Partnership with Persistence

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Abstract

We study a continuous-time model of partnership, with persistence and imperfect state monitoring. Partners exert private efforts to shape the stock of fundamentals, which drives the profits of the partnership, and the profits are the only signal they observe. The near-optimal strongly symmetric equilibria are non-Markovian and are characterized by a novel differential equation that describes the supremum of equilibrium incentives for any level of relational capital. Imperfect monitoring of the fundamentals helps sustain incentives, due to deferred incentives, and increases the partnership's value (*Sand in the wheels*). Good profit outcomes rally the partners to further increase effort when relational capital is low, but lead them to coast and decrease effort when relational capital is high. Even partnerships with high fundamentals may unravel after a short spell of terrible signals (*Beatles' break-up*).

Keywords: partnership, dynamic games, continuous time, relational capital

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1 Introduction

Partnerships are among the main forms of organizing joint economic activity. Characterized by a fixed rule for sharing the benefits, they are common among individuals, constitute one of the dominant forms of structuring a firm—along with corporations and limited liability companies—and are also common among businesses in the form of joint ventures. Yet each partnership is built on an incentive problem: partners exert private effort to contribute to a common good. The success of a joint venture requires everyone to pull his weight, but each partner is tempted to free-ride and blame lack of luck for poor results. The key to the success of a partnership is to properly motivate its members.

The incentive problem is particularly complicated in the case of ongoing, dynamic ventures. To fix ideas, consider a start-up. On a daily basis, each partner devotes his effort to improving the venture's fundamentals: upgrading the quality of the product; broadening the customer base; facilitating access to external capital; improving the internal organization; and more. Each of these fundamentals depends on the entire past stream of efforts and only gradually changes over time. Moreover, none of the fundamentals needs to be directly observed by the partners, who see only how they are reflected in profits, customer reviews, or internal audits. In such an environment, where actions have persistent effects and the state is imperfectly monitored, the scope for free-riding widens: a partner can shirk today, observe the profit or customer review outcomes, and try to catch up if those are flagging. At the same time, the range of potential motivating mechanisms widens as well.

In this paper, we present a dynamic model of partnership where effort has persistent effect, and the state—the *fundamentals*—is imperfectly monitored. We first develop a method to characterize near-optimal strongly symmetric equilibria of the game. They are characterized by a one-dimensional differential equation that describes the supremum of incentives achievable in an equilibrium, for any level of *relational capital*: an endogenous state variable capturing the "soft" capital—goodwill or mutual trust—in the partnership. Second, we show how imperfect monitoring of fundamentals helps to incentivize partners. Partnership cannot overcome the free-riding problem when fundamentals are observed

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(Sannikov and Skrzypacz [2007]; Proposition 1 below) or closely monitored (Proposition 5). Imperfect monitoring mitigates the ratcheting effect and allows rewards offered in the future to motivate today's effort. Finally, we use the tractable characterization to generate novel predictions about the dynamics of effort, fundamentals, and profits. Below we elaborate on the method, as well as on the mechanisms that drive incentives and on the equilibrium dynamics.

In our continuous-time model, at any point in time, partners privately choose costly effort and evenly split the profits of their venture. Fundamentals are the sum of past total efforts, discounted by the depreciation rate and, in turn, determine the expected profit flow. Consequently, the private marginal benefit of effort due to the direct effect on profits (Markov incentives) is constant, and equals half of the social marginal benefit of effort. In the main model neither efforts nor fundamentals are observable, and profits, which follow Brownian diffusion, are the partners' only publicly available information.

Our minimal monitoring structure does not allow the signals to separately identify each partner's effort (Fudenberg et al. [1994]). Consequently, we focus on the strongly symmetric equilibria (SSE), without asymmetric punishments of presumed deviators. Those modeling assumptions are restrictive. Investment in the monitoring technology and separately policing each of the partners would be an alternative way to address the moral hazard problem, in the spirit of Alchian and Demsetz [1972]. In contrast, we are interested in the incentives that can be sustained by the information that is readily available in any venture, the profits. Intuitively, instead of punishing the likely deviators, partners coordinate on relatively efficient effort levels after good outcomes, indicative of high effort in the past, and coordinate on relatively inefficient effort after bad outcomes. This provides the additional incentives (relational incentives) to motivate effort.

