \frac{\bar{z}_b}{\bar{z}_s} e^{(\lambda_b - \lambda_s^{v1})(T^* - t^n) - \lambda_s^{v1} t^n} \leq L e^{(\lambda_b - \lambda_s^{v1})(T^* - t^n) - \lambda_s^{v1} t^n}$$

where the right hand side must converge to 0 given that $\lambda_b < \lambda_s^{v1}$ and $(T^* - t^n) \to \infty$. Clearly, this gives a contradiction to $(1 - F_b(\bar{t}^n) - E_b(\bar{t}^n)) \geq \varepsilon > 0$ for large n. □

Proof of Propositions 2 and 3. First notice that by choosing $\varepsilon' > 0$ sufficiently small, a ε' rich commitment type space must have $\underline{p} \leq \varepsilon' < \min\{v - w : (v, w) \in \Theta\}$. Moreover, let $\tilde{p}_s = \max\{p_s \in P_s : p_s \leq \min\{p^*, v - w : (v, w) \in \Theta\}\}$, then for small enough $\varepsilon' > 0$, we have $\tilde{p}_s > \underline{p}$, and the seller will always demand $p_s \in P_s$ such that $p_s \geq \tilde{p}_s$. To see this, notice that demanding $\tilde{p}_s$ guarantees that $x^{\tilde{p}_s} = 1$ and since $\tilde{p}_s \leq p^*$, any counterdemand $p_b \in P$ will imply $\lambda_s^{\underline{v}, p_b, \tilde{p}_s} > \lambda_b^{p_b, \tilde{p}_s}$ given $p_b < \tilde{p}_s \leq \min\{p^*, \underline{v} - \underline{w}\}$. After any counterdemand $p_b \in P$ the buyer makes with positive limit probability (for some subsequence) we must have $\bar{z}_b^{p_b, \tilde{p}_s} / \bar{z}_s^{\tilde{p}_s} \leq L'$ for some constant L' and for all n sufficiently large. Hence, by Lemma 6, the buyer must concede with probability approaching one in the limit. This would guarantee the seller a payoff of at least $\tilde{p}_s$ in the limit and so she certainly won't demand less. Moreover, she will never demand $p_s \notin P_s$ as then the highest limit payoff she could expect would be $\underline{p}$. Nor will a rational seller ever demand $p_s > \overline{v} - \underline{w}$ as she would then need to immediately concede against any counterdemand (Lemma 1, part (a)), again giving her a limit payoff of $\underline{p}$.

Suppose then that the seller demanded some price $p_s \in P_s$ with positive limit probability such that $\underline{\lambda}^{v^{0,p_s}, \underline{w}, p_s, \underline{p}} < \lambda_s^{v^{1,p_s}, p_s, \underline{p}}$ for some type $(v^{0,p_s}, \underline{w}) \in \Theta^{e,p_s}$, where recall that $v^{1,p_s} = \min\{v \in V : v > p_s + \underline{w}\}$, $v^{0,p_s} = \max\{v \in V : v < v^{1,p_s}\}$. Lemma 4, implies that $(v^{0,p_s}, \underline{w})$ always counter-demands $\underline{p}$. Hence, by Lemma 5, we know that the seller must then concede with probability approaching one in the limit, providing a limit payoff of $\underline{p}$ (as all buyers will then demand $\underline{p}$). This is again a contradiction as we have already established that she can guarantee a payoff of $\tilde{p}_s > \underline{p}$. Hence, the seller can never make such a demand with positive probability and we can restrict attention to seller demands, $p_s \in P_s^*$ where $\underline{\lambda}^{v^{0,p_s}, \underline{w}, p_s, \underline{p}} > \lambda_s^{v^{1,p_s}, p_s, \underline{p}}$. Also notice that for sufficiently small $\varepsilon' > 0$ we must also have that $(p_s - \underline{p})g(v, w) > \underline{p}(1 - g(v, w))$ for all $p_s \geq \tilde{p}_s$ and $(v, w) \in \Theta$.

