Forecasting with Shadow-Rate VARs*

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Abstract

Interest rate data are an important element of macroeconomic forecasting. Projections of future interest rates are not only an important product themselves, but also typically matter for forecasting other macroeconomic and financial variables. A popular class of forecasting models are linear Vector Autoregressions (VARs) that include shorter- and longer-term interest rates. However, in a number of economies, at least shorter-term interest rates have now been stuck for years at or near their effective lower bound (ELB), with longer-rates drifting toward the constraint as well. In such an environment, linear forecasting models that ignore the ELB constraint on nominal interest rates appear inept. To handle the ELB on interest rates, we model observed rates as censored observations of a latent shadow-rate process in an otherwise standard VAR setup. The shadow rates are assumed to be equal to observed rates, when above the ELB. Point and density forecasts for interest rates (short-term and long-term) constructed from a shadow-rate VAR for the US since 2009 are superior to predictions from a standard VAR that ignores the ELB. For other indicators of financial conditions and measures of economic activity and inflation, the accuracy of forecasts from our shadow-rate specification is on par with a standard VAR that ignores the ELB.

Keywords: Macroeconomic forecasting, effective lower bound, term structure, censored observations
JEL classification codes: C34, C53, E17, E37, E43, E47

PRELIMINARY AND INCOMPLETE

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1 Introduction

Interest rate data are an important element of macroeconomic forecasting. Projections of future interest rates are not only an important product themselves, but also typically matter for forecasting other macroeconomic and financial variables. A popular class of forecasting models are linear Vector Autoregressions (VARs) that include shorter- and longer-term interest rates. However, in a number of economies, at least shorter-term interest rates have now been stuck for years at or near their effective lower bound (ELB), with longer-rates drifting toward the constraint as well. In such an environment, linear forecasting models that ignore the ELB constraint on nominal interest rates can be problematic along various dimensions.

For concreteness, we consider the case of the US, where the Federal Open Market Committee (FOMC) has set the target range for the federal funds rate no lower than 0-25 basis points. The Committee maintained this target range over the seven-year stretch from December 2008 through December 2015, after the Great Recession, and has again maintained this target range since March 2020, when the COVID-19 pandemic initiated a recession, and indicated an intention to maintain the range for an extended period. Considering other economies, the ELB may even be a bit below zero with several central banks pursing so-called negative interest rate policies (NIRP), albeit still at levels close to zero.[1] Similar to the US experience, policy rates observed under NIRP so far appear constrained to fall much below zero.

In such an environment, a fundamental challenge for forecasting models is to appropriately capture the existence of an ELB on interest rates and the resulting asymmetries in predictive densities not only for interest rates, but likely also other economic variables. The

[1] For example, the Swiss National Bank is targeting a level of −75 basis points for its policy rate, while the European Central Bank has maintained a deposit rate of −50 basis points since September 2019, the culmination of a series of steps starting in December 2011 to gradually lower the rate from 25 basis points. The repo rate of the Swedish Riksbank has been at or below 25 basis points since October 2014, bottoming out at −50 basis points from February 2016 to January 2019, and remaining at 0 through 2020. One of the most extensive episodes of monetary policy near the ELB has occurred in Japan, where policy rates have been near zero since 2008, with the current policy rate at −10 basis points since 2016.

likelihood of a binding ELB may also affect economic dynamics and co-movements between different variables more broadly. At a mechanical level, the existence of an ELB calls for treating nominal interest rates as variables whose observations are censored at their lower bound.[2] So far the literature has discussed a number of potential remedies to ELB complications, in some cases taking a short-cut that avoids dealing with censoring. From a macroeconomic perspective, Swanson and Williams (2014) have argued that it may be sufficient to track longer-term nominal interest rates, as long as their dynamics have remained unaffected by a binding ELB on shorter-term rates; and this has been done, for example, by Debortoli, Gali, and Gambetti (2019). However, by 2020, even 10-year US Treasury yields have fallen below 1 percent, with 5-year yields hovering just above 25 basis points.

In contrast, the finance literature has derived important implications of the ELB for the entire term structure of interest rates. Following the seminal work of Black (1995), the term structure literature views the ELB as a censoring constraint on nominal interest rates (as we do), from which no-arbitrage restrictions are derived for yields of all maturities (which we do not). The resulting restrictions are, however, non-trivial, and have mostly been implemented for models with state dynamics that are affine, homoskedastic, and time-invariant; see for example, Christensen and Rudebusch (2015), Krippner (2015), Bauer and Rudebusch (2016), and Wu and Xia (2016).[3]

An upshot of the term structure literature is the availability of shadow-rate estimates, such as those regularly updated by Wu and Xia (2016). Indeed, one possible choice for applied work is to plug in these shadow-rate estimates as data points for the nominal short-term interest during an ELB episode. However, while convenient, this plug-in approach risks a generated regressor problem that could be substantial, as argued, for example, by Krippner (2020). Mavroeidis (2020) notes that a plug-in approach rules out consistent estimation and

[2] Alternatively, a bounded process for the nominal interest rate could be specified as in Bäurle, Kaufmann, Kaufmann, and Strachan (2016), Chan and Strachan (2014), Iwata and Wu (2006) and Nakajima (2011) without a role for an uncensored state variable to drive the nominal interest rate.

[3] Kim and Singleton (2012) also consider a quadratic-Gaussian specification with a shadow rate and find that it fits data for Japan from 1995 to 2008 as well as a shadow-rate specification in the tradition of Black (1995).

valid inference with a VAR, due to estimation error in the shadow rate that is often highly auto-correlated and not asymptotically negligible. Shadow-rate estimates are model-specific objects, fitted to best capture the dynamics of observed data through the lens of the model, and can be quite sensitive to model choices (Christensen and Rudebusch (2015); Krippner (2020)). An obvious remedy to these considerations is to integrate the shadow-rate inference into the forecasting model.

In this paper, we develop a shadow-rate approach for accommodating the ELB in macroeconomic VARs commonly used in forecasting. To do so, we extend the unobserved components model of Johannsen and Mertens (2021) to the general VAR setting. To handle the ELB on interest rates, we model observed rates as censored observations of a latent shadow-rate process in an otherwise standard VAR setup. The shadow rates are assumed to be equal to observed rates, when above the ELB. Our approach is made feasible by the development of a shadow rate algorithm more computationally efficient than that of Johannsen and Mertens (2021). In particular, we use a sequential procedure, that is embedded in an MCMC sampler, to generate posterior draws from the latent shadow rate process that is computationally much more efficient than the rejection sampling used by Johannsen and Mertens (2021). We apply our shadow-rate approach to a medium-scale Bayesian VAR (BVAR) with stochastic volatility that has already been shown to generate competitive forecasts when ignoring the ELB (e.g., Carriero, Clark, and Marcellino (2019)).[4]

In our results, forecasts for interest rates obtained from a shadow-rate VAR for the US since 2009 are clearly superior, both in terms of point and density forecasts, to predictions from a standard VAR that ignores the ELB. These interest rates include not only the federal funds rate but also longer-term bond yields. For other indicators of financial conditions and measures of economic activity and inflation, the accuracy of forecasts from our shadow-rate

[4]Apart from modeling interest rates at the ELB treatment, our setup follows Carriero, Clark, and Marcellino (2019), who describe efficient MCMC methods for the estimation of a VAR with stochastic shock volatilities when applied to a larger variable vector as in our application. Other studies documenting the relevance of heteroskedasticity in VARs are Clark (2011), D'Agostino, Gambetti, and Giannone (2013), Clark and Ravazzolo (2015), and Chan and Eisenstat (2018).

specification is on par with a standard VAR that ignores the ELB. Overall, our shadow-rate specification successfully addresses the ELB and improves interest rate forecasts without harming a standard VAR's ability to forecast a range of other variables. In this respect, our proposed approach could be seen as a helpful tool for preserving the practical value of VARs for forecasting. In practical settings, presented with forecasts from standard VARs in which interest rates fall below the ELB, consumers of forecasts could question the reliability or plausibility of the forecasts of the other variables of interest. To these consumers, forecasts of macroeconomic variables from a shadow-rate VAR that obey the ELB could be seen as more reliable or plausible even if their historical accuracy was no greater than that achieved by a standard VAR ignoring the ELB.

As a simpler alternative to our shadow-rate VAR, a researcher might be interested in estimating a standard VAR and merely truncating its predictive densities for nominal interest rates to capture the ELB.[5] Indeed, in terms of average forecast accuracy for the 2009-2020 period, we find important benefits for federal funds forecasts from such an approach. But, when the policy rate is at the ELB, such an approach tends to place non-negligible odds on an imminent departure from the bound at every period, which has not been borne out by the relatively long-lived ELB episode seen in the US after 2008 (and other countries, outside our sample, as well). Moreover, in average forecast accuracy, this approach does not improve the accuracy of forecasts of other interest rates. In addition, we compare forecasts from our shadow-rate VAR against those obtained from the aforementioned plug-in approach, where external shadow-rate estimates, like those from Wu and Xia (2016) or Krippner (2013, 2015), are used as data, in place of the actual short-term interest rate, in an otherwise standard VAR. As reported below, we find consistent benefits for point and density forecasting from using the shadow-rate VAR across a wide range of variables.

To relate our approach to other shadow rate work, we share with the term structure

[5] In this truncated VAR, forecasts for other variables are also affected by the truncation of predictive densities for nominal interest rates through their effects in dynamic simulations of future values of all variables included in the VAR system.

literature on shadow-rate models the approach of modeling nominal interest rates as censored variables, but we do not enforce any specific no-arbitrage (or other structural) restrictions. As such, our approach is part of the literature that uses VARs (or other reduced form models) to derive forecasts and expectational errors of financial and economic variables without imposing restrictions of a specific structural model (such as an affine term structure or DSGE model). Should the data satisfy such restrictions, they will also be embodied in estimates derived from a more generic reduced form model. The potential loss in the efficiency of forecasts that do not expressively enforce such restrictions can be offset by a gain in robustness obtained from not imposing restrictions that are false. In fact, as argued by Joslin, Le, and Singleton (2013), the possible gains for forecasting from imposing restrictions from the true term structure model may be small. Moreover, as in Johannsen and Mertens (2021), economic forecasters may be interested in using time series models that allow for features, such as time-varying parameters and stochastic volatilities, that may be harder to embed in a formal no-arbitrage model.

In the context of structural VAR models (SVARs), Mavroeidis (2020) and Aruoba, Mlikota, Schorfheide, and Villalvazo (2021) consider shadow-rate approaches to identify and estimate impulse responses to monetary policy shocks. Mavroeidis (2020) discusses various specification choices for the underlying reduced-form VAR model, similar to some that we also evaluate. In contrast, Aruoba et al. (2021) limit attention to settings where VAR forecasts depend on lagged actual rates, but not lagged shadow rates.[6] We differ from these in focusing on the implementation of the shadow-rate approach in a reduced-form Bayesian VAR (with stochastic volatility), and we evaluate its application to a medium-scale forecasting problem.[7] To focus on this question, we abstract from uncertainty and possible drift in

[6] In Aruoba et al. (2021), the shadow rate arises only contemporaneously when the VAR vector is shocked. Similarly, Iwata and Wu (2006), Berg (2017), and Chan and Strachan (2014) consider only censoring of the VAR's left-hand side variables, without tracking the underlying, uncensored, shadow rate as potential predictor. The inclusion of lagged shadow rates as VAR predictors could, however, be potentially relevant as a means of tracking make-up policies at the ELB, as discussed, among others, by Reifschneider and Williams (2000), Gust, Herbst, López-Salido, and Smith (2017), and Billi (2020).

[7] Johannsen and Mertens (2021) provide an out-of-sample forecast evaluation for short- and long-term nominal interest rates in a model smaller than our VARs, and find their unobserved components shadow-

the level of the ELB, which appears to be a reasonable approach at least in the context of the US.[8]

The remainder of this paper is structured as follows: Section 2 describes modeling and estimation of our shadow-rate VARs. Section 3 details the data used in our empirical application. Section 4 presents shadow-rate estimates and resulting interest rate projections. Section 5 provides a forecast evaluation, and Section 6 concludes.

2 Shadow-rate VARs

This section contrasts our shadow-rate approach with a conventional VAR, as well as related alternatives. Throughout, we take the value of the lower bound, denoted ELB, as a given and known constant. For brevity, we use the singular to refer to "the" nominal interest rate, i_t, and its associated shadow rate, s_t. The framework is easily extended to the cases where the ELB is binding for N_s interest rates of multiple maturities, which might arise, for example, in case of aggressive forward guidance, or yield curve control.