Our main results, Theorems 1 and 2, characterize the expected utilities that can be obtained in an SSE, and construct near-optimal equilibria of the partnership game. We begin by proposing a state variable—relational capital—which captures the purely relational component of partners' expected utility, net of the value of inherited fundamentals. It generalizes the notion of continuation value from the i.i.d. setting, and serves a simi-

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lar, dual purpose: It is both an accounting device for how well the partnership is doing (maximizing expected utility is equivalent to maximizing relational capital), but also, by responding to profit flows, it provides relational incentives for exerting effort. Unlike in the i.i.d. settings (bang-bang results, Abreu et al. [1986]), however, in our equilibria relational capital and partners' incentives change smoothly in response to the news about the profits. Indeed, the characterization and construction in Theorems 1 and 2 are based on a one-dimensional differential equation, whose solution parametrizes the supremum of relational incentives deliverable in SSE, for any level of relational capital.

While the importance of marginal benefit of effort as an additional state variable in models with persistence is well understood, our approach of treating it as an objective function is novel. Moreover, given that the goal is to maximize utility in an SSE, maximizing incentives might seem like a strange choice. In particular, since the efficient level of effort is interior, it is possible to overincentivize the partners. Our approach is based on the idea of tracing out the upper boundary of the set of relational capital-incentives pairs achievable in SSE, as a way of getting to the rightmost point of supremum relational capital. In Theorem 1 we establish that this boundary satisfies the novel differential equation, and that the supremum relational capital is unattainable. In Theorem 2 we show that approximate solutions are self-generating, and thus give rise to near-optimal equilibria.

Our approach gives rise to a novel technical problem. The differential equation in Theorem 1 is not an HJB equation associated with a dynamic stochastic control problem, since the change in relational capital depends on efforts, which depend on the value function (incentives). Nevertheless, our verification results establish that an HJB-like characterization is also valid in our setting. Another difficulty is familiar: our differential equation is a solution to a relaxed problem, only under local incentive constraints. In a separate result, Theorem 3, we provide conditions on the primitives—roughly, the cost of effort being convex enough—so that the constructed strategies are fully incentive-compatible.

The characterization gives us a convenient tool for analyzing the value of a partner-

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ship, the dynamics of effort and fundamentals, and the underlying incentive mechanisms.

First, we investigate the impact of monitoring on the value of a partnership. Equilibria with nontrivial relational incentives exist – as long as partners are patient and fundamentals are persistent, imperfectly monitored, and evolve with little noise (Proposition 4). Surprisingly, improved monitoring of the fundamentals may hurt partners, and eventually shrinks the equilibrium payoffs to just the repeated static Nash payoffs (Sand in the wheels, Proposition 5). This comparative statics is driven by a novel effect of *extreme ratcheting*, in the following way. Partners use profit outcomes not only to provide relational incentives, but also to estimate fundamentals.[1] Good outcomes drive up the estimate – the benchmark against which the relational incentives are provided – and partners will be held to higher standards in the future (ratchet effect, see the discussion in Section 1.1 below). When monitoring becomes precise, ratcheting becomes extreme and eliminates incentive provision. Intuitively, given the relatively large swings of fundamentals due to noise, high profits are credited not to high effort, but mostly to a lucky change in fundamentals.

Second, we consider the dynamics of efforts and underlying incentive mechanisms in a near-optimal equilibrium. Effort depends on the level of relational capital in a non-monotone way (Corollary 1). When relational capital is low, partnership is close to unraveling and reverting to the inefficient repeated static Nash equilibrium. In this case, high profit realizations, which always boost mutual trust and increase relational capital, *rally* the partnership away from the brink and encourage higher effort. This complementarity of efforts across time is related to the encouragement effect in the literature on experimentation in teams (Bolton and Harris [1999]), whereby good signals make partners believe that the exogenous success of the project is more likely. Here, good signals make partners believe that the endogenous failure (unraveling of the partnership) is less likely.