We first seek to establish the upper bound on the buyer's payoff. Recall that if the seller demands $p_s \in P_s$, with $p_s \geq \tilde{p}_s$, then any buyer with $(v, w) \in \Theta^{e,p_s}$ will counterdemand $\underline{p} \leq \varepsilon' \leq \delta$. Hence, the best possible case for the seller who demands price p_s is that all types $(v, w) \in \Theta^{c,p_s}$ accept her demand and all types $(v, w) \in \Theta^{e,p_s}$ demand $\underline{p}$, giving her a payoff of at most $(1 - H(p_s))p_s + \delta$. For the seller to obtain a limit payoff larger than $\max_{p \in [0, p^* + 2\varepsilon]}(1 - H(p))p + \delta$, therefore, we must assume she makes some demand $p_s > p^* + \varepsilon$ with positive limit probability.

Define $\hat{p}_b^{p_s} = \min\{p_b \in P_b : p_b > v^{1,p_s} - p_s\}$. I claim that $\hat{p}_b^{p_s}$ is well defined and $\hat{p}_b^{p_s} < p^* + 2\varepsilon$. There are two cases to consider, (a) $p_s < \underline{v} - \underline{w}$ and (b) $p_s > \underline{v} - \underline{w}$. First consider case (a) where $v^{1,p_s} = \underline{v}$. If $\underline{w} \geq \underline{v}/2$ then we would have $p_s < \underline{v} - \underline{w} \leq \underline{w}$ which would contradict $p_s > p^* + 2\varepsilon$, and so we must in fact have $\underline{w} < \underline{v}/2 < p_s - 2\varepsilon$. And so, $\underline{v} - p_s < p_s - 4\varepsilon$. Given $\varepsilon' \leq \varepsilon/2$, there exists $p_b \in [\underline{v} - p_s, \underline{v} - p_s + \varepsilon] \cap P_b$ in any ε' rich commitment type space, and so $\hat{p}_b^{p_s}$ is not only well defined but $\hat{p}_b^{p_s} \leq \underline{v} - p_s + \varepsilon \leq \underline{v}/2 - \varepsilon < p^*$.

Next consider case (b). Let $\hat{\varepsilon} = \max_{d \in [\underline{v}, \overline{v}]} \min_{v \in V} |d - v|$. Given that a rational buyer's type space is ε rich, we must have that $\hat{\varepsilon} < \varepsilon$. Since $v^{1,p_s} \neq \underline{v}$ we have v^{0,p_s} well-defined. Moreover, since $v^{1,p_s} - 2\hat{\varepsilon} \leq v^{0,p_s} < p_s + \underline{w}$, we must have $v^{1,p_s} - p_s < \underline{w} + 2\hat{\varepsilon}$. Given $\varepsilon' \leq \varepsilon - \hat{\varepsilon}$, there must be some $p_b \in [\underline{w} + 2\hat{\varepsilon}, \underline{w} + 2\varepsilon) \cap P_b$ in a ε' rich commitment type space, and hence $\hat{p}_b^{p_s}$ is well-defined with $\hat{p}_b^{p_s} < \underline{w} + 2\varepsilon < p_s$. Also notice that $\hat{p}_b^{p_s} > v^{1,p_s} - p_s > \underline{w} > \underline{p}$.

Without loss of generality, I will assume that types $(v^{1,p_s}, w) \in \Theta^c$ never concede with positive probability at time 0^2. They will certainly never do so if the buyer concedes at time 0^3 or 0^4 for some counterdemand, but if $F_s^{p_s,p_b}(0^4) = 0$ for all $p_b < p_s$ then conceding at 0^4 is no different from conceding at 0^2.