A central element of our approach is to relate actual and shadow rate via a censoring equation known from Black (1995):

$$i_t = \max(ELB, s_t). \tag{1}$$

As in the no-arbitrage literature on the term structure of interest rates (surveyed in Section 1), the censoring function (1) implies that the shadow rate is observed and equal to the actual interest rate when the latter is above the ELB.[9] When the ELB is binding, so

rate model to be competitive to the no-arbitrage model of Wu and Xia (2016), but do not consider forecasts of other variables. Gonzalez-Astudillo and Laforte (2020) embed a shadow-rate model in an unobserved components model and report improved point forecasts for economic and financial variables from the shadow-rate approach.

[8]In our empirical application on US data, we consider the ELB to have a constant and known value of 25 basis points, consistent with other studies, such as Bauer and Rudebusch (2016), Wu and Xia (2016), and Johannsen and Mertens (2021). Considering the euro area, for example, Wu and Xia (2020) model and estimate a stochastic downward drift in the ELB level.

[9]The property that the shadow rate is identical to the actual rate when above the ELB makes our approach

that $i_t = ELB$, the shadow rate is a latent variable that can only take values below (or equal to) ELB, which will inform our inference about s_t. Before turning to our VAR-based specification of a process for s_t, we describe the conventional VAR approach.

2.1 Conventional VAR

A conventional VAR is a linear model for the evolution of a vector of observed data, y_t. Omitting intercepts, we have the following system of N_y equations for a VAR with p lags:

$$y_t = \sum_{j=1}^{p} A_j \, y_{t-j} + v_t \,, \qquad \text{with} \quad v_t \sim N(0, \Sigma_t). \tag{2}$$

Anticipating our subsequent application, we assume time-invariant transition matrices, A_j but allow for time-varying shock volatilities, Σ_t, as in Clark (2011) and Carriero, Clark, and Marcellino (2019).[10] However, at this stage the system could also be represented more generally as a time-varying parameter VAR with stochastic volatility in the tradition of Cogley and Sargent (2005), Primiceri (2005), and Cogley, Primiceri, and Sargent (2010). Critically, VAR errors are typically assumed to have a symmetric distribution with unbounded support. When y_t includes the nominal interest rate, i_t, the resulting predictive densities will fail to incorporate the effects of the effective lower bound, with particularly detrimental effect when i_t is close to ELB. As a special case of (2), consider the case of a random walk for the nominal interest rate, $i_t = i_{t-1} + v_t$.[11] When $i_t = ELB$, the k-period ahead point forecast still satisfy the ELB, since $E_t i_{t+k} = ELB$. But, the associated density forecasts have 50 percent of their mass below ELB as the linear model ignores the ELB constraint.

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based on Black (1995) distinct from others, like Lombardi and Zhu (2014), that define shadow rates more broadly as a common factor of interest rates and possibly other variables intended to capture the stance of monetary policy.

[10] In our empirical application, we follow Carriero, Clark, and Marcellino (2019) and assume that $v_t = A^{-1}\Lambda_t^{-0.5}\varepsilon_t$, where A is a lower unit-triangular matrix, Λ_t is a diagonal matrix, and the vector of its diagonal elements is denoted λ_t, with $\log\lambda_t = \log\lambda_{t-1} + \eta_t$, $\eta \sim N(0, \Phi)$and $\varepsilon_t \sim N(0, I)$. Other forms of heteroskedasticity could also be specified.

[11] The random walk for i_t is a special case of the VAR in (2) with $y_t = i_t$, $p = 1$, $A_1 = 1$, and $A_j = 0$ $\forall\, i > 1$.

2.2 Truncated VAR

An applied fix to the ELB problem could be to estimate a standard VAR that ignores the ELB at the estimation stage, but then truncate the predictive densities for interest rates in the simulation stage. This approach is adopted, for example, by Schorfheide and Song (2020) in what they refer to as a poor man's version of the shadow-rate approach. We include this truncated VAR setup in our model evaluation.

When the ELB binds, the truncated VAR has a tendency to place substantial odds on a subsequent rise in i_t above ELB. To see this, consider again the special case of a random-walk model for i_t. In this case, a forecast jump-off with $i_t = ELB$ leads to a heavily skewed predictive density that combines a point mass at ELB and a truncated normal distribution for values above the bound. At the one-step ahead horizon, the odds of the nominal interest rate rising above the ELB are 50 percent (and increasing for longer horizons).[12] The resulting tendency to expect an imminent departure from the ELB contrasts with the shadow-rate VAR that is described next. In the basic version of the shadow-rate approach, the VAR vector includes the shadow rate, s_t, instead of the actual interest rate, i_t, and with $s_t < ELB$, predictions of future interest rates will need to see projections of s_t to rise above the ELB to expect the same for i_t.

2.3 Shadow-rate VAR

The shadow-rate approach does not posit a VAR for the vector of observed variables, y_t, which contains the actual interest rate, i_t. Instead, a VAR is posited for a hypothetical data vector, z_t, that is identical to y_t except for replacing i_t by s_t. Without loss of generality, partition y_t in a vector of $N_x = N_y - N_s$ other variables, x_t, that have unbounded support,

[12] While the probability of $s_{t+k} > ELB$ remains at 50 percent at all horizons $k > 0$, the odds of $i_{t+k} > ELB$ are increasing with k, as the truncation $i_{t+i} = \max(s_{t+i}, ELB)$ is imposed at every step $i = 1, 2, \ldots, k$.

and the nominal interest rate i_t with x_t ordered on top:

$$y_t = \begin{bmatrix} x_t \\ i_t \end{bmatrix} \quad \text{and let} \quad z_t = \begin{bmatrix} x_t \\ s_t \end{bmatrix} \quad \text{with} \quad i_t = \max\left(ELB, s_t\right). \tag{3}$$

In the shadow-rate VAR approach we posit VAR dynamics for the partially latent vector z_t.[13] Analogously to (2) we have:

$$z_t = \sum_{j=1}^{p} A_j \, z_{t-j} + v_t \,, \qquad \text{with} \quad v_t \sim N(0, \Sigma_t). \tag{4}$$

The shadow-rate VAR system is a non-linear state space model that consists of the kinked measurement equation (3) and the linear state evolution described by the VAR in (4).[14]

Considering a standard VAR, Bernanke and Blinder (1992) proposed to interpret the policy-rate equation of the VAR as a feedback rule that describes monetary policy.[15] In a similar spirit, the shadow-rate equation of the VAR in (4) model can be thought of as embedding a monetary policy reaction function that relates the shadow rate to the variables included in the VAR (4).[16] The actual policy rate follows the same reaction function, except that the actual rate is constrained to not fall below the ELB. As a result, the policy prescriptions from the model — evident in out-of-sample forecasts — obey the ELB on actual policy rates. In contrast, in a standard VAR ignoring the ELB, the implied reaction function relates the actual policy rate to the model's variables and allows the reaction function to prescribe a policy rate below the ELB.

Researchers might also be interested in allowing for potential time variation in parameters of the VAR. For example, in (4), VAR residuals have time-varying volatility. We leave po-

[13]The extension to higher order systems is straightforward and described in Appendix A.

[14]In addition, the shadow-rate VAR system includes any state equations needed to track parameter drift, such as the time-varying volatilities embedded in Σ_t in the case of our application.

[15]The idea to capture the systematic behavior of monetary policy by the policy-rate equation of a VAR has spawned a rich literature, including Christiano, Eichenbaum, and Evans (1996, 1999), and Rotemberg and Woodford (1997).

[16]Using a smaller model in an unobserved components form, Johannsen and Mertens (2021) identify monetary policy shocks from surprises to the shadow rate, using short-run restrictions.

tential extensions to time variation in the VAR's regressions coefficients, A_j, to future work. Identification of time-varying slope coefficients may become an issue since the shadow-rate components of z_t are latent when the ELB binds. Moreover, as noted by Mavroeidis (2020), the constant-parameter version of (4) is consistent with work by Swanson and Williams (2014), Debortoli, Gali, and Gambetti (2019), and Wu and Zhang (2016) that sees monetary policy as unconstrained by the ELB (for example through the use of unconventional policies) so that economic dynamics remain unaffected by the ELB.

In reduced form, our shadow-rate VAR corresponds to what Mavroeidis (2020) refers to as "censored SVAR (CSVAR)." The truncated VAR corresponds to the reduced form of a "kinked VAR" in the terminology of Mavroeidis (2020), which is also used by Aruoba et al. (2021). However, as discussed above, our implementation of the truncated VAR consciously disregards the implications of censoring for estimation of the VAR parameters, while the shadow-rate and hybrid VARs explicitly include interest rate censoring.

2.4 Estimation and forecasting

Each of our models is estimated with an MCMC sampler, based on the methods of Carriero, Clark, and Marcellino (2019) for large BVAR-SV models, with details provided therein. As in their work, we use a Minnesota prior for the VAR coefficients A_j and follow their other choices for priors as far as applicable, too.[17] Throughout, we use $p = 12$ lags in a monthly data set, which is described in further detail in Section 3. Here we briefly explain the algorithm adjustments needed to handle the shadow-rate as a latent process whose posterior is truncated *from above* when the ELB binds.

Provided data on s_t and thus z_t were always observed, estimation of the shadow-rate VAR

[17] All VAR coefficients, A_j, have independent normal priors; all are centered around means of zero, except for the first-order own lags of certain variables as listed in Table 1. As usual, different degrees of shrinkage are applied to own- and cross-lag coefficients. Prior variances of the jth-order own lag are set to θ_1/j^{θ_4}. The cross-lag of the coefficient on variable m in equation n has prior variance equal to $\theta_1/j^{\theta_4} \cdot \theta_2 \cdot \hat{\sigma}_n^2/\hat{\sigma}_n^2$. The intercept of equation n has prior variance $\theta_3 \cdot \hat{\sigma}_n^2$. In all of these settings, $\hat{\sigma}_n^2$ is the OLS estimate of residual variance of variable n in an AR(1) estimated over the entire sample. The shrinkage parameters are $\theta_1 = 0.05$, $\theta_2 = 0.5$, $\theta_3 = 100$, $\theta_4 = 2$, and $\theta_5 = 1$.

in (4) would be straightforward to do with existing Bayesian MCMC methods for VARs.[18] However, when the data includes observations for which the ELB is binding, not only does s_t become a latent variable, but also it is subject to the constraint that $s_t \leq ELB$ when $i_t = ELB$.

The shadow-rate VAR system consisting of (3) and (4) belongs to a class of conditionally Gaussian unobserved components models, for which Johannsen and Mertens (2021) have derived a generic shadow-rate sampling approach that can be nested inside an otherwise standard MCMC sampler for the VAR estimation. The Johannsen-Mertens approach employs the conditionally linear, Gaussian structure of the model to derive a truncated normal posterior for the vector of unobserved shadow rates in the system, given draws of other model parameters, such as the VAR coefficients A_j, and the stochastic volatilities captured by Σ_t.[19] Crucially, this truncated normal posterior pertains to the entire trajectory of unobserved shadow rates (or ensemble of trajectories in case of multiple ELB periods), necessitating draws from a multivariate truncated normal. Johannsen and Mertens (2021) successfully employ rejection sampling to generate joint draws from this multivariate shadow-rate posterior. However, in more general applications, rejection sampling can become computationally tedious and highly inefficient.[20]

Specifically, consider the following setup for the shadow-rate VAR given by (3) and (4): Values for the VAR coefficients $\{A_j\}_{j=1}^p$ and error variances $\{\Sigma_t\}_{t=1}^T$ are given and data for $\{x_t\}_{t=1}^T$ is known. We further assume that at $t = 1$, p lags of data for x_t are known, and that

[18] General textbook treatment are provided, for example, the textbooks of Koop (2003) and Canova (2007). For the case of a medium-scale system with stochastic volatilities in the VAR residuals, as used in our application described further below, efficient methods are described by Carriero, Clark, and Marcellino (2019).

[19] For the remainder of this section, references to the shadow-rate posterior are understood as pointing to the posterior distribution of shadow rates conditional on other model parameters and other latent states, such as the sequence of time-varying variance of covariance matrices for the residuals, $\{\Sigma_t\}_{t=1}^T$.