In contrast, when relational capital is high, partners *coast*: high profit realizations

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[1]In Proposition 6 we show that when fundamentals are deterministic, and so profits are used only to provide incentives and not to estimate fundamentals, we recover the intuitive result that better monitoring results in better equilibria.

decrease their effort. This is because when relational capital is close to the bliss point, it can no longer increase after good public outcomes. In models in which today's effort is incentivized using only today's signals, this prevents any relational incentives (Sannikov and Skrzypacz [2007]). In our case, effort is motivated by *deferred incentives:* discounted relational incentives from the future, when the relational capital drifts down. Intuitively, even at a bliss point, partners work to push the fundamentals unexpectedly higher, as it will result in unexpectedly high fundamentals tomorrow, and throughout the future – together with a stream of high profits rewarded by relational capital.

The mechanism of deferred incentives also helps justify why partners may work inefficiently much in equilibrium (*over-working*). While this may not happen in models with no persistence, over-working may be optimal in a setting like ours, in which incentives are linked across time. The perspective of very high incentives in the future might help sustain reasonable incentives today.

Lastly, the equilibrium dynamics is non-Markovian. Stage game payoffs and profitability are driven by the fundamentals, whereas efforts are driven by the endogenous relational capital. The basic empirical implication is that the dynamics of profitability does not determine the dynamics of effort. For example, when fundamentals are (close to) deterministic, beliefs about the fundamentals are sluggish and change only as the effects of effort accrue over time. Relational capital, and so the total partnership value, responds to the profit outcomes and is more volatile. One consequence is that a spell of sharp negative shocks will drain mutual trust and unravel the partnership, with hardly any effect on its fundamentals. Even very profitable partnerships collapse (Beatles' break-up, Proposition *7).*

1.1 Related Literature

This paper belongs to the literature on free-riding in groups, in dynamic environments.[2] The repeated partnership game was first studied in Radner [1985] and Radner et al.

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[2]See Olson [1971], Alchian and Demsetz [1972], Holmstrom [1982] as well as Legros and Matthews [1993] and Winter [2004] for the seminal contributions in static settings.

[1986], who demonstrate inefficiency of equilibria, and Fudenberg et al. [1994], who pin down the identifiability conditions violated in the model. Symmetric equilibria in this setting feature a "bang-bang" property (Abreu et al. [1986]), with effort changing only once on the equilibrium path. Lack of identifiability also hampers incentive provision in our model. However, it features true, gradual equilibrium dynamics, due to persistence and imperfect state monitoring, as the signals about effort accrue slowly over time.

Abreu et al. [1991] and Sannikov and Skrzypacz [2007] show how increasing the frequency of interactions may have detrimental effect on incentives. In particular, the Brownian model[3] of partnership or collusion in Sannikov and Skrzypacz [2007], which is closely related to ours but has no persistence or has perfectly monitored state, features only trivial repeated static Nash equilibrium.[4] Faingold and Sannikov [2011] and Bohren [2018] establish related results in models with one long-lived player in a competitive market setting. We show that when actions have a persistent effect and there is any amount of noise in monitoring, nontrivial equilibria may exist (when players are patient, fundamentals are persistent and evolve with little noise), with efforts motivated by deferred incentives. On the other hand, relational incentives disappear even when monitoring is imperfect, yet very precise, due to the novel effect of extreme ratcheting.[5] Rahman [2014] shows how relational incentives may be restored in the presence of a mediator, using secret monitoring and infrequent coordination.