Notice that if following some demand $p_b \in P_b$, with $p_b \geq \hat{p}_b^{p_s} > \underline{p}$, we have $\lim_n \bar{g}(v^{1,p_s}, w) > 0$ for some type $(v^{1,p_s}, w) \in \Theta^{c,p_s}$, then by Lemma 5, then the seller must concede with probability one in the limit. We reach a similar conclusion if $\lim_n \bar{z}_b^{p_s,p_b} > 0$. Clearly if $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) = 1$ then no buyer would imitate $p_b > \hat{p}_b^{p_s}$ in the limit and the seller's payoff would certainly be less than $\hat{p}_b^{p_s}(1 - H(\hat{p}_b^{p_s})) + \delta$, establishing the desired payoff bound. Suppose then that $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) < 1$, and so $\lim_n \bar{z}_b^{p_s,\hat{p}_b^{p_s}} = 0$ and $\lim_n \bar{g}(v^{1,p_s}, w) = 0$ for all $(v^{1,p_s}, w) \in \Theta^{c,p_s}$.

Clearly, we must have that v^{2,p_s} is well defined and that $t^{2,p_s,\hat{p}_b^{p_s}} \to \infty$. Since without loss of generality, all types $(v^{1,p_s}, w) \in \Theta^{c,p_s}$ must make some counterdemand and they cannot demand $\hat{p}_b^{p_s}$ with positive limit probability, suppose that they instead demand $p_b' > \hat{p}_b^{p_s}$ with positive limit probability, giving a limit payoff of $v^{1,p_s} - p_b'$. In this case we must have $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4) = (p_s - p_b')/(p_s - \hat{p}_b^{p_s}) < 1$, by Lemma 4. Since $\hat{p}_b^{p_s} > \underline{p}$, and all $(v, w) \in \Theta^{e,p_s}$ never imitate demands $p_b > \underline{p}$, we have $F_b^{p_s,\hat{p}_b^{p_s}}(0^4) = 0$. And so,

$$z_b^{p_s,\hat{p}_b^{p_s}} + (1 - z_b^{p_s,\hat{p}_b^{p_s}}) \sum_{(v^{1,p_s}, w) \in \Theta^{c,p_s}} \bar{g}^{p_s,\hat{p}_b^{p_s}}(v^{1,p_s}, w) = 1 - F_b^{p_s,\hat{p}_b^{p_s}}(t^{2,p_s,\hat{p}_b^{p_s}}) = e^{-\lambda_b t^{2,p_s,\hat{p}_b^{p_s}}}$$

converges to zero, implying $t^{2,p_s,\hat{p}_b^{p_s}} \to \infty$. Hence, by imitating $\hat{p}_b^{p_s}$ type $(v^{1,p_s}, w) \in \Theta^{c,p_s}$ secures a payoff of at least

$$\begin{aligned} &(v^{1,p_s} - \hat{p}_b^{p_s})\Big(F_s^{p_s,\hat{p}_b^{p_s}}(0^4) + \int_{0^4 < t < t^{2,p_s,\hat{p}_b^{p_s}}} e^{-rt} dF_s^{p_s,\hat{p}_b^{p_s}}(t)\Big) \qquad (1) \\ \geq &(v^{1,p_s} - \hat{p}_b^{p_s})\Big(F_s^{p_s,\hat{p}_b^{p_s}}(0^4) + \frac{v^{2,p_s} - p_s}{v^{2,p_s} - \hat{p}_b^{p_s}}(1 - F_s^{p_s,\hat{p}_b^{p_s}}(0^4) - e^{-rt^{2,p_s,\hat{p}_b^{p_s}}}(1 - F_s^{p_s,\hat{p}_b^{p_s}}(t^{2,p_s,\hat{p}_b^{p_s}})))\Big) \end{aligned}$$

where the second inequality follows from the fact that v^{2,p_s} would find it optimal to concede at $t^{2,p_s,\hat{p}_b^{p_s}}$ conditional on demanding $\hat{p}_b^{p_s}$, that is:

