Monotone Additive Statistics*

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Abstract

The expectation is an example of a descriptive statistic that is monotone with respect to stochastic dominance, and additive for sums of independent random variables. We provide a complete characterization of such statistics, and explore a number of applications to models of individual and group decision-making. These include a representation of stationary, monotone time preferences, extending the work of Fishburn and Rubinstein (1982) to time lotteries, as well as a characterization of risk-averse preferences over monetary gambles that are invariant to mean-zero background risks.

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1 Introduction

How should a random quantity be summarized by a single number? In Bayesian statistics, point estimators capture an entire posterior distribution. In finance, risk measures quantify the risk in a distribution of returns. And in economics, certainty equivalents characterize an agent's preference for uncertain outcomes.

We use the term *descriptive statistic*, or simply *statistic*, to refer to a map that assigns a number to each bounded random variable (Bickel and Lehmann, 1975a). We study statistics that are monotone with respect to first-order stochastic dominance, and additive for sums of independent random variables. An example of a monotone additive statistic is the expectation. The median is monotone but not additive, while the variance is additive, but not monotone.

Monotonicity is a well studied property of statistics (see, e.g., Bickel and Lehmann, 1975a,b), and holds, for example, for certainty equivalents of monotone preferences over lotteries. Additivity is a stronger assumption. We focus on this property because of its conceptual simplicity and because it serves as a baseline assumption in many settings. As we argue, additivity corresponds to stationarity in the context of preferences over time lotteries (see §3). In the context of choices over monetary gambles it corresponds to a form of separability across independent decision problems (see §4).

Beyond the expectation, an additional example of a monotone additive statistic is the map K_a that, given $a \in \mathbb{R}$, assigns to each random variable X the value

$$K_a(X) = \frac{1}{a} \log \mathbb{E}\left[e^{aX}\right]. \tag{1}$$

In the fields of probability and statistics, this function is known as the (normalized) cumulant generating function evaluated at a. In economics, it corresponds to the certainty equivalent of an expected utility maximizer who exhibits constant absolute risk aversion (CARA) over gambles. For bounded random variables, the essential minimum and maximum provide further examples of such statistics; as we explain later, they are the limits of K_a as a approaches $\pm\infty$. The expectation is equal to K_0, the limit of K_a as a approaches 0.

Our main result establishes that these examples, and their weighted averages, are the only monotone additive statistics. That is, we show that every monotone additive statistic Φ is of the form

$$\Phi(X) = \int K_a(X)\,\mathrm{d}\mu(a)$$

for some probability measure μ. This result provides a simple representation of a natural family of statistics, which one may a priori have expected to be much richer.

Our first application is to time preferences. The starting point for our analysis is the work by Fishburn and Rubinstein (1982), who study preferences over dated rewards: a monetary reward, together with the time at which it will be received. They show that exponential discounting of time arises from a set of axioms, of which the most substantial is *stationarity*. Their stationarity axiom postulates that preferences between two dated rewards are unaffected by the addition of a common delay.

We extend the analysis of Fishburn and Rubinstein (1982) to that of *time lotteries*: a monetary reward x, together with a random time T at which it will be received. In this setting, we too introduce a stationarity axiom that requires preferences to be invariant with respect to random independent delays. As we argue in the main text, this stationarity axiom captures a basic requirement of dynamic consistency, together with the idea that preferences do not depend on calendar time.

We show that a monotone and stationary preference over time lotteries admits a representation of the form

$$u(x)\mathrm{e}^{-r\Phi(T)}$$

where Φ is a monotone additive statistic (Theorem 3). Thus, $\Phi(T)$ is the certainty equivalent of the random time T, i.e. the deterministic time that is as desirable as T.

Over deterministic dated rewards, the above representation coincides with standard discounted utility. General time lotteries are reduced to deterministic ones through the certainty equivalent Φ. By our main representation theorem, it takes the form $\Phi(T) = \int K_a(T)\,\mathrm{d}\mu(a)$. In this context, each $K_a(T)$ is the certainty equivalent of T under an expected discounted preference with discount rate $-a$. The different certainty equivalents are then averaged according to the measure μ.

Our representation theorem for monotone and stationary time preferences has implications for understanding the relation between stationarity and risk attitudes toward time. How people choose among prospects that involve risk over time has been studied both theoretically and experimentally (Chesson and Viscusi, 2003; Onay and Öncüler, 2007; Ebert, 2020; DeJarnette et al., 2020; Ebert, 2021). A basic paradox these papers highlight is that many subjects display risk aversion over the time dimension, even though the standard theory of expected discounted utility predicts that people are risk-seeking with respect to time lotteries. Our analysis shows that expected discounted utility is only one of many ways to extend exponential discounting from dated rewards to time lotteries. In particular, monotone and stationary time preferences can accommodate risk aversion over time, as well as more nuanced preferences that display risk-aversion or risk-seeking behavior depending on the choice at hand.

We further apply the characterization of monotone stationary preferences to the problem of aggregating heterogeneous time preferences. It is well known that directly averaging exponential discounting utilities leads to present bias (see Jackson and Yariv, 2014, 2015). Based on this observation, the literature concludes that within expected discounted utility, it is impossible to aggregate individual preferences into a social preference unless the latter is dictatorial.

We show that this difficulty is not due to stationarity, but rather to an insistence on the idea that the social preference should conform to expected discounted utility. If the social preference is allowed to be a general monotone stationary preference, then this difficulty vanishes. In fact, a social preference obtained by averaging the certainty equivalents of the individuals satisfies Pareto efficiency and stationarity. Moreover, every Paretian and stationary social preference is obtained in this way (Theorem 4).

Monotone additive statistics also find natural applications to models of choice over monetary gambles. It is well known that an expected utility agent whose preferences are invariant to independent background risks must have CARA preferences. This invariance property makes CARA utility functions useful modeling tools when the analyst does not observe the agents' wealth level or the additional risks they face (see, e.g., Barseghyan et al., 2018). Beyond expected utility, preferences that are invariant to background risks have certainty equivalents that are monotone additive statistics, and thus, by our representation theorem, are weighted averages of CARA certainty equivalents. We then investigate two separate weakenings of this invariance property: invariance to the level of wealth, and invariance to mean-zero risks. While wealth invariance and the latter form of risk invariance are equivalent under expected utility, we prove that for general preferences these properties are independent, and study their respective implications.

Our final application concerns group decision-making under risk. We consider a firm that employs multiple agents, each of whom makes decisions following an individual preference relation, or a rule prescribed by the firm. We show that in order for the agents' independent choices to not violate stochastic dominance when combined, it is sufficient and necessary that their preferences are represented by the same monotone additive statistic. Thus, these are the only preferences with the property that decentralized decisions cannot result in stochastically dominated outcomes for the organization.

1.1 Related Literature

A large literature in statistics studies descriptive statistics of probability distributions. A representative example is the work of Bickel and Lehmann (1975a,b), who study location statistics using an axiomatic, non-parametric approach that is similar to ours. This literature has however focused on different properties, and, to the best of our knowledge, does not contain a similar characterization of additivity and monotonicity. The mathematics literature has studied additive statistics as homomorphisms from the convolution semigroup to the real numbers (see Ruzsa and Székely, 1988; Mattner, 1999, 2004), but without imposing monotonicity.

In finance and actuarial sciences, the certainty equivalent $K_a(X)$ is also known as an *entropic risk measure*, and is used to assess the riskiness of a financial position X. It is a canonical example of a coherent risk measure (see Föllmer and Schied, 2002, 2011; Föllmer and Knispel, 2011). In this literature, Goovaerts, Kaas, Laeven, and Tang (2004) study additive statistics that are monotone with respect to all entropic risk measures, i.e. those with the property that $K_a(X) \geq K_a(Y)$ for all $a \in \mathbb{R}$ implies $\Phi(X) \geq \Phi(Y)$, and show that they must be weighted averages of entropic risk measures, as in our main representation. Taken in isolation, their monotonicity assumption does not admit a natural interpretation. In contrast, we show that this condition is implied by monotonicity and additivity of Φ. Proving this implication is a crucial step in our analysis.

In an earlier paper, Pomatto, Strack, and Tamuz (2020) show that on the domain of random variables that have all moments, the only monotone additive statistic is the expectation. In this paper we consider a smaller domain and obtain a larger class of possible statistics. These different results can be reconciled by observing that for any $a \neq 0$, the statistic $K_a(X)$ takes infinite value for some random variable X that has all moments but thick tails. Fritz, Mu, and Tamuz (2020) prove that the expectation remains the unique monotone additive statistic on the domain L^p, for any $p \geq 1$, while there are no monotone additive statistics on L^p with $p < 1$, or on the domain of all random variables.

Monotone additive statistics also relate to what we called *additive divergences* in Mu, Pomatto, Strack, and Tamuz (2021). An additive divergence is a map defined over Blackwell experiments that satisfies monotonicity with respect to the Blackwell order and additivity for product experiments. Our characterization of additive divergences in that paper is reminiscent of the one we provide here, with Rényi divergences playing the role of the cumulant generating functions.

Decision criteria that aggregate multiple certainty equivalents have appeared before in the literature. Myerson and Zambrano (2019) advocate the maximization of a sum of certainty equivalents as an effective rule for risk sharing. Chambers and Echenique (2012) formalize and characterize this rule as a social welfare functional, using axioms quite different in nature from the ones we study. Our representation also bears resemblance to cautious expected utility theory (Cerreia-Vioglio, Dillenberger, and Ortoleva, 2015), in which a gamble is evaluated according to the minimum certainty equivalent across a family of utility functions. Our axioms are different in that we study invariance properties of the preference when adding an independent gamble, while Cerreia-Vioglio, Dillenberger, and Ortoleva (2015) consider the effect of mixing with another gamble.

The properties of risk and wealth invariance we study in §4 are related to Safra and Segal (1998), who consider preferences that are invariant to shifting and scaling gambles. Those preferences are characterized by certainty equivalents Ψ that satisfy

$\Psi(\alpha(X + \beta)) = \alpha(\Psi(X) + \beta)$ for all constants $\alpha > 0$ and β. Certainty equivalents of the form $K_a(X)$ are not positively homogeneous, but satisfy our more restrictive additivity property. As a consequence, the set of preferences characterized by Safra and Segal (1998) is neither included in, nor includes the set of preferences represented by monotone additive statistics.

The remainder of the paper is organized as follows. In §2 we introduce monotone additive statistics and state our main result. In §3 we apply this result to time lotteries, and in §4 we apply it to monetary gambles. Comparative risk attitudes of monotone additive statistics are discussed in §5, while §6 provides an overview of the proof of our main result. The appendix and online appendix contain omitted proofs for the results in the main text.

2 Monotone Additive Statistics

We denote by L^∞ the collection of bounded real random variables, defined over a nonatomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will identify each $c \in \mathbb{R}$ with the corresponding constant random variable $X(\omega) = c$.

We say that a map $\Phi\colon L^\infty \to \mathbb{R}$ is a *statistic* if it satisfies (i) $\Phi(X) = \Phi(Y)$ whenever $X, Y \in L^\infty$ have the same distribution, and (ii) $\Phi(c) = c$ for every $c \in \mathbb{R}$; that is, Φ assigns c to the constant random variable c. We are interested in statistics that satisfy monotonicity with respect to first-order stochastic dominance and additivity for sums of independent random variables. Formally, Φ is

- *additive* if $\Phi(X + Y) = \Phi(X) + \Phi(Y)$ whenever X and Y are independent, and
- *monotone* if $X \geq_1 Y$ implies $\Phi(X) \geq \Phi(Y)$, where $\geq_1$ denotes first-order stochastic dominance.

Since, by assumption, the value $\Phi(X)$ depends only the distribution of the random variable X, monotonicity is equivalent to the requirement that $\Phi(X) \geq \Phi(Y)$ whenever $X \geq Y$ almost surely. This equivalence is based on the well-known fact that $X \geq_1 Y$ if and only if there are random variables $\tilde{X}, \tilde{Y}$ such that X and $\tilde{X}$ are identically distributed, Y and $\tilde{Y}$ are identically distributed, and $\tilde{X} \geq \tilde{Y}$ almost surely.[1]

We denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ the extended real numbers. Given $X \in L^\infty$ and

[1] An alternative, equivalent definition for a statistic is to let the domain of Φ be the set of probability distributions on $\mathbb{R}$ with bounded support. In this domain, additivity would be defined with respect to convolution. We choose to have the domain consist of random variables, as this approach offers some notational advantages.

$a \in \overline{\mathbb{R}} \setminus \{0, \pm\infty\}$, we consider the statistic

$$K_a(X) = \frac{1}{a} \log \mathbb{E}\left[\mathrm{e}^{aX}\right]. \tag{2}$$

The value $K_a(X)$ is the certainty equivalent of X for a CARA utility function with coefficient of risk aversion $-a$. In probability and statistics, $K_a(X)$ is known as the *(normalized) cumulant generating function* of X, evaluated at a. If X and Y are independent, then $\mathbb{E}\left[\mathrm{e}^{a(X+Y)}\right] = \mathbb{E}\left[\mathrm{e}^{aX}\right]\mathbb{E}\left[\mathrm{e}^{aY}\right]$, and hence K_a is additive. It is also monotone.

We additionally define $K_0(X), K_\infty(X), K_{-\infty}(X)$ to be the expectation, the essential maximum, and the essential minimum of X, respectively. This choice of notation makes $a \mapsto K_a(X)$ a continuous function from $\overline{\mathbb{R}}$ to $\mathbb{R}$. Our main result is a representation theorem for monotone additive statistics:

Theorem 1. *$\Phi\colon L^\infty \to \mathbb{R}$ is a monotone additive statistic if and only if there exists a unique Borel probability measure μ on $\overline{\mathbb{R}}$ such that for every $X \in L^\infty$*

$$\Phi(X) = \int_{\overline{\mathbb{R}}} K_a(X)\, \mathrm{d}\mu(a). \tag{3}$$

Each K_a satisfies monotonicity and additivity, and it is immediate that these two properties are preserved under convex combinations. Theorem 1 says that the one-parameter family $\{K_a\}$ forms the extreme points of the set of monotone additive statistics; every such statistic must be a weighted average obtained by mixing over this family. In §6 we provide an overview of the proof of Theorem 1.

Theorem 1 can be extended to other domains of random variables. We consider the set L^∞_+ of bounded non-negative random variables, the set $L^\infty_{\mathbb{N}}$ of bounded non-negative integer-valued random variables, and the set L_M of random variables X for which $K_a(X)$, as defined in (2), is finite for all $a \in \mathbb{R}$. The domain L_M contains those unbounded random variables whose distributions have sub-exponential tails, as in the case of the normal distribution.

Theorem 2. *Let L be either L^∞_+, $L^\infty_{\mathbb{N}}$ or L_M. Then $\Phi\colon L \to \mathbb{R}$ is a monotone additive statistic if and only if it admits a (unique) representation of the form (3) where, in the case of L_M, the measure μ has compact support in $\mathbb{R}$.*

The domains L^∞_+ and $L^\infty_{\mathbb{N}}$ will be important in §3 for studying preferences over time lotteries in continuous or discrete time, where a random variable X corresponds to the stochastic future time at which a reward is obtained by a decision maker. The extension of Theorem 1 to the larger domain L_M adds to the applicability of our representation, as it includes distributions with unbounded support, such as Gaussian or Poisson, for which the function K_a has closed-form expressions. For example, Theorem 2 implies that for a Gaussian random variable Z, a monotone additive statistic Φ defined on L_M takes the simple mean-variance form $\Phi(Z) = \mathbb{E}[Z] + c\text{Var}[Z]/2$, where $c \in \mathbb{R}$ is the mean of the measure μ characterizing Φ.

To prove Theorem 2 for the cases of $L = L^\infty_+$ and $L = L^\infty_{\mathbb{N}}$, we first prove that any monotone additive statistic defined on these smaller domains can be extended to L^∞, and then invoke Theorem 1. The case of the larger domain L_M is more difficult, and its proof requires some different techniques.

A few additional remarks are in order. A possible strengthening of our additivity condition requires $\Phi(X + Y) = \Phi(X) + \Phi(Y)$ to hold for all pairs of random variables, rather than just the independent ones. As is well known, the only statistic additive in this more restrictive sense is the expectation (see, for example, de Finetti, 1970). A different strengthening is additivity with respect to uncorrelated random variables. It follows from the analysis of Chambers and Echenique (2020) that on a finite probability space the expectation is, again, the only monotone statistic that is additive for uncorrelated random variables.

Finally, one could also consider a weakening of additivity to a *sub-additivity* condition, i.e. statistics that satisfy $\Phi(X + Y) \le \Phi(X) + \Phi(Y)$ for independent X, Y. In the supplementary appendix §I, we develop this extension and obtain a characterization of monotone sub-additive statistics which generalizes Theorem 1.

3 Monotone Stationary Time Preferences

Next, we apply monotone additive statistics to the study of time preferences. We consider decision problems where an agent is asked to choose between time lotteries that pay a fixed reward at a future random time, as in the case of a driver choosing between different routes, where some routes are more likely than others to face heavy traffic, or a company choosing between projects with different payout times. We argue that in this context additivity is connected to a notion of stationarity, according to which a choice between future rewards is not affected by the addition of an independent delay. In this section we study preferences over time lotteries that are monotone and stationary, characterize them using monotone additive statistics, discuss the risk attitudes they can model, and apply them to the problem of aggregating heterogeneous time preferences.

3.1 Domain and Axioms

A *time lottery* is a monetary reward received by a decision maker at a future, random time. Formally, it consists of a pair (x, T), where $x \in \mathbb{R}_{++}$ is a positive payoff and $T \in L^\infty_+$ is the random time at which it realizes. Thus, time is non-negative and continuous. Our primitive is a complete and transitive binary relation $\succeq$ on the domain $\mathbb{R}_{++} \times L^\infty_+$. We denote by $\sim$ the indifference relation induced by $\succeq$. To avoid notational confusion, in the rest of this section x and y always denote monetary payoffs, t, s and d denote deterministic times, and T, S, D denote random times.

We say that a preference relation $\succeq$ on $\mathbb{R}_{++} \times L^\infty_+$ is a *monotone stationary time preference* (henceforth, MSTP) if it satisfies the following axioms:

Axiom 3.1 (More is Better)**.** *If $x > y$ then $(x, T) \succ (y, T)$.*

Axiom 3.2 (Earlier is Better)**.** *If $s > t$ then $(x, t) \succ (x, s)$, and if $S \geq_1 T$ then $(x, T) \succeq (x, S)$.*

Axiom 3.3 (Stationarity)**.** *If $(x, T) \succeq (y, S)$ then $(x, T + D) \succeq (y, S + D)$ for any D that is independent from T and S.*

Axiom 3.4 (Continuity)**.** *For any (y, S), the sets $\{(x, t) \colon (x, t) \succeq (y, S)\}$ and $\{(x, t) \colon (x, t) \preceq (y, S)\}$ are closed in $\mathbb{R}_{++} \times \mathbb{R}_+$.*

The first two Axioms 3.1 and 3.2 are standard conditions that directly generalize the properties studied by Fishburn and Rubinstein (1982). They require the decision maker to prefer higher payoffs, and to prefer (stochastically) earlier times. Axiom 3.4 is a standard continuity assumption. It does not require a choice of topology over random times.

The most substantive condition is stationarity. In the absence of risk, it was shown by Halevy (2015) that stationarity can be understood as the implication of two more basic principles: that preferences are not affected by calendar time, and that the decision maker is dynamically consistent. As we next argue, Axiom 3.3 extends the same logic to time lotteries.

Consider first a deterministic delay d added to both S and T. Reasoning as in Halevy (2015), we can imagine an enlarged framework where the decision maker is endowed with a profile $(\succeq_t)$ of preferences over time lotteries, with $\succeq_t$ representing the preference the decision maker expresses at time t. If preferences are not affected by calendar time, then the ranking $(x, T) \succeq_0 (y, S)$ at time zero must imply the same ranking $(x, T + d) \succeq_d (y, S + d)$ at time d. Moreover, dynamic consistency requires that a choice between $(x, T + d)$ and $(y, S + d)$, when evaluated at time zero, must be the same choice the decision maker would in fact make at time d. Hence, $(x, T) \succeq_0 (y, S)$ implies $(x, T + d) \succeq_0 (y, S + d)$, as required by Axiom 3.3.

Consider now a delay D that is random and independent of S and T. As argued above, dynamic consistency and time invariance imply the ranking $(x, T + d) \succeq_0 (y, S + d)$ for each deterministic time d. We can then imagine that prior to making a choice between $(x, T + D)$ and $(y, S + D)$, the decision maker is informed of the actual realization d of D. Regardless of what the value d is, this information should not change the decision maker's preference of $(x, T+d)$ over $(y, S+d)$, since D is independent of T and S. So, dynamic consistency with respect to this piece of information requires the decision maker to prefer $(x, T+D)$ to $(y, S+D)$. While this latter form of dynamic consistency is suggestive of expected discounted utility, we will in fact derive alternative representations that also satisfy this consistency condition.

3.2 Representation

Our next result characterizes monotone stationary time preferences:

Theorem 3. *A preference relation $\succeq$ over time lotteries is an MSTP if and only if there exist a monotone additive statistic Φ, a constant $r > 0$, and a continuous and increasing function $u \colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $\succeq$ is represented by*

$$V(x, T) = u(x) \cdot \mathrm{e}^{-r\Phi(T)}. \tag{4}$$

As in Fishburn and Rubinstein (1982), the parameter r can be normalized to be any arbitrary positive constant by applying a monotone transformation to the representation V. We will often set r appropriately to simplify the form of the representation. In contrast, the monotone additive statistic Φ is uniquely determined by the preference.

Over the domain of deterministic time lotteries, V coincides with an exponentially discounted utility representation with discount rate r. For general time lotteries, $\Phi(T)$ is the certainty equivalent of T, i.e. the deterministic time that satisfies $(x, T) \sim (x, \Phi(T))$. The monotonicity and continuity axioms ensure that such a certainty equivalent exists, and it is an implication of stationarity that $\Phi(T)$ is independent of the reward x.

As we show in the proof of Theorem 3, the monotonicity and stationarity axioms formally translate into the certainty equivalent Φ being a monotone additive statistic. Thus, by Theorem 2, every MSTP has a representation of the form

$$V(x, T) = u(x) \cdot \mathrm{e}^{-r \int K_a(T)\, \mathrm{d}\mu(a)}. \tag{5}$$

We recover expected discounted utility when μ is a point mass concentrated on a point $-a < 0$, in which case Φ takes the form

$$\Phi(T) = K_{-a}(T) = \frac{1}{-a} \log \mathbb{E}\left[\mathrm{e}^{-aT}\right].$$

This certainty equivalent, with the normalization $r = a$, yields the familiar representation $V(x, T) = u(x)\mathbb{E}[\mathrm{e}^{-aT}]$. For a general measure μ, the statistic $\Phi(T) = \int K_a(T)\, \mathrm{d}\mu(a)$ aggregates different discount rates by mixing over their corresponding certainty equivalents. The resulting representation is behaviorally distinct from expected discounted utility whenever μ is not a point mass, since in this case it violates the independence axiom, as we formally prove in §F in the online appendix.

The representation in Theorem 3 can be extended to a discrete-time setting, which we consider in the supplementary appendix §G. One difficulty that arises is that a discrete time lottery need not have a certainty equivalent that is an integer time. Because of this, an additional step is required to first relate each time lottery to a deterministic dated reward.

3.3 Implications for Risk Attitudes toward Time

Theorem 3 demonstrates that there are many ways to extend discounted utility to the domain of time lotteries, while maintaining stationarity. As is well known, standard expected discounted utility preferences are *risk-seeking* over time, in the sense that a decision maker prefers receiving a reward at a random time T rather than at the deterministic expected time $t = \mathbb{E}[T]$. But other monotone additive statistics lead to stationary time preferences that are not risk-seeking. As an example, for every $a > 0$ the statistic

$$\Phi(T) = K_a(T) = \frac{1}{a} \log \mathbb{E}\left[\mathrm{e}^{aT}\right]$$

leads, with the normalization $r = a$, to the representation

$$V(x, T) = \frac{u(x)}{\mathbb{E}\left[\mathrm{e}^{aT}\right]},$$

which is in fact *risk-averse* over time. Under this preference, the decision maker applies a *negative* discount rate $-a$ within the monotone additive statistic Φ, and yet is impatient. These two aspects are compatible because in the representation $u(x)\mathrm{e}^{-r\Phi(T)}$ the statistic Φ controls the risk attitude, while the decision maker still prefers receiving prizes earlier rather than later, since Φ appears with a negative coefficient.

Another key distinctive property of monotone stationary time preferences is their flexibility in allowing for risk attitudes that are not uniform across time lotteries. To illustrate this point, consider two decision problems with a fixed common reward $x = \$1000$, where in the first problem the choice is between

(I) receiving the reward after 1 day for sure, versus

(II) receiving the reward immediately with 99% probability and after 100 days with 1% probability.

In the second decision problem the choice is between

(I') receiving the reward after 99 days for sure, versus

(II') receiving the reward immediately with 1% probability and after 100 days with 99% probability.

In both problems, the times at which the safe options I and I' deliver the prize are equal to the expected delay of the lotteries II and II', and thus a decision maker who is globally risk-averse or risk-seeking toward time must either choose the safe options or the risky options in both problems. Nevertheless, it does not seem unreasonable for a person to choose I over II in order to avoid the risk of a long delay, but also choose II' to I', since the time lottery offers at least a chance of avoiding an otherwise very long delay.[2]

Preferences based on monotone additive statistics are not necessarily globally risk-averse or risk-seeking, and can accommodate the aforementioned behavior. For example, the statistic

$$\Phi(T) = \frac{1}{2}K_1(T) + \frac{1}{2}K_{-1}(T) = \frac{1}{2}\log \mathbb{E}\left[\mathrm{e}^{T}\right] - \frac{1}{2}\log \mathbb{E}\left[\mathrm{e}^{-T}\right]$$

leads the decision maker to choose the safe option I in the first problem and the risky option II' in the second.

Empirically, both risk-averse and risk-seeking behavior over time lotteries are observed. For example, in their experiment, DeJarnette, Dillenberger, Gottlieb, and Ortoleva (2020) find that out of 5 different choices over time lotteries, only 2.9% of subjects are always risk-seeking and only 12.4% are always risk-averse. Thus 84.7% of subjects exhibit behavior that is sometimes risk-seeking and sometimes risk-averse.[3] Similarly, the experiment by Ebert (2021) finds that there are risk-seeking and risk-averse subjects: "Overall, therefore, and in contrast to the evidence on wealth risk preferences, there is substantial heterogeneity in preferences toward delay risk."

In §5 we provide a detailed analysis of the risk attitudes of preferences represented by monotone additive statistics, including a characterization of those statistics that give rise to mixed risk attitudes as in the above example.

3.4 Aggregation of Preferences over Time Lotteries

In this section we apply our findings to collective decision problems, and study how a group, rather than an individual, may choose among time lotteries.

A company choosing among projects with different expected completion dates, a public agency choosing which research projects to fund, or a family deciding which highway to take, are all examples of social decisions where the alternatives at hand can be seen as time lotteries. In such situations, even if individuals share similar views about the desirability of different outcomes, there is still a need to compromise between conflicting degrees of

[2]We are grateful to Weijie Zhong for suggesting this example to us.

[3]See Table 1 in DeJarnette et al. (2020). These percentages are for a treatment with maximal delay of 12 weeks across all questions. DeJarnette et al. (2020) also measured risk preferences for time lotteries with a shorter maximal delay of 5 weeks. In this case an even higher number of 86.8% percent of subjects is neither purely risk-seeking nor purely risk-averse across all choices.

impatience. For instance, a politician appointed to a public office and who is seeking re-election may be more impatient for short-term results than a seasoned public servant.

We formulate a collective decision problem as a problem of aggregating preferences over time lotteries that display different degrees of impatience. We take as primitive a group represented by n preference relations $\succeq_1, \ldots, \succeq_n$ over time lotteries, each admitting a standard expected discounted utility representation $u(x)\mathbb{E}[\mathrm{e}^{-r_i T}]$, where $u\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a common utility function that is increasing and continuous, and $r_i > 0$ is agent i's discount rate. These preferences must be aggregated into a social preference relation $\succeq$. We require $\succeq$ to be an MSTP, and to agree with individual preferences whenever there is a consensus among the agents:

Axiom 3.5 (Pareto)**.** *If $(x,T) \succeq_i (y,S)$ for every i, then $(x,T) \succeq (y,S)$.*

The next result characterizes the implications of the Pareto Axiom:

Theorem 4. *Let $(\succeq_1, \ldots, \succeq_n, \succeq)$ be preference relations over time lotteries, where each $\succeq_i$ is represented by $u(x)\mathbb{E}[\mathrm{e}^{-r_i T}]$ and $\succeq$ is an MSTP. The Pareto axiom is satisfied if and only if there exists a probability vector $(\lambda_1, \ldots, \lambda_n)$ such that $\succeq$ can be represented by $u(x)\mathrm{e}^{-r\Phi(T)}$ with*

$$\Phi = \sum_{i=1}^{n} \lambda_i K_{-r_i} \quad \textit{and} \quad \frac{1}{r} = \sum_{i=1}^{n} \frac{\lambda_i}{r_i}.$$

Thus, under the Pareto axiom, the social preference admits a representation with the same utility function as the individuals, and a certainty equivalent Φ that is an average of the individual certainty equivalents.

A first implication of Theorem 4 is that any social preference $\succeq$ that satisfies the Pareto axiom and admits an expected discounted utility representation must coincide with one of the individual preferences. This follows from the general observation that an average $\sum_i \lambda_i K_{-r_i}$ of multiple certainty equivalents does not satisfy the von Neumann-Morgenstern independence axiom unless λ is a point mass. Thus, dictatorship becomes the only admissible aggregation procedure if one insists that the social preference must conform to expected discounted utility. Similar impossibility results have been obtained, in the setting of preferences over consumption streams, by Gollier and Zeckhauser (2005), Zuber (2011), Jackson and Yariv (2014, 2015), and Feng and Ke (2018). Our result shows that the tension between the Pareto principle and having a stationary social discount rate already emerges when considering time lotteries.[4]

[4] A certain richness in the choice domain is necessary for this type of impossibility result to hold. Dictatorship need not be the only solution in the smaller domain of deterministic dated rewards: if $u(x)\mathrm{e}^{-r_i t} \geq u(y)\mathrm{e}^{-r_i s}$ for every i, then $u(x)\mathrm{e}^{-rt} \geq u(y)\mathrm{e}^{-rs}$ holds for every social discount rate r that lies between $\min_i\{r_i\}$ and $\max_i\{r_i\}$.