Our paper ties into the literature on experimentation in teams, either in the exponential bandit (Keller et al. [2005], Keller and Rady [2010], Klein and Rady [2011], and Bonatti and Hörner [2011]) or Brownian model (Bolton and Harris [1999], Georgiadis [2014], and Cetemen et al. [2017]).[6] In those models, productivity of effort depends on the exogenous observable state, common to all, or on public beliefs about the state.

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[3]More precisely, Sannikov and Skrzypacz [2007] consider models with short period lengths, approximating the Brownian model.

[4]With a perfectly monitored state, the sufficient signal for efforts is the instantaneous *change* in the state, much like the instantaneous public signal in a model without persistence; see Section 6 in Sannikov and Skrzypacz [2007].

[5]Ratchet effect also prevents nontrivial effort in Bhaskar [2014], but for entirely different reasons. There, impossibility relies on a setting that combines continuous and discrete choices.

[6]See, also, Décamps and Mariotti [2004], Rosenberg et al. [2007], Murto and Välimäki [2011], and Hopenhayn and Squintani [2011] for related stopping games with incomplete information.

The literature studies effects of payoff or information externalities on incentives, and focuses on Markov equilibria, with no relational component. Our paper is complementary: It has production technology independent of history, with constant Markov incentives, but we focus on optimal equilibria, which rely on relational incentives. Our equilibrium characterization is equally tractable, with incentives driven by the endogenous relational capital of the partnership. Working to rally the partnership is related to the encouragement effect identified by Bolton and Harris [1999], and coasting is reminiscent of the work-shirk-work dynamics in the reputation model of Board and Meyer-ter Vehn [2013].

Beyond partnerships, persistence plays an important role in agency problems, most importantly in dynamic moral hazard models with learning, where it gives rise to ratchet effect (see Holmstrom [1982] and Cisternas [2017] for models without, and Williams [2011], Prat and Jovanovic [2014], DeMarzo and Sannikov [2016], and He et al. [2017], for models with commitment, in a Brownian setting similar to ours). In particular, Jarque [2010], Sannikov [2014], and Prat [2015] analyze payments in the optimal commitment contracts, when effort has a persistent effect. Although the questions and the incentive mechanisms are different from ours, the literature has long recognized the difficulty of accounting for the marginal benefits of deviations, or incentives, as well as verifying global incentive compatibility. Our solution method is new and is based on maximizing incentives, rather than including them as an additional state variable. Moreover, we provide conditions on the primitives of the model (in our case, the convexity of costs), so that the solution of the relaxed problem is fully incentive-compatible (see Edmans et al. [2012] and Cisternas [2017] for related results).[7]

Finally, there is a large literature on strategic management, documenting the recent growth in partnering and external collaboration between coorporations, as well as the particularities of managing joint ventures given the risks of shirking associated with those enterprises. For instance, see Powell et al. [1996], Luo [2002], and Reuer and Arino [2007]. In particular, Madhok [2006] argues that overemphasis on the outcome of joint ventures has led to neglect of the importance of trust for the quality of the relationships. In

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[7]See also Williams [2011], Sannikov [2014], and Prat [2015], who provide analytical conditions on the solution of the relaxed problem, under which the first order approach is valid.

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a similar vein, a large literature in social psychology focuses on free-riding in teams. For instance, see Gersick [1988], McGrath [1991], Smith [2001], or Levi [2015] for a survey of team theory and the dynamics of teamwork. McGrath's Time, Interaction, and Performance (TIP) theory emphasizes that different teams follow different paths to reach the same point. This resonates with the non-Markovian nature of our equilibria (see Figure 4). In contrast, Hackman [1987] explores different criteria to evaluate team success, with emphasis on (i) completing the task, (ii) maintaining social relations, and (iii) benefiting the individual. While the first is related to the partnership's fundamentals in our model, the other two connect to our concept of relational capital.