$$\int_{0^4<t<t^{2,p_s,\hat{p}_b^{p_s}}} e^{-rt} dF_s^{p_s,\hat{p}_b^{p_s}}(t) \geq \frac{v^{2,p_s}-p_s}{v^{2,p_s}-\hat{p}_b^{p_s}}(1-F_s^{p_s,\hat{p}_b^{p_s}}(0^4)-e^{-rt^{2,p_s,\hat{p}_b^{p_s}}}(1-F_s^{p_s,\hat{p}_b^{p_s}}(t^{2,p_s,\hat{p}_b^{p_s}})))$$

but as $t^{2,p_s,\hat{p}_b^{p_s}} \to \infty$, the right hand side of (1) converges to

$$(v^{1,p_s}-\hat{p}_b^{p_s})\Big(\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4)+\frac{v^{2,p_s}-p_s}{v^{2,p_s}-\hat{p}_b^{p_s}}(1-\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4))\Big). \tag{2}$$

However, this is strictly larger than $v^{1,p_s}-p'_b$ given that $(v-p_s)/(v-p_b)$ is strictly increasing in v and $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4)=(p_s-p'_b)/(p_s-\hat{p}_b^{p_s})$, which contradicts the optimality of demanding p'_b.

Next suppose that type $(v^{1,p_s},\underline{w}) \in \Theta^{c,p_s}$ imitates $p'_b \in (\underline{p},\hat{p}_b^{p_s}) \cap P_b$ with positive limit probability, then since $(v,w)\in\Theta^{e,p_s}$ never imitate demands $p'_b > \underline{p}$, we must have that $p'_b < v^{1,p_s}-p_s$ and so the buyer must concede with probability approaching one (by Lemma 6), to give $(v^{1,p_s},w)\in\Theta^{c,p_s}$ a limit payoff of $v^{1,p_s}-p_s$. However, since $t^{2,p_s,\hat{p}_b^{p_s}} \to \infty$ such a buyer could secure a limit payoff of at least 2 by imitating $\hat{p}_b^{p_s}$ which exceeds $v^{1,p_s}-p_s$ even if $\lim_n F_s^{p_s,\hat{p}_b^{p_s}}(0^4)=0$ (since $(v-p_s)/(v-p_b)$ is strictly increasing in v). This is again, a contradiction.

The final possibility is that $\lim_n \mu_b^{p_s,v^{1,p_s},w}(\underline{p})=1$ for all $(v^{1,p_s},w)\in\Theta^{c,p_s}$. If v^{0,p_s} is well defined then by assumption we have $\underline{\lambda}^{v^{0,p_s},\underline{w},p_s,\underline{p}} < \lambda_s^{v^{1,p_s},p_s,\underline{p}}$, or else the demand would not have been made (as argued above). Hence, since $(p_s-\underline{p})g(v^{1,p_s},\underline{w}) > \underline{p}(1-g(v^{1,p_s},\underline{w}))$, the buyer must either concede or exit immediately with probability approaching one in the limit by Lemma 6, giving $(v^{1,p_s},w)\in\Theta^{c,p_s}$ a payoff of $v^{1,p_s}-p_s$, which is again strictly less than the payoff of (2) she could have obtained by demanding $\hat{p}_b^{p_s}$. This contradiction ensures that $F_s^{p_s,\hat{p}_b^{p_s}}(0^4)$ approaches 1 in the limit, so that no buyer proposes a higher price and the seller's payoff is at most $(1-H(p_s))\hat{p}_b^{p_s}+\delta \leq \max_{p\in[0,p^*+2\varepsilon]}(1-H(p))p+\delta$ since $\hat{p}_b^{p_s} < p^*+2\varepsilon < p_s$. Whether or not the seller demands $p_s > p^*+2\varepsilon$ with positive limit probability, therefore, the buyer enjoys a limit payoff of at least $\max\{v-(p^*+2\varepsilon),w\}$, establishing the weaker of the two lower bounds on the seller's payoffs.