At the same time, Theorem 4 provides a solution to this impossibility result. It demonstrates that Paretian aggregation and stationarity are compatible, and do not necessarily result in a dictatorship, if we allow the social preference to belong to the larger class of MSTPs. By averaging across the different certainty equivalents, the social preference described by Theorem 4 can aggregate different discount rates without violating stationarity. This approach complements alternative solutions that have been proposed in the literature to resolve the tension between Paretian aggregation and stationarity.[5]

We now comment on the relation between Theorem 4 and Harsanyi's theorem on utilitarianism (Harsanyi, 1955), and briefly discuss our proof method. Both results establish that linear aggregation is a consequence of a Pareto axiom, but linearity takes different meanings in the two theorems. Harsanyi's theorem concerns the question of how to aggregate expected utility preferences with utility functions (v_i) into a social expected utility preference v, and implies that under the Pareto axiom, v must be a positive linear combination of (v_i).

In our setting, for any fixed prize x, each individual preference over time lotteries (x, T) can be seen as an expected utility preference over random times with discounting function $v_i(t) = \mathrm{e}^{-r_i t}$. But applying Harsanyi's theorem to this domain leads to the aforementioned impossibility result that Paretian and stationary aggregation are incompatible under expected utility: if the social preference over random times has an expected utility form, then its discounting function must be a linear combination $\sum_i \lambda_i \mathrm{e}^{-r_i t}$; such a preference is not stationary unless λ is a point mass.

In order to derive the linear aggregation of certainty equivalents from the Pareto axiom, we first consider a fixed prize x and deduce that the social certainty equivalent Φ must respect the individual certainty equivalents $\Phi_i = K_{-r_i}$, in the sense that $\Phi(T) \le \Phi(S)$ whenever $\Phi_i(T) \le \Phi_i(S)$ for every i. Because Φ and each Φ_i are linear functionals of the certainty equivalents $K_a(T)$, this Pareto property implies through a duality argument that Φ must be a linear aggregation of (Φ_i). The rest of the proof employs the full Pareto axiom (for different prizes) to deduce that the social utility function over prizes coincides with the individuals. In this step we provide a novel construction of time lotteries (x, T) and (y, S) that are indifferent under every individual preference, and use their indifference under the social preference to pin down the form of the social utility function.

[5] For example, Feng and Ke (2018) define a different notion of Pareto efficiency that takes into account the preferences of individuals across generations. They show that a standard expected discounted social preference can satisfy this weaker Pareto axiom so long as it is more patient than all the individuals. Chambers and Echenique (2018) study a number of representations that weaken stationarity and generalize expected discounted utility.

4 Preferences over Gambles

In the theory of risk, CARA utility functions form a restrictive but useful class of expected utility preferences. Their usefulness stems partly from the analytical tractability of the exponential form, as well as from their invariance properties.

CARA utility functions are invariant to changes in wealth, so that a prospect X is preferred to Y if and only if $X + w$ is preferred to $Y + w$ for all wealth levels w. They are also invariant to the addition of background risks: X is preferred to Y if and only if $X + Z$ is preferred to $Y + Z$ for every independent gamble Z with zero mean. We refer to these properties as *wealth invariance* and *risk invariance*.

Both properties make CARA utility functions useful modeling tools in applications. Wealth invariance is a good approximation whenever stakes are small, and commonly used in empirical settings in which wealth is unknown. For example, when estimating risk preferences from insurance choices, the CARA family "*has the advantage that it implies a household's prior wealth w, which frequently is unobserved, is irrelevant to the household's decisions.*" (Barseghyan, Molinari, O'Donoghue, and Teitelbaum, 2018). Risk invariance is equally important, since households' additional background risks—arising from, say, investments in the stock market or health conditions—is often unobservable. Thus, risk invariance becomes a convenient assumption.

The invariance properties of CARA utility functions are conceptually distinct from the assumption that preferences obey the expected utility axioms. In this section, we apply monotone additive statistics to study preferences that are monotone with respect to stochastic dominance, and satisfy wealth invariance, or risk invariance, or both. A first observation is that monotone preferences that satisfy both forms of invariance are represented by monotone additive statistics. We further illustrate how forfeiting either form of invariance leads to strictly larger classes of preferences.

In the second part of this section we turn our attention to group decision-making. Our main conclusion is that preferences represented by monotone additive statistics are the only class of preferences with the property that choices over independent domains can be decentralized without violating first-order stochastic dominance.

4.1 Wealth and Risk Invariance

We consider a preference relation $\succeq$ over the set L^∞ of bounded gambles, that is complete, transitive, and non-trivial (i.e., $X \succ Y$ for some pair of gambles). Throughout, X, Y, Z denote bounded gambles and x, y, z, c denote constants. We maintain two basic conditions on the preference:

Axiom 4.1 (Monotonicity)**.** *If $X \geq_1 Y$ then $X \succeq Y$.*

Axiom 4.2 (Continuity)**.** *If $X \succ Y$ then there exists $\varepsilon > 0$ such that $X \succ Y + \varepsilon$ and $X - \varepsilon \succ Y$.*

The next axioms capture the aforementioned forms of invariance:

Axiom 4.3 (Wealth Invariance)**.** *$X \succeq Y$ if and only if $X + z \succeq Y + z$ for all $z \in \mathbb{R}$.*

Axiom 4.4 (Risk Invariance)**.** *Suppose Z is mean-zero and independent of X and Y. Then $X \succeq Y$ if and only if $X + Z \succeq Y + Z$.*

As we argue next, preferences that satisfy all four axioms are represented by monotone additive statistics. Monotonicity and continuity imply that to each gamble X we can associate a certainty equivalent $\Phi(X)$. This certainty equivalent satisfies, by wealth and risk invariance, the property $X + Y \sim \Phi(X) + Y$ for any two independent random variables X and Y. Likewise, by wealth invariance, $Y + \Phi(X) \sim \Phi(Y) + \Phi(X)$. Combining the two indifferences yields

$$X + Y \sim \Phi(X) + \Phi(Y).$$

So, the certainty equivalent of $X + Y$ is given by the sum $\Phi(X) + \Phi(Y)$, and thus Φ is an additive, and by assumption monotone, statistic that represents the preference relation. By Theorem 1, the certainty equivalent Φ is then a weighted average of the certainty equivalents of multiple CARA expected utility agents.

As noted earlier, the representation $\Phi(X) = \int K_a(X)\,\mathrm{d}\mu(a)$ is not of the expected utility form, unless μ is a point mass. However, we show in §F in the online appendix that a special class of measures μ gives rise to a representation that is compatible with the well-known betweenness axiom proposed by Dekel (1986) and Chew (1989).

4.2 Disentangling the Two Invariance Properties

Under a monotone expected utility preference, wealth invariance and risk invariance each imply CARA utility functions, and in particular imply each other (see the supplementary appendix §H for details). For general preferences over gambles, the two properties are logically independent.

Wealth invariance imposes very little structure on its own. For example, preferences represented by the median, or any other quantile, satisfies wealth invariance, as do the preferences in the class studied by Safra and Segal (1998), as well as monotone additive statistics. These representations are very different in nature, and do not admit a common parametric generalization. The example of the median, moreover, illustrates that wealth invariance does not in general imply risk invariance.

At the same time, risk invariance does not imply wealth invariance. An example of such a preference is given by the monotone representation $V(X) = (\mathbb{E}[X])^3 + K_a(X)$ for some

$a \neq 0$.[6] Despite this, our next theorem proves that risk invariance by itself significantly constrains the preference. Specifically, we characterize all preferences that are invariant to mean-zero background risks, and additionally exhibit risk aversion as formalized in the following two axioms. The first axiom is standard:

Axiom 4.5 (Second-order Monotonicity)**.** *$X \succeq Y$ whenever X dominates Y in second-order stochastic dominance.*

To state the next condition, for each $c > 0$ we denote by W_c a random variable that is equal to $\pm c$ with equal probabilities. As c becomes large, W_c is a mean-zero risk of increasing magnitude. We require the decision maker to regard W_c as arbitrarily undesirable as $c \to \infty$:

Axiom 4.6 (Archimedeanity)**.** *For every $x \in \mathbb{R}$ there exists $c > 0$ such that $W_c \prec x$ and $x + W_c \prec 0$.*

This condition imposes a lower bound on the decision maker's degree of risk aversion. It is, in particular, satisfied by any expected utility preference with a concave utility function whose Arrow-Pratt coefficient is bounded away from 0.

Theorem 5. *A preference $\succeq$ on L^∞ satisfies Axioms 4.2, 4.4, 4.5, 4.6 (i.e., continuity, risk invariance, second-order monotonicity and Archimedeanity) if and only if there exist a continuous and non-decreasing function $v \colon \mathbb{R} \to \mathbb{R}$ and a probability measure μ supported on $[-\infty, 0)$ such that $\succeq$ is represented by*

$$V(X) = v(\mathbb{E}[X]) + \int_{[-\infty,0)} K_a(X)\, \mathrm{d}\mu(a).$$

Thus, a risk-averse preference that is invariant to mean-zero risks evaluates each gamble in an additively separable manner. The first part of the representation is a monotone function of the expectation, which is unaffected by adding mean-zero risks, and the second part is a monotone additive statistic.[7]

[6] Theorem 5 below implies that this preference satisfies risk invariance. To see that it does not satisfy wealth invariance, we can choose a pair of X, Y such that $V(X) = V(Y)$ but $\mathbb{E}[X] \neq \mathbb{E}[Y]$, which is possible whenever $a \neq 0$. Then there exists $c \in \mathbb{R}$ such that $(\mathbb{E}[X] + c)^3 - (\mathbb{E}[X])^3 \neq (\mathbb{E}[Y] + c)^3 - (\mathbb{E}[Y])^3$. For this c we have $V(X + c) \neq V(Y + c)$ even though $V(X) = V(Y)$. Thus $X + c \not\sim Y + c$ despite $X \sim Y$, violating wealth invariance.

[7] It is a natural conjecture that if we only require monotonicity with respect to first-order stochastic dominance (i.e. Axiom 4.1 instead of Axiom 4.5), then the preference is represented by $v(\mathbb{E}[X]) + \int K_a(X)\, \mathrm{d}\mu(a)$ where the measure μ can be supported on $[-\infty, 0) \cup (0, \infty]$. We do not immediately see a way to adapt our current argument to that case, and leave the verification of this conjecture for future work.

4.3 Combined Choices

In large organizations, risky prospects are not always chosen through a deliberate, centralized process. Rather, they are combinations of independent choices, often carried out with limited coordination among the different actors.

Consider, for example, a bank that employs two workers. The first is a trader who must choose between two contracts, the Lean Hog futures X and X'. The second is an administrator who must choose between two insurance policies Y or Y' for the bank's building. Assuming the first worker chooses X and the second Y, the resulting revenue for the bank is given by the random variable $X + Y$. When, as in this example, the agents face choice problems that belong to independent domains, so that X and X' are stochastically independent from Y and Y', it is natural to ask to what extent coordination is necessary for the organization.

In this section we make this question precise by asking under what conditions the agents' combined choices respect first-order stochastic dominance. Our main result shows this is true if and only if individual preferences are identical and represented by a monotone additive statistic. Thus, this is the only class of preferences with the property that choices over independent domains can be decentralized without obvious harm to the organization.

We study the following model. We are given two preference relations $\succeq_1$ and $\succeq_2$ over L^∞, the set of bounded gambles, that are complete and transitive (our result immediately generalizes to three or more agents). As in the example above, we think of each preference relation as describing the choices of a different agent, so that $X \succeq_i X'$ if agent i chooses X over X'. These preferences can be interpreted as being endogenous or as the result of exogenous incentives; for example, the bank trader's preferences could be driven by her contract with the employer.

Our main axiom requires that whenever the two agents face independent decision problems, their choices, when combined, do not violate stochastic dominance:

Axiom 4.7 (Consistency of Combined Choices)**.** *Suppose X, X' are independent of Y, Y'. If $X \succ_1 X'$ and $Y \succ_2 Y'$, then $X' + Y'$ does not strictly dominate $X + Y$ in first-order stochastic dominance.*

If we interpret $\succeq_1$ and $\succeq_2$ as decision-making rules that are determined by the organization, then Axiom 4.7 requires such rules to never result in an outcome that is stochastically dominated. That collective choices should not violate stochastic dominance is clearly a desirable requirement for a rational organization. A similar axiom was first introduced by Rabin and Weizsäcker (2009) in the context of a model of narrow framing.

In addition to this axiom, we assume individual preference relations $\succeq_i$ satisfy a basic continuity condition, Axiom 4.2, as well as the next monotonicity assumption:

Axiom 4.8 (Responsiveness)**.** *$X + \varepsilon \succ_i X$ for every $\varepsilon > 0$.*

We next show that under these axioms, the two preference relations must be represented by monotone additive statistics. Moreover, the statistic must be the same for both agents.

Theorem 6. *Two preference $\succeq_1, \succeq_2$ on L^∞ satisfy Axioms 4.2, 4.7 and 4.8 (i.e., continuity, consistency of combined choices and responsiveness) if and only if there exists a monotone additive statistic that represents both $\succeq_1$ and $\succeq_2$.*

Thus, when individual choices are not coordinated, their combination will, in general, lead to violations of stochastic dominance, even when agents' choices concern independent decision problems. The theorem singles out preferences represented by monotone additive statistics as the only class of preferences that are robust to this lack of coordination.

Theorem 6 admits an alternative interpretation, closely related to the work of Rabin and Weizsäcker (2009) on narrow framing. In their paper, a decision maker faces multiple decisions and engages in "narrow bracketing" by choosing separately, in each problem, according to a fixed preference relation $\succeq$ over gambles. This is a special case of our model where $\succeq \; = \; \succeq_1 \; = \; \succeq_2$. They show that the decision maker's combined choices result in dominated outcomes whenever $\succeq$ is not wealth invariant (i.e. if $X \succ Y$ but $Y + c \succ X + c$ for some X, Y and $c \in \mathbb{R}$), but leave open the question of characterizing the class of preferences, beyond expected utility, that satisfy Axiom 4.7. Theorem 6 provides a complete characterization of those preferences over gambles for which narrow framing does not lead to dominated choices.

5 Comparative Risk Attitudes

In this section we characterize risk-averse and risk-seeking behavior for preferences that are represented by monotone additive statistics. A preference relation $\succeq$ over gambles is risk-averse if its certainty equivalent Φ satisfies $\Phi(X) \leq \mathbb{E}[X]$ for every gamble X, and risk-seeking if the opposite inequality holds. In the domain of time lotteries, since the decision maker prefers lower waiting times, risk aversion corresponds to the opposite inequality $\Phi(T) \geq \mathbb{E}[T]$ for every random time T.

Risk aversion translates into a property of the support of the corresponding mixing measure μ:

Proposition 1. *A monotone additive statistic satisfies $\Phi(X) \leq \mathbb{E}[X]$ for every $X \in L^\infty$ if and only if*

$$\Phi(X) = \int_{\overline{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu(a)$$

for a Borel probability measure μ supported on $[-\infty, 0]$. Symmetrically, $\Phi(X) \geq \mathbb{E}[X]$ for every X if and only if the measure μ is supported on $[0, \infty]$.

In other words, a risk-averse decision maker ranks gambles by aggregating the certainty equivalents of risk-averse CARA utility functions. For the setting of time lotteries, the second part of the result shows that risk aversion, which corresponds to $\Phi(T) \geq \mathbb{E}[T]$, happens if and only if the measure μ is supported on $[0, \infty]$.[8] Thus, risk aversion toward time lotteries occurs whenever the decision maker aggregates the certainty equivalents of preferences that are represented by $u(x)/\mathbb{E}[\mathrm{e}^{aT}]$, across different $a > 0$. Mixed risk attitude, as discussed in §3, occurs when μ assigns positive mass to both negative and positive values.

A corollary of Proposition 1 is that an additive statistic Φ is monotone with respect to *second-order* (or any higher-order) stochastic dominance if and only if $\Phi(X) = \int K_a(X)\,\mathrm{d}\mu(a)$ for a probability measure μ supported on $[-\infty, 0]$. To see this, note that monotonicity in higher-order stochastic dominance implies risk aversion and thus constrains the support of μ. Conversely, for each $a \leq 0$, the statistic $K_a(X) = \frac{1}{a}\log\mathbb{E}[\mathrm{e}^{aX}]$ satisfies higher-order monotonicity because the function e^{ax} has derivatives of all orders that alternate signs. By linearity, $\int K_a(X)\,\mathrm{d}\mu(a)$ is also higher-order monotone whenever μ is supported on $[-\infty, 0]$.

We now proceed to compare the risk attitudes expressed by different monotone additive statistics. For two preference relations $\succeq_1$ and $\succeq_2$ over gambles, with corresponding certainty equivalents Φ_1 and Φ_2, the preference $\succeq_1$ is *more risk-averse* than $\succeq_2$ if $\Phi_1(X) \leq \Phi_2(X)$ for every gamble X. That is, if the first decision maker assigns to every gamble a lower certainty equivalent. The next proposition characterizes comparative risk aversion for preferences represented by monotone additive statistics:

Proposition 2. *Let Φ_1, Φ_2 be monotone additive statistics, characterized by measures μ_1 and μ_2 respectively. Then $\Phi_1(X) \leq \Phi_2(X)$ for all $X \in L^\infty$ if and only if*

(i) For every $b > 0$, $\int_{[b,\infty]} \frac{a-b}{a}\,\mathrm{d}\mu_1(a) \leq \int_{[b,\infty]} \frac{a-b}{a}\,\mathrm{d}\mu_2(a)$.

(ii) For every $b < 0$, $\int_{[-\infty,b]} \frac{a-b}{a}\,\mathrm{d}\mu_1(a) \geq \int_{[-\infty,b]} \frac{a-b}{a}\,\mathrm{d}\mu_2(a)$.

Since the certainty equivalent $K_a(X)$ increases with the parameter a, a sufficient condition for μ_1 to induce a more risk-averse preference is the first-order stochastic dominance of μ_2 over μ_1. This can be seen from Proposition 2, since for $b > 0$ the integrand $\frac{a-b}{a}$ in condition (i) is an increasing function of a over the interval $[b, \infty]$, and similarly for condition (ii).

But, as Proposition 2 states, first-order stochastic dominance of the mixing measures is not necessary for comparative risk aversion. The reason is that the functions of the

[8] A small twist is that in the time setting, risk aversion only requires $\Phi(T) \geq \mathbb{E}[T]$ for every non-negative random variable T. But our proof shows that Proposition 1 also holds for the smaller domain L^∞_+. This comment also applies to Proposition 2 below.

form $K_{(\cdot)}(X)$, as we vary X, do not span in their cone the collection of all increasing functions.[9] For a concrete example, consider μ_1 to be a point mass at $a = 2$ and μ_2 to have 1/4 mass at $a = 1$ and 3/4 mass at $a = 3$. Condition (ii) in Proposition 2 is trivially satisfied, whereas condition (i) reduces to $\frac{1}{2}(2-b)^+ \leq \frac{1}{4}(1-b)^+ + \frac{1}{4}(3-b)^+$, which holds because the function $(a-b)^+ = \max\{a-b, 0\}$ is convex in a.

6 Overview of the Proof of Theorem 1

Our approach to the proof of Theorem 1 is via a stochastic order known as the *catalytic stochastic order* (see Fritz, 2017, and references therein). Given $X, Y \in L^\infty$, we say that X dominates Y in the catalytic stochastic order on L^∞ if there exists a $Z \in L^\infty$, independent of X and Y, such that $X + Z$ dominates $Y + Z$ in first-order stochastic dominance.

The applicability of this order to our problem is immediate. If X dominates Y in the catalytic stochastic order then

$$\Phi(X + Z) \geq \Phi(Y + Z)$$

for some Z, independent of X and Y. If Φ is also additive, then $\Phi(X + Z) = \Phi(X) + \Phi(Z)$ and $\Phi(Y + Z) = \Phi(Y) + \Phi(Z)$, and so we have that $\Phi(X) \geq \Phi(Y)$. Thus, any monotone additive Φ is monotone with respect to this order.

Clearly, if $X \geq_1 Y$ then X also dominates Y in the catalytic stochastic order, as one can take $Z = 0$. A priori, one may conjecture that this is also a necessary condition. But as Figure 1 shows, it is easy to give examples of two random variables X and Y that are not ranked with respect to first-order stochastic dominance, but are ranked with respect to the catalytic stochastic order.[10] The random variable X equals 1 with probability 1/3 and 0 with probability 2/3, while Y is uniformly distributed on $[-\frac{3}{5}, \frac{2}{5}]$. As the figure shows, their c.d.f.s are not ranked, and hence they are not ranked in terms of first-order stochastic dominance.[11]

However, if we let Z assign probability half to $\pm\frac{1}{5}$, then $X + Z >_1 Y + Z$. Intuitively, since the c.d.f. of $X + Z$ is the average of the two translations (by $\pm\frac{1}{5}$) of the c.d.f. of X, and since the same holds for the c.d.f. of Y, the result of adding Z is the disappearance of the small "kink" in which the ranking of the c.d.f.s is reversed. This is depicted in Figure 2.

[9] In particular, it is well known that $a \cdot K_a(X) = \log \mathbb{E}\left[e^{aX}\right]$ is a convex function in a for any X.

[10] We are indebted to the late Kim Border for helping us construct this example.

[11] Pomatto, Strack, and Tamuz (2020) give examples of random variables X and Y that are not ranked in stochastic dominance, but are ranked after adding an *unbounded* independent Z. In fact, they show that this is possible whenever $\mathbb{E}[X] > \mathbb{E}[Y]$. As we explain below, this result no longer holds when Z is required to be bounded.

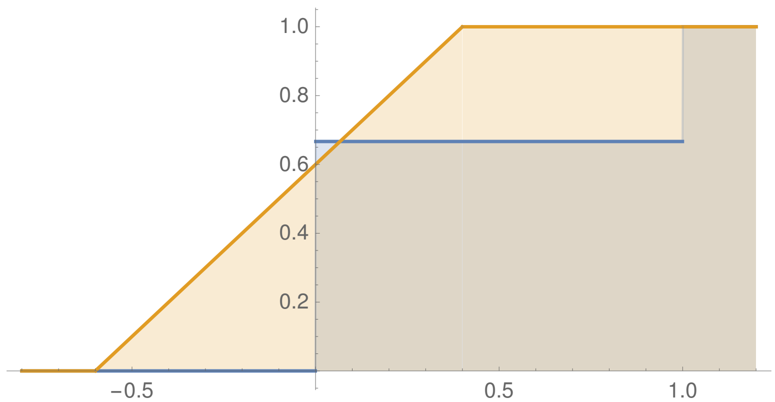

Figure 1: The c.d.f.s of X (blue) and Y (orange).

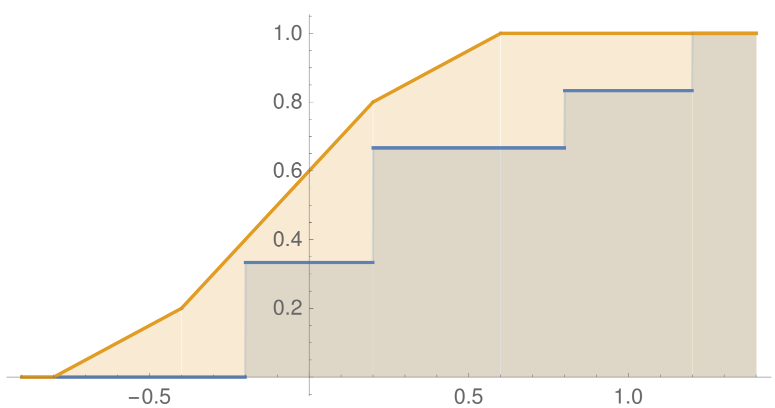

Figure 2: The c.d.f.s of $X + Z$ (blue) and $Y + Z$ (orange).

Every monotone additive statistic Φ provides an obstruction to dominance in the catalytic stochastic order. That is, if $\Phi(X) < \Phi(Y)$ then it is impossible that $X+Z \geq_1 Y+Z$ for some independent Z, since monotonicity would imply that $\Phi(X+Z) \geq \Phi(Y+Z)$, and additivity would then imply that $\Phi(X) \geq \Phi(Y)$. In particular, considering the statistic K_a yields that $K_a(X) \geq K_a(Y)$ for all $a \in \overline{\mathbb{R}}$ is necessary for there to exist some Z that makes X stochastically dominate Y.[12] The following result shows that the statistics K_a are, in a sense, the only obstructions:[13]

Theorem 7. *Let $X, Y \in L^\infty$ satisfy $K_a(X) > K_a(Y)$ for all $a \in \overline{\mathbb{R}}$. Then there exists an independent $Z \in L^\infty$ such that $X + Z \geq_1 Y + Z$.*

To prove Theorem 7 we explicitly construct Z as a truncated Gaussian with appropriately chosen parameters. The idea behind the proof is as follows. Denote by F and G the c.d.f.s of X and Y, respectively, and suppose that they are supported on $[-N, N]$. Let $h(x) = \frac{1}{\sqrt{2\pi V}} \mathrm{e}^{-\frac{x^2}{2V}}$ be the density of a Gaussian Z. Then the c.d.f.s of $X+Z$ and $Y+Z$ are given by the convolutions $F * h$ and $G * h$, and their difference is equal to

$$\begin{aligned}[G * h - F * h](y) &= \int_{-N}^{N} [G(x) - F(x)] \cdot h(y - x) \,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi V}} \mathrm{e}^{-\frac{y^2}{2V}} \cdot \int_{-N}^{N} \underbrace{[G(x) - F(x)] \cdot \mathrm{e}^{\frac{y}{V} \cdot x}}_{(*)} \cdot \underbrace{\mathrm{e}^{-\frac{x^2}{2V}}}_{(**)} \,\mathrm{d}x\end{aligned}$$

If we denote $a = \frac{y}{V}$, then by integration by parts, the integral of just $(*)$ is equal to $\frac{1}{a}\left(\mathbb{E}\left[\mathrm{e}^{aX}\right] - \mathbb{E}\left[\mathrm{e}^{aY}\right]\right)$, which is positive by the assumption that $K_a(X) > K_a(Y)$ and is in fact bounded away from zero. The term $(**)$ can be made arbitrarily close to 1—uniformly on the integral domain $[-N, N]$—by making V large. This implies that $[G*h - F*h](y) \geq 0$ for all y, and we further show that the inequality still holds if we modify Z by truncating its tails, ensuring that it is in L^∞.

Theorem 7 leads to the following lemma, which is a key component of the proof of Theorem 1:

Lemma 1. *Let $\Phi\colon L^\infty \to \mathbb{R}$ be a monotone additive statistic. If $K_a(X) \geq K_a(Y)$ for all $a \in \overline{\mathbb{R}}$ then $\Phi(X) \geq \Phi(Y)$.*

[12] In fact, except for the trivial case where X and Y have the same distribution, it is necessary to have the strict inequality $K_a(X) > K_a(Y)$ for all $a \in \mathbb{R}$. This is because $X + Z \geq_1 Y + Z$ implies the strict inequality $K_a(X+Z) > K_a(Y+Z)$ whenever $X+Z$ and $Y+Z$ have different distributions. Thus, Theorem 7 implies that for distributions with different minima and maxima, the condition $K_a(X) > K_a(Y)$ for all $a \in \overline{\mathbb{R}}$ is both necessary and sufficient for dominance in the catalytic stochastic order.

[13] A result similar to Theorem 7 holds if we demand a weaker conclusion that $X + Z$ *second-order* stochastically dominates $Y + Z$. See Proposition 3 in the appendix.

Proof. Suppose $K_a(X) \geq K_a(Y)$ for all $a \in \overline{\mathbb{R}}$. For any $\varepsilon > 0$, consider $\hat{X} = X + \varepsilon$. Then $K_a(\hat{X}) = K_a(X) + \varepsilon > K_a(Y)$ for all a, and by Theorem 7 there is an independent $Z \in L^\infty$ such that $\hat{X} + Z \geq_1 Y + Z$. Hence, by monotonicity of Φ, $\Phi(\hat{X} + Z) \geq \Phi(Y + Z)$, and by additivity $\Phi(\hat{X}) \geq \Phi(Y)$. This means that $\Phi(X) + \varepsilon \geq \Phi(Y)$ for all $\varepsilon > 0$, and hence $\Phi(X) \geq \Phi(Y)$. □

An alternative proof of Lemma 1 can be given based on a different stochastic order, known as the *large numbers order*. Given two random variables X and Y, let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be i.i.d. copies of X and Y, respectively. We say that X dominates Y in large numbers if

$$X_1 + \cdots + X_n \geq_1 Y_1 + \cdots + Y_n$$

for all n large enough. Using large-deviations techniques, it was shown by Aubrun and Nechita (2008) that if $K_a(X) > K_a(Y)$ for all $a \in \overline{\mathbb{R}}$, then X dominates Y in large numbers. This implies Lemma 1 since, by the additivity of Φ, $\Phi(X) \geq \Phi(Y)$ holds if and only if $n\Phi(X) = \Phi(X_1 + \cdots + X_n) \geq \Phi(Y_1 + \cdots + Y_n) = n\Phi(Y)$. Compared to this alternative argument, our proof of Lemma 1, which is based on Theorem 7, is self-contained and more elementary. Furthermore, Theorem 7 may be of independent interest, and is directly useful for proving some of our other results, including in particular Theorem 5.

Once we have established Lemma 1, the remainder of the proof uses functional analysis techniques (in particular the Riesz Representation Theorem) to deduce the integral representation in Theorem 1. See §A in the appendix for the complete proof.

References

C. D. Aliprantis and K. Border. *Infinite dimensional analysis: A hitchhiker's guide*. Springer, 2006.

G. Aubrun and I. Nechita. Catalytic majorization and ℓ_p norms. *Communications in Mathematical Physics*, 278(1):133–144, 2008.

L. Barseghyan, F. Molinari, T. O'Donoghue, and J. C. Teitelbaum. Estimating risk preferences in the field. *Journal of Economic Literature*, 56(2):501–64, 2018.