[20] For example, in an application like ours with monthly data for the US covering the years 2009 through 2015, the shadow-rate posterior is a multivariate truncated normal with (at least) 72 elements, necessitating a rejection whenever a single element out of these 72 should lie above the ELB. For illustrative purpose, consider the case where the shadow rate draws were *iid* with an individual probability of being below ELB of $\pi = 0.95$. The probability of all 72 draws laying below the ELB is then merely $0.95^{72} = 0.02$. Of course, in reality, we can expect positive serial correlation amongst adjacent shadow rates, but not every element's probability of falling below the ELB need be as high as 0.95, either.

the intiial p lags of the shadow rate vector, s_0, s_{-1}, …, s_{-p+1} are known.[21]

The shadow-rate s_t is unknown at least for some t.[22] For ease of notation, we normalize time subscripts so that the first time the ELB is binding occurs at $t = 1$. In addition, denote the last ELB observation by $T^* \leq T$ (where T is the length of the data sample), so that s_t is unknown for $1 \leq t \leq T^*$.[23] For simplicity we refer to the entire sequence $\{s_t\}_{t=1}^{T^*}$ as "unobserved," which corresponds to the case of a single ELB episode. However, the procedures described below apply also when multiple ELB episodes occur between $t = 1$ and T^*, so that only some, but not all, values of s_t in this window are unobserved. In addition, we define the vector $\bar{y}_t$ that contains the observed data except for observations of the actual interest rate at the ELB; as noted above, the vector of all observed variables is y_t.[24]

For ease of reference, we collect all unobserved shadow rates in a vector $\boldsymbol{S}$ and all observations of $\bar{y}_t$ in a vector $\bar{\boldsymbol{Y}}$, and observations of y_t in a vector $\boldsymbol{Y}$:[25]

$$\boldsymbol{S} = \begin{bmatrix} s_{T^*} \\ s_{T^*-1} \\ \vdots \\ s_2 \\ s_1 \end{bmatrix}, \quad \text{and} \quad \bar{\boldsymbol{Y}} = \begin{bmatrix} \bar{y}_T \\ \bar{y}_{T-1} \\ \vdots \\ \bar{y}_0 \\ \bar{y}_{-p+1} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{Y} = \begin{bmatrix} y_T \\ y_{T-1} \\ \vdots \\ y_0 \\ y_{-p+1} \end{bmatrix}. \tag{5}$$

The task of the shadow-rate sampler is then to sample $\boldsymbol{S} \,|\, \boldsymbol{Y}$, which includes the information that $\boldsymbol{S} \leq ELB$ (where the inequality is element-wise). Following Johannsen and Mertens (2021), the shadow-rate sampler builds on solving the "missing value" problem of characterizing $\boldsymbol{S} \,|\, \bar{\boldsymbol{Y}}$. The missing-value problem does not condition on information that the ELB

[21]Assuming that the ELB has not been binding for $t < 1$, we have observations on $s_t = i_t$ for $t = 0, -1, \ldots, -p+1$.

[22]Recall that the shadow-rate is known (and identical to the actual rate) when $i_t > ELB$.

[23]Using more general notation, we could denote the time index of the first observation with a binding ELB by $T_0 + 1$, and consider the setup laid out here as normalizing the time index at $T_0 = 0$.

[24]We have $\bar{y}_t = x_t$ when the ELB is binding, and $\bar{y}_t = [x_t' \, s_t']'$ otherwise.

[25]The vector $\boldsymbol{S}$ is intended to capture only *unobserved* shadow rates. In case of a single ELB episode lasting from $t = 1$ through T^*, $\boldsymbol{S}$, consistent of the entire sequence $\{s_t\}_{t=1}^{T^*}$. In case of multiple ELB episodes, observations where $s_t = i_t > ELB$ are excluded from entering $\boldsymbol{S}$.

has been binding for certain observations, and thus does not impose $\boldsymbol{S} \leq ELB$. As shown in Johannsen and Mertens (2021), the linear structure of the model and its Gaussian error distribution results in a posterior of the missing value problem that is a multivariate normal, with the solution of the shadow-rate sampler being given by a corresponding truncated multivariate normal:[26]

$$\boldsymbol{S} \,|\, \bar{\boldsymbol{Y}} \sim N\left(\boldsymbol{\mu}, \boldsymbol{\Omega}\right) \qquad \Rightarrow \qquad \boldsymbol{S} \,|\, \boldsymbol{Y} \sim TN\left(\boldsymbol{\mu}, \boldsymbol{\Omega}, -\infty, ELB\right) . \tag{6}$$

The moments $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$ can be recursively computed using a standard Kalman smoother, and draws can be generated via a corresponding smoothing sampler.[27] Our paper extends the Johannsen-Mertens approach to a generic VAR with details provided in Appendix A.

A further contribution of our paper is the implementation of the shadow-rate sampler via Gibbs sampling, following Geweke (1991), and adapted to the variance-covariance structure of the VAR(p) case, rather than the rejection sampling employed by Johannsen and Mertens (2021). Depending on parameter values, a (well known) issue with rejection sampling from the truncated normal is a possibly low acceptance rate. In our case, the acceptance probability in sampling from (6) critically depends on VAR parameters and the observed data for macroeconomic and financial variables (other than the federal funds rate). As reported further below, when VAR parameters are drawn from the eventual posterior of our shadow-rate VAR, the acceptance probability for draws from the missing-value problem to lie below ELB is fairly high. However, this need not be the case in general, and does not hold, for example, when our VAR is estimated while treating observations for the federal funds rate as missing (rather than censored) data when the ELB binds. Our adaptation of the Gibbs sampling approach of Geweke (1991) to the VAR(p) case, with details described in Appendix A, provides a more efficient solution to the shadow-rate sampling problem.

[26]The notation $\boldsymbol{S} \sim TN(\boldsymbol{\mu}, \boldsymbol{\Omega}, a, b)$ denotes a truncated multivariate normal distribution for the random vector $\boldsymbol{S}$, with typical elements s_j, where $a \leq s_j \leq b \;\forall\, j$, and where $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$ are mean vector and variance-covariance matrix of the underlying normal distribution.

[27]Alternatively, the moments $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}$ could be computed using the sparse methods of Chan and Jeliazkov (2009).

In out-of-sample forecasting, for every model considered (standard/truncated/shadow rate), we simulate draws from the predictive density of y_{t+k} at forecast origin t by recursive simulations. In each case, to generate draws from the h-step ahead density, VAR residuals, v_{t+k}, are drawn for $k = 1, 2 \ldots, h$.[28] In case of the standard VAR, conditional on current and lagged data for y_t, the simulation is standard and iterates over (2). For the truncated VAR, the iteration also proceeds using (2), but applies the censoring function (1) to predictions for interest rates at every step of the forecast simulation.[29] In contrast, for the shadow-rate VAR, simulation of the predictive densities jumps off MCMC draws for s_t, s_{t-1}, $\ldots s_{t-p+1}$ that are used to initialize recursions over (4). In case of the baseline shadow-rate VAR, censoring of predicted interest rates is applied only at the level of the measurement equation (1), while uncensored draws of lagged shadow rates are fed into the VAR equation (4) to simulate subsequent predictions of y_{t+k}.

3 Data

Our data set consists of monthly observations for 18 macroeconomic and financial variables for the sample 1959:03 to 2020:09, taken from the October 2020 vintage of the FRED-MD database maintained by the Federal Reserve Bank of St. Louis. The variables and their transformation to logs or log-differences are listed in Table 1. Reflecting the raw sample, transformations, and lag specification, the sample for model estimation always begins with 1960:04. Critically, the data set includes the federal funds rate, which has been constrained by the ELB from late 2008 through late 2015, and then again starting in March 2020. In addition to the federal funds rate, our data set contains two longer-term interest rates, measuring the yields on Treasury bonds with 5 and 10 years maturity. Data for the federal

[28] As described, for example, in Carriero, Clark, and Marcellino (2019), draws from $v_{t+k} \sim N(0, \Sigma_{t+k})$ are conditioned on an MCMC draw of underlying model parameters and SV states and involve forward simulation of the SV processes.

[29] In our application, there are three interest rate variables, the federal funds rate, plus nominal yields on 5 and 10 year Treasury bonds. Censoring is applied to predictions of all three of them in case of the truncated VAR as well as the shadow-rate VAR.

funds rate and these two longer-term yields is shown in Figure 1. During and following the Great Recession, longer-term bond yields remained at higher levels, solidly or well above the ELB. The 10-year (5-year) Treasury yield declined from 2.4 percent (1.5 percent) in December 2008 to a low of 1.5 percent (0.6 percent) in July 2012 and then moved higher. Since the COVID-19 outbreak and the FOMC's quick and substantial easing of monetary policy, bond yields have been lower than following the Great Recession and much closer to the ELB. From April through September 2020, the 10-year (5-year) Treasury yield averaged 0.7 percent (0.3 percent).

In our application with US postwar data, the value of ELB is set to 25 basis points, which has been the upper end of the FOMC's target range for the federal funds rate between late 2008 and 2015, and then again since spring 2020.[30] As a matter of consistency with this convention, we set readings for the federal funds rate to 25 basis points when estimating the shadow-rate VAR (not when including the federal funds rate in a standard VAR that ignores the lower bound constraint). Yields with maturity of five years and longer stayed above 25 basis points in the data and can thus be treated as part of the vector x_t, defined in Section 2, for the purpose of model estimation.[31]

4 Shadow-rate estimates

Figure 2 reports our shadow rate estimates associated with the federal funds rate, along with a comparison to measures from Krippner (2013, 2015) and Wu and Xia (2016) based on affine term structure models. Panel (a) of the figure compares full-sample estimates using data through September 2020 (black/red line with credible set indicated by gray shading) to quasi-real-time estimates (solid red line with credible set indicated by dotted lines).[32] The quasi-real-time estimates are the end-of-sample estimates produced by recursive estimation

[30] See, for example, Wu and Xia (2016) and Johannsen and Mertens (2021).

[31] The lower bound constraint is an issue when simulating the predictive density for these yields, but it is not relevant for estimating the VAR.

[32] To be clear, in the full sample case, the model is estimated with data for 1960:04 through 2020:09, but the figure omits the period of 1960-2008 during which the ELB did not bind.

of the model starting in January 2009. Panel (b) compares our full-sample estimate to the Krippner and Wu-Xia measures.

The full-sample estimates show the shadow rate dropping sharply starting in 2009, reaching a nadir of about -1.7 percent in late 2011. The rate then gradually rose and reached the ELB in early 2016, following the Federal Reserve's first increase in the federal funds rate in December 2015 (when the FOMC raised the target range from 0-25 basis points to 25-50 basis points). The rate dropped precipitously in early 2020, with the posterior median reaching about -90 basis points in May 2020, and has hovered near that level through the end of our sample in September 2020. As might be expected, the quasi-real-time estimates have more time variability than do the full sample estimates, but follow a quite similar contour. As might also be expected, the quasi-real-time estimates are less precise, with credible sets wider than those of the full sample estimate (more so for the 2009-2015 period than 2020, as might be expected, given that, at this time, little history is available on the current ELB episode).

Although our VAR does not impose the restrictions of an affine term structure model, our shadow rate estimates have some similarities to the Krippner and Wu-Xia measures based on affine term structure models. As indicated in Panel (b) of Figure 2, our estimate and the Wu-Xia series move together from 2009 through 2013. But over the remainder of the ELB episode following the Great Recession, as our estimate gradually rose to the ELB over the course of 2014 and 2015, the Wu-Xia series fell and then rose sharply. Our estimate also follows the same general contours as the Krippner measure, although the Krippner series shows much sharper declines.

Figure 3 provides some comparisons to assess the effects of shadow rate modeling and enforcement of the ELB in model estimation. Panel (a) compares shadow-rate (black) and missing-data (red) draws for the shadow rate s_t obtained from the posterior of our baseline shadow-rate VAR. Shadow-rate draws are obtained from the truncated posterior for s_t that satisfies the ELB, and described by the problem of drawing from $\boldsymbol{S}|\boldsymbol{Y}$ in (6). Missing-data draws are obtained from the underlying (and un-truncated) posterior of the missing data problem, that ignores the ELB, and correspond to draws from $\boldsymbol{S}|\bar{\boldsymbol{Y}}$ in (6). For much of the sample, the posteriors obtained from these alternatives are very similar.[33] These results might suggest that an approach, which treats observed policy rates at the ELB as missing values, might be a close alternative to shadow-rate sampling that explicitly accounts for the ELB.[34] However, such a conclusion would neglect the effects of enforcing the ELB as part of the shadow-rate sampling on inference for other VAR parameters and state variables (like SV). To illustrate these effects, Panel (b) compares missing-data posteriors obtained from two sets of VAR estimates: In the baseline (red), parameter and SV draws reflect shadow-rate sampling (as shown also in Panel (a)). In the alternative version (blue), parameters and SV are drawn while treating the policy rate at the ELB as missing data and without requiring that missing data draws lie below the ELB. The comparison highlights the non-negligible effects of shadow-rate sampling, which takes into account observations of interest rates at the ELB, on model estimates of parameters and SV. Without forcing the draws of missing interest rate observations to lay at or below the ELB, the upper bound of the posterior credible rises sharply above the ELB for much of the 2009-2011 period and again in 2020, which contradicts observations of the federal funds rate that were at the ELB during those times. In contrast, the use of shadow-rate sampling, as opposed to a missing-data approach, leads to estimates of parameters and SV that increase the odds of obtaining missing-data draws for the shadow rate that lie below the ELB (for observations when the ELB binds).

5 Forecast evaluation

We conduct an out-of-sample forecast evaluation in quasi-real time, where we simulate forecasts made from January 2009 through September 2020. For every forecast origin, each

[33]Early in the Great Recession episode shown and in the early months of the COVID-19 episode, using a missing data approach without fully enforcing the ELB led to draws of interest rates above the ELB, whereas with the full shadow rate treatment, the interest rate distributions remained at or below 25 basis points.