Summarizing, we contribute to the literature in the following way. Persistence and imperfect monitoring are important for applications, yet known to lead to intractable solutions. First, on the theoretical side, we provide a model of partnership that includes those two features, and we show that (i) it can sustain nontrivial relational incentives; (ii) it has a tractable solution, characterized by a one-dimensional differential equation; and (iii) it features true equilibrium dynamics. The solution method is new, and goes beyond the application of the stochastic optimal control. It allows for verification of global incentive compatibility under conditions directly on model's primitives. Second, we show how imperfect monitoring mitigates ratcheting and helps incentive provision (Sand in the wheels). Third we uncover the underlying relational incentive mechanisms (deferred incentives and encouragement effect), and we provide empirical predictions on the relational, non-Markovian dynamics of partnerships (rallying, over-working, coasting, and Beatles' break-up).

2   Model

Two partners play a stochastic game with imperfect monitoring in continuous time. At every moment in time, $t \in [0, \infty)$, each partner i privately and independently chooses effort a_t^i, $i = 1, 2$, from an interval $[0, A]$.[8] Formally, each $\{a_t^i\}$ is a process measurable

[8]The upper bound on effort is used for technical reasons, to guarantee boundedness of continuation value, in Propositions 1 and 2. In all the simulations, as well as Theorem 4 the bound A is large

with respect to a filtration $\{\mathcal{F}_t\}$ of public information, which includes the sigma-algebra generated by the process of cumulative profits $\{Y_t\}$ and allows for public randomization.[9] Time t total effort, $a_t^1 + a_t^2$, contributes to the fundamentals μ_t of the partnership. The stock of fundamentals depreciates at a constant rate $\alpha > 0$, and, in the main model, is unobservable to the partners. At any point in time, it determines the publicly observable flow of partnership profits dY_t,

$$d\mu_t = (r + \alpha)(a_t^1 + a_t^2)dt - \alpha\mu_t dt + \sigma_\mu dB_t^\mu, \tag{1}$$

$$dY_t = \mu_t dt + \sigma_Y dB_t^Y,$$

where $\{B_t^\mu\}_{t \geq 0}$ and $\{B_t^Y\}_{t \geq 0}$ are two independent Brownian Motions.

Exerting effort a entails a private flow cost $c(a)$, where $c(\cdot)$ is a twice differentiable, strictly convex cost of effort function, and we normalize $c(0) = 0$. (Later on, in the construction of near-optimal equilibria, and the verification of global incentive compatibility, we further restrict the cost function to be quadratic.) At each point in time, partners split the profits evenly. Both are risk-neutral, and discount the future at a constant common rate $r > 0$. Thus, for fixed effort choices of both partners, $i's$ expected discounted continuation payoffs are given by

$$W_\tau^i = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(\frac{\mu_t}{2} - c(a_t^i) \right) dt \right].$$

Given discounting and mean reversion, the marginal social net present value of increased fundamentals is $\frac{1}{r+\alpha}$. Since each parter captures only half of it, and the efforts are private, partnership gives rise to an obvious free-riding problem. Formally, as private effort bumps fundamentals up by a factor of $r + \alpha$, the efficient effort level a_{EF} and the effort a_{NE} in the unique stationary equilibrium satisfy

$$c'(a_{EF}) = 1, \quad c'(a_{NE}) = \frac{1}{2}, \tag{2}$$

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enough, so that the equilibrium efforts are interior (see also Lemma 5 in the Appendix, which bounds the relational incentives and so efforts in equilibrium).

[9]Unless otherwise specified, the processes are indexed by time t, $t \in [0, \infty)$.

as long as the boundary constraints are slack.

In the paper we let $a_{NE} = 0$ (by setting $c'(0) = \frac{1}{2}$), so that the effort in our model is the effort in excess of the stationary level, absent any relational incentives. This implies that the corresponding continuation value is $W_\tau^{NE} = \frac{\bar{\mu}_\tau}{2(r+\alpha)}$. With some abuse of terminology, we call a pair of strategies that play 0 after every history a repeated static Nash equilibrium. A partnership *unravels* if, from that point on, partners exert no more effort—that is, play the repeated static Nash equilibrium. An alternative interpretation is that the partners stop the partnership and sell the fundamentals, recovering value $\frac{\bar{\mu}_\tau}{2(r+\alpha)}$ (see Section 5).