P. Bickel and E. Lehmann. Descriptive statistics for nonparametric models I. introduction. *Annals of Statistics*, 3(5):1038–1044, 1975a.

P. Bickel and E. Lehmann. Descriptive statistics for nonparametric models II. location. *Annals of Statistics*, 3(5):1045–1069, 1975b.

V. I. Bogachev. *Measure theory*, volume 1. Springer Science & Business Media, 2007.

S. Cerreia-Vioglio, D. Dillenberger, and P. Ortoleva. Cautious expected utility and the certainty effect. *Econometrica*, 83(2):693–728, 2015.

C. P. Chambers and F. Echenique. When does aggregation reduce risk aversion? *Games and Economic Behavior*, 76(2):582–595, 2012.

C. P. Chambers and F. Echenique. On multiple discount rates. *Econometrica*, 86(4): 1325–1346, 2018.

C. P. Chambers and F. Echenique. Spherical preferences. *Journal of Economic Theory*, 189:105086, 2020.

H. W. Chesson and W. K. Viscusi. Commonalities in time and ambiguity aversion for long-term risks. *Theory and Decision*, 54(1):57–71, 2003.

S. H. Chew. Axiomatic utility theories with the betweenness property. *Annals of Operations Research*, 19(1):273–298, 1989.

J. H. Curtiss. A note on the theory of moment generating functions. *The Annals of Mathematical Statistics*, 13(4):430–433, 1942.

B. de Finetti. *Theory of Probability*. Wiley, 1970.

P. DeJarnette, D. Dillenberger, D. Gottlieb, and P. Ortoleva. Time lotteries and stochastic impatience. *Econometrica*, 88(2):619–656, 2020.

E. Dekel. An axiomatic characterization of preferences under uncertainty: Weakening the independence axiom. *Journal of Economic Theory*, 40:304–318, 1986.

S. Ebert. Decision making when things are only a matter of time. *Operations Research*, 2020.

S. Ebert. Prudent discounting: Experimental evidence on higher-order time risk preferences. *International Economic Review*, 2021.

T. Feng and S. Ke. Social discounting and intergenerational pareto. *Econometrica*, 86(5): 1537–1567, 2018.

P. C. Fishburn and A. Rubinstein. Time preference. *International Economic Review*, 23: 677–694, 1982.

H. Föllmer and T. Knispel. Entropic risk measures: Coherence vs. convexity, model ambiguity and robust large deviations. *Stochastics and Dynamics*, 11(02n03):333–351, 2011.

H. Föllmer and A. Schied. Convex measures of risk and trading constraints. *Finance and stochastics*, 6(4):429–447, 2002.

H. Föllmer and A. Schied. *Stochastic finance: an introduction in discrete time*. Walter de Gruyter, 2011.

T. Fritz. Resource convertibility and ordered commutative monoids. *Mathematical Structures in Computer Science*, 27(6):850–938, 2017.

T. Fritz, X. Mu, and O. Tamuz. Monotone homomorphisms on convolution semigroups, 2020. Working Paper.

C. Gollier and R. Zeckhauser. Aggregation of heterogeneous time preferences. *Journal of Political Economy*, 113(4):878–896, 2005.

M. J. Goovaerts, R. Kaas, R. J. Laeven, and Q. Tang. A comonotonic image of independence for additive risk measures. *Insurance: Mathematics and Economics*, 35(3):581–594, 2004.

Y. Halevy. Time consistency: Stationarity and time invariance. *Econometrica*, 83(1): 335–352, 2015.

J. C. Harsanyi. Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *Journal of political economy*, 63(4):309–321, 1955.

M. O. Jackson and L. Yariv. Present bias and collective dynamic choice in the lab. *American Economic Review*, 104(12):4184–4204, 2014.

M. O. Jackson and L. Yariv. Collective dynamic choice: the necessity of time inconsistency. *American Economic Journal: Microeconomics*, 7(4):150–78, 2015.

L. Kantorovich. On the moment problem for a finite interval. In *Dokl. Akad. Nauk SSSR*, volume 14, pages 531–537, 1937.

L. Mattner. What are cumulants? *Documenta Mathematica*, 4:601–622, 1999.

L. Mattner. Cumulants are universal homomorphisms into hausdorff groups. *Probability theory and related fields*, 130(2):151–166, 2004.

X. Mu, L. Pomatto, P. Strack, and O. Tamuz. From Blackwell dominance in large samples to Rényi divergences and back again. *Econometrica*, 89(1):475–506, 2021.

R. B. Myerson and E. Zambrano. *Probability models for economic decisions*. Mit Press, 2019.

S. Onay and A. Öncüler. Intertemporal choice under timing risk: An experimental approach. *Journal of Risk and Uncertainty*, 34(2):99–121, 2007.

L. Pomatto, P. Strack, and O. Tamuz. Stochastic dominance under independent noise. *Journal of Political Economy*, 128(5):1877–1900, 2020.

M. Rabin and G. Weizsäcker. Narrow bracketing and dominated choices. *American Economic Review*, 99(4):1508–43, 2009.

I. Ruzsa and G. J. Székely. *Algebraic probability theory*. John Wiley & Sons Inc, 1988.

Z. Safra and U. Segal. Constant risk aversion. *Journal of economic theory*, 83(1):19–42, 1998.

T. Strzalecki. Axiomatic foundations of multiplier preferences. *Econometrica*, 79(1):47–73, 2011.

S. Zuber. Can social preferences be both stationary and paretian? *Annals of Economics and Statistics*, 101/102:347–360, 2011.

Appendix

The appendix contains the omitted proofs for most of the results that have been explicitly stated in the main text, in the order in which they appeared. The only exceptions are Theorem 2 and Theorem 5, whose proofs are more technical and relegated to the online appendix. Additional results and proofs are presented in a separate supplementary appendix.

In the proofs we often use the notation

$$K_X(a) = K_a(X),$$

so that K_X is a map from $\overline{\mathbb{R}}$ to $\mathbb{R}$. Throughout the proofs we will often apply the following standard facts.

Lemma 2. *Let $X, Y \in L^\infty$.*

1. *$K_X : \overline{\mathbb{R}} \to \mathbb{R}$ is well defined, non-decreasing and continuous.*
2. *If $K_X = K_Y$ then X and Y have the same distribution.*

Proof. See Curtiss (1942). □

For a random variable X, we will also use the notation X^{*n} to denote the random variable that is the sum of n i.i.d. copies of X.

A Proof of Theorem 1

We follow the proof outline in §6 of the main text and first establish the catalytic stochastic order stated in Theorem 7.

A.1 Proof of Theorem 7

First, we can add the same constant b to both X and Y so that $\min[Y + b] = -N$ and $\max[X + b] = N$ for some $N > 0$. Since translating both X and Y leaves the existence of an appropriate Z unchanged (and also does not affect $K_X > K_Y$), we henceforth assume without loss of generality that $\min[Y] = -N$, and $\max[X] = N$. Since $K_X > K_Y$, we know that $\min[X] > -N$ and $\max[Y] < N$.

Denote the c.d.f.s of X and Y by F and G, respectively. Let $\sigma(x) = G(x) - F(x)$. Note that σ is supported on $[-N, N]$ and bounded in absolute value by 1. Moreover, by choosing $\varepsilon > 0$ sufficiently small, we have that $\min[X] > -N + \varepsilon$ and $\max[Y] < N - \varepsilon$. So $\sigma(x)$ is positive on $[-N, -N + \varepsilon]$ and on $[N - \varepsilon, N]$. In fact, there exists $\delta > 0$ such that

$\sigma(x) \geq \delta$ whenever $x \in [-N + \frac{\varepsilon}{4}, -N + \frac{\varepsilon}{2}]$ and $x \in [N - \frac{\varepsilon}{2}, N - \frac{\varepsilon}{4}]$. We also fix a large constant A such that

$$\mathrm{e}^{\frac{\varepsilon A}{4}} \geq \frac{8N}{\varepsilon \delta}.$$

Define

$$M_\sigma(a) = \int_{-N}^{N} \sigma(x) \mathrm{e}^{ax} \, \mathrm{d}x.$$

Note that for $a \neq 0$, integration by parts shows $M_\sigma(a) = \frac{1}{a}\left(\mathbb{E}\left[\mathrm{e}^{aX}\right] - \mathbb{E}\left[\mathrm{e}^{aY}\right]\right)$, and that $M_\sigma(0) = \mathbb{E}[X] - \mathbb{E}[Y]$. Therefore, since $K_X > K_Y$, we have that M_σ is strictly positive everywhere. Since $M_\sigma(a)$ is clearly continuous in a, it is in fact bounded away from zero on any compact interval.

We will use these properties of σ to construct a truncated Gaussian density h such that

$$[\sigma * h](y) = \int_{-N}^{N} \sigma(x) h(y - x) \, \mathrm{d}x \geq 0$$

for each $y \in \mathbb{R}$. If we let Z be a random variable independent from X and Y, whose distribution has density function h, then $\sigma * h = (G - F) * h$ is the difference between the c.d.f.s of $Y + Z$ and $X + Z$. Thus $[\sigma * h](y) \geq 0$ for all y would imply $X + Z \geq_1 Y + Z$.

To do this, we write $h(x) = \mathrm{e}^{-\frac{x^2}{2V}}$ for all $|x| \leq T$, where V is the variance and T is the truncation point to be chosen.[14] We will show that given the above constants N and A, $[\sigma * h](y) \geq 0$ holds for each y when V is sufficiently large and $T \geq AV + N$.

First consider the case where $y \in [-AV, AV]$. In this region, $|y - x| \leq T$ is automatically satisfied when $x \in [-N, N]$. So we can compute the convolution $\sigma * h$ as follows:

$$\int \sigma(x) h(y - x) \, \mathrm{d}x = \mathrm{e}^{-\frac{y^2}{2V}} \cdot \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V} \cdot x} \cdot \mathrm{e}^{-\frac{x^2}{2V}} \, \mathrm{d}x. \tag{6}$$

Note that $\frac{y}{V}$ in the exponent belongs to the compact interval $[-A, A]$. So for our fixed choice of A, the integral $M_\sigma(\frac{y}{V}) = \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V} \cdot x} \, \mathrm{d}x$ is uniformly bounded away from zero when y varies in the current region. Thus,

$$\begin{aligned} \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V} \cdot x} \cdot \mathrm{e}^{-\frac{x^2}{2V}} \, \mathrm{d}x &= M_\sigma\left(\frac{y}{V}\right) - \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V} \cdot x} \cdot (1 - \mathrm{e}^{-\frac{x^2}{2V}}) \, \mathrm{d}x \\ &\geq M_\sigma\left(\frac{y}{V}\right) - 2N \cdot \mathrm{e}^{AN} \cdot (1 - \mathrm{e}^{\frac{-N^2}{2V}}), \end{aligned} \tag{7}$$

which is positive when V is sufficiently large. So the right-hand side of (6) is positive.

Next consider the case where $y \in (AV, T + N - \varepsilon]$; the case where $-y$ is in this range can be treated symmetrically. Here the convolution can be written as

$$[\sigma * h](y) = \int_{\max\{-N, y - T\}}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{-(y-x)^2}{2V}} \, \mathrm{d}x.$$

[14] In general we need a normalizing factor to ensure h integrates to one, but this multiplicative constant does not affect the argument.

We break the range of integration into two sub-intervals: $I_1 = [\max\{-N, y-T\}, N-\varepsilon]$ and $I_2 = [N-\varepsilon, N]$. On I_1 we have $\sigma(x) = G(x) - F(x) \geq -1$, so

$$\int_{x \in I_1} \sigma(x) \cdot \mathrm{e}^{\frac{-(y-x)^2}{2V}} \,\mathrm{d}x \geq -2N \cdot \mathrm{e}^{\frac{-(y-N+\varepsilon)^2}{2V}}.$$

On I_2 we have $\sigma(x) \geq 0$ by our choice of ε, and furthermore $\sigma(x) \geq \delta$ when $x \in [N-\frac{\varepsilon}{2}, N-\frac{\varepsilon}{4}]$. Thus

$$\int_{x \in I_2} \sigma(x) \cdot \mathrm{e}^{\frac{-(y-x)^2}{2V}} \,\mathrm{d}x \geq \frac{\varepsilon}{4} \cdot \delta \cdot \mathrm{e}^{\frac{-(y-N+\frac{\varepsilon}{2})^2}{2V}} \geq 2N \cdot \mathrm{e}^{\frac{-(y-N+\frac{\varepsilon}{2})^2}{2V} - \frac{\varepsilon A}{4}},$$

where the second inequality holds by the choice of A. Observe that when $y > AV$ and V is large, the exponent $\frac{-(y-N+\frac{\varepsilon}{2})^2}{2V} - \frac{\varepsilon A}{4}$ is larger than $\frac{-(y-N+\varepsilon)^2}{2V}$. Summing the above two inequalities then yields the desired result that $[\sigma * h](y) \geq 0$.

Finally, if $y \in (T+N-\varepsilon, T+N]$, then the range of integration in computing $[\sigma * h](y)$ is from $x = y-T$ to $x = N$, where $\sigma(x)$ is always positive. So the convolution is positive. And if $y > T+N$, then clearly the convolution is zero. These arguments symmetrically apply to $-y \in (T+N-\varepsilon, T+N]$ and $-y > T+N$. We therefore conclude that $[\sigma * h](y) \geq 0$ for all y, completing the proof.

A.2 Integral Representation

For fixed X, $K_X(a) = K_a(X)$ is a function of a, from $\overline{\mathbb{R}}$ to $\mathbb{R}$. Let $\mathcal{L}$ denote the set of functions $\{K_X : X \in L^\infty\}$. If Φ is a monotone additive statistic and $K_X = K_Y$, then X and Y have the same distribution and $\Phi(X) = \Phi(Y)$. Thus there exists some functional $F\colon \mathcal{L} \to \mathbb{R}$ such that $\Phi(X) = F(K_X)$. It follows from the additivity of Φ and the additivity of K_a that F is additive: $F(K_X + K_Y) = F(K_X) + F(K_Y)$.[15] Moreover, F is monotone in the sense that $F(K_X) \geq F(K_Y)$ whenever $K_X \geq K_Y$ (i.e., $K_X(a) \geq K_Y(a)$ for all $a \in \overline{\mathbb{R}}$); this follows from Lemma 1 which in turn is proved by Theorem 7 (see §6 in the main text).

The rest of this proof is analogous to the proof of Theorem 2 in Mu, Pomatto, Strack, and Tamuz (2021), but for completeness we provide the details below. The main goal is to show that the monotone additive functional F on $\mathcal{L}$ can be extended to a positive linear functional on the entire space of continuous functions $\mathcal{C}(\overline{\mathbb{R}})$. We first equip $\mathcal{L}$ with the sup-norm of $\mathcal{C}(\overline{\mathbb{R}})$ and establish a technical claim.

Lemma 3. *$F\colon \mathcal{L} \to \mathbb{R}$ is 1-Lipschitz:*

$$|F(K_X) - F(K_Y)| \leq \|K_X - K_Y\|.$$

[15] We note that $\mathcal{L}$ is closed under addition. To see this observe that $K_X + K_Y = K_{X'} + K_{Y'}$ where X', Y' are independently distributed random variables with the same distribution as X, Y. Such random variables exist as the probability space is non-atomic, see for example Proposition 9.1.11 in Bogachev (2007). We thus have that for $K_X, K_Y \in \mathcal{L}$ it holds that $K_X + K_Y = K_{X'} + K_{Y'} = K_{X'+Y'}$ where $K_{X'+Y'} \in \mathcal{L}$.

Proof. Let $\|K_X - K_Y\| = \varepsilon$. Then $K_{X+\varepsilon} = K_X + \varepsilon \geq K_Y$. Hence, by Lemma 1, $F(K_Y) \leq F(K_{X+\varepsilon})$, and so

$$F(K_Y) - F(K_X) \leq F(K_{X+\varepsilon}) - F(K_X) = F(K_\varepsilon) = \Phi(\varepsilon) = \varepsilon.$$

Symmetrically we have $F(K_X) - F(K_Y) \leq \varepsilon$, as desired. □

Lemma 4. *Any monotone additive functional F on $\mathcal{L}$ can be extended to a positive linear functional on $\mathcal{C}(\overline{\mathbb{R}})$.*

Proof. First consider the rational cone spanned by $\mathcal{L}$:

$$\mathrm{Cone}_{\mathbb{Q}}(\mathcal{L}) = \{qL : q \in \mathbb{Q}_+, L \in \mathcal{L}\}.$$

Define $G\colon \mathrm{Cone}_{\mathbb{Q}}(\mathcal{L}) \to \mathbb{R}$ as $G(qL) = qF(L)$, which is an extension of F. The functional G is well defined: If $\frac{m}{n}K_1 = \frac{r}{n}K_2$ for $K_1, K_2 \in \mathcal{L}$ and $n, m, r \in \mathbb{N}$, then, using the fact that $\mathcal{L}$ is closed under addition, we obtain $mF(K_1) = F(mK_1) = F(rK_2) = rF(K_2)$, hence $\frac{m}{n}F(K_1) = \frac{r}{n}F(K_2)$. G is also additive, because

$$G\left(\frac{m}{n}K_1\right) + G\left(\frac{r}{n}K_2\right) = \frac{m}{n}F\left(K_1\right) + \frac{r}{n}F\left(K_2\right) = \frac{1}{n}F(mK_1 + rK_2) = G\left(\frac{m}{n}K_1 + \frac{r}{n}K_2\right).$$

In the same way we can show G is positively homogeneous over $\mathbb{Q}_+$ and monotone.

Moreover, G is Lipschitz: Lemma 3 implies

$$\left|G\left(\frac{m}{n}K_1\right) - G\left(\frac{r}{n}K_2\right)\right| = \frac{1}{n}\left|F(mK_1) - F\left(rK_2\right)\right| \leq \frac{1}{n}\left\|mK_1 - rK_2\right\| = \left\|\frac{m}{n}K_1 - \frac{r}{n}K_2\right\|.$$

Thus G can be extended to a Lipschitz functional H defined on the closure of $\mathrm{Cone}_{\mathbb{Q}}(\mathcal{L})$ with respect to the sup norm. In particular, H is defined on the convex cone spanned by $\mathcal{L}$:

$$\mathrm{Cone}(\mathcal{L}) = \{\lambda_1 K_1 + \cdots + \lambda_k K_k : k \in \mathbb{N} \text{ and for each } 1 \leq i \leq k, \lambda_i \in \mathbb{R}_+, K_i \in \mathcal{L}\}.$$

It is immediate to verify that the properties of additivity, positive homogeneity (now over $\mathbb{R}_+$), and monotonicity extend, by continuity, from G to H.

Consider the vector subspace $\mathcal{V} = \mathrm{Cone}(\mathcal{L}) - \mathrm{Cone}(\mathcal{L}) \subset \mathcal{C}(\overline{\mathbb{R}})$ and define $I\colon \mathcal{V} \to \mathbb{R}$ as

$$I(g_1 - g_2) = H(g_1) - H(g_2)$$

for all $g_1, g_2 \in \mathrm{Cone}(\mathcal{L})$. The functional I is well defined and linear (because H is additive and positively homogeneous). Moreover, by monotonicity of H, $I(f) \geq 0$ for any non-negative function $f \in \mathcal{V}$.

The result then follows from the next theorem of Kantorovich (1937), a generalization of the Hahn-Banach Theorem. It applies not only to $\mathcal{C}(\overline{\mathbb{R}})$ but to any Riesz space (see Theorem 8.32 in Aliprantis and Border, 2006).

Theorem. *If $\mathcal{V}$ is a vector subspace of $\mathcal{C}(\overline{\mathbb{R}})$ with the property that for every $f \in \mathcal{C}(\overline{\mathbb{R}})$ there exists a function $g \in \mathcal{V}$ such that $g \geq f$. Then every positive linear functional on $\mathcal{V}$ extends to a positive linear functional on $\mathcal{C}(\overline{\mathbb{R}})$.*

The "majorization" condition $g \geq f$ is satisfied because every function in $\mathcal{C}(\overline{\mathbb{R}})$ is bounded and $\mathcal{V}$ contains all of the constant functions. □

The integral representation in Theorem 1 now follows from Lemma 4 by the Riesz-Markov-Kakutani Representation Theorem.

A.3 Uniqueness of Measure

We complete the proof of Theorem 1 by showing that the measure μ in the representation is unique. The following result shows that uniqueness holds even on the smaller domain $L^\infty_{\mathbb{N}}$ of non-negative integer-valued random variables.

Lemma 5. *Suppose μ and ν are two Borel probability measures on $\overline{\mathbb{R}}$ such that*

$$\int_{\overline{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu(a) = \int_{\overline{\mathbb{R}}} K_a(X)\,\mathrm{d}\nu(a).$$

for all $X \in L^\infty_{\mathbb{N}}$. Then $\mu = \nu$.

Proof. We first show $\mu(\{\infty\}) = \nu(\{\infty\})$. For any $\varepsilon > 0$, consider the Bernoulli random variable X_ε that takes value 1 with probability ε. It is easy to see that as ε decreases to zero, $K_a(X_\varepsilon)$ also decreases to zero for each $a < \infty$ whereas $K_\infty(X_\varepsilon) = \max[X_\varepsilon] = 1$. Since $K_a(X_\varepsilon)$ is uniformly bounded in $[0, 1]$, the Dominated Convergence Theorem implies

$$\lim_{\varepsilon \to 0} \int_{\overline{\mathbb{R}}} K_a(X_\varepsilon)\,\mathrm{d}\mu(a) = \mu(\{\infty\}).$$

A similar identity holds for the measure ν, and $\mu(\{\infty\}) = \nu(\{\infty\})$ follows from the assumption that $\int_{\overline{\mathbb{R}}} K_a(X_\varepsilon)\,\mathrm{d}\mu(a) = \int_{\overline{\mathbb{R}}} K_a(X_\varepsilon)\,\mathrm{d}\nu(a)$.

We can symmetrically apply the above argument to the Bernoulli random variable that takes value 1 with probability $1 - \varepsilon$. Thus $\mu(\{-\infty\}) = \nu(\{-\infty\})$ holds as well.

Next, for each $n \in \mathbb{N}_+$ and real number $b > 0$, let $X_{n,b} \in L^\infty_{\mathbb{N}}$ satisfy

$$\begin{aligned}\mathbb{P}\left[X_{n,b} = n\right] &= \mathrm{e}^{-bn}\\ \mathbb{P}\left[X_{n,b} = 0\right] &= 1 - \mathrm{e}^{-bn}.\end{aligned}$$

Then $K_a(X_{n,b}) = \frac{1}{a}\log\left[(1 - \mathrm{e}^{-bn}) + \mathrm{e}^{(a-b)n}\right]$, and so

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} K_a(X_{n,b}) &= \lim_{n\to\infty} \frac{1}{n}\frac{1}{a}\log\left[1 - \mathrm{e}^{-bn} + \mathrm{e}^{(a-b)n}\right]\\ &= \begin{cases} 0 & \text{if } a < b\\ \frac{a-b}{a} & \text{if } a \geq b.\end{cases}\end{aligned}$$

This result holds also for $a = 0, \pm\infty$.

Note that $\frac{1}{n}K_a(X_{n,b})$ is uniformly bounded in $[0,1]$ for all values of n, b, a, since $K_a(X_{n,b})$ is bounded between $\min[X_{n,b}] = 0$ and $\max[X_{n,b}] = n$. Thus, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{n} K_a(X_{n,b}) \, \mathrm{d}\mu(a) = \int_{[b,\infty]} \frac{a-b}{a} \, \mathrm{d}\mu(a), \tag{8}$$

and similarly for ν. It follows that for all $b > 0$,

$$\int_{[b,\infty]} \frac{a-b}{a} \, \mathrm{d}\mu(a) = \int_{[b,\infty]} \frac{a-b}{a} \, \mathrm{d}\nu(a).$$

As $\mu(\{\infty\}) = \nu(\{\infty\})$, we in fact have

$$\int_{[b,\infty)} \frac{a-b}{a} \, \mathrm{d}\mu(a) = \int_{[b,\infty)} \frac{a-b}{a} \, \mathrm{d}\nu(a).$$

This common integral is denoted by $f(b)$.

We now define a measure $\hat{\mu}$ on $(0,\infty)$ by the condition $\frac{\mathrm{d}\hat{\mu}(a)}{\mathrm{d}\mu(a)} = \frac{1}{a}$; note that $\hat{\mu}$ is a positive measure, but need not be a probability measure. Then

$$f(b) = \int_{[b,\infty)} \frac{a-b}{a} \, \mathrm{d}\mu(a) = \int_{[b,\infty)} (a-b) \, \mathrm{d}\hat{\mu}(a) = \int_b^\infty \hat{\mu}([x,\infty)) \, \mathrm{d}x,$$

where the last step uses Tonelli's Theorem. Hence $\hat{\mu}([b,\infty])$ is the negative of the left derivative of $f(b)$ (this uses the fact that $\hat{\mu}([b,\infty])$ is left continuous in b). In the same way, if we define $\hat{\nu}$ by $\frac{\mathrm{d}\hat{\nu}(a)}{\mathrm{d}\nu(a)} = \frac{1}{a}$, then $\hat{\nu}([b,\infty])$ is also the negative of the left derivative of $f(b)$. Therefore $\hat{\mu}$ and $\hat{\nu}$ are the same measure on $(0,\infty)$, which implies that μ and ν coincide on $(0,\infty)$.

By a symmetric argument (with $n - X_{n,b}$ in place of $X_{n,b}$), we deduce that μ and ν also coincide on $(-\infty, 0)$. Finally, since they are both probability measures, μ and ν must have the same mass at 0, if any. So $\mu = \nu$. □

B Catalytic Order for Second-order Stochastic Dominance

In this appendix we generalize Theorem 7 to obtain an analogous characterization of the catalytic stochastic order for *second-order* stochastic dominance. This will be a key step toward the proof of Theorem 5, which we present in the online appendix.

Proposition 3. *Let $X, Y \in L^\infty$ satisfy $K_a(X) > K_a(Y)$ for all $a \in [-\infty, 0]$. Then there exists an independent $Z \in L^\infty$ such that $X + Z \geq_2 Y + Z$.*

Proof. As is well known, X dominates Y in second-order stochastic dominance if and only if their c.d.f.s F and G satisfy $\int_{-\infty}^{z}(G(y)-F(y))\,\mathrm{d}y \geq 0$ for every $z \in \mathbb{R}$. Thus, if we let Z be an independent random variable with density h, then $X+Z \geq_2 Y+Z$ if and only if

$$\int_{-\infty}^{z}[\sigma * h](y)\,\mathrm{d}y \geq 0 \quad \forall z \in \mathbb{R}.$$

Here, as in the proof of Theorem 7, σ denotes the difference $G-F$ and is supported on $[-N,N]$. Since $K_{-\infty}(X) > K_{-\infty}(Y)$, we have $\min[X] > \min[Y]$. So we can choose $\varepsilon, \delta > 0$ such that $\sigma(x) \geq 0$ for $x \in [-N, -N+\varepsilon]$ and $\sigma(x) \geq \delta$ for $x \in [-N+\frac{\varepsilon}{4}, -N+\frac{\varepsilon}{2}]$. We again fix constant A such that $\mathrm{e}^{\frac{\varepsilon A}{4}} \geq \frac{8N}{\varepsilon\delta}$.

Now let $h(x) = \mathrm{e}^{\frac{-x^2}{2V}}$ for $|x| \leq T$, where V is a large variance and $T = AV + N$. Then, as in the proof of Theorem 7, we have

$$[\sigma * h](y) \geq 0 \quad \forall y \leq -AV.$$

This simply uses the fact that σ is positive near the minimum of its support.

Moreover, by assumption $K_a(X) > K_a(Y)$ for $a \leq 0$. So by continuity there exists small $\gamma > 0$ such that $K_a(X) > K_a(Y)$ for $a \leq \gamma$. It follows that

$$M_\sigma(a) = \int_{-N}^{N}\sigma(x)\mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}\left(\mathbb{E}\left[\mathrm{e}^{aX}\right] - \mathbb{E}\left[\mathrm{e}^{aY}\right]\right) > 0 \quad \forall a \leq \gamma.$$

By continuity, we can find $\eta > 0$ such that

$$M_\sigma(a) \geq \eta \quad \forall a \in [-A, \gamma].$$

Thus, when $y \in [-AV, \gamma V]$, we can follow the calculation in (6) and (7) to obtain

$$\begin{aligned}
[\sigma * h](y) &= \mathrm{e}^{-\frac{y^2}{2V}} \cdot \int_{-N}^{N}\sigma(x)\cdot \mathrm{e}^{\frac{y}{V}\cdot x}\cdot \mathrm{e}^{-\frac{x^2}{2V}}\,\mathrm{d}x \\
&\geq \mathrm{e}^{-\frac{y^2}{2V}} \cdot \left(\eta - 2N\cdot \mathrm{e}^{AN}\cdot (1-\mathrm{e}^{\frac{-N^2}{2V}})\right) \\
&\geq \mathrm{e}^{-\frac{y^2}{2V}} \cdot \frac{\eta}{2},
\end{aligned}$$

where the last step holds when V is sufficiently large.