[34]Such a missing data approach has been used, for example, by Del Negro, Giannone, Giannoni, and Tambalotti (2017).

model is re-estimated based on growing samples of data that start in 1959:03. (As indicated in tables provided in the supplementary appendix, we obtain very similar results when we shorten the sample to end in December 2017 to avoid the unusual volatility of the COVID-19 pandemic).[35] Of course, in either case the evaluation window is relatively short, and largely informed only by a single ELB episode. Forecasts made prior to 2009 are not considered, due to the absence of observed interest rates at the ELB in post-war US data. All data are taken from the October 2020 vintage of FRED-MD; we abstract from issues related to real-time data collection.

5.1 Average performance 2009–2020

Tables 2, 3, and 4 provide results on point and density forecast accuracy, measured by root mean squared error (RMSE), median absolute deviations (MAD) and continuous ranked probability score (CRPS), respectively. The reported forecast horizons are $h = 3$, 6, 12, and 24 months.

To facilitate comparisons, we report RMSE, MAD, and CRPS results as relative to the baseline of a standard VAR that simply takes the forecasts as given and does nothing to obey ELB constraints, so that entries of less (more) than 1 mean a given forecast is more (less) accurate than the baseline. To roughly gauge significance of differences with respect to the baseline, we use t-tests as in Diebold and Mariano (1995) and West (1996), denoting significance in the tables with asterisks. In light of the concerns of Bauer and Rudebusch (2016) with the use of RMSE to measure the accuracy of forecasts in the presence of ELB constraints, we also report results for median forecasts evaluated with a median absolute error loss function (MAD). Measured by MAD, the performance of median forecasts is qualitatively similar to the RMSE performance of the corresponding mean predictions. However,

[35] In companion work we investigate the use of outlier-adjusted versions of the SV model to handle the particular swings in data seen since the outbreak of COVID-19 (Carriero, Clark, Marcellino, and Mertens, 2021). Through the use of latent states to capture outliers, the outlier-adjusted procedures discussed there retain a conditionally Gaussian representation, and combination with the shadow-rate sampling methods described here is straightforward. For the sake of parsimony, we maintain a standard SV specification in the present paper, which should, however, not materially affect the relative comparisons shown here.

quantitatively, the gains from applying the shadow-rate VAR are even more substantial in the case of interest rates.

As a starting point, consider the simplest possible approach to obeying the constraints of the ELB: simply truncating interest rate forecasts to rule out values below the ELB. As indicated in columns 2-5 of the tables, this simple approach is helpful in one respect but harmful or of little consequence in others. In particular, the truncated specification materially improves federal funds rate forecasts, with RMSE and CRPS ratios of roughly 0.5 or 0.6 for $h = 3$, 6, and 12, though only an MAD ratio of 0.76 for $h = 3$ (and close to 1 otherwise). But the truncated approach harms the accuracy of Treasury bond yields at horizons of 6 months and more. For example, with $h = 12$, the RMSE, MAD and CRPS ratios for the 10-year Treasury yield are 1.34, 1.42 and 1.24, respectively. For indicators of economic activity, measures of inflation, and other financial indicators, the truncated approach has little consistent effect on accuracy. In a few cases, the truncated approach yields forecasts more accurate than the standard VAR baseline (e.g., for PCE inflation), whereas in some others, the truncated forecasts are less accurate than the baseline (e.g., the unemployment rate).

Our proposed shadow-rate specification for accommodating the ELB perform better in forecasting than does simple truncation. Results for these specifications are covered in columns 10-13 of Tables 2, 3, and 4. Compared to the standard VAR baseline, the shadow-rate specification significantly improve forecasts of not only the federal funds rate (FFR) but also bond yields, without harming the forecasts of indicators of economic activity, measures of inflation, and other financial indicators. With our two preferred approaches to accommodating the ELB, RMSE ratios for FFR forecasts range from 0.36 (h=1) to 0.52 (h=24), and the CRPS ratios range from 0.28 to 0.34, with statistical significance of all of the CRPS gains. MAD gains are even maximal, since the median forecasts correctly predict the FFR outcome for at least half the times during our evaluation window for the reported forecast horizons, leading to perfect MAD scores of 0. The funds rate forecasts from the shadow-rate specification are more accurate than those from the truncated specification. In addition, unlike the approach based on truncation, the shadow-rate VAR improve forecasts of 5- and 10-year Treasury yields, more so at longer horizons than shorter horizons. For example, the shadow rate model's RMSE ratios for the 5-year yield decline from 0.93 at h=3 to 0.70 at h=24, with very similar results of density forecast accuracy as measured by the CRPS. In the case of the Baa-Treasury spread, the shadow-rate specification performs much better than the truncated specification, and its forecasts are quite a bit more accurate than those obtained from the standard VAR for h = 6,12,24 (and on par for h=3). Finally, for the indicators of economic activity, stock price returns, and the exchange rate, the treatment of the ELB on interest rates does not seem to bear consistently and importantly on forecast accuracy. RMSE and CRPS ratios for the truncated, and shadow-rate specifications are often close to 1. In fact, to take real consumption and nonfarm payrolls as examples, the RMSE ratios are all 1.00 (for each of four forecast horizons and three specifications). In some cases, our preferred shadow-rate specification yield forecasts a little more accurate than the standard VAR (e.g., 24-months-ahead point forecasts for hourly earnings). These specifications also tend to improve forecasts of housing starts, perhaps the most interest rate-sensitive activity indicator in the model. In a few cases, our preferred forecasts are somewhat less accurate than the baseline (e.g., 24-months-ahead forecasts for capacity utilization).

In addition we compare point and density forecasts from our preferred shadow-rate VAR against those obtained from a plug-in approach, where external shadow-rate estimates, specifically from Wu and Xia (2016) and Krippner (2013, 2015), are used as data, in place of the actual short-term interest rate, in an otherwise standard VAR. Following the spirit of the shadow-rate literature, and different from the truncaetd VAR discussed before, forecasts from the plug-in VAR are simulated without censoring the resulting (shadow) interest rate projections. Similar to the case of our shadow-rate VAR, forecasts for nominal interest rates are censored only after the dynamic simulations for all variables (and all horizons) have been done. As reported in Tables 5 and 6, we find consistent benefits for point and density forecasting from using the shadow-rate VAR across a wide range of variables

In addition, our online results appendix reports various robustness checks, with fairly similar results to what is reported here. In particular, in light of concerns raised by Krippner (2020), we replace various interest rates with alternative measures of similar maturities. Changing the 10-year Treasury yield, as used in our baseline, against the 20-year Treasury yield, has little effect on our results. Using the 3-month Treasury bill rate, instead of the federal funds rate, delivers also broadly similar forecast comparisons; if anything favoring the shadow-rate VAR a little more than what is reported here. We also consider alternative versions of the truncated and shadow-rate VAR based on setting the ELB to 12.5 basis points (rather than 25 basis points). While the ELB tends to bind a little less often in this case, the forecast comparisons tend to display similar patterns as in our baseline results. Finally, the online results appendix reports alternative forecast comparisons, derived from a slightly shorter evaluation period, ending in 2017:12, to avoid data related to the outbreak of the COVID-19 pandemic in 2020 to affect the results. While the economic effects of the pandemic left a heavy mark on readings of macroeconomic and financial variables in 2020 — see, for example, our companion work in Carriero et al. (2021) — those did not materially affect the relative model forecast comparisons reported here.

5.2 Interest rate forecasts made at selected origins since 2009

To this point, the results presented have focused on the average performance of the various models over the entire evaluation sample. In broad terms, these comparisons show that our proposed shadow-rate specification performs best for forecasting the federal funds rate, with the truncated approach not quite as good, and the standard VAR materially worse. To get a better understanding of this relative performance, and also to get a glance at the absolute performance, it is instructive to compare the point and density forecasts of the federal funds rate for selected forecast origins. Figure 4 reports a set of point forecasts (medians) and 68 percent bands of distributions, at horizons of 1 through 24 months, in December of 2013, 2014, 2015 (when the FOMC raised the funds rate), 2016, and 2018. In this figure, the actual path of the federal funds rate is represented by green dots.

To illustrate the effects of ignoring the ELB, Panel (a) of Figure 4 compares forecasts from a standard VAR ignoring the ELB with forecasts from the truncation approach, using a forecast origin of 2013:12, two years before the FOMC actually raised the funds rate target from the ELB.[36] In this case, the point forecast from ignoring the ELB (solid red line) proves a little more accurate than the forecast obtained with the truncation approach (solid blue line), although this do-nothing point forecast is negative for the entire forecast horizon, in contrast with the ELB. The uncertainty around the do-nothing point forecast is also generally larger than for the truncated model, in particular at longer horizons.

The remaining panels of Figure 4 compare forecasts from the truncation approach (black line with gray shading) to those from our shadow-rate specification (blue lines), for forecast origins in December 2013, 2014, 2015, 2016, and 2018. In the examples for the years before the FOMC raised the funds rate target above the ELB, the point forecasts from the shadow-rate specification are much more accurate than those from the model using truncation. The differences are clearly large in 2013, 2014, and 2015, when the shadow-rate forecast is at or just slightly above the ELB throughout the forecast horizon, whereas the truncation-based forecast rises throughout the horizon. Since the median forecasts from the shadow-rate model correctly predict the FFR to stay at the ELB most of the time during our evaluation window, these forecasts also achieve a perfect MAD score of 0 as reported in Table 3 for most forecast horizons. Considering forecasts made in 2013, 2014, and 2015, it is also striking that the forecast intervals are much narrow with the shadow-rate specification. Although the same basic patterns prevail in subsequent years, the specifics of the pictures evolve. In the case of forecasts made in 2015:12, both the shadow rate and truncation approaches show increases in the funds rate, but the shadow rate's increase is later and much smaller

[36]For brevity, our discussion will abstract from nuances of the real-time data flow, and simply refer to forecasts being "made" at (or even "in" the month of) a particular forecast origin, even though the underlying data would have been available in FRED-MD only in a subsequent month.

than that projected by the truncated specification. From a mechanical perspective, the tendency of the truncated VAR to place larger odds on interest rate increases near the ELB reflects its dependence on lagged actual rates, which are censored, rather than the uncensored shadow rates, as discussed in Section 2. From an economic perspective, the shadow-rate VAR can capture lower-for-longer or make-up elements of the Federal Reserve's monetary policy strategy through the dependence of predicted interest rates on lagged notional rates as suggested, for example, by the models of Reifschneider and Williams (2000), Gust et al. (2017) and Billi (2020). Moreover, the shadow-rate estimates are informed by observed data on longer-term yields and economic conditions, which enables the estimates to pick up on the effects of unconventional policies, such as forward guidance and asset purchases, through these channels.

In the 2016:12 and 2018:12 cases, both coming after the FOMC had raised the funds rate off the ELB, forecasts from the shadow rate and truncated specifications are relatively similar. In Panel (e), depicting forecasts made in 2016:12, both models under-predict the increase in the funds rate that eventually happens. Given the earlier behavior of the FFR, neither model had enough information to predict the sharp increase in the FFR that would have taken place in the following months. Finally, Panel (f) shows forecasts made in 2018:12 — with the forecast horizon extending out to 2020:12 — with the Covid-19 period included in the evaluation sample. Both specifications predict a decline in the funds rate over the first 9 months that was sharper than actually occurred, but starting in March 2020, the funds rate fell much faster than the models predicted. Of course, no model could have predicted — 15 months ahead — the outbreak of the pandemic and the easing of monetary policy that followed.

5.3 Forecasts made since the outbreak of COVID-19 in 2020

The period following the outbreak of the COVID-19 pandemic in the US and the aggressive easing of monetary policy by the FOMC provides an opportunity for a case study of predicted interest rate dynamics from our shadow-rate VAR as compared to a standard VAR that ignores the ELB and a VAR approach that relies on truncation. Figure 5 shows the evolution of federal funds rate forecasts over selected origins between January and September 2020. In January 2020, prior to the outbreak of COVID-19 in the US, forecasts from all model variants could, of course, not yet foresee the outbreak of COVID-19, and predicted the FFR to hover around its then-value of about 1.5 percent for the next two years.[37] Once the outbreak of COVID-19 hit the US economy, in March and April, the point forecast from a standard VAR puts the funds rate well below the ELB for the entire forecast horizon, with substantial probability mass on very negative rates. At subsequent forecast origins, the point forecast was close to the ELB, but substantial mass in the predictive distribution remained in negative territory. At the other extreme, the approach of truncating federal funds rate predictions at the ELB resulted in point forecasts that had the federal funds rate gradually rising over the forecast horizon, with substantial probability mass on quite high rates. These results reflect the dependence of predicted values in the truncated VAR on lagged actual rates, which are censored, rather than the uncensored shadow rates.