Turning to the lower bound on payoffs, let $\hat{p} \in \arg\max_{p\in[0,p^*]}(1-H(p))$ and $\check{p}(p)=\mathbb{1}_{p\leq\underline{v}-\underline{w}}p+\mathbb{1}_{p>\underline{v}-\underline{w}}\max\{v-\underline{w}\leq p : v\in V\}$, where $\check{p}(p)\in[p-2\hat{\varepsilon},p]$ and $\hat{\varepsilon}=\max_{d\in[\underline{v},\overline{v}]}\min_{v\in V}|d-v|<\varepsilon$. Let $\overline{p}_s=\max\{p_s\in P_s : p_s<\check{p}(\hat{p}_s)\}$, where certainly $\overline{p}_s \geq \tilde{p}_s < \min\{v-w : (v,w)\in\Theta\}$ and in fact given any $\varepsilon' \leq \varepsilon-\hat{\varepsilon}$ rich commitment type space, we must have $\overline{p}_s \in [\hat{p}_s-2\varepsilon,\check{p}(\hat{p}_s))\cap P_s$.

We want to show that $\overline{p}_s \in P_s^*$. If $\overline{v}=\underline{v}$ then since $\hat{p}_s \leq \overline{v}-\underline{w}$ we have $\check{p}(\hat{p}_s)=p\leq\underline{v}-\underline{w}$, otherwise let $\check{\varepsilon}=\min\{v-v' : v\neq v'\in V\}\in(0,\overline{v})$ and assume that $2\varepsilon' \leq \underline{w}\check{\varepsilon}/(\overline{v}-\check{\varepsilon})$. Notice that if $\check{p}(\hat{p}_s)\leq\underline{v}-\underline{w}$ then clearly $v^{1,\overline{p}_s}=\underline{v}$. On the other hand, suppose that $\check{p}(\hat{p}_s)>\underline{v}-\underline{w}$ and so $v^{1,\overline{p}_s}-\underline{w}=\check{p}(\hat{p}_s)<\overline{p}_s+2\varepsilon'$ and $v^{0,\overline{p}_s}\leq v^{1,\overline{p}_s}-\check{\varepsilon}$. In this case we have,

$$
\begin{aligned}
&(v^{1,\overline{p}_s} - \overline{p}_s)(v^{0,\overline{p}_s} - \underline{w}) - \underline{w}\overline{p}_s < (\underline{w} + 2\varepsilon')(v^{0,\overline{p}_s} - \underline{w}) - \underline{w}(v^{1,\overline{p}_s} - \underline{w} - 2\varepsilon') \\
\leq &(\underline{w} + 2\varepsilon')(v^{1,\overline{p}_s} - \check{\varepsilon} - \underline{w}) - \underline{w}(v^{1,\overline{p}_s} - \underline{w} - 2\varepsilon') \leq (\underline{w} + 2\varepsilon')(\overline{v} - \check{\varepsilon} - \underline{w}) - \underline{w}(\overline{v} - \underline{w} - 2\varepsilon') \leq 0
\end{aligned}
$$

where the first inequality follows from $\overline{p}_s > v^{1,\overline{p}_s} - \underline{w} - 2\varepsilon'$, the second from $v^{0,\overline{p}_s} \leq v^{1,\overline{p}_s} - \check{\varepsilon}$, the third from $v^{1,\overline{p}_s} \leq \overline{v}$ and the fourth from $2\varepsilon' \leq \underline{w}\check{\varepsilon}/(\overline{v} - \check{\varepsilon})$. Furthermore notice that

$$
(v^{1,\overline{p}_s} - \overline{p}_s)(v^{0,\overline{p}_s} - p_b - \underline{w}) - \underline{w}(\overline{p}_s - p_b) \tag{3}
$$