Therefore, $[\sigma * h](y) \geq 0$ for all $y \leq \gamma V$, and clearly $\int_{-\infty}^{z}[\sigma * h](y)\,\mathrm{d}y \geq 0$ also holds for $z \leq \gamma V$. Below we consider $z > \gamma V$. The idea here is that $\int_{-\infty}^{\gamma V}[\sigma * h](y)\,\mathrm{d}y$ is sufficiently positive to compensate for the possible negative contribution from $\int_{\gamma V}^{z}[\sigma * h](y)\,\mathrm{d}y$. Specifically, using the above lower bound for $[\sigma * h](y)$, we have

$$\int_{-\infty}^{\gamma V}[\sigma * h](y)\,\mathrm{d}y \geq \int_{-\frac{\gamma V}{2}}^{\frac{\gamma V}{2}}[\sigma * h](y)\,\mathrm{d}y \geq \frac{\eta\gamma V}{2}\cdot \mathrm{e}^{-\frac{\gamma^2 V}{8}}.$$

On the other hand, when $y > \gamma V$ we can bound the magnitude of $[\sigma * h](y)$ as follows:

$$|[\sigma * h](y)| \leq \int_{-N}^{N} |\sigma(x)h(y-x)| \,\mathrm{d}x \leq \int_{-N}^{N} \mathrm{e}^{-\frac{(y-x)^2}{2V}} \,\mathrm{d}x \leq 2N \cdot \mathrm{e}^{-\frac{(\gamma V - N)^2}{2V}}$$

Since σ is supported on $[-N, N]$ and h is supported on $[-AV - N, AV + N]$, we know that $\sigma * h$ is supported on $[-AV - 2N, AV + 2N]$. Thus for $z > \gamma V$,

$$\int_{\gamma V}^{z} [\sigma * h](y) \,\mathrm{d}y \geq -\int_{\gamma V}^{AV+2N} |[\sigma * h](y)| \,\mathrm{d}y \geq -(AV + 2N - \gamma V) \cdot \mathrm{e}^{-\frac{(\gamma V - N)^2}{2V}}.$$

It is easy to see that for sufficiently large V,

$$\frac{\eta \gamma V}{2} \cdot \mathrm{e}^{-\frac{\gamma^2 V}{8}} > (AV + 2N - \gamma V) \cdot \mathrm{e}^{-\frac{(\gamma V - N)^2}{2V}}.$$

Hence the above estimates imply that $\int_{-\infty}^{z} [\sigma * h](y) \,\mathrm{d}y \geq 0$ also holds for $z > \gamma V$. So $X + Z \geq_2 Y + Z$ as we desire to show. $\square$

C Additional Proofs

C.1 Proof of Theorem 3

It is straightforward to check that the representation satisfies the axioms, so we focus on the other direction of deriving the representation from the axioms. In the first step, we fix any reward $x > 0$. Then by monotonicity in time and continuity, for each (x, T) there exists a (unique) deterministic time $\Phi_x(T)$ such that $(x, \Phi_x(T)) \sim (x, T)$. Clearly, when T is a deterministic time, $\Phi_x(T)$ is simply T itself. Note also that if S first-order stochastically dominates T, then

$$(x, \Phi_x(T)) \sim (x, T) \succeq (x, S) \sim (x, \Phi_x(S)),$$

so that $\Phi_x(S) \geq \Phi_x(T)$. We next show that for any T and S that are independent, $\Phi_x(T+S) = \Phi_x(T) + \Phi_x(S)$. Indeed, by stationarity, $(x, \Phi_x(T)) \sim (x, T)$ implies $(x, \Phi_x(T) + S) \sim (x, T + S)$ and $(x, \Phi_x(S)) \sim (x, S)$ implies $(x, \Phi_x(T) + \Phi_x(S)) \sim (x, \Phi_x(T) + S)$. Taken together, we have

$$(x, \Phi_x(T) + \Phi_x(S)) \sim (x, T + S).$$

Since $\Phi_x(T) + \Phi_x(S)$ is a deterministic time, the definition of Φ_x gives $\Phi_x(T) + \Phi_x(S) = \Phi_x(T + S)$ as desired.

In the second step, note that our preference $\succeq$ induces a preference on $\mathbb{R}_{++} \times \mathbb{R}_+$ consisting of deterministic dated rewards. By Theorem 2 in Fishburn and Rubinstein

(1982), for any given $r > 0$ we can find a continuous and strictly increasing utility function $u\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that for deterministic times $t, s \geq 0$

$$(x,t) \succeq (y,s) \quad \text{if and only if} \quad u(x) \cdot \mathrm{e}^{-rt} \geq u(y) \cdot \mathrm{e}^{-rs}.$$

By definition, $(x,T) \sim (x, \Phi_x(T))$ for any random time T. Thus we obtain that the decision maker's preference is represented by

$$(x,T) \succeq (y,S) \quad \text{if and only if} \quad u(x) \cdot \mathrm{e}^{-r\Phi_x(T)} \geq u(y) \cdot \mathrm{e}^{-r\Phi_y(S)}.$$

It remains to show that for all $x, y > 0$, Φ_x and Φ_y are the same statistic. For this we choose deterministic times t and s such that $(x,t) \sim (y,s)$, i.e., $u(x) \cdot \mathrm{e}^{-rt} = u(y) \cdot \mathrm{e}^{-rs}$. For any random time T, stationarity implies $(x, t+T) \sim (y, s+T)$, so that

$$u(x) \cdot \mathrm{e}^{-r\Phi_x(t+T)} = u(y) \cdot \mathrm{e}^{-r\Phi_y(s+T)}.$$

Using the additivity of Φ_x and Φ_y, we can divide the above two equalities and obtain $\Phi_x(T) = \Phi_y(T)$ as desired. Since this holds for all T and all $x, y > 0$, we can write $\Phi_x(T) = \Phi(T)$ for a single monotone additive statistic Φ. This completes the proof.

C.2 Proof of Theorem 4

We first prove that the proposed representation for the social preference relation $\succeq$ satisfies the Pareto axiom. If $(x,T) \succeq_i (y,S)$ for every i, then $u(x)\mathbb{E}[\mathrm{e}^{-r_i T}] \geq u(y)\mathbb{E}[\mathrm{e}^{-r_i S}]$, which in turn can be rewritten as

$$K_{-r_i}(T) - K_{-r_i}(S) \leq \frac{1}{r_i} \log \frac{u(x)}{u(y)}.$$

Summing across i using the weights λ_i we obtain

$$\sum_{i=1}^{n} \lambda_i (K_{-r_i}(T) - K_{-r_i}(S)) \leq \log \frac{u(y)}{u(x)} \sum_{i=1}^{n} \frac{\lambda_i}{r_i}.$$

If $\Phi = \sum_{i=1}^n \lambda_i K_{-r_i}$ and $\frac{1}{r} = \sum_{i=1}^n \frac{\lambda_i}{r_i}$, then $\Phi(T) - \Phi(S) \leq \frac{1}{r} \log \frac{u(x)}{u(y)}$ which is equivalent to $u(x)\mathrm{e}^{-r\Phi(T)} \geq u(y)\mathrm{e}^{-r\Phi(S)}$. Thus $(x,T) \succeq (y,S)$. The Pareto axiom is therefore satisfied.

We now show that the Pareto axiom implies the desired representation for $\succeq$. By Theorem 3, $\succeq$ admits a representation $V(x,t) = \tilde{v}(x)\mathrm{e}^{-\tilde{r}\Phi(T)}$. If $K_{-r_i}(T) \leq K_{-r_i}(S)$ for every i, then $(x,T) \succeq_i (x,S)$ for every i and thus, by the Pareto axiom, $\Phi(T) \leq \Phi(S)$. From this property we next show that Φ must be an average of the statistics $K_{-r_1}, \ldots, K_{-r_n}$. In fact, we have a stronger result in the next lemma. We say that $(\Phi_1, \ldots, \Phi_n, \Phi)$ have the *Pareto property* if for every S, T in their domain it holds that $\Phi_i(T) \leq \Phi_i(S)$ for every i implies $\Phi(T) \leq \Phi(S)$.

Lemma 6. *Let $(\Phi_1, \ldots, \Phi_n, \Phi)$ be monotone additive statistics defined on L_+^∞, and suppose that they have the Pareto property. Then there exists a probability vector $(\lambda_1, \ldots, \lambda_n)$ such that $\Phi = \sum_{i=1}^n \lambda_i \Phi_i$.*

Proof. Let $(\mu_1, \ldots, \mu_n, \mu)$ be the measures on $\overline{\mathbb{R}}$ that correspond to the monotone additive statistics $(\Phi_1, \ldots, \Phi_n, \Phi)$. Define the linear functionals $(I_1, \ldots, I_n, I)$ on $\mathcal{C}(\overline{\mathbb{R}})$ as $I_i(f) = \int_{\overline{\mathbb{R}}} f \mathrm{d}\mu_i$ and $I(f) = \int_{\overline{\mathbb{R}}} f \mathrm{d}\mu$.

We call a set of functions $\mathcal{D} \subseteq \mathcal{C}(\overline{\mathbb{R}})$ a *Pareto domain* if for every $f, g \in \mathcal{D}$,

$$I_i(f) \geq I_i(g) \quad i = 1, \ldots, n \implies I(f) \geq I(g).$$

The Pareto property implies $\mathcal{L}_+ = \{K_X : X \in L_+^\infty\}$ is a Pareto domain. Define, as in the proof of Theorem 1, the set $\mathcal{L} = \{K_X : X \in L^\infty\}$ as well as the rational cone spanned by $\mathcal{L}$:

$$\mathrm{cone}_{\mathbb{Q}}(\mathcal{L}) = \{qL : q \in \mathbb{Q}_+, L \in \mathcal{L}\} = \bigcup_{n=1}^{\infty} \frac{1}{n}\mathcal{L}$$

We show that $\mathcal{L}$ and $\mathrm{cone}_{\mathbb{Q}}(\mathcal{L})$ are both Pareto domains. Given $X, Y \in L^\infty$, let c be a large enough positive constant such that $X + c \geq 0$ and $Y + c \geq 0$. If $I_i(K_X) \geq I_i(K_Y)$ for all i then $I_i(K_X + c) \geq I_i(K_Y + c)$ for all i since each I_i is linear. Thus, by the Pareto property and the linearity of I, $I(K_X + c) \geq I(K_Y + c)$ and $I(K_X) \geq I(K_Y)$. This shows $\mathcal{L}$ is a Pareto domain. As for $\mathrm{cone}_{\mathbb{Q}}(\mathcal{L})$, observe that $I_i(\frac{1}{m}K_X) \geq I_i(\frac{1}{n}K_Y)$ for all i is equivalent to $I_i(nK_X) \geq I_i(mK_Y)$ for all i, which implies $I(nK_X) \geq I(mK_Y)$ given that $\mathcal{L}$ is a Pareto domain and is closed under addition. This shows $I(\frac{1}{m}K_X) \geq I(\frac{1}{n}K_Y)$ as desired.

Next we show that the closure of $\mathrm{cone}_{\mathbb{Q}}(\mathcal{L})$ (with respect to the usual sup norm) is also a Pareto domain. Let f, g be in the closure, such that $I_i(f) \geq I_i(g)$ for all i. Let (f_n) and (g_n) be sequences in $\mathrm{cone}_{\mathbb{Q}}(\mathcal{L})$ converging to f and g. Let $\varepsilon_{i,n} = |I_i(f) - I_i(f_n)| + |I_i(g_n) - I_i(g)|$ and $\varepsilon_n = \max_i \varepsilon_{i,n}$. Then $I_i(f_n) \geq I_i(g_n) - \varepsilon_n = I_i(g_n - \varepsilon_n)$ for every i. Note that $g_n - \varepsilon_n$ belongs to $\mathrm{cone}_{\mathbb{Q}}(\mathcal{L})$ since the latter contains all the constant functions and is closed under addition. Thus by the fact that $\mathrm{cone}_{\mathbb{Q}}(\mathcal{L})$ is a Pareto domain, $I_i(f_n) \geq I_i(g_n - \varepsilon_n)$ for every i implies $I(f_n) \geq I(g_n - \varepsilon_n) = I(g_n) - \varepsilon_n$ for every n. Continuity of the functionals (I_i) yields $\varepsilon_n \to 0$. Continuity of I thus yields $I(f) \geq I(g)$.

This proves that the closure of $\mathrm{cone}_{\mathbb{Q}}(\mathcal{L})$ is a Pareto domain. Since the subset of a Pareto domain is a Pareto domain, we conclude that $\mathrm{cone}(\mathcal{L})$ (i.e. the cone generated by $\mathcal{L}$) is a Pareto domain as well.

Now define $\mathcal{V} = \mathrm{cone}(\mathcal{L}) - \mathrm{cone}(\mathcal{L})$ to be the vector space generated by the cone. It is immediate to verify, using the linearity of the integral, that $\mathcal{V}$ is a Pareto domain as well. In particular, for any $f \in \mathcal{V}$, $I_i(f) \leq 0$ for every i implies $I(f) \leq 0$. Corollary 5.95 in Aliprantis and Border (2006) thus implies there exist non-negative scalars $\lambda_1, \ldots, \lambda_n$ such that $I = \sum_{i=1}^n \lambda_i I_i$ on V. In particular, $I(K_X) = \sum_{i=1}^n \lambda_i I_i(K_X)$ for every $X \in L^\infty$, which implies

$$\Phi(X) = I(K_X) = \sum_{i=1}^n \lambda_i I_i(K_X) = \sum_{i=1}^n \lambda_i \Phi_i(X).$$

If X is a constant, then $\Phi(X) = \sum_i \lambda_i \Phi_i(X)$ implies $\sum_i \lambda_i = 1$. This concludes the proof of the lemma. □

We conclude that the certainty equivalent Φ of the social preference $\succeq$ satisfies $\Phi = \sum_{i=1}^n \lambda_i K_{-r_i}$. Let $r > 0$ be defined by $\frac{1}{r} = \sum_{i=1}^n \frac{\lambda_i}{r_i}$, and define $v(x) = (\tilde{v}(x))^{(r/\tilde{r})}$. Then

$$V(x,T) = (\tilde{V}(x,T))^{\frac{r}{\tilde{r}}} = v(x)\mathrm{e}^{-r\Phi(T)}$$

also represents the social preference $\succeq$. In the last step we show that $v = \alpha u$ for some positive constant α, and thus $u(x)\mathrm{e}^{-r\Phi(T)}$ represents $\succeq$.

To establish this fact we need another technical lemma:

Lemma 7. *Given $r_1, \dots, r_n \in \mathbb{R}_{++}$, there exists a constant $\eta \in (0,1)$ with the property that for any $c \in (1-\eta, 1)$, one can find $T, S \in L_+^\infty$ such that $\mathbb{E}[\mathrm{e}^{-r_i T}] = c \cdot \mathbb{E}[\mathrm{e}^{-r_i S}]$ holds for every i.*

Proof. It is without loss of generality to assume all r_i are distinct. We first prove that there exist real numbers $d_1, \dots, d_{n+1}$ such that

$$\sum_{k=1}^{n+1} d_k \mathrm{e}^{-r_i \cdot k} = 0 \quad \text{for every} \quad 1 \le i \le n \quad \text{and} \quad \sum_{k=1}^{n+1} d_k = 1.$$

Indeed, we can let $r_{n+1} = 0$, define $z_i = \mathrm{e}^{-r_i}$ and define an $(n+1) \times (n+1)$ matrix M whose (i,j)-th entry is $\mathrm{e}^{-r_i \cdot j} = z_i^j$. Then M is equal to the Vandermonde matrix given by the distinct numbers $z_1, \dots, z_{n+1}$, multiplied by z_i on each row i. It follows that the determinant of M is non-zero. Thus M is invertible, and there exists a vector $d = (d_1, \dots, d_{n+1})$ such that $M \cdot d' = (0, \dots, 0, 1)'$ (where the apostrophe denotes transpose). This is what we wanted to show.

With these d_k fixed, we now construct T and S such that $\mathbb{E}[\mathrm{e}^{-r_i T}] = c \cdot \mathbb{E}[\mathrm{e}^{-r_i S}]$. For each $1 \le k \le n+1$, we let $T = k$ with probability a_k, and let $S = k$ with probability b_k. These probabilities a_k, b_k are chosen to satisfy the following: $a_k, b_k \ge 0 \ \forall k$, $\sum_k a_k = \sum_k b_k = 1$, and there exists λ such that $a_k - cb_k = \lambda d_k$ for all k. The last condition ensures that

$$\mathbb{E}[\mathrm{e}^{-r_i T}] - c \cdot \mathbb{E}[\mathrm{e}^{-r_i S}] = \sum_k a_k \mathrm{e}^{-r_i \cdot k} - c \sum_k b_k \mathrm{e}^{-r_i \cdot k} = \lambda \sum_k d_k \mathrm{e}^{-r_i \cdot k} = 0.$$

To show that such $\{a_k\}, \{b_k\}$ exist, note first that $a_k - cb_k = \lambda d_k$ for each k implies, by summing across k, that $\lambda = 1 - c$. Thus, for each k where $d_k \ge 0$, we can choose any $a_k \geq (1-c)d_k$ and define $b_k \geq 0$ to satisfy $a_k - cb_k = (1-c)d_k$. And if $d_k < 0$, then we can choose any $a_k \geq 0$ and determine b_k accordingly. The upshot is that so long as we choose $a_k \geq \max\{(1-c)d_k, 0\}$, the corresponding $b_k = (a_k - (1-c)d_k)/c$ will also be non-negative, and $\sum_k b_k = 1$ whenever $\sum_k a_k = 1$. Thus it suffices to have $a_k \geq \max\{(1-c)d_k, 0\}$ and $\sum_k a_k = 1$. This is possible whenever $1-c$ is sufficiently small so that $\sum_k \max\{(1-c)d_k, 0\} \leq 1$. Hence the lemma is proved. □

We can now relate the social utility function v to the individual utility function u. Consider any pair of rewards $x > y > 0$ with $\frac{u(y)}{u(x)} > 1-\eta$. By Lemma 7, there exist T and S such that $\mathbb{E}[\mathrm{e}^{-r_i T}] = \frac{u(y)}{u(x)}\mathbb{E}[\mathrm{e}^{-r_i S}]$ for every i. That is, $(x, T) \sim_i (y, S)$ for every i. As shown in the first part of the proof, the representation $u(x)\mathrm{e}^{-r\Phi(T)}$, with $\Phi = \sum_{i=1}^n \lambda_i K_{-r_i}$ and $\frac{1}{r} = \sum_{i=1}^n \frac{\lambda_i}{r_i}$, satisfies the Pareto axiom. Thus,

$$u(x)\mathrm{e}^{-r\Phi(T)} = u(y)\mathrm{e}^{-r\Phi(S)}.$$

On the other hand, the Pareto axiom also implies $(x, T) \sim (y, S)$ for the social preference relation. Therefore, its representation must satisfy

$$v(x)\mathrm{e}^{-r\Phi(T)} = v(y)\mathrm{e}^{-r\Phi(S)}.$$

Dividing the above two equalities yields $\frac{v(x)}{u(x)} = \frac{v(y)}{u(y)}$ whenever x is slightly larger than y, in the sense that $\frac{u(y)}{u(x)} \in (1-\eta, 1)$.

For general x and y we can find a sequence $x_0 = x > x_1 > \cdots > x_m = y$ such that $\frac{u(x_{j+1})}{u(x_j)} \in (1-\eta, 1)$ for each j. This follows from the fact that u is increasing and continuous. Thus, repeated applications of the previous argument yield

$$\frac{v(x)}{u(x)} = \frac{v(x_0)}{u(x_0)} = \frac{v(x_1)}{u(x_1)} = \cdots = \frac{v(x_m)}{u(x_m)} = \frac{v(y)}{u(y)}.$$

Therefore $v(x)$ is equal to $u(x)$ up to a multiplicative constant, as we desired to show. This concludes the proof.

C.3 Proof of Theorem 6

The "if" direction is straightforward: if $\succeq_1$ and $\succeq_2$ are both represented by a monotone additive statistic Φ, then they satisfy responsiveness and continuity. In addition, combined choices are not stochastically dominated because if $X \succ_1 X'$ and $Y \succ_2 Y'$ then $\Phi(X) > \Phi(X')$ and $\Phi(Y) > \Phi(Y')$. Thus $\Phi(X+Y) > \Phi(X'+Y')$ and $X'+Y'$ cannot stochastically dominate $X+Y$.

Turning to the "only if" direction, we suppose $\succeq_1$ and $\succeq_2$ satisfy the axioms. We first show that these preferences are the same. Suppose for the sake of contradiction that $X \succeq_1 Y$ but $Y \succ_2 X$ for some X, Y. Then by continuity, there exists $\varepsilon > 0$ such that

$Y \succ_2 X + \varepsilon$. By responsiveness, we also have $X \succeq_1 Y \succ Y - \frac{\varepsilon}{2}$. Thus $X \succ_1 Y - \frac{\varepsilon}{2}$, $Y \succ_2 X + \varepsilon$, but $X + Y$ is strictly stochastically dominated by $Y - \frac{\varepsilon}{2} + X + \varepsilon = X + Y + \frac{\varepsilon}{2}$, contradicting Axiom 4.7.

Henceforth we denote both $\succeq_1$ and $\succeq_2$ by $\succeq$. We next show that for any X and any $\varepsilon > 0$, $\max[X] + \varepsilon \succ X \succ \min[X] - \varepsilon$. To see why, suppose for contradiction that X is weakly preferred to $\max[X] + \varepsilon$ (the other case can be handled similarly). Then we obtain a contradiction to Axiom 4.7 by observing that $X \succ \max[X] + \frac{\varepsilon}{2}$, $\frac{\varepsilon}{4} \succ 0$ but $X + \frac{\varepsilon}{4} <_1 \max[X] + \frac{\varepsilon}{2} + 0$.

Given these upper and lower bounds for X, we can define $\Phi(X) = \sup\{c \in \mathbb{R} : \ c \preceq X\}$, which is well-defined and finite. By definition of the supremum and responsiveness, for any $\varepsilon > 0$ it holds that $\Phi(X) - \varepsilon \prec X \prec \Phi(X) + \varepsilon$. Thus by continuity, $\Phi(X) \sim X$ is the (unique) certainty equivalent of X.

It remains to show that Φ is a monotone additive statistic. For this we show that $X \sim Y$ implies $X + Z \sim Y + Z$ for any independent Z. Suppose for contradiction that $X + Z \succ Y + Z$. Then by continuity we can find $\varepsilon > 0$ such that $X + Z \succ Y + Z + \varepsilon$. By responsiveness, it also holds that $Y + \frac{\varepsilon}{2} \succ Y \sim X$. But the sum $(X + Z) + (Y + \frac{\varepsilon}{2})$ is stochastically dominated by $(Y + Z + \varepsilon) + X$, contradicting Axiom 4.7.

Therefore, from $X \sim \Phi(X)$ and $Y \sim \Phi(Y)$ we can apply the preceding result twice to obtain $X + Y \sim \Phi(X) + Y \sim \Phi(X) + \Phi(Y)$ whenever X, Y are independent, so that $\Phi(X + Y) = \Phi(X) + \Phi(Y)$ is additive. Finally, we show Φ is monotone. Consider any $Y \geq_1 X$, and suppose for contradiction that $X \succ Y$. Then there exists $\varepsilon > 0$ such that $X \succ Y + \varepsilon$. This leads to a contradiction since $X \succ Y + \varepsilon$, $\frac{\varepsilon}{2} \succ 0$, but $X + \frac{\varepsilon}{2}$ is stochastically dominated by $Y + \varepsilon + 0$.

This completes the proof that both preferences $\succeq_1$ and $\succeq_2$ are represented by the same certainty equivalent $\Phi(X)$, which is a monotone additive statistic.

C.4 Proof of Proposition 1

The result can be derived as a corollary of Proposition 2, but we also provide a direct proof here. We focus on the "only if" direction because the "if" direction follows immediately from the monotonicity of $K_a(X)$ in a. Suppose μ is not supported on $[-\infty, 0]$, we will show that the resulting monotone additive statistic Φ does not always exhibit risk aversion. Since μ has positive mass on $(0, \infty]$, we can find $\varepsilon > 0$ such that μ assigns mass at least ε to $(\varepsilon, \infty]$. Now consider a gamble X which is equal to 0 with probability $\frac{n-1}{n}$ and equal to n with probability $\frac{1}{n}$, for some large positive integer n. Then $\mathbb{E}[X] = 1$ and $K_a(X) \geq \min[X] = 0$ for every $a \in \bar{\mathbb{R}}$. Moreover, for $a \geq \epsilon$ we have

$$K_a(X) \geq K_\varepsilon(X) = \frac{1}{\varepsilon} \log\left(\frac{n-1}{n} + \frac{1}{n}\mathrm{e}^{\varepsilon n}\right) \geq \frac{n}{2}$$

whenever n is sufficient large. Thus

$$\Phi(X) = \int_{\bar{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu(a) \geq \int_{[\varepsilon,\infty]} K_a(X)\,\mathrm{d}\mu(a) \geq \frac{n}{2}\varepsilon.$$

We thus have $\Phi(X) > 1 = \mathbb{E}[X]$ for all large n, showing that the preference represented by Φ sometimes exhibits risk seeking.

Symmetrically, if μ is not supported on $[0,\infty]$, then Φ must sometimes exhibit risk aversion (by considering X equal to 0 with probability $\frac{1}{n}$ and equal to n with probability $\frac{n-1}{n}$). This completes the proof.

C.5 Proof of Proposition 2

We first show that conditions (i) and (ii) are necessary for $\int_{\bar{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu_1(a) \leq \int_{\bar{\mathbb{R}}} K_a(Y)\,\mathrm{d}\mu_2(a)$ to hold for every X. This part of the argument closely follows the proof of Lemma 5. Specifically, by considering the same random variables $X_{n,b}$ as defined there, we have the key equation (8). Since the limit on the left-hand side is smaller for μ_1 than for μ_2, we conclude that for every $b > 0$, $\int_{[b,\infty]} \frac{a-b}{a}\,\mathrm{d}\mu_1(a)$ on the right-hand side must be smaller than the corresponding integral for μ_2. Thus condition (i) holds, and an analogous argument shows condition (ii) also holds.

To complete the proof, it remains to show that when conditions (i) and (ii) are satisfied,

$$\int_{\bar{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu_1(a) \leq \int_{\bar{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu_2(a)$$

holds for every X. Since μ_1 and μ_2 are both probability measures, we can subtract $\mathbb{E}[X]$ from both sides and arrive at the equivalent inequality

$$\int_{\bar{\mathbb{R}}_{\neq 0}} (K_a(X) - \mathbb{E}[X])\,\mathrm{d}\mu_1(a) \leq \int_{\bar{\mathbb{R}}_{\neq 0}} (K_a(X) - \mathbb{E}[X])\,\mathrm{d}\mu_2(a). \tag{9}$$

Note that we can exclude $a = 0$ from the range of integration because $K_a(X) = \mathbb{E}[X]$ there. Below we show that condition (i) implies

$$\int_{(0,\infty]} (K_a(X) - \mathbb{E}[X])\,\mathrm{d}\mu_1(a) \leq \int_{(0,\infty]} (K_a(X) - \mathbb{E}[X])\,\mathrm{d}\mu_2(a). \tag{10}$$

Similarly, condition (ii) gives the same inequality when the range of integration is $[-\infty, 0)$. Adding these two inequalities would yield the desired comparison in (9).

To prove (10), we let $L_X(a) = a \cdot K_a(X) = \log \mathbb{E}\left[\mathrm{e}^{aX}\right]$ be the cumulant generating function of X. It is well known that $L_X(a)$ is convex in a, with $L_X'(0) = \mathbb{E}[X]$ and $\lim_{a\to\infty} L_X'(a) = \max[X]$. Then the integral on the left-hand side of (10) can be calculated as follows:

$$\begin{aligned}\int_{(0,\infty]} (K_a(X) - \mathbb{E}[X])\,\mathrm{d}\mu_1(a) &= \int_{(0,\infty)} (K_a(X) - \mathbb{E}[X])\,\mathrm{d}\mu_1(a) + (\max[X] - \mathbb{E}[X]) \cdot \mu_1(\{\infty\}) \\ &= \int_{(0,\infty)} (L_X(a) - a\mathbb{E}[X])\,\mathrm{d}\frac{\mu_1(a)}{a} + (\max[X] - \mathbb{E}[X]) \cdot \mu_1(\{\infty\})\end{aligned}$$

Note that since the function $g(a) = L_X(a) - a\mathbb{E}[X]$ satisfies $g(0) = g'(0) = 0$, it can be written as

$$g(a) = \int_0^a g'(t)\,\mathrm{d}t = \int_0^a \int_0^t g''(b)\,\mathrm{d}b\,\mathrm{d}t = \int_0^a g''(b)\cdot(a-b)\,\mathrm{d}b.$$

Plugging back to the previous identity, we obtain

$$\begin{aligned}
&\int_{(0,\infty]} (K_a(X) - \mathbb{E}[X])\,\mathrm{d}\mu_1(a) \\
&= \int_{(0,\infty)} \int_0^a L_X''(b)\cdot(a-b)\,\mathrm{d}b\,\mathrm{d}\frac{\mu_1(a)}{a} + (\max[X] - \mathbb{E}[X])\cdot\mu_1(\{\infty\}) \\
&= \int_0^\infty L_X''(b) \int_{[b,\infty)} (a-b)\,\mathrm{d}\frac{\mu_1(a)}{a}\,\mathrm{d}b + (L_X'(\infty) - L_X'(0))\cdot\mu_1(\{\infty\}) \\
&= \int_0^\infty L_X''(b) \int_{[b,\infty)} \frac{a-b}{a}\,\mathrm{d}\mu_1(a)\,\mathrm{d}b + \int_0^\infty L_X''(b)\cdot\mu_1(\{\infty\})\,\mathrm{d}b \\
&= \int_0^\infty L_X''(b) \int_{[b,\infty]} \frac{a-b}{a}\,\mathrm{d}\mu_1(a)\,\mathrm{d}b,
\end{aligned}$$

where the last step uses $\frac{a-b}{a} = 1$ when $a = \infty > b$.

The above identity also holds when μ_1 is replaced by μ_2. We then see that (10) follows from condition (i) and $L_X''(b) \geq 0$ for all b. This completes the proof.

Online Appendix

D Proof of Theorem 2

D.1 Proof for L^∞_+ and $L^\infty_{\mathbb{N}}$

It suffices to show that a monotone additive statistic defined on L^∞_+ or $L^\infty_{\mathbb{N}}$ can be extended to a monotone additive statistic defined on L^∞. First suppose Φ is defined on L^∞_+, the collection of non-negative random variables. Then for any bounded random variable X, we can define

$$\Psi(X) = \min[X] + \Phi(X - \min[X]),$$

where we note that $X - \min[X]$ is a non-negative random variable.

Clearly Ψ is a statistic that depends only on the distribution of X (as Φ does), and $\Psi(c) = c + \Phi(0) = c$ for constants c. When X is non-negative, the additivity of Φ gives $\Phi(X) = \Phi(\min[X]) + \Phi(X - \min[X]) = \min[X] + \Phi(X - \min[X])$, so Ψ is an extension of Φ. Moreover, Ψ is additive because $\min[X + Y] = \min[X] + \min[Y]$, and $\Phi(X + Y - \min[X + Y]) = \Phi(X - \min[X]) + \Phi(Y - \min[Y])$ by the additivity of Φ. Finally, to show Ψ is monotone, suppose X and Y are bounded random variables satisfying $X \geq_1 Y$. Then we can choose a sufficiently large n such that $X + n$ and $Y + n$ are both non-negative, and $X + n \geq_1 Y + n$. Since Φ is monotone for non-negative random variables, $\Phi(X + n) \geq \Phi(Y + n)$. Thus $\Psi(X + n) \geq \Psi(Y + n)$ by the fact that Ψ extends Φ, and $\Psi(X) \geq \Psi(Y)$ by the additivity of Ψ. This proves that Ψ is a monotone additive statistic on L^∞ that extends Φ.