Through the first half of the year, the point forecast of the funds rate from the shadow-rate approach remained at the ELB throughout the forecast horizon. In later months, the point forecast from the shadow-rate VAR shows a small increase in the funds rate after 12 months or so. In all cases, the predictive distributions from the shadow-rate VAR are considerably narrower than those obtained with the truncation approach.

Forecasts of the federal funds rate from the shadow-rate VARs reflect the predicted evolution of the shadow rate (not shown in the interest of chart readability). As noted before, shadow rates reflect the unconstrained policy-rate prescriptions of the feedback rule for monetary policy that is implied by the VAR in (4).[38] As a reference point, the Federal

[37] Nevertheless, uncertainty bands generated from the standard VAR assigned odds of over 30 percent to the event of the funds rate falling below the ELB after a year and a half.

[38] The interpretation of the shadow-rate VAR as embedding the monetary feedback rule for the federal funds rate extends arguments made, for example, by Bernanke and Blinder (1992), Christiano, Eichenbaum, and Evans (1996, 1999), or Rotemberg and Woodford (1997) in the context of a standard VAR to the shadow-rate case.

Reserve Bank of Cleveland regularly publishes a set of policy path prescriptions obtained from monetary policy rules in the tradition of Taylor (1993, 1999). Prescriptions are derived from seven different rules and for three alternative sets of forecasts for economic conditions; similar to our shadow-rate concept, all prescriptions ignore the ELB.[39] Some of these rules are so-called "inertial" rules that generate prescriptions with strong lagged dependence on past policy rates. At the ELB, inertia with respect to lagged unconstrained prescriptions (or so-called notional rates) can capture lower-for-longer policies as discussed, for example, by Billi (2020), and resemble the form of the feedback rule embedded in our shadow-rate VAR.

With data available as of December 1, 2020, the median rule prescription calculated by the Federal Reserve Bank of Cleveland puts the federal funds rate at about −50 basis points in 2020:Q4 and −70 basis points for the first three quarters of 2021. In comparison, our shadow-rate estimates for the COVID-19 period from April through September 2020 are modestly negative (about −40 basis points in September 2020), which broadly aligns with unconstrained prescriptions of common policy rules.[40] Moreover, the shallow funds-rate path predicted by the shadow-rate VAR is also much closer to survey expectations obtained from Blue Chip Financial Forecasts and the Survey of Professional Forecasters (SPF).[41]

6 Conclusion

Motivated by the prevalence of lower bound constraints on nominal interest rates, this paper develops a tractable approach to including a shadow-rate specification in medium-scale VARs commonly used in macroeconomic forecasting. Our model treats interest rates as censored

[39] The rule results and documentation are available at `https://www.clevelandfed.org/en/our-research/indicators-and-data/simple-monetary-policy-rules.aspx`. Prescriptions from a similar set of policy rules are also computed by staff at the Board of Governors and presented to the FOMC ahead of each of its meetings as part of Tealbook Book B; see also Board of Governors of the Federal Reserve System (2020).

[40] We refer to rule prescriptions that ignore the ELB on the federal funds rate as "unconstrained."

[41] For example, the third-quarter SPF of 2020 does not see any significant rise in the 3-month Treasury rate before the end of 2023, which is an even shallower path than the funds-rate projections from the shadow-rate VAR shown in Panel (e) of Figure 5 for September 2020. The first-quarter SPF of 2021 sees a modest rise in short-term interest rates over the course of 2023.

observations of a latent shadow-rate process in an otherwise standard VAR setup, with the shadow rate allowed to go below the ELB when the actual interest rate is at the ELB, and with the shadow rate equal to the observed interest rate when the ELB is not binding. Our approach extend the specific unobserved components model of Johannsen and Mertens (2021) to the general VAR setting. By using a computationally more efficient shadow-rate sampling algorithm than Johannsen and Mertens (2021), together with the recursive methods of Carriero, Clark, and Marcellino (2019) for efficient estimation of Bayesian VARs with stochastic volatility, our approach is easily applied to a medium-scale VAR system.

We use our shadow rate approach to form forecasts from a medium-scale BVAR with stochastic volatility. In our results, forecasts for interest rates obtained from a shadow-rate VAR for the US since 2009 are clearly superior, both in terms of point and density forecasts, to predictions from a standard VAR that ignores the ELB. These interest rates include not only the federal funds rate but also longer-term bond yields. For other indicators of financial conditions and measures of economic activity and inflation, the accuracy of forecasts from our shadow-rate specification is broadly on par with a standard VAR that ignores the ELB. Overall, our shadow-rate specification successfully addresses the ELB and improves interest rate forecasts without harming a standard VAR's ability to forecast a range of other variables.

A Shadow-rate sampling

This appendix provides further details on the application of a shadow-rate sampler in a VAR context that can be embedded into an otherwise standard MCMC estimation. Throughout, we take as given values of all parameters (incl. SV) of the shadow-rate VAR in (4). These parameter values can be obtained from standard MCMC steps (and based on previously sampled "data" for $\{z_t\}_{t=1}^{T}$) as described, for example in Carriero, Clark, and Marcellino (2019). Here we focus on the MCMC step concerned with sampling from the shadow-rate problem stated in (6) for given values of the VAR parameters $\{A_j\}_{j=1}^{p}$ and $\{\Sigma_t\}_{t=1}^{T}$. For ease of notation, references to $\{A_j\}_{j=1}^p$ and $\{\Sigma_t\}_{t=1}^T$ will be suppressed from the conditioning sets described below. Moreover, the value of ELB is a known constant. For ease of exposition, we continue to focus on the case of a scalar shadow rate, s_t, which can, however, be easily generalized to the case of $N_s > 1$.

A.1 Gibbs sampling from the truncated multivariate normal

While the joint density from a multivariate normal (MVN) distribution can be factorized into a product of univariate normal densities, the same property does not generally extend to the truncated MVN density, as discussed, for example, by Geweke (1991). In our context, this means that draws from the missing value problem, $\boldsymbol{S} \,|\, \bar{\boldsymbol{Y}} \sim N(\boldsymbol{\mu}, \boldsymbol{\Omega})$ in (6), could be recursively obtained by a sequence of univariate normal draws, but not so for the corresponding shadow-rate problem $\boldsymbol{S} \,|\, \boldsymbol{Y} \sim TN(\boldsymbol{\mu}, \boldsymbol{\Omega}, -\infty, ELB)$. However, consider a single element of $\boldsymbol{S}$, denoted s_t, and let $s_{1:t-1}$ and $s_{t+1:T}$ the vectors of all elements of $\boldsymbol{S}$ that precede and follow s_t, respectively.[42] Conditional on $s_{1:t-1}$ and $s_{t+1:T}$ (as well as $\boldsymbol{Y}$), s_t has, however, a univariate truncated normal distribution,

$$s_t \,\middle|\, s_{1:t-1}, s_{t+1:T}, \boldsymbol{Y} \sim TN\left(\mu_{1:t-1,t+1:T}, \omega_{1:t-1,t+1:T}, -\infty, ELB\right) , \tag{7}$$

with moment parameters $\mu_{1:t-1,t+1:T}$ and $\omega_{1:t-1,t+1:T}$ identical to those obtained from the corresponding missing value problem

$$s_t \,\middle|\, s_{1:t-1}, s_{t+1:T}, \bar{\boldsymbol{Y}} \sim N\left(\mu_{1:t-1,t+1:T}, \omega_{1:t-1,t+1:T}\right) . \tag{8}$$

For details, see, for example, Horrace (2005), and Chopin (2011).[43]

[42] In the case of $N_s > 1$, so that s_t is not scalar, the argument made here applies to a single scalar element of s_t conditional on values for the remainder of the shadow-rate vector, as well as $s_{1:t-1}$, $s_{t+1:T}$, and $\boldsymbol{Y}$.

[43] The argument extends also to the case where the truncation bounds vary from one element of the vector to another, which allows us to handle also where the sequence $s_{1:T}$ covers observations where the ELB does not bind so that of $s_t = i_t > ELB$. To handle those cases, the shadow-rate samplign problem can be stated more generally as $\boldsymbol{S} \,|\, \boldsymbol{Y} \sim TN(\boldsymbol{\mu}, \boldsymbol{\Omega}, -\infty, \boldsymbol{I})$ where $\boldsymbol{I}$ is the vector of all actual rates, and the inequality $\boldsymbol{S} \leq \boldsymbol{I}$ applies elementwise. Note that observations of i_t for which the ELB does not bind are included in

As discussed already by Geweke (1991), Gelfand, Smith, and Lee (1992) and references therein, the fact that conditional distributions of the truncated MVN are also truncated normals, lends the problem of obtaining draws from the truncated MVN to a Gibbs sampler, which we also pursue.[44]

Our Gibbs sampler employed to draw from the shadow-rate problem in (6) exploits the particular structure of our VAR(p) setting to derive the conditional moments, $\mu_{1:t-1,t+1:T}$, $\omega_{1:t-1,t+1:T}$ without ever having to compute the entire mean vector, $\boldsymbol{\mu}$, and variance-covariance matrix, $\boldsymbol{\Omega}$, of the full shadow-rate problem in (6).[45]

To derive $\mu_{1:t-1,t+1:T}$, $\omega_{1:t-1,t+1:T}$, we now focus on the moments of the missing value problem stated in (8), understanding that draws for s_t are to be generated from the shadow-rate problem in (7). We draw from the (univariate) truncated normal distribution directly by application of uniform-inverse-transform sampling. Alternatively, rejection sampling could be used, or a combination of both approaches that considers the optimal acceptance probability for application of the rejection sampling approach; see, for example, Geweke (1991), Chopin (2011), or Botev (2017).[46]

$\bar{\boldsymbol{Y}}$ so that the distribution of the missing-value problem in (6) collapses on a point mass, $s_t = i_t$, for those observations.

[44]Specifically, adopting language from Geweke (1991), let $\boldsymbol{S}^{(j)}$ denote a draw of $\boldsymbol{S}$ from the jth pass of our MCMC sampler over all states and parameters of the shadow-rate VAR. In order to generate a typical element of $\boldsymbol{S}^{(j)}$, denoted $s_t^{(j)}$, we iterate from $t = 1, 2 \ldots, T$ to generate draws from $s_t^{(j)} \mid s_{1:t-1}^{(j)}, s_{t+1:T}^{(j-1)}, \bar{\boldsymbol{Y}}$.

[45]By exploiting sparsities and the recursive structure of the VAR's state space representation, our approach echoes recent advances in the field of sampling form the truncated MVN distribution made by Cong, Chen, and Zhou (2017), albeit specialized to the VAR(p) that we intend to investigate further. Other advances in Gibbs sampling from the truncated MVN distribution are discussed by Robert (1995), Damien and Walker (2001), Chopin (2011), and Botev (2017).

[46]In our application, potential gains from applying a combination approach appeared so far, however, to be limited if not negative. At least in our MATLAB programming environment, direct application of the `trandn.m` routine provided by Botev (2017) underperformed relative to uniform-inverse-transform sampling. A likely cause for the somewhat surprising performance of the latter, appears to be our use of large pregenerated random arrays as opposed to generating pseudo-random value one-at-a-time as done in the case of `trandn.m`.

A.2 Companion form of the shadow-rate VAR

As the joint dynamics of s_t and x_t in the shadow-rate VAR system in (4) are characterized by a VAR(p) it is sufficient to consider no more than p lags and leads of z_t in the derivation of $\mu_{1:t-1,t+1:T}$, $\omega_{1:t-1,t+1:T}$. To allow for more compact notation consider the following companion form notation for the VAR (omitting intercepts), adapted to the partitioning of z_t into x_t and s_t:

$$Z_t = \boldsymbol{A} Z_{t-1} + \boldsymbol{B}_t w_t \,, \qquad\qquad w_t \sim N(0, I) \,, \tag{9}$$

and let C_x and C_s be selection matrices so that $x_t = C_x Z_t$ and $s_t = C_s Z_t$.

To construct this companion form, consider for concreteness the case of a second-order system, with $p = 2$, and let

$$X_t = \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}, \qquad S_t = \begin{bmatrix} s_t \\ s_{t-1} \end{bmatrix}, \qquad Z_t = \begin{bmatrix} X_t \\ S_t \end{bmatrix} = \begin{bmatrix} x_t \\ x_{t-1} \\ s_t \\ s_{t-1} \end{bmatrix}. \tag{10}$$

$$\text{with} \quad \boldsymbol{A} = \begin{bmatrix} A^1_{xx} & A^2_{xx} & A^1_{xs} & A^2_{xs} \\ I & 0 & 0 & 0 \\ A^1_{sx} & A^2_{sx} & A^1_{ss} & A^2_{ss} \\ 0 & 0 & I & 0 \end{bmatrix}, \qquad \boldsymbol{B}_t = \begin{bmatrix} B_{x,t} \\ 0 \\ B_{s,t} \\ 0 \end{bmatrix}, \tag{11}$$

where $B_{x,t}$ and $B_{s,t}$ are conformable partitions of a factorization $\Sigma_t^{0.5}$ of the variance-covariance matrix of VAR residuals in (4), so that $\Sigma_t = \Sigma_t^{0.5} \left(\Sigma_t^{0.5}\right)'$ and $\Sigma_t^{0.5} = \begin{bmatrix} B'_{x,t} & B'_{s,t} \end{bmatrix}'$.[47]

[47]Without loss of generality, we can, but do not have to, assume that $\Sigma_t^{0.5}$ is lower triangular. In our application, based on the VAR-SV model of Carriero, Clark, and Marcellino (2019) we have $\Sigma_t^{0.5} = A_0^{-1} \Lambda_t^{0.5}$ where A_0 is a unit-lower-triangular, and $\Lambda_t^{0.5}$ is a diagonal matrix of stochastic volatilities.