is decreasing in p_b given $v^{1,\overline{p}_s} - \overline{p}_s > \underline{w}$ and so must be negative for any $p_b \in (0, \overline{p}_s)$. Given this, I claim that $\lambda_s^{\overline{p}_s, \underline{p}, v^{1,\overline{p}_s}} < \underline{\lambda}^{\overline{p}_s, \underline{p}, v, w}$ for all $(v, w) \in \Theta^{e,\overline{p}_s}$; in other words, if $(v^{1,\overline{p}_s}, \underline{w})$ concedes at time t then all $(v, w) \in \Theta^{e,\overline{p}_s}$ must have exited before t. Clearly, we have just established the claim for $(v^{0,\overline{p}_s}, \underline{w})$, and noticing that (3) is increasing in $v^{0,\overline{p}_s}$, it must likewise hold for any $(v, \underline{w})$ with $v < v^{0,\overline{p}_s}$. Since (3) is decreasing in $\underline{w}$, the claim must also hold for all $(v, w) \in \Theta$ with $v \leq v^{0,\overline{p}_s}$ and $w \geq \underline{w}$.

A buyer with value $v \geq v^{1,\overline{p}_s}$ must be indifferent to conceding on some interval after the counterdemand $\underline{p}$ when the concession rate is $r(v - \overline{p}_s)/(\overline{p}_s - \underline{p}) = \lambda_s^{\overline{p}_s, \underline{p}, v} \geq \lambda_s^{\overline{p}_s, \underline{p}, v^{1,\overline{p}_s}}$, where this concession rate is strictly decreasing in $\overline{p}_s$. Hence, if $\overline{p}_s > v - w$ (so that $(v, w) \in \Theta^{e,\overline{p}_s}$), we must have $r(v - \overline{p}_s)/(\overline{p}_s - \underline{p}) < rw/(v - w - \underline{p}) = \underline{\lambda}^{\overline{p}_s, \underline{p}, v, w}$, which establishes the claim. And so, $\overline{p}_s \in P_s^*$.

Hence, suppose the seller demands $\overline{p}_s$, then since $\overline{p}_s \leq p^*$ for any counterdemand $p_b < \overline{p}_s$ we must have $v - \overline{p}_s > p_b$ for all $(v, w) \in \Theta^{c,\overline{p}_s}$. To see this, notice that if $\underline{v}/2 \geq \overline{p}_s$ then $v - \overline{p}_s \geq \underline{v}/2 \geq \overline{p}_s > p_b$ whereas if $\underline{w} \geq \overline{p}_s$ then $v - \overline{p}_s > w \geq \underline{w} \geq \overline{p}_s > p_b$ for $(v, w) \in \Theta^{c,\overline{p}_s}$. Hence, if a buyer with$(v, w) \in \Theta^{c,\overline{p}_s}$ demands $p_b > \underline{p}$ with positive probability in the limit, she must subsequently immediately concede with probability one in the limit by Lemma 6 to give her payoffs of $(v - \overline{p}_s)$ for all large n (since buyers with $(v, w) \in \Theta^{e,\overline{p}_s}$ always demand $\underline{p}$.

As argued previously, it is without loss of generality to assume that type $(v^{1,\overline{p}_s}, \underline{w})$ always makes some counterdemand $p_b \in P_b$, and in this case, therefore, without loss to assume she counterdemands $\underline{p}$ with probability approaching one in this limit. However, in this case Lemma 6 tells us that in the limit the buyer concedes with probability approaching $\lim_n x^{\overline{p}_s, \underline{p}}$ and exits with probability approaching $1 - \lim_n x^{\overline{p}_s, \underline{p}}$ at time 0[4] given that $(\overline{p}_s - \underline{p})g(v^{1,\overline{p}_s}, \underline{w}) > \underline{p}(1 - g(v^{1,\overline{p}_s}, \underline{w}))$. And so, this demand secures a limit payoff of at least $(1 - H(\overline{p}_s))\overline{p}_s \geq \max_{p \in [0,p^*]}(1 - H(p))p - 2\varepsilon$ where the inequality follows from $\overline{p}_s \in [\hat{p}_s - 2\varepsilon, \hat{p}_s]$.