In what follows, we consider the other case where Φ is initially defined on $L^\infty_{\mathbb{N}}$, the collection of non-negative integer-valued random variables. Given what has been shown above, we just need to extend Φ to a monotone additive statistic on L^∞_+. We denote by $\lfloor X \rfloor$ the random variables that equals X rounded down to the nearest (non-negative) integer. Note that $\lfloor X + 1 \rfloor \geq_1 X \geq_1 \lfloor X \rfloor$. We thus have

$$\lfloor X + Y \rfloor \geq_1 \lfloor X \rfloor + \lfloor Y \rfloor, \tag{11}$$

$$\lfloor X + 1 \rfloor + \lfloor Y + 1 \rfloor \geq_1 \lfloor X + Y + 1 \rfloor. \tag{12}$$

Given a monotone additive statistic $\Phi\colon L^\infty_{\mathbb{N}} \to \mathbb{R}$, define $\Psi\colon L^\infty_+ \to \mathbb{R}$ by

$$\Psi(X) = \lim_{n\to\infty} \frac{1}{n}\Phi(\lfloor X^{*n} + 1 \rfloor) = \lim_{n\to\infty} \frac{1}{n}\Phi(\lfloor X^{*n} \rfloor),$$

where we recall X^{*n} denotes the sum of n i.i.d. copies of X. The first limit exists because $b_n = \Phi(\lfloor X^{*n} + 1 \rfloor)$ is a non-negative sequence which is sub-additive by (12) and by monotonicity and additivity of Φ, and thus $\lim_{n\to\infty} b_n/n = \inf_n b_n/n$ is well-known to exist. That the two limits above coincide follows from the additivity of Φ.

Ψ is a statistic because $\Psi(c) = \lim_{n\to\infty} \frac{1}{n}\Phi(\lfloor nc \rfloor) = \lim_{n\to\infty} \frac{1}{n}\lfloor nc \rfloor = c$ for every constant $c \geq 0$. It is also immediate to see that for integer-valued random variables X,

$$\Psi(X) = \lim_{n\to\infty} \frac{1}{n}\Phi(\lfloor X^{*n} \rfloor) = \lim_{n\to\infty} \frac{1}{n}\Phi(X^{*n}) = \lim_{n\to\infty} \frac{1}{n} n\Phi(X) = \Phi(X).$$

So Ψ extends Φ.

Moreover, if $X \geq_1 Y$, then $\lfloor X^{*n} \rfloor \geq_1 \lfloor Y^{*n} \rfloor$ for each n. This implies $\Psi(X) \geq \Psi(Y)$ by the above definition, so Ψ is monotone. Finally, to check Ψ is additive, we suppose X and Y be independent random variables. Then using (12), we have that for each n,

$$\lfloor (X+Y)^{*n} + 1 \rfloor \leq_1 \lfloor X^{*n} + 1 \rfloor + \lfloor Y^{*n} + 1 \rfloor.$$

Together with the monotonicity and additivity of Φ, this implies

$$\begin{aligned} \Psi(X+Y) &= \lim_{n\to\infty} \frac{1}{n}\Phi(\lfloor (X+Y)^{*n} + 1 \rfloor) \\ &\leq \lim_{n\to\infty} \frac{1}{n}\Phi(\lfloor X^{*n} + 1 \rfloor) + \lim_{n\to\infty} \frac{1}{n}\Phi(\lfloor Y^{*n} + 1 \rfloor) \\ &= \Psi(X) + \Psi(Y). \end{aligned}$$

Symmetrically, we can use the other definition of $\Psi(X+Y)$ and (11) to show that $\Psi(X+Y) \geq \Psi(X) + \Psi(Y)$. Hence equality holds, and Ψ is a monotone additive statistic that extends Φ. This completes the proof.

D.2 Proof for L_M

We break the proof into several steps below:

D.2.1 Step 1: Catalytic Order on L_M

We first establish a generalization of Theorem 7 to unbounded random variables. For two random variables X and Y with c.d.f. F and G respectively, we say that X *dominates* Y *in both tails* if there exists a positive number N with the property that

$$G(x) > F(x) \quad \text{for all } |x| \geq N.$$

In particular, X needs to be unbounded from above, and Y unbounded from below.

Lemma 8. *Suppose $X, Y \in L_M$ satisfy $K_a(X) > K_a(Y)$ for every $a \in \mathbb{R}$. Suppose further that X dominates Y in both tails. Then there exists an independent random variable $Z \in L_M$ such that $X + Z \geq_1 Y + Z$.*

Proof. We will take Z to have a normal distribution, which does belong to L_M. Following the proof of Theorem 7, we let $\sigma(x) = G(x) - F(x)$, and seek to show that $[\sigma * h](y) \geq 0$ for every y when h is a Gaussian density with sufficiently large variance. By assumption, $\sigma(x)$ is strictly positive for $|x| \geq N$. Thus there exists $\delta > 0$ such that $\int_{N+1}^{N+2} \sigma(x)\,\mathrm{d}x > \delta$, as well as $\int_{-N-2}^{-N-1} \sigma(x)\,\mathrm{d}x > \delta$. We fix $A > 0$ that satisfies $\mathrm{e}^A \geq \frac{4N}{\delta}$.

Similar to (6), we have for $h(x) = \mathrm{e}^{-\frac{x^2}{2V}}$ that

$$\mathrm{e}^{\frac{y^2}{2V}} \int \sigma(x) h(y-x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V}\cdot x} \cdot \mathrm{e}^{-\frac{x^2}{2V}}\,\mathrm{d}x. \tag{13}$$

The variance V is to be determined below.

We first show that the right-hand side is positive if $V \geq (N+2)^2$ and $\frac{y}{V} \geq A$. Indeed, since $\sigma(x) > 0$ for $|x| \geq N$, this integral is bounded from below by

$$\begin{aligned}
& \int_{-N}^{N} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V}\cdot x} \cdot \mathrm{e}^{-\frac{x^2}{2V}}\,\mathrm{d}x + \int_{N+1}^{N+2} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V}\cdot x} \cdot \mathrm{e}^{-\frac{x^2}{2V}}\,\mathrm{d}x \\
\geq \; & -2N \cdot \mathrm{e}^{\frac{y}{V}\cdot N} + \delta \cdot \mathrm{e}^{\frac{y}{V}\cdot (N+1)} \cdot \mathrm{e}^{-\frac{(N+2)^2}{2V}} \\
= \; & \mathrm{e}^{\frac{y}{V}\cdot N} \cdot (-2N + \delta \cdot \mathrm{e}^{\frac{y}{V}} \cdot \mathrm{e}^{-\frac{(N+2)^2}{2V}}) \\
> \; & 0,
\end{aligned}$$

where the last inequality uses $\mathrm{e}^{\frac{y}{V}} \geq \mathrm{e}^A \geq \frac{4N}{\delta}$ and $\mathrm{e}^{-\frac{(N+2)^2}{2V}} \geq \mathrm{e}^{-\frac{1}{2}} > \frac{1}{2}$. By a symmetric argument, we can show that the right-hand side of (13) is also positive when $\frac{y}{V} \leq -A$.

It remains to consider the case where $\frac{y}{V} \in [-A, A]$. Here we rewrite the integral on the right-hand side of (13) as

$$\int_{-\infty}^{\infty} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V}\cdot x} \cdot \mathrm{e}^{-\frac{x^2}{2V}}\,\mathrm{d}x = M_\sigma(\frac{y}{V}) - \int_{-\infty}^{\infty} \sigma(x) \cdot \mathrm{e}^{\frac{y}{V}\cdot x} \cdot (1 - \mathrm{e}^{-\frac{x^2}{2V}})\,\mathrm{d}x,$$

where $M_\sigma(a) = \int_{-\infty}^{\infty} \sigma(x) \cdot \mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}\mathbb{E}\left[\mathrm{e}^{aX}\right] - \frac{1}{a}\mathbb{E}\left[\mathrm{e}^{aY}\right]$ is by assumption strictly positive for all a. By continuity, there exists some $\varepsilon > 0$ such that $M_{\sigma(a)} > \varepsilon$ for all $|a| \leq A$. So it only remains to show that when V is sufficiently large,

$$\int_{-\infty}^{\infty} \sigma(x) \cdot \mathrm{e}^{ax} \cdot (1 - \mathrm{e}^{-\frac{x^2}{2V}})\,\mathrm{d}x < \varepsilon \quad \text{for all } |a| \leq A. \tag{14}$$

To estimate this integral, note that $M_\sigma(A) = \int_{-\infty}^{\infty} \sigma(x) \cdot \mathrm{e}^{Ax}\,\mathrm{d}x$ is finite. Since $\sigma(x) > 0$ for $|x|$ sufficiently large, we deduce from the Monotone Convergence Theorem that $\int_{-\infty}^{T} \sigma(x) \cdot \mathrm{e}^{Ax}\,\mathrm{d}x$ converges to $M_\sigma(A)$ as $T \to \infty$. In other words, $\int_{T}^{\infty} \sigma(x) \cdot \mathrm{e}^{Ax}\,\mathrm{d}x \to 0$. We can thus find a sufficiently large $T > N$ such that $\int_{T}^{\infty} \sigma(x) \cdot \mathrm{e}^{Ax}\,\mathrm{d}x < \frac{\varepsilon}{4}$, and likewise $\int_{-\infty}^{-T} \sigma(x) \cdot \mathrm{e}^{-Ax}\,\mathrm{d}x < \frac{\varepsilon}{4}$.

As $1 - \mathrm{e}^{-\frac{x^2}{2V}} \geq 0$ and $\mathrm{e}^{ax} \leq \mathrm{e}^{A|x|}$ when $|a| \leq A$, we deduce that

$$\int_{|x| \geq T} \sigma(x) \cdot \mathrm{e}^{ax} \cdot (1 - \mathrm{e}^{-\frac{x^2}{2V}})\,\mathrm{d}x < \frac{\varepsilon}{2} \quad \text{for all } |a| \leq A.$$

Moreover, for this fixed T, we have $\mathrm{e}^{-\frac{T^2}{2V}} \to 1$ when V is large, and thus

$$\int_{|x|\le T} \sigma(x)\cdot \mathrm{e}^{ax}\cdot(1-\mathrm{e}^{-\frac{x^2}{2V}})\,\mathrm{d}x < 2T\mathrm{e}^{AT}(1-\mathrm{e}^{-\frac{T^2}{2V}}) < \frac{\varepsilon}{2} \quad \text{for all } |a|\le A.$$

These estimates together imply that (14) holds for sufficiently large V. This completes the proof. □

D.2.2 Step 2: A Perturbation Argument

With Lemma 8, we know that if Φ is a monotone additive statistic defined on L_M, then $K_a(X) \ge K_a(Y)$ for all $a \in \mathbb{R}$ implies $\Phi(X) \ge \Phi(Y)$ *under the additional assumption that* X dominates Y in both tails (same proof as for Lemma 1). Below we deduce the same result without this extra assumption. To make the argument simpler, assume X and Y are unbounded both from above and from below; otherwise, we can add to them an independent Gaussian random variable without changing either the assumption or the conclusion. In doing so, we can further assume X and Y admit probability density functions.

We first construct a heavy right-tailed random variable as follows:

Lemma 9. *For any $Y \in L_M$ that is unbounded from above and admits densities, there exists $Z \in L_M$ such that $Z \ge 0$ and $\frac{\mathbb{P}[Z>x]}{\mathbb{P}[Y>x]} \to \infty$ as $x \to \infty$.*

Proof. For this result, it is without loss assume $Y \ge 0$ because we can replace Y by $|Y|$ and only strengthen the conclusion. Let $g(x)$ be the probability density function of Y. We consider a random variable Z whose p.d.f. is given by $cxg(x)$ for all $x \ge 0$, where $c > 0$ is a normalizing constant to ensure $\int_{x\ge 0} cxg(x)\,\mathrm{d}x = 1$. Since the likelihood ratio between $Z = x$ and $Y = x$ is cx, it is easy to see that the ratio of tail probabilities also diverges. Thus it only remains to check $Z \in L_M$. This is because

$$\mathbb{E}\left[\mathrm{e}^{aZ}\right] = c\int_{x\ge 0} xg(x)\mathrm{e}^{ax}\,\mathrm{d}x,$$

which is simply c times the derivative of $\mathbb{E}\left[\mathrm{e}^{aY}\right]$ with respect to a. It is well-known that the moment generating function is smooth whenever it is finite. So this derivative is finite, and $Z \in L_M$. □

In the same way, we can construct heavy left-tailed distributions:

Lemma 10. *For any $X \in L_M$ that is unbounded from below and admits densities, there exists $W \in L_M$, such that $W \le 0$ and $\frac{\mathbb{P}[W\le x]}{\mathbb{P}[X\le x]} \to \infty$ as $x \to -\infty$.*

With these technical lemmata, we now construct "perturbed" versions of any two random variables X and Y to achieve dominance in both tails. For any random variable $Z \in L_M$ and every $\varepsilon > 0$, let Z_ε be the random variable that equals Z with probability ε, and 0 with probability $1 - \varepsilon$. Note that Z_ε also belongs to L_M.

Lemma 11. *Given any two random variables $X, Y \in L_M$ that are unbounded on both sides and admit densities. Let $Z \geq 0$ and $W \leq 0$ be constructed from the above two lemmata. Then for every $\varepsilon > 0$, $X + Z_\varepsilon$ dominates $Y + W_\varepsilon$ in both tails.*

Proof. For the right tail, we need $\mathbb{P}[X + Z_\varepsilon > x] > \mathbb{P}[Y + W_\varepsilon > x]$ for all $x \geq N$. Note that $W_\varepsilon \leq 0$, so $\mathbb{P}[Y + W_\varepsilon > x] \leq \mathbb{P}[Y > x]$. On other hand,

$$\mathbb{P}[X + Z_\varepsilon > x] \geq \mathbb{P}[X \geq 0] \cdot \mathbb{P}[Z_\varepsilon > x] = \mathbb{P}[X \geq 0] \cdot \varepsilon \cdot \mathbb{P}[Z > x].$$

Since by assumption X is unbounded from above, the term $\mathbb{P}[X \geq 0] \cdot \varepsilon$ is a strictly positive constant that does not depend on x. Thus for sufficiently large x, we have

$$\mathbb{P}[X \geq 0] \cdot \varepsilon \cdot \mathbb{P}[Z > x] > \mathbb{P}[Y > x]$$

by the construction of Z. This gives dominance in the right tail. The left tail is similar. □

D.2.3 Step 3: Monotonicity w.r.t. K_a

The next result generalizes the key Lemma 1 to our current setting:

Lemma 12. *Let $\Phi \colon L_M \to \mathbb{R}$ be a monotone additive statistic. If $K_a(X) \geq K_a(Y)$ for all $a \in \mathbb{R}$ then $\Phi(X) \geq \Phi(Y)$.*

Proof. As discussed, we can without loss assume X, Y are unbounded on both sides, and admit densities. Let Z and W be constructed as above, then for each $\varepsilon > 0$, $X + Z_\varepsilon$ dominates $Y + W_\varepsilon$ in both tails, and $K_a(X + Z_\varepsilon) > K_a(X) \geq K_a(Y) > K_a(Y + W_\varepsilon)$ for every $a \in \mathbb{R}$, where the inequalities are strict as Z, W are not identically zero.

Thus the pair $X + Z_\varepsilon$ and $Y + W_\varepsilon$ satisfy the assumptions in Lemma 8. We can then find an independent random variable $V \in L_M$ (depending on ε), such that

$$X + Z_\varepsilon + V \geq_1 Y + W_\varepsilon + V.$$

Monotonicity and additivity of Φ then imply $\Phi(X) + \Phi(Z_\varepsilon) \geq \Phi(Y) + \Phi(W_\varepsilon)$, after canceling out $\Phi(V)$. The desired result $\Phi(X) \geq \Phi(Y)$ follows from the lemma below, which shows that our perturbations only slightly affect the statistic value. □

Lemma 13. *For any $Z \in L_M$ with $Z \geq 0$, it holds that $\Phi(Z_\varepsilon) \to 0$ as $\varepsilon \to 0$. Similarly $\Phi(W_\varepsilon) \to 0$ for any $W \in L_M$ with $W \leq 0$.*

Proof. We focus on the case for Z_ε. Suppose for contradiction that $\Phi(Z_\varepsilon)$ does not converge to zero. Note that as ε decreases, Z_ε decreases in first-order stochastic dominance. So $\Phi(Z_\varepsilon) \geq 0$ also decreases, and non-convergence must imply there exists some $\delta > 0$ such that $\Phi(Z_\varepsilon) > \delta$ for every $\varepsilon > 0$. Let μ_ε be image measure of Z_ε. We now choose a sequence ε_n that decreases to zero very fast, and consider the measures

$$\nu_n = \mu_{\varepsilon_n}^{*n},$$

which is the n-th convolution power of μ_{ε_n}. Thus the sum of n i.i.d. copies of Z_{ε_n} is a random variable whose image measure is ν_n. We denote this sum by U_n.

For each n we choose ε_n sufficiently small to satisfy two properties: (i) $\varepsilon_n \leq \frac{1}{n^2}$, and (ii) it holds that

$$\mathbb{E}\left[\mathrm{e}^{nU_n} - 1\right] \leq 2^{-n}.$$

This latter inequality can be achieved because $\mathbb{E}\left[\mathrm{e}^{nU_n}\right] = \left(\mathbb{E}\left[\mathrm{e}^{nZ_{\varepsilon_n}}\right]\right)^n$, and as $\varepsilon_n \to 0$ we also have $\mathbb{E}\left[\mathrm{e}^{nZ_{\varepsilon_n}}\right] = 1 - \varepsilon_n + \varepsilon_n \mathbb{E}\left[\mathrm{e}^{nZ}\right] \to 1$ since $Z \in L_M$.

For these choices of ε_n and corresponding U_n, let $H_n(x)$ denote the c.d.f. of U_n, and define $H(x) = \inf_n H_n(x)$ for each $x \in \mathbb{R}$. Since $H_n(x) = 0$ for $x < 0$, the same is true for $H(x)$. Also note that each $H_n(x)$ is a non-decreasing and right-continuous function in x, and so is $H(x)$.

We claim that $\lim_{x\to\infty} H(x) = 1$. Indeed, recall that U_n is the n-fold sum of Z_{ε_n}, which has mass $1-\varepsilon_n$ at zero. So U_n has mass at least $(1-\varepsilon_n)^n \geq (1-\frac{1}{n^2})^n \geq 1-\frac{1}{n}$ at zero. In other words, $H_n(0) \geq 1 - \frac{1}{n}$. By considering the finitely many c.d.f.s $H_1(x), H_2(x), \ldots, H_{n-1}(x)$, we can find N such that $H_i(x) \geq 1 - \frac{1}{n}$ for every $i < n$ and $x \geq N$. Together with $H_i(x) \geq H_i(0) \geq 1 - \frac{1}{i} \geq 1 - \frac{1}{n}$ for $i \geq n$, we conclude that $H_i(x) \geq 1 - \frac{1}{n}$ whenever $x \geq N$, and so $H(x) \geq 1 - \frac{1}{n}$. Since n is arbitrary, the claim follows. The fact that $H_n(x) \geq 1 - \frac{1}{n}$ also shows that in the definition $H(x) = \inf_n H_n(x)$, the "inf" is actually achieved as the minimum.

These properties of $H(x)$ imply that it is the c.d.f. of some non-negative random variable U. We next show $U \in L_M$, i.e., $\mathbb{E}\left[\mathrm{e}^{aU}\right] < \infty$ for every $a \in \mathbb{R}$. Since $U \geq 0$, we only need to consider $a \geq 0$. To do this, we take advantage of the following identity based on integration by parts:

$$\mathbb{E}\left[\mathrm{e}^{aU_n} - 1\right] = -\int_{x\geq 0} (\mathrm{e}^{ax} - 1)\, \mathrm{d}(1 - H_n(x)) = a \int_{x\geq 0} \mathrm{e}^{ax}(1 - H_n(x))\, \mathrm{d}x.$$

Now recall that we chose U_n so that $\mathbb{E}\left[\mathrm{e}^{nU_n} - 1\right] \leq 2^{-n}$. So $\mathbb{E}\left[\mathrm{e}^{aU_n} - 1\right] \leq 2^{-n}$ for every positive integer $n \geq a$. It follows that the sum $\sum_{n=1}^{\infty} \mathbb{E}\left[\mathrm{e}^{aU_n} - 1\right]$ is finite for every $a \geq 0$. Using the above identity, we deduce that

$$a \int_{x\geq 0} \mathrm{e}^{ax} \sum_{n=1}^{\infty} (1 - H_n(x))\, \mathrm{d}x < \infty,$$

where we have switched the order of summation and integration by the Monotone Convergence Theorem. Since $H(x) = \min_n H_n(x)$, it holds that $1 - H(x) \le \sum_{n=1}^{\infty}(1 - H_n(x))$ for every x. And thus

$$\mathbb{E}\left[\mathrm{e}^{aU} - 1\right] = a \int_{x \ge 0} \mathrm{e}^{ax}(1 - H(x))\,\mathrm{d}x < \infty$$

also holds. This proves $U \in L_M$.

We are finally in a position to deduce a contradiction. Since by construction the c.d.f. of U is no larger than the c.d.f. of each U_n, we have $U \ge_1 U_n$ and $\Phi(U) \ge \Phi(U_n)$ by monotonicity of Φ. But $\Phi(U_n) = n\Phi(Z_{\varepsilon_n}) > n\delta$ by additivity, so this leads to $\Phi(U)$ being infinite. This contradiction proves the desired result. □

D.2.4 Step 4: Functional Analysis

To complete the proof of the case of L_M in Theorem 2, we also need to modify the functional analysis step in our earlier proof of Theorem 1. One difficulty is that for an unbounded random variable X, $K_a(X)$ takes the value ∞ as $a \to \infty$. Thus we can no longer think of $K_X(a) = K_a(X)$ as a real-valued continuous function on $\overline{\mathbb{R}}$.

We remedy this as follows. Note first that if Φ is a monotone additive statistic defined on L_M, then it is also monotone and additive when restricted to the smaller domain of bounded random variables. Thus Theorem 1 gives a probability measure μ on $\mathbb{R} \cup \{\pm\infty\}$ such that

$$\Phi(X) = \int_{\overline{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu(a)$$

for all $X \in L^{\infty}$. In what follows, μ is fixed. We just need to show that this representation also holds for $X \in L_M$.

As a first step, we show μ does not put any mass on $\pm\infty$. Indeed, if $\mu(\{\infty\}) = \varepsilon > 0$, then for any bounded random variable $X \ge 0$, the above integral gives $\Phi(X) \ge \varepsilon \cdot \max[X]$. Take any $Y \in L_M$ such that $Y \ge 0$ and Y is unbounded from above. Then monotonicity of Φ gives $\Phi(Y) \ge \Phi(\min\{Y, n\}) \ge \varepsilon \cdot n$ for each n. This contradicts $\Phi(Y)$ being finite. Similarly we can rule out any mass at $-\infty$.

The next lemma gives a way to extend the representation to certain unbounded random variables.

Lemma 14. *Suppose $Z \in L_M$ is bounded from below by 1 and unbounded from above, while $Y \in L_M$ is bounded from below and satisfies $\lim_{a\to\infty} \frac{K_a(Y)}{K_a(Z)} = 0$, then*

$$\Phi(Y) = \int_{(-\infty,\infty)} K_a(Y)\,\mathrm{d}\mu(a).$$

Proof. Given the assumptions, $K_a(Z) \ge 1$ for all $a \in \mathbb{R}$, with $\lim_{a\to\infty} K_a(Z) = \infty$. Let L_M^Z be the collection of random variables $X \in L_M$ such that X is bounded from below, and $\lim_{a\to\infty} \frac{K_a(X)}{K_a(Z)}$ exists and is finite. L_M^Z includes all bounded X (in which case $\lim_{a\to\infty} \frac{K_a(X)}{K_a(Z)} = 0$), as well as Y and Z itself. L_M^Z is also closed under adding independent random variables.

Now, for each $X \in L_M^Z$, we can define

$$K_{X|Z}(a) = \frac{K_a(X)}{K_a(Z)},$$

which reduces to our previous definition of $K_X(a)$ when Z is the constant 1. This function $K_{X|Z}(a)$ extends by continuity to $a = -\infty$, where its value is $\frac{\min[X]}{\min[Z]}$, as well as to $a = \infty$ by definition of L_M^Z. Thus $K_{X|Z}(\cdot)$ is a continuous function on $\overline{\mathbb{R}}$.

Since Φ induces an additive statistic when restricted to L_M^Z, and $K_{X|Z} + K_{Y|Z} = K_{X+Y|Z}$, we have an additive functional F defined on $\mathcal{L} = \{K_{X|Z} : X \in L_M^Z\}$, given by

$$F(K_{X|Z}) = \frac{\Phi(X)}{\Phi(Z)}.$$

Because $Z \geq 1$ implies $\Phi(Z) \geq 1$, F is well-defined, and $F(1) = 1$. By Lemma 12, F is also monotone in the sense that $K_{X|Z}(a) \geq K_{Y|Z}(a)$ for each $a \in \mathbb{R}$ implies $F(K_{X|Z}) \geq F(K_{Y|Z})$.

Likewise we can show F is 1-Lipschitz. Note that $K_{X|Z}(a) \leq K_{Y|Z}(a) + \frac{m}{n}$ is equivalent to $K_a(X) \leq K_a(Y) + \frac{m}{n} K_a(Z)$ and equivalent to $K_a(X^{*n}) \leq K_a(Y^{*n} + Z^{*m})$. If this holds for all a, then by Lemma 12 we also have $\Phi(X^{*n}) \leq \Phi(Y^{*n} + Z^{*m})$, and thus $\Phi(X) \leq \Phi(Y) + \frac{m}{n}\Phi(Z)$ by additivity. An approximation argument shows that for any *real* number $\varepsilon > 0$, $K_{X|Z}(a) \leq K_{Y|Z}(a) + \varepsilon$ for all a implies $\Phi(X) \leq \Phi(Y) + \varepsilon\Phi(Z)$. Thus the functional F is 1-Lipschitz.

Given these properties, we can exactly follow the proof of Theorem 1 to extend the functional F to be a positive linear functional on the space of all continuous functions over $\overline{\mathbb{R}}$ (the majorization condition is again satisfied by constant functions, as $K_{Z|Z} = 1$). Therefore, by the Riesz Representation Theorem, we obtain a probability measure μ_Z on $\overline{\mathbb{R}}$ such that for all $X \in L_M^Z$,

$$\frac{\Phi(X)}{\Phi(Z)} = \int_{\overline{\mathbb{R}}} \frac{K_a(X)}{K_a(Z)} \,\mathrm{d}\mu_Z(a).$$

In particular, for any X bounded from below such that $\lim_{a\to\infty} \frac{K_a(X)}{K_a(Z)} = 0$, it holds that

$$\Phi(X) = \int_{[-\infty,\infty)} K_a(X) \cdot \frac{\Phi(Z)}{K_a(Z)} \,\mathrm{d}\mu_Z(a),$$

where we are able to exclude ∞ from the range of integration (this is useful below).

If we define the measure $\hat{\mu}_Z$ by $\frac{\mathrm{d}\hat{\mu}_Z}{\mathrm{d}\mu_Z}(a) = \frac{\Phi(Z)}{K_a(Z)} \leq \Phi(Z)$, then since $K_a(X)$ is finite for $a < \infty$, we have

$$\Phi(X) = \int_{[-\infty,\infty)} K_a(X) \,\mathrm{d}\hat{\mu}_Z(a).$$

This in particular holds for all bounded X, so plugging in $X = 1$ gives that $\hat{\mu}_Z$ is a probability measure. But now we have two probability measures μ and $\hat{\mu}_Z$ on $\overline{\mathbb{R}}$ that lead to the same integral representation for bounded random variables, so Lemma 5 implies that $\hat{\mu}_Z$ coincides with μ and is supported on the standard real line. Plugging in $X = Y$ in the above display then yields the desired result. □

The next lemma further extends the representation:

Lemma 15. *For every $X \in L_M$ that is bounded from below,*

$$\Phi(X) = \int_{(-\infty,\infty)} K_a(X)\,\mathrm{d}\mu(a).$$

Proof. It suffices to consider X that is unbounded from above. Moreover, without loss we can assume $X \geq 0$,, since we can add any constant to X. Given the previous lemma, we just need to construct $Z \geq 1$ such that $\lim_{a\to\infty} \frac{K_a(X)}{K_a(Z)} = 0$. Note that $\mathbb{E}\left[\mathrm{e}^{aX}\right]$ strictly increases in a for $a \geq 0$. This means we can uniquely define a sequence $a_1 < a_2 < \cdots$ by the equation $\mathbb{E}\left[\mathrm{e}^{a_n X}\right] = \mathrm{e}^n$. This sequence diverges as $n \to \infty$. We then choose any increasing sequence b_n such that $b_n > n$ and $a_n b_n > 2n^2$.

Consider the random variable Z that is equal to b_n with probability $\mathrm{e}^{-\frac{a_n b_n}{2}}$ for each n, and equal to 1 with remaining probability. To see that $Z \in L_M$, we have

$$\mathbb{E}\left[\mathrm{e}^{aZ}\right] \leq \mathrm{e}^a + \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{a_n b_n}{2}} \cdot \mathrm{e}^{a b_n} = \mathrm{e}^a + \sum_{n=1}^{\infty} \mathrm{e}^{(a-\frac{a_n}{2})\cdot b_n}.$$

For any fixed a, $\frac{a_n}{2}$ is eventually greater than $a+1$. This, together with the fact that $b_n > n$, implies the above sum converges.