A.3 Moments of the missing-value problem

Given the VAR(p) structure of the model, and using the companion-form notation introduced above, the posterior density of the Gaussian missing-value problem in (8) simplifies as follows:

$$f\left(s_t \,\middle|\, s_{1:t-1}, s_{t+1:T}, \bar{\boldsymbol{Y}}\right) = f\left(s_t \,\middle|\, s_{t-p:t-1}, s_{t+1:t+p}, x_{t-p:t+p}\right) \tag{12}$$

$$= f\left(s_t \,\middle|\, Z_{t-1}, Z_{t+p}, x_t\right) \tag{13}$$

$$= f\left(s_t \,\middle|\, Z_{t-1}, Z_{t+p} - Z_{t+p|t-1}, v_t^x\right) \tag{14}$$

where $Z_{t+p|t-1} = E(Z_{t+p}|Z_{t-1}) = \boldsymbol{A}^{p+1} Z_{t-1}$ and $v_t^x = x_t - E(x_t|Z_{t-1}) = C_x v_t$.

As stated above, we assume to have observations for at least p initial lags of s_t at $t = 1$, and can thus always condition on $s_{t-p:t-1}$. Now, consider observations for t such that $t+p \leq T$ and let

$$\boldsymbol{\mathcal{Z}}_{t+p} = \begin{bmatrix} Z_{t+p} - Z_{t+p|t-1} \\ v_t^x \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{p} \boldsymbol{A}^{p-j} \boldsymbol{B}_{t+j}\, w_{t+j} \\ C_x \boldsymbol{B}_t\, w_t \end{bmatrix} . \tag{15}$$

The moments $\mu_{1:t-1,t+1:T}$ and $\omega_{1:t-1,t+1:T}$ follow from standard Gaussian signal extraction arguments:

$$\mu_{1:t-1,t+1:T} = E(s_t|Z_{t-1}, Z_{t+1}, x_t) = E(s_t|Z_{t-1}) + \boldsymbol{J}_t \boldsymbol{\mathcal{Z}}_{t+p} \tag{16}$$

$$\text{with} \quad \boldsymbol{J}_t = \operatorname{Cov}\left(s_t, \boldsymbol{\mathcal{Z}}_{t+p} \,\middle|\, Z_{t-1}\right) \left(\operatorname{Var}\left(\boldsymbol{\mathcal{Z}}_{t+p} \,\middle|\, Z_{t-1}\right)\right)^{-1} , \tag{17}$$

$$\operatorname{Var}\left(\boldsymbol{\mathcal{Z}}_{t+p} \,\middle|\, Z_{t-1}\right) = \begin{bmatrix} \sum_{j=0}^{p} \boldsymbol{A}^{p-j} \boldsymbol{B}_{t+j} \boldsymbol{B}_{t+j}' \left(\boldsymbol{A}^{p-j}\right)' & \boldsymbol{A}^p \boldsymbol{B}_t \boldsymbol{B}_t' C_x' \\ C_x \boldsymbol{B}_t \boldsymbol{B}_t' \left(\boldsymbol{A}^p\right)' & C_x \boldsymbol{B}_t \boldsymbol{B}_t' C_x' \end{bmatrix} , \tag{18}$$

$$\operatorname{Cov}\left(s_t, \boldsymbol{\mathcal{Z}}_{t+p} \,\middle|\, Z_{t-1}\right) = \begin{bmatrix} C_s \boldsymbol{B}_t \boldsymbol{B}_t' \left(\boldsymbol{A}^p\right)' \\ C_s \boldsymbol{B}_t \boldsymbol{B}_t' C_x' \end{bmatrix} . \tag{19}$$

$$\text{and}\quad \omega_{1:t-1,t+1:T} = \text{Var}\left(s_t | Z_{t-1}, Z_{t+1}, x_t\right) = C_s \boldsymbol{B}_t \boldsymbol{B}_t' C_s'$$
$$- \text{Cov}\left(s_t, \boldsymbol{\mathcal{Z}}_{t+p} \,\middle|\, Z_{t-1}\right) \left(\text{Var}\left(\boldsymbol{\mathcal{Z}}_{t+p} \,\middle|\, Z_{t-1}\right)\right)^{-1} \text{Cov}\left(s_t, \boldsymbol{\mathcal{Z}}_{t+p} \,\middle|\, Z_{t-1}\right). \quad (20)$$

To compute the Kalman-smoothing gain $\boldsymbol{J}_t$ and residual variance $\text{Var}\left(s_t | Z_{t-1}, Z_{t+1}, x_t\right)$ efficiently, and robustly to numerical round-off errors (which could otherwise imply non-positive-definite values for variance-covariance matrices) we employ a QR factorization, that builds on some of the fast-array algorithms presented by Kailath, Sayed, and Hassibi (2000).

For observations t with $t + p > T$, the signal vector must be limited to include only leads of s_t and x_t up to T. Specifically, let $t = T - k$ (with $k < p$) and the adapted signal vector becomes

$$\boldsymbol{\mathcal{Z}}_T = \begin{bmatrix} Z_T - Z_{T|t-k-1} \\ v_t^x \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{k} \boldsymbol{A}^{p-j} \, \boldsymbol{B}_{t+j} \, w_{t+j} \\ C_x \, \boldsymbol{B}_t \, w_t \end{bmatrix}. \quad (21)$$

and the expressions for $\boldsymbol{J}_t$ and $\text{Var}\left(s_t | Z_{t-1}, Z_{t+1}, x_t\right)$ in (17) and (20) are adjusted accordingly.

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Table 1: List of variables

Variable	FRED-MD code	transformation	Minnesota prior
Real Income	RPI	$\Delta\log(x_t)\cdot 1200$	0
Real Consumption	DPCERA3M086SBEA	$\Delta\log(x_t)\cdot 1200$	0
IP	INDPRO	$\Delta\log(x_t)\cdot 1200$	0
Capacity Utilization	CUMFNS		1
Unemployment	UNRATE		1
Nonfarm Payrolls	PAYEMS	$\Delta\log(x_t)\cdot 1200$	0
Hours	CES0600000007		0
Hourly Earnings	CES0600000008	$\Delta\log(x_t)\cdot 1200$	0
PPI (Fin. Goods)	WPSFD49207	$\Delta\log(x_t)\cdot 1200$	1
PPI (Metals)	PPICMM	$\Delta\log(x_t)\cdot 1200$	1
PCE Prices	PCEPI	$\Delta\log(x_t)\cdot 1200$	1
Federal Funds Rate	FEDFUNDS		1
Housing Starts	HOUST	$\log(x_t)$	1
S&P 500	SP500	$\Delta\log(x_t)\cdot 1200$	0
USD / GBP FX Rate	EXUSUKx	$\Delta\log(x_t)\cdot 1200$	0
5-Year Yield	GS5		1
10-Year Yield	GS10		1
Baa Spread	BAAFFM		1

Note: Data obtained from the 2020-10 vintage of FRED-MD. Monthly observations from 1959:M03 to 2020:M09. Entries in the column “Minnesota prior” report the prior mean on the first own-lag coefficient of the corresponding variable in each BVAR. Prior means on all other VAR coeffcients are set to zero.

Table 2: Relative RMSE

					Relative to Standard …							
	Standard				Truncated				Shadow rate			
Variable / Horizon	3	6	12	24	3	6	12	24	3	6	12	24
Real Income	15.66	15.55	15.68	15.53	1.00	1.00*	1.00	1.00	1.00	1.00	1.00	0.99
Real Consumption	19.17	18.55	18.58	18.71	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
IP	18.58	18.76	18.95	17.95	0.99	1.01	1.01	0.99*	1.00	1.00	1.01	1.01
Capacity Utilization	1.41	2.31	2.78	2.99	1.00	1.03	1.16	1.30	1.01	1.01	1.08	1.30***
Unemployment	0.90	1.32	1.56	1.76	1.00	1.00	1.01**	1.11**	1.00	1.00	0.99	1.00
Nonfarm Payrolls	16.79	17.38	16.62	16.28	1.00	1.00*	1.00	1.00	1.00	1.00	1.00	1.00
Hours	0.32	0.39	0.45	0.46	1.01	1.02*	1.13	1.35	1.06	1.01	1.00	1.08
Hourly Earnings	2.88	2.57	2.45	2.32	1.00	0.99	0.97	0.92**	0.99	1.01	1.00	0.96***
PPI (Fin. Goods)	8.53	8.50	8.54	8.47	1.00	0.99	0.97	0.94	1.02	1.03	1.03	0.99
PPI (Metals)	40.83	39.81	38.34	37.92	1.01	1.00	0.99	1.01	1.01	1.01	1.01	0.99**
PCE Prices	2.47	2.53	2.40	2.48	0.98*	0.95	0.90	0.77	1.06**	1.07*	1.09	1.06
Federal Funds Rate	0.21	0.41	0.59	0.77	0.46*	0.48	0.58	1.02	0.36*	0.36	0.34	0.52**
Housing Starts	0.08	0.10	0.11	0.12	1.03	1.00	1.00	1.08	1.05	1.06	1.04	0.95
S&P 500	47.90	45.54	45.71	43.55	1.01	1.02	1.03	1.04	0.98*	0.99	0.99	0.99
USD / GBP FX Rate	25.67	26.33	26.20	25.47	1.01	1.00	1.00	1.01	0.99	0.99*	0.99	0.97***
5-Year Yield	0.27	0.43	0.52	0.62	1.01	1.05	1.29	1.93	0.93	0.92	0.77**	0.70**
10-Year Yield	0.24	0.39	0.48	0.59	1.05	1.06**	1.34	2.27	0.98	0.95	0.81	0.64**
Baa Spread	0.33	0.62	0.87	1.09	0.96	1.04	1.25	2.26	1.00	0.97	0.89	0.83

Note: Comparison of "Standard" (baseline, in denominator of relative comparisons) against "Truncated" and "Shadow rate." Values below one indicate improvement over baseline. Evaluation window from 2009:M01 through 2020:M09. Significance assessed by Diebold-Mariano-West test using Newey-West standard errors with $h+1$ lags. Due to the close behavior of some of the models compared, and rounding of the report values, a few comparisons show a significant Relative RMSE of 1.00. These cases arise from persistent differences in performance that are, however, too small to be relevant after rounding.

Table 3: Relative MAD

	Standard				Relative to Standard … Truncated				Shadow rate			
Variable / Horizon	3	6	12	24	3	6	12	24	3	6	12	24
Real Income	2.42	2.43	2.46	2.55	1.06**	1.01	1.15	1.03	1.00	1.04	0.99	1.07
Real Consumption	2.15	1.91	1.76	1.77	1.05	1.03	1.06	0.89	1.06	1.01	1.11	0.95
IP	4.48	4.38	4.30	4.46	0.90*	1.02	1.09	0.85	1.06*	1.06	1.15**	1.10**
Capacity Utilization	0.27	0.41	0.45	0.62	1.04	1.04	1.09	1.15	1.16***	1.24**	1.39**	1.92***
Unemployment	0.09	0.12	0.14	0.16	1.05	1.05	1.20***	1.39***	1.08**	1.13	1.13	1.00
Nonfarm Payrolls	0.67	0.70	0.76	0.87	1.02	1.03*	1.19**	1.32***	1.05	1.09	1.30**	1.30**
Hours	0.11	0.13	0.15	0.15	1.00	1.01*	0.99	1.37*	1.02*	1.10	1.06	1.29**
Hourly Earnings	1.35	1.34	1.47	1.43	1.01	1.00	0.80	0.87**	1.05	1.00	1.10	0.99*
PPI (Fin. Goods)	5.02	4.92	4.85	4.75	1.03	1.00	0.99	0.91	1.06	0.94	0.94	0.89
PPI (Metals)	20.89	18.46	20.38	20.67	1.00	1.02	1.00	1.02	0.99	1.00	0.98	1.07
PCE Prices	1.11	1.29	1.33	1.47	1.03**	0.93*	0.86*	0.85*	1.07	1.05	1.02	1.08
Federal Funds Rate	0.05	0.09	0.15	0.19	0.76**	1.01	1.11	1.10	0.00***	0.00**	0.00**	0.00***
Housing Starts	0.05	0.05	0.06	0.06	0.98	0.97	0.84	1.44	0.95	1.07	0.87	0.84
S&P 500	21.59	23.37	23.17	21.22	1.01	1.07	1.02	1.08	0.97**	1.03**	0.95	1.07
USD / GBP FX Rate	15.51	15.34	16.17	15.41	1.01**	1.02	1.00	1.04	0.97	0.99*	0.97	0.96**
5-Year Yield	0.14	0.23	0.30	0.40	1.02	0.98	1.42	1.09	0.81	0.89	0.76	0.60*
10-Year Yield	0.13	0.21	0.30	0.36	0.97*	0.97**	1.42	1.40	0.96	0.97	0.88	0.82
Baa Spread	0.14	0.25	0.32	0.39	1.01	0.99	0.84	0.86	1.07	1.01	0.96	0.96

Note: Comparison of "Standard" (baseline, in denominator of relative comparisons) against "Truncated" and "Shadow rate." Values below one indicate improvement over baseline. Evaluation window from 2009:M01 through 2020:M09. Significance assessed by Diebold-Mariano-West test using Newey-West standard errors with $h + 1$ lags.