I now turn to establishing the stronger predictions when $\hat{p}_s \in \arg\max_{p \in [0,p^*+2\varepsilon]}(1 - H(p))p = \arg\max_{p \in [0,p^*]}(1 - H(p))p$, and $\check{p}(\hat{p}_s)(1 - H(\check{p}(\hat{p}_s))) > \max_{p \in [0,\check{p}(\hat{p}_s)) \cup (\hat{p}_s, p^*+2\varepsilon]} p(1 - H(p)$. Then as above let $\overline{p}_s = \max\{p_s \in P_s : p_s < \check{p}(\hat{p}_s)\} \in (\check{p}(\hat{p}_s) - 2\varepsilon', p^*]$. As argued above, for $\varepsilon' > 0$ small enough $\overline{p}_s \in P_s^*$. Moreover, for any $p_s \in P_s^*$ with $p_s \in [\tilde{p}_s, p^*]$, all buyers with $v - w > p_s$

immediately concede with probability approaching one in the limit and all buyers with $v-w < p_s$ exit by Lemma 6. Hence, the seller's limit payoff from such a demand is exactly $(1 - H(\overline{p}_s))p_s$. And so, the buyer's payoff from demanding $\overline{p}_s$ is at least $(\check{p}(\hat{p}_s))(1 - H(\check{p}(\hat{p}_s))) - 2\varepsilon'$.

If $\overline{p}_s > \min\{v - w : (v, w) \in \Theta\}$ then let $\dot{p} = \max\{p < \check{p}(\hat{p}_s) : H(p) < H(\check{p}(\hat{p}_s))\}$; notice that H is constant on the non-degenerate interval $(\dot{p}, \check{p}(\hat{p}_s))$ and $(1 - H(p_s))p_s$ is increasing in p_s on this interval so that $(1 - H(p_s))p_s < (1 - H(\overline{p}_s))\overline{p}_s$ for $p_s \in (\dot{p}, \overline{p}_s)$. Let $2\varepsilon' < \check{p}(\hat{p}_s)(1 - H(\check{p}(\hat{p}_s))) - \max_{p \leq \dot{p}} p(1-H(p))$ (where the right hand side is strictly positive by assumption) then the seller's payoff from demanding $p_s \leq \dot{p}$ is certainly less than from demanding $\overline{p}_s$. Hence, the seller will certainly never demand $p_s < \overline{p}_s$ in the limit.

On the other hand, suppose that the seller demands $p_s > p^* + 2\varepsilon$, then we showed above that for small $\varepsilon' > 0$ the buyer will counterdemand $p_b \leq p^* + 2\varepsilon$, which the seller will immediately concede to with strictly positive probability, giving her a payoff less than $(p^* + 2\varepsilon)(1 - H(p^* + 2\varepsilon)) + \varepsilon'$. And so, the seller's best possible limit payoff from demanding $p_s > \hat{p}_s$, is always less than $\max_{p \in (\hat{p}_s, p^*+2\varepsilon]} p(1 - H(p)) + \varepsilon'$. This payoff is strictly then less than her payoff from proposing $\overline{p}_s$ whenever $3\varepsilon' < \check{p}(\hat{p}_s)(1 - H(\check{p}(\hat{p}_s))) - \max_{p \in (\hat{p}_s, p^*+2\varepsilon]} p(1 - H(p))$ (where the right hand side is strictly positive by assumption). Hence, the seller will never demand $p_s > \hat{p}_s$ in the limit. Hence, the seller only demands $p_s \in [\hat{p}_s - 2\varepsilon, \hat{p}_s]$ with positive probability in the limit, and since $\hat{p}_s \leq p^*$, buyers' with $v - w > p_s$ immediately purchase and those with $v - w < p_s$ immediately exit. This completes the proof.

□