Moreover, for any $a \in [a_n, a_{n+1})$, we have

$$\mathbb{E}\left[\mathrm{e}^{aZ}\right] \geq \mathbb{E}\left[\mathrm{e}^{a_n Z}\right] \geq \mathbb{P}[Z = b_n] \cdot \mathrm{e}^{a_n b_n} \geq \mathrm{e}^{\frac{a_n b_n}{2}} > \mathrm{e}^{n^2},$$

whereas $\mathbb{E}\left[\mathrm{e}^{aX}\right] \leq \mathbb{E}\left[\mathrm{e}^{a_{n+1} X}\right] \leq \mathrm{e}^{n+1}$. Thus

$$\frac{K_a(X)}{K_a(Z)} = \frac{\log \mathbb{E}\left[\mathrm{e}^{aX}\right]}{\log \mathbb{E}\left[\mathrm{e}^{aZ}\right]} \leq \frac{n+1}{n^2},$$

which converges to zero as a (and thus n) approaches infinity. □

D.2.5 Step 5: Wrapping Up

By a symmetric argument, the representation $\Phi(X) = \int_{(-\infty,\infty)} K_a(X)\,\mathrm{d}\mu(a)$ also holds for all X bounded from above. In the remainder of the proof, we will use an approximation argument to generalize this to all $X \in L_M$. We first show a technical lemma:

Lemma 16. *The measure μ is supported on a compact interval of $\mathbb{R}$.*

Proof. Suppose not, and without loss assume the support of μ is unbounded from above. We will construct a non-negative $Y \in L_M$ such that $\Phi(Y) = \infty$ according to the integral representation. Indeed, by assumption we can find a sequence $2 < a_1 < a_2 < \cdots$ such that $a_n \to \infty$ and $\mu([a_n, \infty)) \geq \frac{1}{n}$ for all large n. Let Y be the random variable that equals n with probability $\mathrm{e}^{-\frac{a_n \cdot n}{2}}$ for each n, and equals 0 with remaining probability. Then similar to the above, we can show $Y \in L_M$. Moreover, $\mathbb{E}\left[\mathrm{e}^{a_n Y}\right] \geq \mathrm{e}^{\frac{a_n \cdot n}{2}}$, implying that $K_{a_n}(Y) \geq \frac{n}{2}$. Since $K_a(Y)$ is increasing in a, we deduce that for each n,

$$\int_{[a_n,\infty)} K_a(Y)\,\mathrm{d}\mu(a) \geq K_{a_n}(Y) \cdot \mu([a_n, \infty)) \geq \frac{n}{2} \cdot \frac{1}{n} = \frac{1}{2}.$$

The fact that this holds for $a_n \to \infty$ contradicts the assumption that $\Phi(Y) = \int_{(-\infty,\infty)} K_a(Y)\,\mathrm{d}\mu(a)$ is finite. □

Thus we can take N sufficiently large so that μ is supported on $[-N, N]$. To finish the proof, consider any $X \in L_M$ that may be unbounded on both sides. For each positive integer n, let $X_n = \min\{X, n\}$ denote the truncation of X at n. Since $X \geq_1 X_n$, we have

$$\Phi(X) \geq \Phi(X_n) = \int_{[-N,N]} K_a(X_n)\,\mathrm{d}\mu(a)$$

Observe that for each $a \in [-N, N]$, $K_a(X_n)$ converges to $K_a(X)$ as $n \to \infty$. Moreover, the fact that $K_a(X_n)$ increases both in n and in a implies that for all a and all n,

$$|K_a(X_n)| \leq \max\{|K_a(X_1)|, |K_a(X)|\} \leq \max\{|K_{-N}(X_1)|, |K_N(X_1)|, |K_{-N}(X)|, |K_N(X)|\}.$$

As $K_a(X_n)$ is uniformly bounded, we can apply the Dominated Convergence Theorem to deduce

$$\Phi(X) \geq \lim_{n\to\infty} \int_{[-N,N]} K_a(X_n)\,\mathrm{d}\mu(a) = \int_{[-N,N]} K_a(X)\,\mathrm{d}\mu(a).$$

On the other hand, if we truncate the left tail and consider $X^{-n} = \max\{X, -n\}$, then a symmetric argument shows

$$\Phi(X) \leq \lim_{n\to\infty} \int_{[-N,N]} K_a(X^{-n})\,\mathrm{d}\mu(a) = \int_{[-N,N]} K_a(X)\,\mathrm{d}\mu(a).$$

Therefore for all $X \in L_M$ it holds that

$$\Phi(X) = \int_{[-N,N]} K_a(X)\,\mathrm{d}\mu(a).$$

This completes the entire proof for the case of L_M in Theorem 2.

E Proof of Theorem 5

Throughout this proof we use the fact that for mean-zero X and Y, X second-order dominates Y if and only if Y is a mean-preserving spread of X.

We first verify that the representation $V(X) = v(\mathbb{E}[X]) + \int_{[-\infty,0)} K_a(X)\,\mathrm{d}\mu(a)$ satisfies the axioms. Monotonicity follows from the fact that $\mathbb{E}[X]$ and each $K_a(X)$ is monotone with respect to first-order stochastic dominance, and v is a non-decreasing function. Continuity follows from the fact that for fixed X, $V(X+\varepsilon)$ is continuous in ε because v is continuous. Risk invariance holds because for any X and mean-zero Z,

$$
\begin{aligned}
V(X+Z) &= v(\mathbb{E}[X+Z]) + \int_{[-\infty,0)} K_a(X+Z)\,\mathrm{d}\mu(a) \\
&= v(\mathbb{E}[X]) + \int_{[-\infty,0)} K_a(X)\,\mathrm{d}\mu(a) + \int_{[-\infty,0)} K_a(Z)\,\mathrm{d}\mu(a) \\
&= V(X) + \int_{[-\infty,0)} K_a(Z).
\end{aligned}
$$

Since a similar identity holds for $V(Y+Z)$, we obtain $V(X+Z) \geq V(Y+Z)$ if and only if $V(X) \geq V(Y)$, as required by risk invariance. This identity also implies the Archimedean axiom, because as $c \to \infty$ the CARA certainty equivalent $K_a(W_c)$ for the gamble W_c converges to $-\infty$. Thus for large c we have $V(W_c) < V(x)$ and $V(x+W_c) < V(0)$. Finally, second-order monotonicity holds because if Y is second-order dominated by X, then $\mathbb{E}[Y] \leq \mathbb{E}[X]$ and $K_a(Y) \leq K_a(X)$ for every $a < 0$. So all the axioms are satisfied by the representation.

We now impose the axioms and derive the representation. We first claim that $W_c \prec 0$ for every $c > 0$. Indeed, by Archimedeanity applied to $x = 0$, there exists $m > 0$ such that $W_m \prec 0$. Now if $W_c \succeq 0$ for some c, then by repeated application of risk invariance we obtain $W_c^{*n} \succeq 0$ for every positive integer n (recall W_c^{*n} is the sum of n i.i.d. copies of W_c). But when n is large W_c^{*n} is a mean-preserving spread of W_m.[16] So we deduce from the second-order monotonicity axiom that $W_m \succeq W_c^{*n} \succeq 0$, leading to a contradiction. Thus $W_c \prec 0$, which further implies $W_{c'} \prec W_c$ for every $c' > c$ because $W_{c'} \preceq W_c + W_{c'-c} \prec W_c$ by second-order monotonicity and risk invariance.

Now consider X, Y being mean-zero risks. We show that if $K_a(X) \geq K_a(Y)$ for all $a \in [-\infty, 0)$, then $X \succeq Y$ according to the preference. To see this, let ε be any positive number. Then $K_a(X+\varepsilon) > K_a(Y)$ holds with strict inequality for all $a \in [-\infty, 0]$. By Proposition 3, we can find Z (depending on ε) such that $X+\varepsilon+Z \geq_2 Y+Z$ in second-order stochastic dominance. Shifting Z by a constant, we can further assume it is mean-zero.

[16] For n large, W_c^{*n} will have probability at least $1/4$ of being larger than $2m$, and probability at least $1/4$ of being smaller than $-2m$. Thus W_c^{*n} will be a mean-preserving spread of the half-half mixture between 0 and W_{2m}, which is in turn a mean-preserving spread of W_m.

Thus $X + \varepsilon + Z \geq_2 Y + Z$ for a mean-zero Z, which implies by second-order monotonicity that $X + \varepsilon + Z \succeq Y + Z$. Thus $X + \varepsilon \succeq Y$ by the risk invariance axiom. Since this holds for every $\varepsilon > 0$, continuity implies that $X \succeq Y$ also holds.

Using this monotonicity property with respect to the risk-averse CARA certainty equivalents, we can characterize the preference $\succeq$ when restricted to mean-zero risks.

Lemma 17. *Under the axioms in Theorem 5, there exists a probability measure ν supported on $[-\infty, 0]$ such that for all mean-zero X, Y,*

$$X \succeq Y \quad \text{if and only if} \quad \int_{[-\infty,0]} \frac{K_a(X)}{K_a(W_1)} \,\mathrm{d}\nu(a) \leq \int_{[-\infty,0]} \frac{K_a(Y)}{K_a(W_1)} \,\mathrm{d}\nu(a).$$

Above, W_1 is the random variable that equals ± 1 with equal probabilities, and the ratio $\frac{K_0(X)}{K_0(W_1)}$ is defined by continuity[17] *to be* $\lim_{a \to 0-} \frac{K_a(X)}{K_a(W_1)} = \mathrm{Var}[X]$.

Proof. For any mean-zero X and any positive integer n, define

$$B_n(X) = \min\{m \in \mathbb{N} : W_1^{*m} \prec X^{*n}\}.$$

$B_n(X)$ is positive because $0 \succeq X^{*n}$ by second-order monotonicity. $B_n(X)$ is also finite because if X^{*n} is supported on $[-c, c]$, then W_c is a mean-preserving spread of X^{*n}. When m is large W_1^{*m} is a mean-preserving spread of W_{2c}, so that $W_1^{*m} \preceq W_{2c} \prec W_c \prec X^{*n}$. Moreover, for fixed X, $\{B_n(X)\}_n$ is a non-decreasing and sub-additive sequence because $W_1^{*m_1} \prec X^{*n_1}$ and $W_1^{*m_2} \prec X^{*n_2}$ imply $W_1^{*m_1+m_2} \prec X^{*n_1+n_2}$ by two applications of the risk invariance axiom.

Thus, for this sub-additive sequence we can define the finite limit

$$B_\infty(X) = \lim_{n\to\infty} \frac{1}{n} B_n(X).$$

Now observe that if $B_n(X) = m$ and $B_n(Y) = m'$ for independent mean-zero X, Y, then $W_1^{*m+m'} \prec X^{*n} + Y^{*n} = (X+Y)^{*n}$ by two applications of risk invariance, so $B_n(X+Y) \leq m + m'$. But by definition we also have $W_1^{*m-1} \succeq X^{*n}$ and $W_1^{*m'-1} \succeq Y^{*n}$. So $W_1^{*m+m'-2} \succeq (X+Y)^{*n}$, which implies $B_n(X+Y) \geq m + m' - 1$. Hence for every n and independent X, Y we have

$$B_n(X) + B_n(Y) - 1 \leq B_n(X+Y) \leq B_n(X) + B_n(Y).$$

The above inequality implies, by taking the limit, that $B_\infty(X+Y) = B_\infty(X) + B_\infty(Y)$. Moreover, whenever $X \succeq Y$ we also have $X^{*n} \succeq Y^{*n}$ by repeated applications of risk invariance. So $W_1^{*m} \prec Y^{*n}$ implies $W_1^{*m} \prec X^{*n}$, which gives $B_n(X) \leq B_n(Y)$ for every n

[17] This follows from L'Hôpital's rule and $K'_Z(0) = \frac{1}{2}\mathrm{Var}[Z]$.

and $B_\infty(X) \le B_\infty(Y)$. Recall that we previously showed $X \succeq Y$ if $K_a(X) \ge K_a(Y)$ for every $a \le 0$. Thus $B_\infty(X) \le B_\infty(Y)$ whenever $K_a(X) \ge K_a(Y)$ for every $a \in [-\infty, 0)$.

We now have a statistic $B_\infty(X)$ defined on the class of mean-zero random variables, which is additive for independent sums, is *anti-tone* with respect to $\{K_a(X)\}_{a\in[-\infty,0)}$ and satisfies $B_\infty(W_1) = 1$ (because $B_n(W_1) = n+1$). We will show there exists a measure ν supported on $[-\infty, 0]$ such that

$$B_\infty(X) = \int_{[-\infty,0]} \frac{K_a(X)}{K_a(W_1)}\, \mathrm{d}\nu(a).$$

Indeed, for each mean-zero X we can define the continuous function $\tilde{K}_X(a) = \frac{K_a(X)}{K_a(W_1)}$ for $a \in [-\infty, 0]$, where the ratio at $a = 0$ is defined by continuity to be $\mathrm{Var}[X]$. Then B_∞ can be viewed as a functional on the class of functions $\{\tilde{K}_X :\ X \in L^\infty, \mathbb{E}[X] = 0\}$. This functional is additive because $\tilde{K}_{X+Y} = \tilde{K}_X + \tilde{K}_Y$ and $B_\infty(X+Y) = B_\infty(X) + B_\infty(Y)$. It is also monotone because $B_\infty(X)$ is anti-tone with respect to $\{K_a(X)\}_{a\in[-\infty,0)}$ and thus monotone with respect to $\{\frac{K_a(X)}{K_a(W_1)}\}_{a\in[-\infty,0)}$ because $K_a(W_1) < 0$. Hence B_∞ is a monotone additive functional on this class of functions $\{\tilde{K}_X\}$, which is a subset of $\mathcal{C}([-\infty, 0])$. The same extension argument as in the proof of Theorem 1 shows that B_∞ can be extended to a monotone additive functional on all of $\mathcal{C}([-\infty, 0])$. The Riesz Representation Theorem then yields the desired integral representation above.

Finally, we need to check that $X \succeq Y$ if and only if $B_\infty(X) \le B_\infty(Y)$. That $X \succeq Y$ implies $B_\infty(X) \le B_\infty(Y)$ has been proved before, so it remains to show $X \succ Y$ implies $B_\infty(X) < B_\infty(Y)$. By continuity, $X \succ Y$ implies $X - \varepsilon \succ Y$ for some $\varepsilon > 0$. Since by monotonicity $X + W_\varepsilon \succeq X - \varepsilon$, we deduce $X + W_\varepsilon \succ Y$. Thus $B_\infty(X + W_\varepsilon) \le B_\infty(Y)$ which gives $B_\infty(X) + B_\infty(W_\varepsilon) \le B_\infty(Y)$. From the integral representation for B_∞ it is clear that $B_\infty(W_\varepsilon) > 0$, and so $B_\infty(X) < B_\infty(Y)$. This completes the proof of the lemma. □

Let $\alpha = \nu(\{0\}) \in [0, 1]$, then the above lemma says that when restricted to mean-zero risks, the preference is represented by $\Phi(X)$, where

$$\Phi(X) = -\alpha\, \mathrm{Var}[X] + \int_{[-\infty,0)} \frac{K_a(X)}{-K_a(W_1)}\, \mathrm{d}\nu(a) \tag{15}$$

for a measure ν supported on $[-\infty, 0)$, with total measure $1 - \alpha$ (the proof shows that $\Phi(X)$ is finite for every mean-zero X). The following lemma sharpens the form of Φ:

Lemma 18. *There exists a constant $C > 0$ and a probability measure μ supported on $[-\infty, 0)$, such that*

$$\Phi(X) = C \int_{[-\infty,0)} K_a(X)\, \mathrm{d}\mu(a)$$

for all mean-zero X.

Proof. Consider any constant $x \leq 0$. Then $x \preceq 0$ by monotonicity, and $x \succeq W_c$ for some c by Archimedeanity. Thus by continuity, $x \sim W_{c(x)}$ for a unique $c(x) \geq 0$.[18] If X is a random variable with $x = \mathbb{E}[X] \leq 0$, then by writing $X = x + (X - x)$ and using the risk invariance axiom, we obtain

$$X \sim W_{c(x)} + (X - x)$$

Hence for X, Y having expectations $x, y \leq 0$, we have $X \succeq Y$ if and only if $W_{c(x)} + (X - x) \succeq W_{c(y)} + (Y - y)$, which occurs precisely when

$$\Phi(W_{c(x)}) + \Phi(X - x) \geq \Phi(W_{c(y)}) + \Phi(Y - y). \tag{16}$$

In particular, this inequality has to hold when $X \geq_1 Y$, by monotonicity.

In what follows we fix $Y = -1$, and consider X to be a random variable with $\min[X] = -1$ and $\mathbb{E}[X] = 0$. Then $x = 0, y = -1, X \geq_1 Y$ and we can study the implications of (16) for the form of Φ. In particular, since $c(x), c(y), Y - y$ are already fixed, from (16) we know that there exists a constant M such that $\Phi(X) \geq -M$ for all X with $\min[X] = -1$ and $\mathbb{E}[X] = 0$.

Now let X_n be equal to -1 with probability $\frac{n}{n+1}$ and equal to n with probability $\frac{1}{n+1}$. Then $\mathrm{Var}[X_n] = n$, and from (15) we see that $\Phi(X_n) \leq -\alpha n$ (because $K_a(X_n)$ and $K_a(W_1)$ are negative for all $a < 0$). Hence $-\alpha n \geq -M$ for every n, forcing α to be zero.

We can then write

$$M \geq -\Phi(X) = \int_{[-\infty,0)} \frac{K_a(X)}{K_a(W_1)} \,\mathrm{d}\nu(a)$$

for a probability measure ν supported on $[-\infty, 0)$. Consider the same X_n as defined above, for which $K_a(X_n) = \frac{1}{a}\log(\frac{n}{n+1}\mathrm{e}^{-a} + \frac{1}{n+1}\mathrm{e}^{an})$. Thus for any $a < 0$, $\lim_{n\to\infty} K_a(X_n) = -1$, and so the positive integrand $\frac{K_a(X)}{K_a(W_1)}$ converges pointwise to the function $\frac{-1}{K_a(W_1)}$. Thus by Fatou's Lemma,

$$0 \leq \int_{[-\infty,0)} \frac{1}{-K_a(W_1)} \,\mathrm{d}\nu(a) \leq \liminf_{n\to\infty}(-\Phi(X_n)) \leq M.$$

Letting $C = \int_{[-\infty,0)} \frac{1}{-K_a(W_1)} \,\mathrm{d}\nu(a) \in (0, \infty)$, we can then write

$$\Phi(X) = \int_{[-\infty,0)} \frac{K_a(X)}{-K_a(W_1)} \,\mathrm{d}\nu(a) = C \cdot \int_{[-\infty,0)} K_a(X) \,\mathrm{d}\mu(a)$$

for the probability measure μ defined by $\frac{\mathrm{d}\mu(a)}{\mathrm{d}\nu(a)} = \frac{1}{-CK_a(W_1)}$. This proves the lemma. ∎

[18] Existence is guaranteed because there always exists $c(x)$ such that $x \succ W_c$ for $c > c(x)$ and $x \prec W_c$ for $c < c(x)$. If $x \succ W_{c(x)}$ then continuity implies $x \succ W_{c(x)} + \varepsilon$ for some $\varepsilon > 0$. But since $W_{c(x)} + \varepsilon$ first-order stochastically dominates $W_{c(x)-\varepsilon}$, monotonicity implies $x \succ W_{c(x)-\varepsilon}$, contradicting the definition of $c(x)$. Likewise it is impossible that $x \prec W_{c(x)}$. Hence $x \sim W_{c(x)}$.

This lemma tells us that when restricted to mean-zero risks, we can represent the preference by $\Psi(X) = \int_{[-\infty,0)} K_a(X)\,\mathrm{d}\mu(a)$ (since C is a positive constant), without a term that depends on the variance as in (15). This already gives the desired representation for mean-zero risks. It is also important for the subsequent analysis that the monotone additive statistic $\Psi(X)$ is not just defined for mean-zero random variables, but naturally extended to all bounded ones. This will allow us to extend the representation to all of L^∞.

To do this extension, let $x \in \mathbb{R}$ be any constant. Then by Archimedeanity, there exists $c_1(x) > 0$ such that $x + W_{c_1(x)} \prec 0$. By Archimedeanity again, there exists $c > 0$ such that $x + W_{c_1(x)} \succ W_c$. Thus by continuity we obtain another constant $c_2(x) > 0$ such that

$$x + W_{c_1(x)} \sim W_{c_2(x)}.$$

Now for each random variable X with mean x, define

$$V(X) = \Psi(W_{c_2(x)}) - \Psi(W_{c_1(x)}) + \Psi(X - x).$$

We will show that $X \succeq Y$ if and only if $V(X) \geq V(Y)$. Indeed, by writing $X + W_{c_1(x)} = x + W_{c_1(x)} + X - x$ we know by risk invariance that $X + W_{c_1(x)} \sim W_{c_2(x)} + X - x$, and thus

$$X + W_{c_1(x)} + W_{c_1(y)} \sim W_{c_1(y)} + W_{c_2(x)} + X - x.$$

Likewise

$$Y + W_{c_1(x)} + W_{c_1(y)} \sim W_{c_1(x)} + W_{c_2(y)} + Y - y.$$

Thus $X \geq Y$ if and only if $X + W_{c_1(x)} + W_{c_1(y)} \succeq Y + W_{c_1(x)} + W_{c_1(y)}$, if and only if $W_{c_1(y)} + W_{c_2(x)} + X - x \succeq W_{c_1(x)} + W_{c_2(y)} + Y - y$. Because the last two gambles are mean-zero, their ranking depends on their Ψ values. It follows that $X \geq Y$ if and only if $\Psi(W_{c_1(y)}) + \Psi(W_{c_2(x)}) + \Psi(X - x) \geq \Psi(W_{c_1(x)}) + \Psi(W_{c_2(y)}) + \Psi(Y - y)$, which is equivalent to $V(X) \geq V(Y)$.

To complete the proof, we note that $c_1(x), c_2(x)$ depend on X only through its expectation x. So $V(X)$ can be alternatively written as

$$v(\mathbb{E}[X]) + \Psi(X) = v(\mathbb{E}[X]) + \int_{[-\infty,0)} K_a(X)\,\mathrm{d}\mu(a),$$

where $v(x) = \Psi(W_{c_2(x)}) - \Psi(W_{c_1(x)}) - \Psi(x)$. This is our desired representation, and it only remains to show that v is non-decreasing and continuous. To prove monotonicity, suppose for the sake of contradiction that $v(x) < v(y)$ for some $x > y$. Let $Y = y$ and let X be such that $\min[X] = y$ and $\mathbb{E}[X] = x$. As the probability that $X = y$ gets closer and closer to one, we have $K_a(X) \to y$ for every $a < 0$ (similar to X_n above). Since all these $K_a(X)$ are bounded between $\min[X] = y$ and $K_0(X) = x$, by the Dominated Convergence Theorem we know that $\int_{[-\infty,0)} K_a(X)\,\mathrm{d}\mu(a) \to y$. Thus we can choose such an X that

$V(X)$ is very close to $v(x) + y$. This is smaller than $v(y) + y = V(y)$, so $V(X) < V(Y)$, contradicting monotonicity of the preference.

Now that we know $v(x)$ is monotone, it is continuous if and only if it does not have any jump discontinuity. Suppose for the sake of contradiction that $\lim_{x' \to x_+} v(x')$ is strictly larger than $v(x)$ (the other case where $\lim_{x' \to x_-} v(x') < v(x)$ can be similarly handled). Let $A = v(x) + x$ and $B = \lim_{x' \to x_+} v(x') + x$, then $V(x) = A$ and $V(x') > B$ for any $x' > x$. Now fix any $y > x$ and consider the random variable $y + W_c$. As c increases from 0 to ∞, $V(y + W_c)$ decreases continuously from above B to $-\infty$. Hence we can choose c such that $V(y + W_c) = \frac{A+B}{2} \in (A, B)$. If we denote $Y = y + W_c$, then $Y \succ x$, but $Y \prec x + \varepsilon$ for every $\varepsilon > 0$. This contradicts the continuity axiom, and completes the proof of the representation.

F Independence and Betweenness Axioms

In this appendix we compare our representation $\Phi(X) = \int_{\overline{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu(a)$ to CARA expected utility, which corresponds to the special case where μ is a point mass at some $a \in \mathbb{R}$. Note that because we are taking an average of certainty equivalents rather than an average of utility functions, the preference represented by $\Phi(X)$ is *not* of expected utility form unless μ is a point mass (which returns to CARA). The following result formalizes this observation by showing that a preference in our bigger class typically violates the independence axiom.

Given two random variables X and Y, we denote by $X_\lambda Y$ a random variable that is equal to X with probability $\lambda \in [0, 1]$ and equal to Y with probability $1 - \lambda$.

Axiom F.1 (Independence)**.** *For all X, Y, Z and all $\lambda \in (0, 1)$, $X \succeq Y$ if and only if $X_\lambda Z \succeq Y_\lambda Z$.*

Proposition 4. *Suppose a preference $\succeq$ is represented by a monotone additive statistic $\Phi(X) = \int_{\overline{\mathbb{R}}} K_a(X)\,\mathrm{d}\mu(a)$. Then $\succeq$ satisfies the independence axiom if and only if μ is a point mass at some $a \in \mathbb{R}$.*

A technical remark is that our Axiom 4.2 is a weaker continuity assumption than the usual mixture continuity assumption needed to derive an expected utility representation. In fact, the preference represented by $\Phi(X)$ will violate mixture continuity when the measure μ has positive mass at $\pm\infty$ (for example, mixture continuity fails when $\Phi(X) = \max[X]$). Thus Proposition 4 does not immediately follow from the Von Neumann-Morgenstern Theorem in conjunction with the known result that within expected utility, only CARA utility satisfies wealth invariance.

One may ask whether our representation is compatible with relaxations of expected utility that have been proposed in the literature. A well known weakening of the independence axiom is the betweenness axiom studied in Dekel (1986) and Chew (1989). Betweenness relaxes expected utility theory by requiring that the decision maker's preference, when seen over the space of probability distributions, displays indifference curves that are straight lines, whereas the standard independence axiom requires the indifference curves to be in addition parallel to each other. Formally,

Axiom F.2 (Betweenness). *For all* X, Y *and all* $\lambda \in (0,1)$, $X \sim Y$ *if and only if* $X_\lambda Y \sim Y$.

Proposition 5. *Suppose a preference* $\succeq$ *is represented by a monotone additive statistic* $\Phi(X) = \int_{\mathbb{R}} K_a(X)\,\mathrm{d}\mu(a)$. *Then* $\succeq$ *satisfies the betweenness axiom if and only if*

1. $\Phi(X) = K_a(X)$ *for some* $a \in \mathbb{R}$, *or*
2. $\Phi(X) = \beta K_{-a\beta}(X) + (1-\beta)K_{a(1-\beta)}(X)$ *for some* $\beta \in (0,1)$ *and* $a \in (0,\infty)$.

The first form of representation is CARA expected utility. The second form is more interesting because it features a mixture between risk aversion and risk seeking, with weights exactly proportional to the levels of risk aversion/seeking. We believe these preferences have other nice properties, which we hope to explore further in future work.

F.1 Proof of Proposition 4

The "if" direction is straightforward, so we focus on the "only if." Note that the independence axiom implies $X \sim Y$ if and only if $X_\lambda Z \sim Y_\lambda Z$. The special case of $Z = Y$ reduces to the betweenness axiom, so we can apply Proposition 5 (see the next section for its proof) to deduce that the certainty equivalent Φ must take one of the following forms:

(i) $\Phi(X) = K_a(X)$ for some $a \in \mathbb{R}$, or

(ii) $\Phi(X) = \beta K_{-a\beta}(X) + (1-\beta)K_{a(1-\beta)}(X)$ for some $\beta \in (0,1)$ and $a \in (0,\infty)$.

We just need to show that form (ii) violates the independence axiom.

Suppose Φ takes form (ii). We choose X and Y such that $\Phi(X) > \Phi(Y)$ but $K_{-a\beta}(X) < K_{-a\beta}(Y)$. For example, let $Y = 1$, and let X be supported on $\{0, k\}$, with $\mathbb{P}[X = k] = \frac{1}{k}$. Then

$$K_b(X) = \frac{1}{b}\log \mathbb{E}\left[1 - \frac{1}{k} + \frac{\mathrm{e}^{bk}}{k}\right].$$

For k tending to infinity, $K_b(X)$ tends to zero if $b < 0$, and to infinity if $b > 0$. Hence, for k large enough, X and Y will have the desired property.

Now let $Z = n$ where n is a large positive integer. Then

$$K_b(Y_\lambda n) = \frac{1}{b}\log \mathbb{E}\left[\lambda \mathbb{E}\left[\mathrm{e}^{bY}\right] + (1-\lambda)\mathrm{e}^{bn}\right]$$
$$K_b(X_\lambda n) = \frac{1}{b}\log \mathbb{E}\left[\lambda \mathbb{E}\left[\mathrm{e}^{bX}\right] + (1-\lambda)\mathrm{e}^{bn}\right]$$

and so

$$K_b(Y_\lambda n) - K_b(X_\lambda n) = \frac{1}{b}\log\left(\frac{\lambda \mathbb{E}\left[\mathrm{e}^{bY}\right] + (1-\lambda)\mathrm{e}^{bn}}{\lambda \mathbb{E}\left[\mathrm{e}^{bX}\right] + (1-\lambda)\mathrm{e}^{bn}}\right).$$

It easily follows that for fixed $\lambda \in (0,1)$ and $b > 0$,

$$\lim_{n\to\infty} K_b(Y_\lambda n) - K_b(X_\lambda n) = 0,$$

whereas for $b < 0$,

$$\lim_{n\to\infty} K_b(Y_\lambda n) - K_b(X_\lambda n) = K_b(Y) - K_b(X).$$

Thus, as n tends to infinity,

$$\begin{aligned}\lim_n \Phi(Y_\lambda n) - \Phi(X_\lambda n) &= \lim_n \beta\left[K_{-a\beta}(Y_\lambda n) - K_{-a\beta}(X_\lambda n)\right] + (1-\beta)\left[K_{a(1-\beta)}(Y_\lambda n) - K_{a(1-\beta)}(X_\lambda n)\right] \\ &= \beta\left[K_{-a\beta}(Y_\lambda n) - K_{-a\beta}(X_\lambda n)\right] > 0.\end{aligned}$$

Therefore, for n large enough, we have found X and Y such that $\Phi(X) > \Phi(Y)$ but $\Phi(X_\lambda n) < \Phi(Y_\lambda n)$. This implies $X \succ Y$ but $X_\lambda n \prec Y_\lambda n$, which contradicts the independence axiom and completes the proof of Proposition 4.