Table 4: Relative CRPS

					Relative to Standard ...							
	Standard				Truncated				Shadow rate			
Variable / Horizon	3	6	12	24	3	6	12	24	3	6	12	24
Real Income	4.75	4.93	5.08	5.10	1.00	1.01	1.01	1.01	0.99	0.99	1.00	1.01
Real Consumption	4.52	4.82	5.07	5.17	1.00	1.00	1.00	1.00	1.00	1.00	1.02**	1.02
IP	5.80	5.89	6.02	6.14	1.00	1.01	1.01	0.99	1.01**	1.01	1.04**	1.07**
Capacity Utilization	0.41	0.63	0.78	0.92	1.01	1.03	1.10	1.15**	1.04***	1.05*	1.12***	1.24***
Unemployment	0.18	0.29	0.37	0.44	1.00	1.00	1.05***	1.18***	1.01*	1.01	1.02	1.03
Nonfarm Payrolls	2.69	2.87	2.94	2.91	1.00	1.01**	1.03*	1.07***	1.00	1.01	1.02**	1.05***
Hours	0.13	0.15	0.17	0.18	1.00	1.02**	1.11*	1.22**	1.04**	1.04*	1.05	1.13**
Hourly Earnings	1.37	1.35	1.37	1.38	1.00	1.00	0.99	0.97***	1.00	1.01	1.00	1.00
PPI (Fin. Goods)	4.44	4.69	4.78	4.72	1.00	1.00	0.99	0.98	1.01	1.01	1.02	1.00
PPI (Metals)	21.09	21.18	20.63	20.50	1.00	1.00	1.00	1.01	1.00	1.00	1.00	1.01*
PCE Prices	1.13	1.27	1.31	1.34	0.99**	0.97*	0.95	0.90*	1.04	1.04	1.06	1.04
Federal Funds Rate	0.08	0.15	0.21	0.28	0.47**	0.55	0.62	0.70	0.28***	0.30**	0.29**	0.34***
Housing Starts	0.05	0.05	0.06	0.07	1.01	1.01	1.02	1.12	1.02	1.00	0.96	0.90
S&P 500	24.02	23.82	24.00	23.16	1.01*	1.01	1.02	1.02	0.99	0.99*	1.00	1.02***
USD / GBP FX Rate	14.42	14.75	14.68	14.45	1.01	1.00	1.00	1.02	0.99	0.99	0.99	1.00
5-Year Yield	0.13	0.22	0.28	0.35	0.99	1.04	1.20	1.40	0.92	0.93	0.81**	0.69***
10-Year Yield	0.12	0.21	0.27	0.34	1.01	1.06**	1.24*	1.60	0.96	0.95	0.86	0.80**
Baa Spread	0.15	0.27	0.37	0.45	0.99	1.00	1.06	1.21	1.01	1.01	0.96	1.00

Note: Comparison of “Standard” (baseline, in denominator of relative comparisons) against “Truncated” and “Shadow rate.” Values below one indicate improvement over baseline. Evaluation window from 2009:M01 through 2020:M09. Significance assessed by Diebold-Mariano-West test using Newey-West standard errors with $h + 1$ lags.

Table 5: Comparison against plug-in VAR with Wu-Xia shadow rates

	RMSE				MAD				CRPS			
Variable / Horizon	3	6	12	24	3	6	12	24	3	6	12	24
Real Income	1.00	1.00**	1.00	0.98***	0.96	1.00*	0.95	0.96	1.00	0.99*	0.99	1.00
Real Consumption	1.00	1.00	1.00	1.00	1.00	0.95	1.00	0.97	1.00	1.00	1.00	1.00
IP	1.00	1.00	1.00	1.00	1.06	1.02	1.09	0.97	1.01	1.01	1.01	1.01
Capacity Utilization	0.97	1.00	1.00	1.03	1.04	1.04	1.02	1.06	0.98	0.99	0.99	1.01
Unemployment	1.00	1.00	1.00	1.04	0.99	0.86***	0.90**	1.11	0.99**	0.97**	0.96**	1.00
Nonfarm Payrolls	1.00	1.00	1.00	1.00	0.95**	1.01*	1.18	1.35	0.99**	0.98**	0.99	1.01
Hours	0.98*	0.97**	0.95	1.02	0.95***	0.92**	0.95	1.04	0.98*	0.95**	0.96	1.05
Hourly Earnings	0.99	0.99	0.99	0.97	1.00	0.97	1.10	0.96	0.99	1.00	0.99	1.01
PPI (Fin. Goods)	1.02	1.01	1.01	0.98	0.99	0.99	1.01*	1.03	1.01	1.01	1.01	0.99
PPI (Metals)	1.00	1.00	1.00	0.99	1.05	0.98	1.00	1.06	1.00	1.00	1.00	1.01*
PCE Prices	1.02	1.00	1.01	0.99	1.04	1.10	0.98	0.98	1.00	0.99	1.00	0.99
Policy Rate	0.80**	0.89**	0.95	1.10**	–.–	–.–	–.–	–.–	0.77**	0.86**	0.94	1.00
Housing Starts	1.02	1.05	1.11	1.13	1.01	0.99	0.79	0.69	0.99	0.99	1.00	1.00
S&P 500	1.01	1.00	0.99	0.99	0.96	1.00**	0.96	1.03	1.00	0.99	0.99	1.01*
USD / GBP FX Rate	0.99	1.00	1.00	0.98	1.13	1.05	1.03	1.02	0.99	1.00	1.00	1.00
5-Year Yield	1.04	1.05	1.02	0.98	0.90	0.93	0.87	0.69	1.02	1.01	0.96	0.90
10-Year Yield	1.01	1.02	0.95	0.79**	1.04	0.91	0.93	0.81*	1.00	1.01	0.94	0.83**
Baa Spread	1.02	1.18	1.07	0.85	0.90	0.80	0.80	0.79	1.02	1.04	0.96	0.96

Note: Comparison of “WuXia (Censored)” (baseline, in denominator) against “Shadow rate.” Values below one indicate improvement over baseline. Evaluation window from 2009:M01 through 2020:M09. Significance assessed by Diebold-Mariano-West test using Newey-West standard errors with $h + 1$ lags. Due to the close behavior of some of the models compared, and rounding of the report values, a few comparisons show significant ratios of 1.00. These cases arise from persistent differences in performance that are, however, too small to be relevant after rounding. In some cases, due to strong performance of the baseline model, relative MAD may involve divisions by zero. These cases are reported as blank entries.

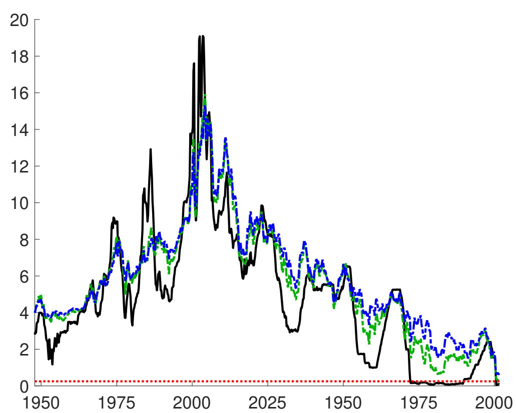
Table 6: Comparison against plug-in VAR with Krippner shadow rates

	RMSE				MAD				CRPS			
Variable / Horizon	3	6	12	24	3	6	12	24	3	6	12	24
Real Income	1.00	1.00*	1.00	0.98***	1.01	1.01	0.93	1.01	0.99**	0.99	0.99*	0.99
Real Consumption	1.00	1.00	1.00	1.00	0.96	0.93*	0.91	0.86**	1.00	0.99**	0.98*	0.98*
IP	1.01	1.00	1.01	0.99	1.03	1.09	1.10	0.93	1.01	1.01	1.01	0.98
Capacity Utilization	0.99	1.03*	1.07	1.12	1.00	1.04	1.12	1.25	1.00	1.03	1.05	1.06
Unemployment	1.00	1.00	1.00	1.06	0.96	0.89*	0.84*	1.03	0.98	0.96*	0.95*	1.01
Nonfarm Payrolls	1.00	1.00	1.00	1.00	0.89	1.10	1.09	1.08	1.00	0.99	0.98***	0.98
Hours	1.01	0.98	0.97	1.04	0.99	0.99	1.16	1.03	0.99	0.97	0.98	1.05
Hourly Earnings	1.00	1.00	1.01	0.96*	1.02	0.97	1.04	0.92	0.99	1.01	1.01	1.01
PPI (Fin. Goods)	0.99	0.98	0.97**	0.97	0.95	0.94	0.88	0.95	0.98	0.98*	0.97*	0.98
PPI (Metals)	0.99	0.99*	0.99	0.99	0.98	1.05	1.01	0.99	0.99	0.99**	0.99	1.00
PCE Prices	1.00	0.96*	0.95*	0.92*	1.02	1.03**	0.94	0.84	0.98	0.96**	0.95	0.95
Policy Rate	0.83*	0.88*	0.89	1.02	−.−	−.−	−.−	−.−	0.84	0.85	0.89	0.99
Housing Starts	1.01	1.04	1.08	1.01	1.00	1.06	0.81	0.71	0.98	0.99	0.96	0.90
S&P 500	1.00	1.00	1.00	0.99	0.94*	0.99	0.93	0.98***	0.98	0.99	0.99	0.99
USD / GBP FX Rate	1.01	1.01	1.00	0.99	1.13*	1.16	1.12	1.03	1.01	1.00	1.00	0.99
5-Year Yield	1.04	1.09*	1.05	0.98	0.96	1.06	0.95	0.64**	1.00	1.02	0.97	0.86**
10-Year Yield	0.98	1.00	0.93	0.80**	1.07	1.00	0.96	0.75**	0.97	0.97	0.90	0.78***
Baa Spread	1.02	1.09	0.97	0.74**	0.78	0.71	0.73	0.66**	0.95	0.94	0.84	0.80***

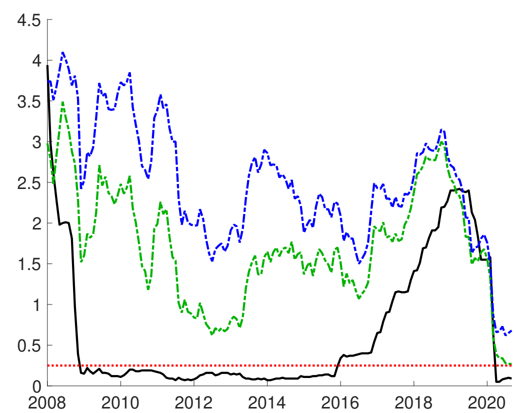
Note: Comparison of "Krippner (Censored)" (baseline, in denominator) against "Shadow rate." Values below one indicate improvement over baseline. Evaluation window from 2009:M01 through 2020:M09. Significance assessed by Diebold-Mariano-West test using Newey-West standard errors with $h + 1$ lags. Due to the close behavior of some of the models compared, and rounding of the report values, one of the comparisons shows a significant ratio of 1.00. This case arises from persistent differences in performance that are, however, too small to be relevant after rounding. In some cases, due to strong performance of the baseline model, relative MAD may involve divisions by zero. These cases are reported as blank entries.