F.2 Proof of Proposition 5

Since the preference $\succeq$ is represented by Φ, the betweenness axiom is equivalent to the following:

$$\Phi(X) = \Phi(Y) \text{ if and only if } \Phi(X_\lambda Y) = \Phi(Y).$$

In this case, we say that the statistic Φ satisfies betweenness. We need to show that $\Phi(X)$ satisfies betweenness if and only if it is equal to $K_a(X)$ for some $a \in \mathbb{R}$ or equal to $\beta K_{-a\beta}(X) + (1-\beta)K_{a(1-\beta)}(X)$ for some $\beta \in (0,1)$ and $a \in (0,\infty)$.

We first show the "if" direction. Specifically, when $\Phi(X) = K_a(X)$ for some $a \in \mathbb{R}$, then the preference is CARA expected utility, which satisfies independence and thus betweenness. When $\Phi(X) = \beta K_{-a\beta}(X) + (1-\beta)K_{a(1-\beta)}(X)$, we can use the definition of K to rewrite it as

$$\Phi(X) = \frac{1}{a}\left(\log \mathbb{E}[\mathrm{e}^{a(1-\beta)X}] - \log \mathbb{E}[\mathrm{e}^{-a\beta X}]\right).$$

Thus $\Phi(X) = \Phi(Y)$ if and only if $\log \mathbb{E}\left[\mathrm{e}^{a(1-\beta)X}\right] - \log \mathbb{E}\left[\mathrm{e}^{-a\beta X}\right] = \log \mathbb{E}\left[\mathrm{e}^{a(1-\beta)Y}\right] - \log \mathbb{E}\left[\mathrm{e}^{-a\beta Y}\right]$, which in turn is equivalent to

$$\frac{\mathbb{E}\left[\mathrm{e}^{a(1-\beta)X}\right]}{\mathbb{E}\left[\mathrm{e}^{a(1-\beta)Y}\right]} = \frac{\mathbb{E}\left[\mathrm{e}^{-a\beta X}\right]}{\mathbb{E}\left[\mathrm{e}^{-a\beta Y}\right]}.$$

Since $\mathbb{E}\left[\mathrm{e}^{bX_\lambda Y}\right] = \lambda\mathbb{E}\left[\mathrm{e}^{bX}\right] + (1-\lambda)\mathbb{E}\left[\mathrm{e}^{bY}\right]$ for every $b \in \mathbb{R}$, it is not difficult to see that the above ratio equality holds if and only if it holds when X is replaced by $X_\lambda Y$. Hence $\Phi(X) = \Phi(Y)$ if and only if $\Phi(X_\lambda Y) = \Phi(Y)$, i.e. betweenness is satisfied.

Turning to the "only if" direction. We will characterize any monotone additive statistic Φ that satisfies a weaker form of betweenness:

$$\Phi(X) = c \;\text{ implies }\; \Phi(X_\lambda c) = c \;\text{ when } c \text{ is a constant.}$$

We will show that this weaker requirement already forces Φ to take one of the two forms considered above, unless $\Phi(X) = \beta \min[X] + (1-\beta)\max[X]$ for some $\beta \in [0,1]$. However, this last form of Φ violates the original betweenness axiom, because for $X = 0$ and any Y supported on ± 1, $X_\lambda Y$ and Y have the same minimum and maximum. So $\Phi(X_\lambda Y) = \Phi(Y)$, but $\Phi(X) = \Phi(Y)$ cannot hold for all Y supported on ± 1.

To study the above weak form of betweenness, the following lemma is key:

Lemma 19. *Suppose $\Phi(X) = \int_{\mathbb{R}} K_a(X)\,\mathrm{d}\mu(a)$ has the property that $\Phi(X) = c$ implies $\Phi(X_\lambda c) \le c$. Then the measure μ restricted to $[0,\infty]$ is either the zero measure, or it is supported on a single point.*

Proof. It suffices to show that if μ puts positive mass on $(0,\infty]$, then that mass is supported on a single point and $\mu(\{0\}) = 0$. For this let $N > 0$ denote the essential maximum of the support of μ; that is, $N = \min\{x : \; \mu((x,\infty]) = 0\}$. We allow $N = \infty$ when the support of μ is unbounded from above, or when μ has a non-zero mass at ∞. For any positive real number $b < N$, consider the same random variable $X_{n,b}$ as in the proof of Lemma 5, given by

$$\begin{aligned}
\mathbb{P}\left[X_{n,b} = n\right] &= \mathrm{e}^{-bn} \\
\mathbb{P}\left[X_{n,b} = 0\right] &= 1 - \mathrm{e}^{-bn}.
\end{aligned}$$

As shown in the proof of Lemma 5, $\frac{1}{n}K_a(X_{n,b})$ is uniformly bounded in $[0,1]$, and

$$\lim_{n\to\infty} \frac{1}{n} K_a(X_{n,b}) = \frac{(a-b)^+}{a}.$$

Thus if we let $c_n = \Phi(X_{n,b})$, then by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \frac{c_n}{n} = \lim_{n\to\infty} \frac{1}{n}\Phi(X_{n,b}) = \lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{n} K_a(X_{n,b})\,\mathrm{d}\mu(a) = \int_{(b,\infty]} \frac{a-b}{a}\,\mathrm{d}\mu(a).$$

Denote $\gamma = \int_{(b,\infty]} \frac{a-b}{a}\,\mathrm{d}\mu(a)$. This number γ is strictly positive because $b < N$ implies $\mu((b,\infty]) > 0$. We can also assume $\gamma < 1$, since otherwise μ must be the point mass at ∞.

Now, as $\Phi(X_{n,b}) = c_n$ we know by assumption that $\Phi(Y_{n,b}) \leq c_n$ for each n, where $Y_{n,b}$ is the mixture between $X_{n,b}$ and the constant c_n (in what follows λ is fixed as n varies):

$$\begin{aligned}\mathbb{P}\left[Y_{n,b} = n\right] &= \lambda \mathrm{e}^{-bn} \\ \mathbb{P}\left[Y_{n,b} = 0\right] &= \lambda(1 - \mathrm{e}^{-bn}) \\ \mathbb{P}\left[Y_{n,b} = c_n\right] &= 1 - \lambda.\end{aligned}$$

Using $\lim_{n\to\infty} c_n/n = \gamma$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} K_a(Y_{n,b}) &= \lim_{n\to\infty} \frac{1}{n}\frac{1}{a} \log\left[\lambda\left(1 - \mathrm{e}^{-bn} + \mathrm{e}^{(a-b)n}\right) + (1-\lambda)\mathrm{e}^{a\cdot c_n}\right] \\ &= \begin{cases} 0 & \text{if } a < 0 \\ (1-\lambda)\gamma & \text{if } a = 0 \\ \gamma & \text{if } 0 < a < \frac{b}{1-\gamma} \\ \frac{a-b}{a} & \text{if } a \geq \frac{b}{1-\gamma}. \end{cases}\end{aligned}$$

Note that the cutoff point $a = \frac{b}{1-\gamma}$ is where $a - b = a\gamma$. When a is smaller than this, the dominant term in the bracketed sum above is $(1-\lambda)\mathrm{e}^{a\cdot c_n}$. Whereas for larger a, the dominant term becomes $\lambda \mathrm{e}^{(a-b)\cdot n}$.

Crucially, $\lim_{n\to\infty} \frac{1}{n} K_a(Y_{n,b}) \geq \frac{(a-b)^+}{a}$ holds for every a, with strict inequality for $a \in [0, \frac{b}{1-\gamma})$. Thus again by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \frac{c_n}{n} \geq \lim_{n\to\infty} \frac{1}{n}\Phi(Y_{n,b}) = \lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{n} K_a(Y_{n,b})\,\mathrm{d}\mu(a) \geq \int_{(b,\infty]} \frac{a-b}{a}\,\mathrm{d}\mu(a).$$

But we know that the far left is equal to the far right. So both inequalities hold equal, and in particular $\lim_{n\to\infty} \frac{1}{n} K_a(Y_{n,b}) = \frac{(a-b)^+}{a}$ holds μ-almost surely.

As discussed, $\lim_{n\to\infty} \frac{1}{n} K_a(Y_{n,b}) > \frac{(a-b)^+}{a}$ for any $a \in [0, \frac{b}{1-\gamma})$. So we can conclude that $\mu([0, \frac{b}{1-\gamma})) = 0$. This must hold for any $b \in (0, N)$ and corresponding γ. Letting b arbitrarily close to N thus yields $\mu([0, N)) = 0$ (since $\frac{b}{1-\gamma} > b$). It follows that when restricted to $[0, \infty]$ the measure μ is concentrated at the single point N, as we desire to show. □

From this lemma, we know that if Φ satisfies the weak form of betweenness, then its associated measure μ can only be supported on one point in all of $[0, \infty]$. By a symmetric argument, μ also has at most one point support in all of $[-\infty, 0]$. Thus either $\mu = \delta_a$ for some $a \in \overline{\mathbb{R}}$, or μ is supported on two points $\{a_1, a_2\}$ with $a_1 < 0 < a_2$. In the former case we are done, so below we study the latter case where μ has two-point support.

Suppose $\Phi(X) = \beta K_{a_1}(X) + (1-\beta) K_{a_2}(X)$ for some $\beta \in (0, 1)$ and $a_1 < 0 < a_2$. If $a_1 = -\infty$ while $a_2 < \infty$, then $\Phi(X) = \beta \min[X] + (1-\beta) K_{a_2}(X)$. Take any non-constant X

and let c denote $\Phi(X)$. Note that since $K_{a_2}(X) > \min[X]$, $c = \beta \min[X] + (1-\beta) K_{a_2}(X)$ lies strictly between $\min[X]$ and $K_{a_2}(X)$. Consider the mixture $X_\lambda c$, then $\min[X_\lambda c] = \min[X]$, whereas

$$K_{a_2}(X_\lambda c) = \frac{1}{a_2} \log\left(\lambda \mathbb{E}\left[\mathrm{e}^{a_2 X}\right] + (1-\lambda)\mathrm{e}^{a_2 c}\right) < \frac{1}{a_2} \log \mathbb{E}\left[\mathrm{e}^{a_2 X}\right] = K_{a_2}(X),$$

where the inequality uses $c < K_{a_2}(X) = \frac{1}{a_2} \log \mathbb{E}\left[\mathrm{e}^{a_2 X}\right]$ and $a_2 > 0$. We thus deduce that

$$\Phi(X_\lambda c) = \beta \min[X_\lambda c] + (1-\beta) K_{a_2}(X_\lambda c) < \beta \min[X] + (1-\beta) K_{a_2}(X) = c,$$

contradicting the betweenness axiom. A symmetric argument rules out the possibility that $a_1 > -\infty$ while $a_2 = \infty$.

Hence, either $a_1 = -\infty$ and $a_2 = \infty$, or $a_1 \in (-\infty, 0)$ and $a_2 \in (0, \infty)$. In the former case $\Phi(X)$ is an average of the minimum and the maximum, so we are again done. It remains to consider the latter case where a_1, a_2 are both finite. In this case we will show that $\beta = \frac{-a_1}{a_2 - a_1}$. Once this is shown, we can let $a = a_2 - a_1$ so that $a_1 = -a\beta$ and $a_2 = a(1-\beta)$. Thus $\Phi(X) = \beta K_{-a\beta}(X) + (1-\beta) K_{a(1-\beta)}(X)$ as desired.

Let us take an arbitrary non-constant X, and let

$$c = \Phi(X) = \frac{\beta}{a_1} \log \mathbb{E}\left[\mathrm{e}^{a_1 X}\right] + \frac{1-\beta}{a_2} \log \mathbb{E}\left[\mathrm{e}^{a_2 X}\right].$$

For an arbitrary $\lambda \in [0, 1]$, we must also have

$$c = \Phi(X_\lambda c) = \frac{\beta}{a_1} \log \mathbb{E}\left[\lambda \mathrm{e}^{a_1 X} + (1-\lambda)\mathrm{e}^{a_1 c}\right] + \frac{1-\beta}{a_2} \log \mathbb{E}\left[\lambda \mathrm{e}^{a_2 X} + (1-\lambda)\mathrm{e}^{a_2 c}\right]. \quad (17)$$

Since (17) holds for every λ, we can differentiate it with respect to λ to obtain

$$0 = \frac{\beta(\mathbb{E}\left[\mathrm{e}^{a_1 X}\right] - \mathrm{e}^{a_1 c})}{a_1 \mathbb{E}\left[\lambda \mathrm{e}^{a_1 X} + (1-\lambda)\mathrm{e}^{a_1 c}\right]} + \frac{(1-\beta)(\mathbb{E}\left[\mathrm{e}^{a_2 X}\right] - \mathrm{e}^{a_2 c})}{a_2 \mathbb{E}\left[\lambda \mathrm{e}^{a_2 X} + (1-\lambda)\mathrm{e}^{a_2 c}\right]}.$$

Plugging in $\lambda = 0$ and $\lambda = 1$ gives, respectively,

$$\frac{\beta(\mathbb{E}\left[\mathrm{e}^{a_1 X}\right] - \mathrm{e}^{a_1 c})}{a_1 \mathrm{e}^{a_1 c}} = -\frac{(1-\beta)(\mathbb{E}\left[\mathrm{e}^{a_2 X}\right] - \mathrm{e}^{a_2 c})}{a_2 \mathrm{e}^{a_2 c}}. \quad (18)$$

$$\frac{\beta(\mathbb{E}\left[\mathrm{e}^{a_1 X}\right] - \mathrm{e}^{a_1 c})}{a_1 \mathbb{E}\left[\mathrm{e}^{a_1 X}\right]} = -\frac{(1-\beta)(\mathbb{E}\left[\mathrm{e}^{a_2 X}\right] - \mathrm{e}^{a_2 c})}{a_2 \mathbb{E}\left[\mathrm{e}^{a_2 X}\right]}. \quad (19)$$

Since $c = \beta K_{a_1}(X) + (1-\beta) K_{a_2}(X)$, the fact that $K_{a_2}(X) > K_{a_1}(X)$ implies c is strictly between $K_{a_1}(X)$ and $K_{a_2}(X)$. Thus, using $a_1 < 0 < a_2$ we deduce $\mathrm{e}^{a_1 c} < \mathbb{E}\left[\mathrm{e}^{a_1 X}\right]$ and $\mathrm{e}^{a_2 c} < \mathbb{E}\left[\mathrm{e}^{a_2 X}\right]$.

We can therefore divide (18) by (19) to obtain

$$\frac{\mathbb{E}\left[\mathrm{e}^{a_1 X}\right]}{\mathrm{e}^{a_1 c}} = \frac{\mathbb{E}\left[\mathrm{e}^{a_2 X}\right]}{\mathrm{e}^{a_2 c}}.$$

Plugging this back to (18), we conclude $\frac{\beta}{a_1} = -\frac{1-\beta}{a_2}$, so $\beta = \frac{-a_1}{a_2 - a_1}$ as we desire to show.

Supplementary Appendix

G Time Lotteries in Discrete Time

In this appendix we characterize monotone stationary time preferences when time is discrete. For this purpose we consider the domain $\mathbb{R}_{++} \times L^\infty_{\mathbb{N}}$ of discrete time lotteries. The original Axioms 3.1, 3.2 and 3.3 for continuous time directly carry over to discrete time, except that in their statements we now restrict to integer-valued random times. We strengthen the continuity assumption, Axiom 3.4, to rule out extreme risk aversion or extreme risk seeking over time. This will turn out to be convenient for working with discrete time lotteries:

Axiom G.1 (Strong Continuity)**.** *Consider any sequence of discrete time lotteries $\{(x_n, T_n)\}$ such that $x_n \to x$, the distributions of T_n weakly converge to that of T, and $\max[T_n]$ is bounded across n. Then for any discrete time lottery (y, S), $(x_n, T_n) \succeq (y, S)$ for every n implies $(x, T) \succeq (y, S)$, and $(x_n, T_n) \preceq (y, S)$ for every n implies $(x, T) \preceq (y, S)$.*

Because the maximum and minimum of a distribution are not continuous with respect to the weak topology, this axiom implies that the measure μ characterizing the certainty equivalent Φ for the preference must be supported on $\mathbb{R}$, rather than the extended real line $\overline{\mathbb{R}}$. We call such monotone additive statistics Φ *strongly monotone*.[19] This definition is reflected in the following result which generalizes Theorem 3:

Proposition 6. *A preference $\succeq$ on $\mathbb{R}_{++} \times L^\infty_{\mathbb{N}}$ satisfies Axioms 3.1, 3.2, 3.3 and G.1 if and only if there exists a strongly monotone additive statistic Φ, an $r > 0$, and a continuous and increasing utility function $u\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$, such that $\succeq$ is represented by*

$$V(x, T) = u(x) \cdot \mathrm{e}^{-r\Phi(T)}.$$

Proof. We first check that the representation satisfies the strong continuity Axiom G.1; the other axioms are straightforward to check. Indeed, suppose

$$\Phi(T) = \int_{\mathbb{R}} K_a(T)\,\mathrm{d}\mu(a)$$

for some probability measure μ supported on $\mathbb{R}$. Then whenever $T_n \to T$ (in terms of their distributions) and $\max[T_n]$ is bounded across n, we can deduce from the definition of weak convergence that $\mathbb{E}\left[\mathrm{e}^{aT_n}\right] \to \mathbb{E}\left[\mathrm{e}^{aT}\right]$ for every $a \in \mathbb{R}$. Thus $K_a(T_n) \to K_a(T)$. Since $K_a(T_n)$ is bounded between 0 and $\max_n\{\max[T_n]\}$, the Dominated Convergence Theorem

[19] We do not use the terminology "strictly monotone" because it suggests a weaker requirement that $\Phi(X) > \Phi(Y)$ whenever X is strictly larger than Y in first-order stochastic dominance. That would correspond to μ having positive mass on $\mathbb{R}$, whereas we require μ to have all mass on $\mathbb{R}$.

implies that $\Phi(T_n) \to \Phi(T)$. This implies $u(x_n) \cdot \mathrm{e}^{-r\Phi(T_n)} \to u(x) \cdot \mathrm{e}^{-r\Phi(T)}$, and thus strong continuity holds.

Turning to the opposite direction, we assume the preference $\succeq$ satisfies the axioms. We first prove the following stronger version of stationarity:

$$(x, T) \succeq (y, S) \quad \text{if and only if} \quad (x, T + D) \succeq (y, S + D)$$

whenever D is independent from T and S. The "only if" direction is assumed, so we focus on the "if". It suffices to show that the strict preference $(x, T) \succ (y, S)$ also implies the strict preference $(x, T + D) \succ (y, S + D)$. By strong continuity there exists $\tilde{x}$ slightly smaller than x such that $(\tilde{x}, T) \succ (y, S)$. Thus by the assumed version of stationarity, $(\tilde{x}, T + D) \succeq (y, S + D)$. Monotonicity in money then yields $(x, T + D) \succ (\tilde{x}, T + D) \succeq (y, S + D)$. This gives the desired result.

Next, as in the proof of Theorem 3, we fix $x > 0$ and define a "certainty equivalent" $\Phi_x(T)$ for every T. However, since $\Phi_x(T)$ will not be an integer in general, we cannot define it using the indifference relation induced by $\succeq$. We instead proceed as follows. For each $T \in L^\infty_{\mathbb{N}}$, define

$$B_x(n, T) = \max\{m \in \mathbb{N} : (x, m) \succeq (x, T^{*n})\}$$

where T^{*n} is the sum of n i.i.d. copies of T. Note that for fixed T, $B_x(n, T)$ is a non-negative super-additive sequence in n. This is because if $(x, m_1) \succeq (x, T^{*n_1})$ and $(x, m_2) \succeq (x, T^{*n_2})$, then applying stationarity twice yields

$$(x, m_1 + m_2) \succeq (x, T^{*n_1} + m_2) \succeq (x, T^{*n_1} + T^{*n_2}) = (x, T^{*n_1+n_2}).$$

Note also that by monotonicity in time, $B_x(n, T) \leq \max[T^{*n}] = n \max[T]$. So we have a well-defined finite limit

$$\Phi_x(T) = \lim_{n\to\infty} \frac{1}{n} B_x(n, T).$$

It is easy to see that Φ_x is a monotone statistic. It is also super-additive because for each n, $(x, m) \succeq (x, T^{*n})$ and $(x, m') \succeq (x, S^{*n})$ imply $(x, m + m') \succeq (x, (T + S)^{*n})$ by two applications of stationarity. Moreover, using

$$B_x(n, T) = \min\{m \in \mathbb{N} : (x, m) \prec (x, T^{*n})\} - 1,$$

we can also show Φ_x is sub-additive. Thus Φ_x is a monotone additive statistic.

We next show that $\Phi_x(S) > \Phi_x(T)$ implies $(x, T) \succ (x, S)$. Indeed, by definition we have $B_x(n, S) > B_x(n, T)$ for sufficiently large n. Thus, for this n, the integer $m = B_x(n, S)$ satisfies

$$(x, T^{*n}) \succ (x, m) \succeq (x, S^{*n}).$$

This implies $(x, T) \succ (x, S)$, because by repeated application of stationarity $(x, S) \succeq (x, T)$ would imply $(x, S^{*n}) \succeq (x, T^{*n})$.

This allows us to show that Φ_x is strongly monotone. Suppose for contradiction that the measure μ associated with Φ_x puts mass more than $\frac{1}{N}$ on $a = \infty$, for some large positive integer N. Consider the random variable T_ε which equals N with probability ε and equals 0 otherwise. Note that $\Phi_x(T_\varepsilon) > \frac{1}{N}\max[T_\varepsilon] = 1$. Thus for any $\varepsilon > 0$, $(x, T_\varepsilon) \prec (x, 1)$ by what we showed above. But since $T_\varepsilon \to 0$ and are uniformly bounded, strong continuity implies $(x, 0) \preceq (x, 1)$, contradicting monotonicity in time. A similar contradiction obtains if μ puts mass more than $\frac{1}{N}$ on $a = -\infty$, by considering the time lotteries $N - T_\varepsilon$ versus the deterministic time $N - 1$.

We can then show that $\Phi_x(S) = \Phi_x(T)$ implies $(x, T) \sim (x, S)$, so that (together with the above result) $(x, T) \succeq (x, S)$ if and only if $\Phi_x(S) \geq \Phi_x(T)$. Indeed, suppose $\Phi_x(S) = \Phi_x(T)$, by symmetry we just need to show $(x, T) \succeq (x, S)$. Let S_ε be equal to S with probability $1 - \varepsilon$, and equal to $\max[S] + 1$ with probability ε. Then by the strong monotonicity of Φ, we have $\Phi_x(S_\varepsilon) > \Phi_x(S) = \Phi_x(T)$, so that $(x, T) \succ (x, S_\varepsilon)$ for every $\varepsilon > 0$. By strong continuity, we thus obtain $(x, T) \succeq (x, S)$ as desired.

Hence, for every $x > 0$ we have constructed a strongly monotone additive statistic Φ_x, such that $(x, T) \succeq (x, S)$ if and only if $\Phi_x(T) \leq \Phi_x(S)$. What remains to be done is to relate the preferences for different rewards x. This is however another new difficulty relative to the proof of Theorem 3. The issue is that we cannot directly reduce the time lottery (x, T) to the deterministic reward $(x, \Phi_x(T))$ by indifference, since the latter need not be in *discrete* time.

To address this problem, we introduce an auxiliary preference $\succeq^*$ defined on the set of deterministic dates rewards $\mathbb{R}_{++} \times \mathbb{R}_+$ in *continuous* time. Specifically, consider any (x, t) and (y, s), where $x, y > 0$ and t and s need not be integers. By the fact that Φ_x, Φ_y satisfy strong continuity, we can find integer-valued bounded random times T, S such that $\Phi_x(T) = t$ and $\Phi_y(S) = s$. We then define $(x, t) \succeq^* (y, s)$ if and only if $(x, T) \succeq (y, S)$. Since we have shown that $(x, T) \sim (x, T')$ whenever $\Phi_x(T) = \Phi_x(T')$, this definition of $\succeq^*$ does not depend on the specific choice of T and S. In addition, it is easy to see that $\succeq^*$ is complete and transitive.

Below we show that the preference $\succeq^*$ satisfies the axioms in Fishburn and Rubinstein (1982). We introduce a key technical lemma that we prove at the end of this section:

Lemma 20. *Let Φ and Ψ be two strongly monotone additive statistics defined on $L^\infty_{\mathbb{N}}$. Then for any real number $d > 0$, there exist two random variables $D, D' \in L^\infty_{\mathbb{N}}$ such that*

$$\Phi(D) - \Phi(D') = d = \Psi(D) - \Psi(D').$$

We use this lemma to prove the stationarity property of $\succeq^*$, namely $(x, t) \succeq^* (y, s)$ if and only if $(x, t+d) \succeq^* (y, s+d)$. Let $T, T', S, S' \in L^\infty_{\mathbb{N}}$ satisfy $\Phi_x(T) = t$, $\Phi_x(T') = t+d$, $\Phi_y(S) = s$, $\Phi_y(S') = s+d$. Also let $D, D' \in L^\infty_{\mathbb{N}}$ be given by Lemma 20, such that

$$\Phi_x(D) - \Phi_x(D') = d = \Phi_y(D) - \Phi_y(D').$$

Suppose $(x, t) \succeq^* (y, s)$, then by definition $(x, T) \succeq (y, S)$. This implies, by stationarity of $\succeq$, that

$$(x, T + D) \succeq (y, S + D).$$

Now observe that

$$\Phi_x(T + D) = \Phi_x(T) + \Phi_x(D) = t + d + \Phi_x(D') = \Phi_x(T' + D').$$

Thus $(x, T + D) \sim (x, T' + D')$ and likewise $(y, S + D) \sim (y, S' + D')$. It follows that

$$(x, T' + D') \succeq (y, S' + D').$$

By stationarity of $\succeq$ again, we conclude that $(x, T') \succeq (y, S')$ and thus $(x, t+d) \succeq^* (y, s+d)$. Moreover, if we have the strict preference $(x, t) \succ^* (y, s)$ to begin with, then the above steps and the conclusion $(x, t+d) \succ^* (y, s+d)$ are also strict. This proves the stationarity of $\succeq^*$.[20]

We now use stationarity to show $\succeq^*$ is monotone in money. Suppose $x > y > 0$, then $(x, 0) \succ (y, 0)$ when viewed as discrete time lotteries, and by definition $(x, 0) \succ^* (y, 0)$ when viewed as dated rewards. Thus stationarity implies $(x, t) \succ^* (y, t)$ for every $t \geq 0$. As for monotonicity in time, suppose $s > t$, then we can find $S >_1 T$ such that $\Phi_x(S) = s$ and $\Phi_x(T) = t$ (start with T and construct S by continuously shifting mass to the right). Since Φ_x represents the preference $\succeq$ when restricted to the reward x, we have $(x, T) \succ (x, S)$. So by definition $(x, t) \succ^* (x, s)$ also holds.

It remains to check $\succeq^*$ is continuous in the sense that if $(x_n, t_n) \to (x, t)$ and $(x_n, t_n) \succeq^* (y, s)$ for every n, then $(x, t) \succeq^* (y, s)$ (and that the same holds for the preferences reversed). To show this, note that $t_n < \lfloor t \rfloor + 1$ for every large n. By strong monotonicity (thus continuity) of Φ_x, we can find a binary integer random variable T_n supported on 0 and $\lfloor t \rfloor + 1$ such that $\Phi_x(T_n) = t_n$. Passing to a sub-sequence if necessary, we can assume T_n has a limit T. Since $(x_n, t_n) \succeq^* (y, s)$, we know by definition that $(x_n, T_n) \succeq (y, S)$ for any S with $\Phi_y(S) = s$. Thus by strong continuity of $\succeq$, we deduce $(x, T) \succeq (y, S)$. Since $\Phi_x(T) = \lim \Phi_x(T_n) = \lim t_n = t$, we have $(x, t) \succeq^* (y, s)$ as desired.

Hence we can apply Theorem 2 in Fishburn and Rubinstein (1982) to deduce that

$$(x, t) \succeq^* (y, s) \quad \text{if and only if} \quad u(x) \cdot \mathrm{e}^{-rt} \geq u(y) \cdot \mathrm{e}^{-rs},$$

[20] This proof would be a little simpler if there exists D such that $\Phi_x(D) = \Phi_y(D) = d$, which would be a stronger statement than Lemma 20. But such integer-valued D might not exist when d is not an integer, and Φ_x is larger than Φ_y in the sense of Proposition 2.

for some continuous and increasing function $u\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$. Since by definition $(x,T) \succeq (y,S)$ if and only if $(x, \Phi_x(T)) \succeq^* (y, \Phi_y(S))$, we obtain

$$(x,T) \succeq (y,S) \quad \text{if and only if} \quad u(x) \cdot \mathrm{e}^{-r\Phi_x(T)} \geq u(y) \cdot \mathrm{e}^{-r\Phi_y(S)}.$$

Once we have this representation, for any $x, y > 0$ we can find $T, S \in L^\infty_{\mathbb{N}}$ such that $u(x) \cdot \mathrm{e}^{-r\Phi_x(T)} = u(y) \cdot \mathrm{e}^{-r\Phi_y(S)}$, so $(x,T) \sim (y,S)$. Then for any independent D, we also have $(x, T+D) \sim (y, S+D)$ so that $u(x) \cdot \mathrm{e}^{-r\Phi_x(T+D)} = u(y) \cdot \mathrm{e}^{-r\Phi_y(S+D)}$. Dividing the two equalities thus yields $\Phi_x(D) = \Phi_y(D)$ for every D. We can therefore write $\Phi_x(T) = \Phi(T)$ for a single strongly monotone additive statistic Φ, which completes the proof. □

Proof of Lemma 20. Suppose for the sake of contradiction that the result is not true. We claim there cannot exist $X, Y, X', Y' \in L^\infty_{\mathbb{N}}$ such that $\Phi(Y) - \Phi(X) = d < \Psi(Y) - \Psi(X)$ and $\Phi(Y') - \Phi(X') = d > \Psi(Y') - \Psi(X')$. Indeed, given such random variables, we may add a large constant to X', Y' so that $X' >_1 X$ and $Y' >_1 Y$, without affecting the assumption. Then as λ varies in $[0,1]$, the statistic value $\Phi(X'_\lambda X)$ increases continuously in λ (where $X'_\lambda X \in L^\infty_{\mathbb{N}}$ is the $(\lambda, 1-\lambda)$-mixture between X' and X). Likewise $\Phi(X'_\lambda X)$ increases continuously in λ. So for any $\lambda \in [0,1]$, there exists a unique $h(\lambda) \in [0,1]$ such that $\Phi(Y'_{h(\lambda)}Y) - \Phi(X'_\lambda X) = d$. This function $h(\lambda)$ is strictly increasing and continuous, and satisfies $h(0) = 0, h(1) = 1$. Note that $\Psi(Y'_{h(\lambda)}Y) - \Psi(X'_\lambda X)$ is larger than d when $\lambda = 0$, but smaller than d when $\lambda = 1$. Thus by continuity, there exists λ such that $\Psi(Y'_{h(\lambda)}Y) - \Psi(X'_\lambda X) = d = \Phi(Y'_{h(\lambda)}Y) - \Phi(X'_\lambda X)$.