Figure 1: Interest rate data

(a) Full data sample

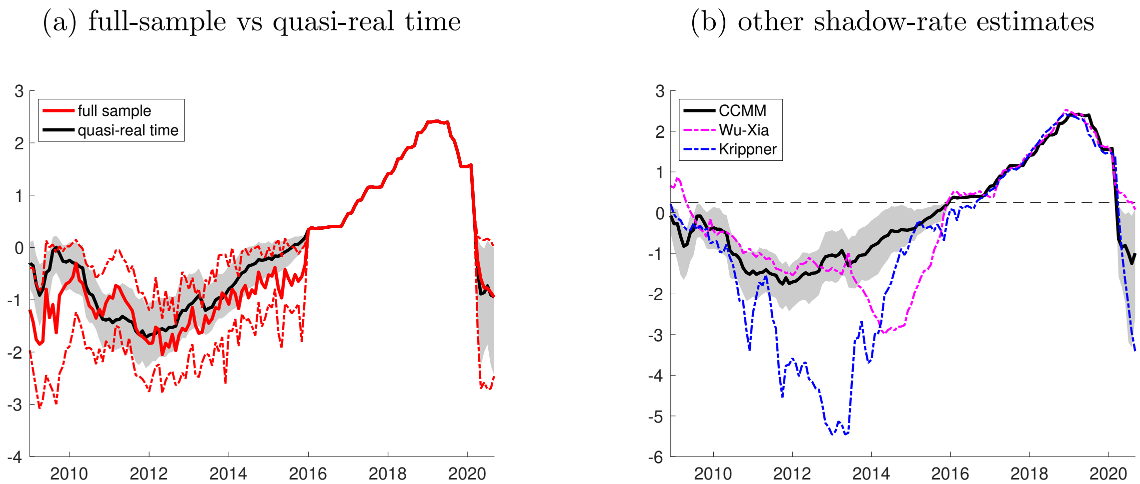


(b) Since the GFC



Note: All interest rates quoted as annualized percentage rates. Data obtained from FRED-MD, for further details see 3.

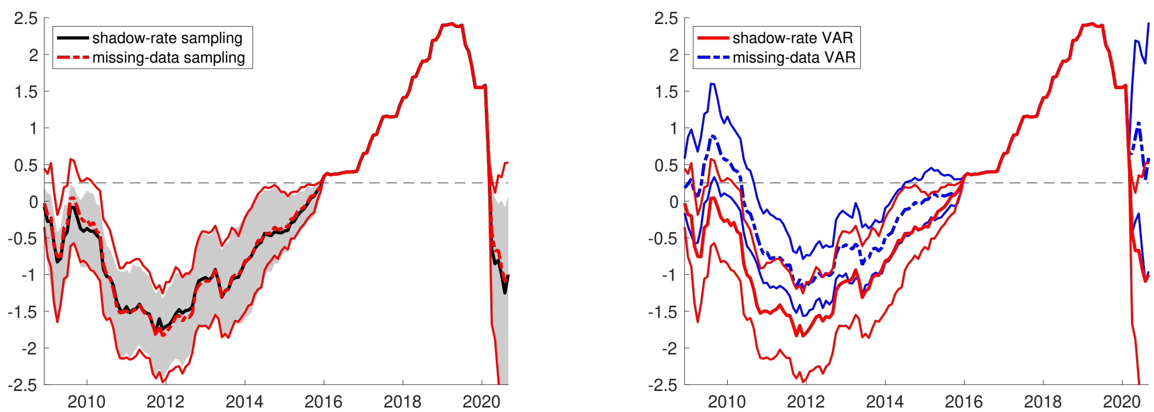
Figure 2: Shadow-rate estimates



Note: Panel (a) compares smoothed and filtered shadow-rate estimates from our baseline shadow-rate VAR. The filtered estimates reflect full re-estimation of the model over growing samples (quasi-real time). Posterior medians are shown as thick lines, grey shaded areas and thin lines depict 90% uncertainty bands. Panel (b) compares the smoothed shadow-rate estimates shown in Panel (a) against updated estimates obtained from Krippner (2013, 2015) and Wu and Xia (2016). Each estimation conditions on available data since 1959:03, but the figure omits the period prior to 2008 during which the ELB did not bind.

Figure 3: Effect of imposing ELB on shadow-rate estimates

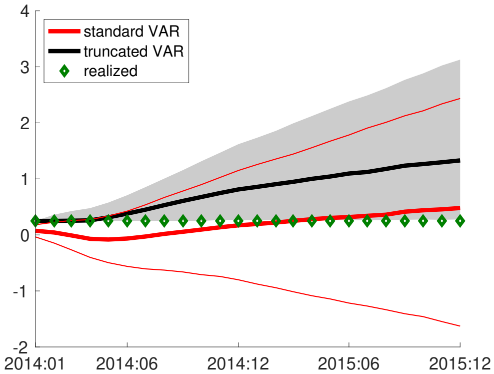
(a) Missing-data and shadow-rate draws (b) Missing-data draws from different VARs



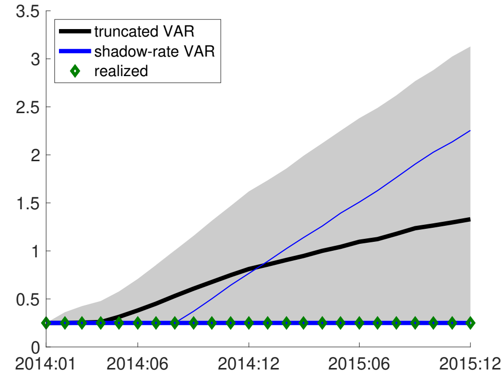
Note: Panel (a) compares shadow-rate (black) and missing-data (red) draws for s_t obtained from the posterior of our baseline shadow-rate VAR. Shadow-rate draws are obtained from the truncated posterior for s_t that satisfies the ELB. Missing-data draws are obtained from the underlying (and untruncated) posterior of the missing data problem that ignores the ELB. Panel (b) displays missing-data posteriors obtained from two sets of VAR estimates: In the baseline (red), parameter and SV draws reflect shadow-rate sampling. In the alternative version (blue), parameters and SV are drawn while treating the policy rate at the ELB as missing data and without requiring that missing data draws lie below the ELB. Medians (thick lines) and 90% uncertainty bands (thin lines and grey shaded area).

Figure 4: Predictive densities for the federal funds rate

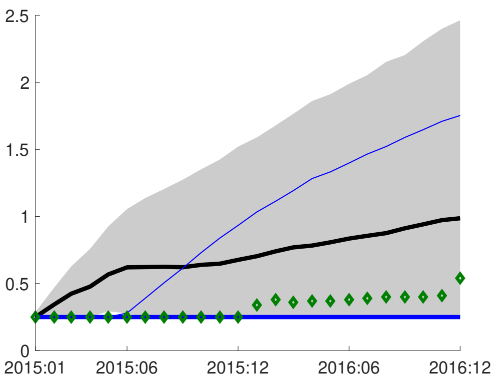
(a) 2013:12



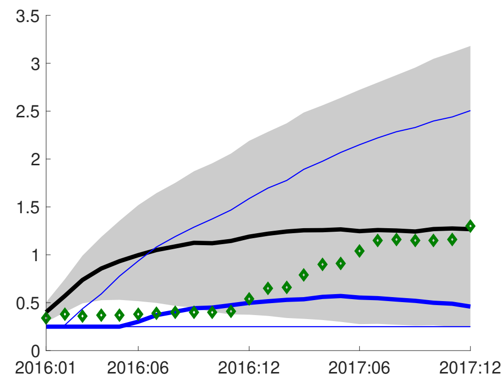
(b) 2013:12



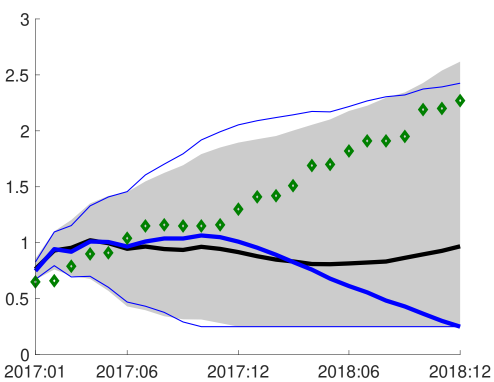
(c) 2014:12



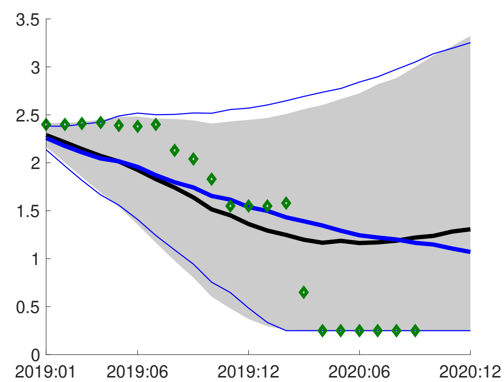
(d) 2015:12



(e) 2016:12

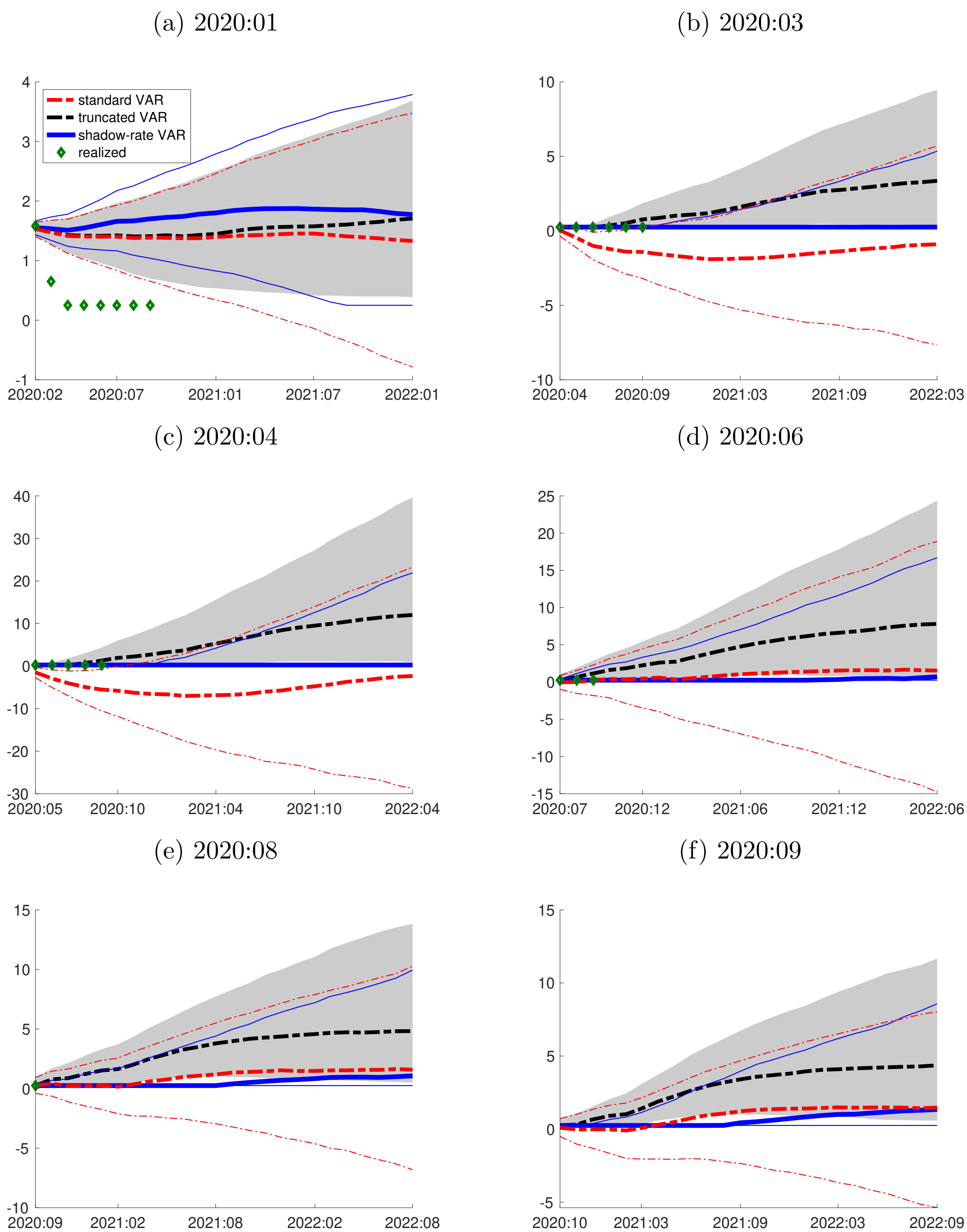


(f) 2018:12



Note: Predictive density for the federal funds rate, simulated out of sample at different jump-off dates for different models. Forecast horizons are reported on the horizontal axis of each panel. Panel (a) compares predictions from the standard VAR against those from the truncated VAR. The remaining panels compare predictions from the truncated VAR against those from the shadow-rate VAR. Realized values for the federal funds rate are shown as green diamonds and were set equal to the ELB value of 25 basis points from 2008:12 until 2015:12, and then again as of 2020:04.

Figure 5: Predictive densities for the federal funds rate in 2020



Note: Predictive density for the federal funds rate, simulated out of sample at different jump-off dates for different models. Realized values for the federal funds rate are shown as green diamonds (set equal to the ELB value of 25 basis points as of 2020:04).