Hence, for the lemma to fail, the only possibility is that $\Psi(Y) - \Psi(X)$ is always larger (or always smaller) than d whenever $\Phi(Y) - \Phi(X) = d$. Below we assume $\Psi(Y) - \Psi(X) > d$, but the opposite case can be symmetrically handled. Choose any positive integer $k > d$, and let $Y^* = k_\lambda 0$ be the unique binary random variable supported on $\{0, k\}$ such that $\Phi(Y^*) = d$. Then $\Psi(Y^*) > d$ by assumption, and we can assume $\Psi(Y^*) = d + \eta$ for some $\eta > 0$. This Y^* and η will be fixed in the subsequent analysis.

Now take any positive integer $m > k$. We can define a continuum of random variables $X_\varepsilon \in L^\infty_{\mathbb{N}}$ for $\varepsilon \in [0,1]$. Specifically, for $\varepsilon \in [0, \lambda]$ we define X_ε to be equal to k with probability ε and equal to 0 with probability $1 - \varepsilon$. And for $\varepsilon \in [\lambda, 1]$, we define X_ε to be equal to m with probability $\frac{\varepsilon - \lambda}{1 - \lambda}$, equal to k with probability $\frac{\lambda(1-\varepsilon)}{1-\lambda}$ and equal to 0 with probability $1 - \varepsilon$. The important thing here is that as ε increases, X_ε increases in first-order stochastic dominance in a continuous way. Thus $\Phi(X_\varepsilon)$ and $\Psi(X_\varepsilon)$ increase continuously with ε. In addition, note that $X_0 = 0$, $X_\lambda = Y^*$ and $X_1 = m$.

Let $n \leq \frac{m}{d}$ be any positive integer. Then we can define $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_n$ by the equations $\Phi(X_{\varepsilon_j}) = j \cdot d$ for every $0 \leq j \leq n$. It is easy to see

$$0 = \varepsilon_0 < \lambda = \varepsilon_1 < \cdots < \varepsilon_n \leq 1.$$

For $1 \leq j \leq n$, we have $\Phi(X_{\varepsilon_j}) - \Phi(X_{\varepsilon_{j-1}}) = d$. So by assumption $\Psi(X_{\varepsilon_j}) - \Psi(X_{\varepsilon_{j-1}}) > d$. Moreover when $j = 1$ we in fact have $\Psi(X_{\varepsilon_j}) - \Psi(X_{\varepsilon_{j-1}}) = \Psi(Y^*) - \Psi(0) = d + \eta$. Summing across j, we thus obtain

$$m = \Psi(X_1) - \Psi(X_0) \geq \Psi(X_{\varepsilon_n}) - \Psi(X_{\varepsilon_0}) = \sum_{j=1}^{n} \left(\Psi(X_{\varepsilon_j}) - \Psi(X_{\varepsilon_{j-1}}) \right) \geq nd + \eta.$$

But we now have a contradiction because the inequality $m \geq nd + \eta$ cannot hold for all sufficiently large integers m and n that satisfy $m \geq nd$. To see this, observe that when d is a rational number, we can choose m, n so that $m = nd$. In that case the inequality $m \geq nd + \eta$ clearly fails. If instead d is an irrational number, then it is well known that the fractional part of nd can be arbitrarily close to one (as implied by the "equidistribution" property). Again we can find large integers m and n such that $nd + \eta > m \geq nd$. Thus a contradiction obtains either way, completing the proof. □

H Equivalence of Wealth and Risk Invariance under EU

In this appendix we prove that wealth invariance and risk invariance are equivalent under a monotone expected utility preference. It is immediate that wealth invariance implies risk invariance under expected utility. It is also well known that wealth invariance together with monotonicity characterize CARA utility (see §A.1.4 in Strzalecki, 2011, for a proof). Conversely, risk invariance and monotonicity also imply CARA utility and thus wealth invariance, but this result appears to be less well known.[21] For completeness we provide a proof here.

Suppose the preference is represented by $\mathbb{E}[u(X)]$ for some non-decreasing and non-constant utility function u. By risk invariance, for any mean-zero Z that is independent of X and Y, we have $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ if and only if $\mathbb{E}[u(X+Z)] \geq \mathbb{E}[u(Y+Z)]$. For each mean-zero Z, let us define a utility function $v_Z : \mathbb{R} \to \mathbb{R}$ by $v_z(x) = \mathbb{E}[u(x+Z)]$. Then $\mathbb{E}[u(X+Z)]$ can be rewritten as $\mathbb{E}[v_Z(X)]$, and thus the risk invariance condition says that $u(x)$ and $v_Z(x)$ are two utility functions that represent the same expected utility preference. As is well known, this implies the existence of constants $\alpha_z > 0$ and β_z, such that

$$\mathbb{E}[u(x+Z)] = v_Z(x) = \alpha_Z \cdot u(x) + \beta_Z \quad \forall x \in \mathbb{R}. \tag{20}$$

Below we will use this property to pin down the form of u.

We first choose Z to be W_1, a random variable that equals ± 1 with equal probabilities. Then (20) implies

$$\frac{1}{2}u(x+1) - \alpha u(x) + \frac{1}{2}u(x-1) = \beta$$

[21] This implication does not hold without the monotonicity assumption. For example, the non-monotone preference represented by $\mathbb{E}[X^2]$ satisfies risk invariance but not wealth invariance.

for some constants $\alpha > 0$ and β. This must hold for all real numbers x, but for now we restrict attention to integers $x = n$ and derive CARA utility on integers. We distinguish three different cases depending on the value of α:

(1) Suppose $\alpha = 1$, then we obtain $(u(n+1) - u(n)) - (u(n) - u(n-1)) = 2\beta$. By monotonicity, $u(n+1) - u(n)$ is non-negative for every integer n, so the only way the previous equation can always hold is when $\beta = 0$. It follows that for every integer n, $u(n+1) - u(n) = u(1) - u(0)$ does not depend on n, and thus $u(n) = an + c$ for some constants $a > 0$ and c. This corresponds to the risk-neutral case of CARA.

(2) Suppose $\alpha > 1$, then the equation $\frac{1}{2}u(n+1) - \alpha u(n) + \frac{1}{2}u(n-1) = \beta$ can be "homogenized" to $\frac{1}{2}\tilde{u}(n+1) - \alpha\tilde{u}(n) + \frac{1}{2}\tilde{u}(n-1) = 0$ by defining $\tilde{u}(n) = u(n) - \frac{\beta}{1-\alpha}$. Thus the sequence $\{\tilde{u}(n)\}_{n\in\mathbb{Z}}$ satisfies a second-order recurrence equation, whose general solution is given by

$$\tilde{u}(n) = a \cdot (\alpha + \sqrt{\alpha^2 - 1})^n + b \cdot (\alpha - \sqrt{\alpha^2 - 1})^n$$

for some real constants a, b. Note that $\lambda_{\pm} := \alpha \pm \sqrt{\alpha^2 - 1}$ are the two real roots of the characteristic polynomial $\frac{1}{2}\lambda^2 - \alpha\lambda + \frac{1}{2}$ associated with the recurrence equation.

Since the two characteristic roots satisfy $\lambda_+ > 1 > \lambda_- > 0$, we see that in order for $\tilde{u}(n)$ to be monotone in n when $n \to \pm\infty$, it must be that $a \geq 0$ and $b \leq 0$. Below we show that only one of a, b can be non-zero. Indeed, if $a > 0 > b$, then we can consider (20) for a different Z that equals 2 with probability 1/3 and equals -1 with probability 2/3. This yields another pair of constants α', β' such that

$$u(n+2) - 3\alpha' u(n) + 2u(n-1) = 3\beta'.$$

Plugging in the expression for $u(n) = \tilde{u}(n) + \frac{\beta}{1-\alpha}$ and collecting terms, we deduce

$$a \cdot \lambda_+^{n-1} \cdot \Big[(\lambda_+)^3 - 3\alpha'\lambda_+ + 2)\Big] + b \cdot \lambda_-^{n-1} \cdot \Big[(\lambda_-)^3 - 3\alpha'\lambda_- + 2)\Big] + d = 0$$

for the constant $d = \frac{3(1-\alpha')\beta}{1-\alpha} - 3\beta'$. For such an equation to hold for $n \to \pm\infty$, it must be that λ_+ and λ_- are both roots of the new characteristic polynomial $\lambda^3 - 3\alpha'\lambda + 2$ (and c must be zero). If we let λ^* be the third root of this polynomial, then we have the polynomial identity

$$\lambda^3 - 3\alpha'\lambda + 2 = (\lambda - \lambda_+)(\lambda - \lambda_-)(\lambda - \lambda^*).$$

Comparing the constant term yields $2 = -\lambda_+\lambda_-\lambda^*$, which gives $\lambda^* = -2$ because $\lambda_+\lambda_- = (\alpha + \sqrt{\alpha^2 - 1})(\alpha - \sqrt{\alpha^2 - 1}) = 1$. But then the LHS above does not have a λ^2 term, whereas the RHS has $-(\lambda_+ + \lambda_- + \lambda^*)\lambda^2 = -(2\alpha - 2)\lambda^2$. As $\alpha > 1$, this leads to a contradiction.

Hence, only one of a, b can be non-zero, which gives $\tilde{u}(n) = a \cdot \lambda_+^n$ for some $a > 0$ or $\tilde{u}(n) = b \cdot \lambda_-^n = b \cdot \lambda_+^{-n}$ for some $b < 0$. If we write $\lambda_+ = \alpha + \sqrt{\alpha^2 - 1} = \mathrm{e}^r$ for some $r > 0$, then $u(n) = a \cdot e^{rn} + \frac{\beta}{1-\alpha}$ or $u(n) = b \cdot e^{-rn} + \frac{\beta}{1-\alpha}$ for every integer n. The former corresponds to risk-seeking CARA, while the latter corresponds to risk-averse CARA.

(3) Finally suppose $\alpha \in (0, 1)$. We will show this is inconsistent with the monotonicity of $u(n)$. Indeed, using the same transformation $\tilde{u}(n) = u(n) - \frac{\beta}{1-\alpha}$, we can still deduce

$$\tilde{u}(n) = a \cdot \lambda_+^n + b \cdot \lambda_-^n$$

for some constants a, b. But the difference here is that the two characteristic roots $\lambda_\pm = \alpha \pm \sqrt{\alpha^2 - 1}$ are complex numbers when $\alpha < 1$. Accordingly, the constants a, b need not be real numbers, either.

However, by plugging into $n = 0$ and $n = 1$ and using the fact that $\tilde{u}(n)$ is a real number, we have $a + b$ and $a\lambda_+ + b\lambda_-$ are real. The latter condition requires $(a+b)\alpha + (a-b)\sqrt{\alpha^2 - 1}$ to be real, and thus $a - b$ is a purely imaginary number. Together with $a+b$ being real, this implies a and b are complex conjugates. Henceforth we write $a = m \cdot \mathrm{e}^{2\pi i \gamma}$ and $b = m \cdot \mathrm{e}^{-2\pi i \gamma}$ for some $m > 0$ and $\gamma \in [0, 1)$, where i denotes $\sqrt{-1}$.

Since $\lambda_\pm = \alpha \pm \sqrt{1 - \alpha^2} i$ are complex numbers with modulus 1, we can also write these two characteristic roots as $\mathrm{e}^{2\pi i \theta}$ and $\mathrm{e}^{-2\pi i \theta}$ for some $\theta \in [0, 1)$. Putting this together, we obtain

$$\tilde{u}(n) = a \cdot \lambda_+^n + b \cdot \lambda_-^n = m \cdot \mathrm{e}^{2\pi i \gamma} \mathrm{e}^{2\pi i n \theta} + m \cdot \mathrm{e}^{-2\pi i \gamma} \mathrm{e}^{-2\pi i n \theta} = 2m \cdot \cos(2\pi(\gamma + n\theta)).$$

This contradicts the monotonicity of $\tilde{u}(n)$ by the following argument: if $\theta = \frac{p}{q}$ is a rational number, then $\tilde{u}(n) = \tilde{u}(0)$ whenever n is a multiple of q. But then the monotonicity of $\tilde{u}$ forces it to be a constant function, contradicting our assumption. If instead θ is an irrational number, then it is well known that the fractional part of $n\theta$ is dense in $(0, 1)$ as the integer n varies. In particular, we can find integers $n_1 < n_2$ such that the fractional part of $\gamma + n_1\theta$ belongs to the interval $(0, \frac{1}{4})$, whereas the fractional part of $\gamma + n_2\theta$ belongs to the interval $(\frac{1}{4}, \frac{1}{2})$. It follows that $\cos(2\pi(\gamma + n_1\theta)) > 0 > \cos(2\pi(\gamma + n_2\theta))$, and so $\tilde{u}(n_1) > 0 > \tilde{u}(n_2)$. This again leads to a contradiction and proves that $\alpha < 1$ is impossible.

To summarize the above three cases, we have shown that when restricted to integers n, $u(n)$ is either equal to $an + c$, or $a \cdot e^{rn} + c$, or $-a \cdot e^{-rn} + c$ for some constants $a, r > 0$ and c. These three cases correspond to u being a risk-neutral/risk-seeking/risk-averse CARA utility function on integers. More generally, for any positive integer k, we can apply essentially the same argument to those x that are integer multiples of $\frac{1}{k}$. We obtain that for $x = \frac{n}{k}$, $u(x)$ is either equal to $a_k x + c_k$, or $a_k \cdot e^{r_k x} + c_k$, or $-a_k \cdot e^{-r_k n} + c_k$. Since integer multiples of $\frac{1}{k}$ include all integers, it is easy to see that which of the three cases obtains for integer multiples of $\frac{1}{k}$ is pinned down by the risk attitude of u on integers. Moreover, the relevant constants a_k, c_k, r_k do not in fact depend on k.

Consequently, we arrive at the conclusion that *for all rational numbers* x, $u(x)$ is either equal to $ax + c$, or $a \cdot e^{rx} + c$, or $-a \cdot e^{-rx} + c$. By monotonicity, this CARA utility form holds for irrational numbers x as well. This completes the proof that within expected utility, risk invariance and monotonicity imply CARA utility.

I Sub- and Super-additive Statistics

In this appendix we replace the additivity condition imposed on a statistic by weaker versions of sub-additivity or super-additivity. Say a statistic Φ is *sub-additive* if $\Phi(X+Y) \leq \Phi(X) + \Phi(Y)$ whenever X, Y are independent bounded random variables, and *super-additive* if the reverse inequality holds. Say Φ is *homogeneous* if the equality $\Phi(X_1 + X_2) = \Phi(X_1) + \Phi(X_2)$ holds when X_1 and X_2 are independent and furthermore *identically distributed*. These properties are all implied by additivity.

The following result characterizes homogeneous and sub-additive (or super-additive) statistics on L^∞:[22]

Theorem 8. *$\Phi \colon L^\infty \to \mathbb{R}$ is monotone, homogeneous and sub-additive if and only if there exists a nonempty closed convex set C of Borel probability measures on $\overline{\mathbb{R}}$, such that for every $X \in L^\infty$ it holds that*

$$\Phi(X) = \max_{\mu \in C} \int_{\overline{\mathbb{R}}} K_a(X) \, \mathrm{d}\mu(a).$$

Likewise, Φ is monotone, homogeneous and super-additive if and only if

$$\Phi(X) = \min_{\mu \in C} \int_{\overline{\mathbb{R}}} K_a(X) \, \mathrm{d}\mu(a).$$

We use a few examples to illustrate that homogeneity and sub-additivity (or super-additivity) are both important for such representations. An example of a monotone statistic that is super-additive but not homogeneous is

$$\Phi(X) = \log\left(\frac{1}{2}\left[\mathrm{e}^{\min[X]} + \mathrm{e}^{\max[X]}\right]\right).$$

[22]The representation in the super-additive case is reminiscent of the cautious expected utility representation of Cerreia-Vioglio, Dillenberger, and Ortoleva (2015), which evaluates a gamble X according to its minimum certainty equivalent across a family of utility functions. The difference is that our agent potentially takes the minimum across averages of certainty equivalents. In fact, since CARA certainty equivalents are increasing in the level of risk-seeking, taking the minimum across these certainty equivalents (and not their averages) in our setting would reduce to a single CARA certainty equivalent.

The super-additivity condition $\Phi(X+Y) \geq \Phi(X) + \Phi(Y)$ is equivalent to

$$2\left[\mathrm{e}^{\min[X]+\min[Y]} + \mathrm{e}^{\max[X]+\max[Y]}\right] \geq \left[\mathrm{e}^{\min[X]} + \mathrm{e}^{\max[X]}\right] \cdot \left[\mathrm{e}^{\min[Y]} + \mathrm{e}^{\max[Y]}\right],$$

which reduces to $(\mathrm{e}^{\max[X]} - \mathrm{e}^{\min[X]}) \cdot (\mathrm{e}^{\max[Y]} - \mathrm{e}^{\min[Y]}) \geq 0$. The same argument shows that $\min[X]$ and $\max[X]$ can be substituted with any pair of monotone additive statistics $\Psi(X)$ and $\Psi'(X)$ satisfying $\Psi \leq \Psi'$ (see Proposition 2). The resulting Φ would also be monotone, super-additive but not homogeneous.

Note that if $\Phi(X)$ is monotone and super-additive, then $-\Phi(-X)$ is monotone and sub-additive. In this way we also have an example of a monotone statistic that is sub-additive but not homogeneous.

As for an example of a monotone statistic that is homogeneous but not sub-additive or super-additive, we consider

$$\Phi(X) = \begin{cases} \min[X] & \text{if } \min[X] + \max[X] \leq 0 \\ \max[X] & \text{otherwise.} \end{cases}$$

This statistic is homogeneous because $\min[X]$ and $\max[X]$ are homogeneous. To see it is monotone, we will show $\Phi(Y) \geq \Phi(X)$ whenever $Y \geq_1 X$. If $\Phi(X) = \min[X]$ then $\Phi(Y) \geq \min[Y] \geq \min[X]$ holds. Otherwise $\Phi(X) = \max[X]$ and $\min[X] + \max[X] > 0$. Since $\max[Y] \geq \max[X]$ and $\min[Y] \geq \min[X]$, we also have $\min[Y] + \max[Y] > 0$. Hence $\Phi(Y) = \max[Y] \geq \Phi(X)$ also holds.

In addition, this statistic Φ is neither sub-additive nor super-additive. To see this, note that if X, Y are non-constant random variables, and if $\min[X] + \max[X] \leq 0 < \min[Y] + \max[Y]$, then whether $\Phi(X+Y) = \min[X+Y]$ or $\max[X+Y]$ depends on the sign of $\min[X] + \max[X] + \min[Y] + \max[Y]$. In the former case $\Phi(X+Y) = \min[X+Y] \leq \min[X] + \max[Y] = \Phi(X) + \Phi(Y)$, whereas in the latter case $\Phi(X+Y) \geq \Phi(X) + \Phi(Y)$. Both situations can occur.

I.1 Proof of Theorem 8

When the domain is all bounded random variables, it is sufficient to focus on the case of sub-additivity. This is because if Φ is monotone, homogeneous and super-additive, then $\Psi(X) = -\Phi(-X)$ is monotone, homogeneous and sub-additive. So the result for super-additivity can be immediately deduced from the result for sub-additivity. We will also omit the proof for the "if" direction of the theorem, which is straightforward.

Below we suppose Φ is sub-additive. Recall X^{*n} denotes the random variable that is the sum of n i.i.d. copies of X. Repeatedly applying sub-additivity, we have $\Phi(X^{*n}) \leq n\Phi(X)$ for each n, and by homogeneity equality holds when n is a power of two. Thus, for each n, if we choose any m with $2^m > n$, then by sub-additivity again

$$2^m\Phi(X) = \Phi(X^{*2^m}) \le \Phi(X^{*n}) + \Phi(X^{*(2^m-n)}) \le n\Phi(X) + (2^m - n)\Phi(X).$$

Thus the above inequalities hold equal, and we conclude that

$$\Phi(X^{*n}) = n\Phi(X), \quad \forall n \in \mathbb{N}_+.$$

We will frequently use this stronger form of homogeneity.

The following lemma generalizes the key Lemma 1:

Lemma 21. *Let Φ be a monotone, homogeneous and sub-additive statistic defined on L^∞ or L^∞_+. If $K_a(X) \ge K_a(Y)$ for all $a \in \overline{\mathbb{R}}$ then $\Phi(X) \ge \Phi(Y)$.*

Proof. It suffices to show $\Phi(X+2\varepsilon) \ge \Phi(Y)$ for any $\varepsilon > 0$, which would imply $\Phi(X)+2\varepsilon \ge \Phi(Y)$ by sub-additivity. Denoting $\tilde{X} = X + \varepsilon$, then $K_a(\tilde{X}) = K_a(X) + \varepsilon > K_a(Y)$ for every $a \in \overline{\mathbb{R}}$. Thus by Theorem 7, there exists a bounded random variable Z such that

$$\tilde{X} + Z \ge_1 Y + Z.$$

Since first-order stochastic dominance is preserved under adding an independent random variable, we have

$$\tilde{X}_1 + \tilde{X}_2 + Z \ge_1 \tilde{X}_1 + Y_2 + Z \ge_1 Y_1 + Y_2 + Z,$$

where $\tilde{X}_1, \tilde{X}_2$ are i.i.d. copies of $\tilde{X}$ and similarly for Y_1, Y_2.

Iterating this procedure, we obtain that for each positive integer n,

$$\tilde{X}^{*n} + Z \ge_1 Y^{*n} + Z.$$

Suppose $|Z| \le N$, then $N \ge_1 Z \ge_1 -N$ which implies

$$\tilde{X}^{*n} + N \ge_1 Y^{*n} + (-N),$$

or equivalently

$$(X + \varepsilon)^{*n} + 2N \ge_1 Y^{*n}.$$

Now, if we choose n so large that $\varepsilon n \ge 2N$, then the above implies

$$(X + 2\varepsilon)^{*n} \ge_1 (X + \varepsilon)^{*n} + 2N \ge_1 Y^{*n}.$$

Thus $\Phi(X + 2\varepsilon) \ge \Phi(Y)$ follows from the monotonicity and homogeneity of Φ. □

Given Lemma 21, we can follow the proof of Theorem 1 and view $\Phi(X)$ as a functional $F(K_X)$, which has the following five properties:

1. constants: $F(c) = c$ for every constant function c;
2. monotonicity: $K_X \geq K_Y$ implies $F(K_X) \geq F(K_Y)$;
3. homogeneity: $F(nK_X) = nF(K_X),\ \forall n \in \mathbb{N}_+$;
4. sub-additivity: $F(K_X + K_Y) \leq F(K_X) + F(K_Y)$;
5. Lipschitz: $|F(K_X) - F(K_Y)| \leq \|K_X - K_Y\|$.

The proof of Lipschitz continuity is essentially the same as Lemma 3, except that we instead have

$$F(K_Y) - F(K_X) \leq F(K_{X+\varepsilon}) - F(K_X) \leq F(K_\varepsilon) = \Phi(\varepsilon) = \varepsilon.$$

The second inequality here uses sub-additivity.

This functional F is initially defined on $\mathcal{L} = \{K_X : X \in L^\infty\}$. We now extend it to all of $\mathcal{C}(\overline{\mathbb{R}})$:

Lemma 22. *Any functional F on $\mathcal{L}$ satisfying the above five properties can be extended to a functional on $\mathcal{C}(\overline{\mathbb{R}})$ maintaining these properties, with homogeneity strengthened to allow for scalar multiplication with any positive real number (instead of n).*

Proof. As in the proof of Lemma 4, we can extend F by homogeneity to the rational cone spanned by $\mathcal{L}$, and then extend by continuity to the entire cone. We thus have a functional H defined on $\text{Cone}(\mathcal{L})$ that satisfies monotonicity, homogeneity (over $\mathbb{R}_+$), sub-additivity and Lipschitz continuity.

To further extend H to all continuous functions, we define for each $g \in \mathcal{C}(\overline{\mathbb{R}})$

$$I(g) = \inf_{f \geq g,\ f \in \text{Cone}(\mathcal{L})} H(f). \tag{21}$$

Note first that $I(g)$ is well-defined and finite. This is because each $\in \mathcal{C}(\overline{\mathbb{R}})$ is bounded, so the constant function $f = \max[g] \in \text{Cone}(\mathcal{L})$ is point-wise greater than g. Moreover, *any* function $f \in \text{Cone}(\mathcal{L})$ that is point-wise greater than g must be point-wise greater than the constant function $\min[g]$. So by monotonicity, $H(f) \geq \min[g]$ for any such f.

Secondly, when $g \in \text{Cone}(\mathcal{L})$ we have $I(g) = H(g)$ by monotonicity of H. So I extends H. It is also easy to see $I(g)$ maintains monotonicity and homogeneity.

Thirdly, we check I is sub-additive. Fix any g_1, g_2 and choose any $\varepsilon > 0$. Then by definition of the infimum, there exists $f_1, f_2 \in \text{Cone}(\mathcal{L})$ such that $f_i \geq g_i$ and $H(f_i) < I(g_i) + \varepsilon$ for $i = 1, 2$. Thus the function $f_1 + f_2 \in \text{Cone}(\mathcal{L})$ and is bigger than $g_1 + g_2$. This implies

$$I(g_1 + g_2) \leq H(f_1 + f_2) \leq H(f_1) + H(f_2) < I(g_1) + I(g_2) + 2\varepsilon,$$

where the second inequality uses the sub-additivity of H. Since ε is arbitrary, I is indeed sub-additive.

Finally, we check I is Lipschitz. Suppose $g_1 \le g_2 + \varepsilon$ for some $\varepsilon > 0$, then for any $f_2 \in \text{Cone}(\mathcal{L})$ that is greater than g_2, we have $f_2 + \varepsilon \in \text{Cone}(\mathcal{L})$ being greater than g_1. So by sub-additivity of H and $H(\varepsilon) = \varepsilon$,

$$I(g_1) \le H(f_2 + \varepsilon) \le H(f_2) + \varepsilon.$$

Letting $H(f_2)$ approach $I(g_2)$ thus yields the desired result $I(g_1) \le I(g_2) + \varepsilon$.

Hence this functional I is the desired extension of F to all of $\mathcal{C}(\overline{\mathbb{R}})$. $\square$

Given this extension I satisfying $I(K_X) = \Phi(X)$, the "only if" direction of Theorem 8 will follow from the next result characterizing such functionals I:

Lemma 23. *Let $I\colon \mathcal{C}(\overline{\mathbb{R}}) \to \mathbb{R}$ be a functional that is monotone, homogeneous, sub-additive and Lipschitz, and maps any constant function to this constant. Then there exists a non-empty closed convex set C of Borel probability measures on $\overline{\mathbb{R}}$, such that for every $g \in \mathcal{C}(\overline{\mathbb{R}})$*

$$I(g) = \max_{\mu \in C} \int_{\overline{\mathbb{R}}} g(a)\, \mathrm{d}\mu(a).$$

Proof. Homogeneity and sub-additivity implies I is *convex*, in the sense that $I(\lambda g_1 + (1 - \lambda) g_2) \le \lambda I(g_1) + (1-\lambda) I(g_2)$ for all $g_1, g_2 \in \mathcal{C}(\overline{\mathbb{R}})$ and $\lambda \in (0,1)$. Thus I is a convex and continuous functional on the normed function space $\mathcal{C}(\overline{\mathbb{R}})$. By Theorem 7.6 in Aliprantis and Border (2006), the functional I coincides with its *convex envelope*, meaning that

$$I(g) = \sup\{J(g) : J \le I \text{ and } J \text{ is an affine and continuous functional}\}. \tag{22}$$

Using the Riesz-Markov-Kakutani Representation Theorem, any such functional J can be written as

$$J(g) = b + \int_{\overline{\mathbb{R}}} g(a)\, \mathrm{d}\mu(a)$$

for some $b \in \mathbb{R}$ and some possibly signed finite measure μ.

Now observe from (22) that $J(0) \le I(0) = 0$, so $b \le 0$. Moreover, since I is homogeneous, we deduce from $J(ng) \le I(ng) = nI(g)$ that $\frac{b}{n} + \int_{\overline{\mathbb{R}}} g(a)\, \mathrm{d}\mu(a) \le I(g)$ for every positive integer n, and thus $\hat{J}(g) = \int_{\overline{\mathbb{R}}} g(a)\, \mathrm{d}\mu(a)$ lies between $J(g)$ and $I(g)$. It follows that we can replace each affine J by the linear functional $\hat{J}$ without affecting (22). So we can rewrite

$$I(g) = \sup_{\mu \in C} \int_{\overline{\mathbb{R}}} g(a)\, \mathrm{d}\mu(a) \tag{23}$$

for some set C of possibly signed measures μ.

Choose $g \le 0$. Then by monotonicity of I we have $I(g) \le 0$. Thus (23) implies that $\int_{\overline{\mathbb{R}}} g(a)\, \mathrm{d}\mu(a) \le I(g) \le 0$. Since this holds for any continuous function $g \le 0$, we conclude that each $\mu \in C$ is a non-negative measure. Moreover, plugging $g = 1$ into (23) yields $|\mu| \leq 1$, whereas plugging $g = -1$ implies $|\mu| \geq 1$. Thus C is a nonempty set of probability measures.

Finally, note that taking the closed convex hull of C does not affect the equality in (23). So we can assume C is closed and convex. In this case the supremum is achieved as maximum, because any sequence of probability measures on the compact metric space $\overline{\mathbb{R}}$ has a weakly convergent sub-sequence, by Prokhorov's Theorem. This proves Lemma 23 and thus Theorem 8. $\square$