The model has a novel structure of a stochastic game with an imperfectly monitored state. Specifically, there is one state variable, the fundamentals μ_τ, which equals

$$\mu_\tau = e^{-\alpha\tau}\mu_0 + \int_0^\tau e^{-\alpha(\tau-t)}\left[\left(a_t^1 + a_t^2\right)dt + \sigma_\mu dB_t^\mu\right].$$

Fundamentals change slowly over time, driven by the efforts of the partners, as well as the production noise $\sigma_\mu dB_t^\mu$. This persistence of fundamentals implies that actions have persistent effect: total effort $a_t^1 + a_t^2$ at time t adds $(r+\alpha)e^{-\alpha s}\left(a_t^1 + a_t^2\right)$ to the fundamentals, and so profit flow dY_t at time $t + s$.[10] Moreover, partners observe only the profit flow dY_t, which is a noisy signal of the fundamentals, subject to the observation noise $\sigma_Y dB_t^Y$.

Persistence and imperfect monitoring of the state have a dramatic effect on the provision of incentives. Without persistence, or when fundamentals are perfectly observed, current change in the observed process (fundamentals or profits) is a sufficient statistic for current efforts, much like in the repeated game literature. In our model, however, current effort must be estimated from the whole path of future profits. Intuitively, this prevents the optimality of the bang-bang symmetric equilibria (see Abreu et al. [1986]), and leads to a smooth dynamic of the partnership.

In addition, generalizing the game to allow for persistence and imperfect state monitoring adds realism to the model of partnership, as follows. Without persistence, profits

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[10]The scaling factor $(r+\alpha)$ guarantees that the present social value of unit effort is one.

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at any time would be determined solely by instantaneous actions. In our model, the profits of a partnership are determined by the company's fundamentals, which is our umbrella term that captures the stock of all the slow moving factors in a venture, shaped partly by partners' effort, and partly by the external circumstances. Heuristically, we allow for the profits of a start-up to be determined not by the lines of codes written at a given moment, but by the overall quality of its app, shaped by all the past efforts. Imperfect state monitoring is an analogue, and partly a consequence of imperfect monitoring of effort. Given the unobserved effort of the partner, the quality of the app (how few bugs there are) is also uncertain. The bugs in the code will be discovered only with some delay. The discovery may happen directly, as a consequence of an internal audit; it may be inferred from the customers' reviews; or, as in our model, it may be inferred from the profit flows.[11]

Strongly Symmetric Equilibria A pair of strategies $\{a_t^1, a_t^2\}$ is a *Perfect Public Equilibrium (PPE)* if for each partner i, at any time $\tau \geq 0$ and after any public history in $\mathcal{F}_\tau$

$$\mathbb{E}_\tau^{\{a_t^i, a_t^{-i}\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(\frac{\mu_t}{2} - c(a_t^i) \right) dt \right] \geq \mathbb{E}_\tau^{\{\widetilde{a}_t^i, a_t^{-i}\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(\frac{\mu_t}{2} - c(\widetilde{a}_t^i) \right) dt \right]. \quad (3)$$

Our partnership game is characterized by a parsimonious information structure. The only information about the partners' effort comes from the joint stream of profits, and the efforts enter profits additively. Consequently, it is not possible to identify which of the partners did, and which one did not, contribute to the common good based solely on the public signals (Fudenberg et al. [1994]). This implies that, as in the classic analysis of repeated duopoly by Green and Porter [1984] or partnerships by Radner et al. [1986], it is not possible to provide incentives by continuation value "transfers" between the agents, shifting resources from likely deviators. Moreover, asymmetric play is inefficient, since cost of effort is convex, and it also does not affect the signals' informativeness of efforts,

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[11]Adding an additional publicly observable signal, beside the profit flows, would hardly affect the results.

from linearity.

The main effect of asymmetric play, thus, is to increase total flow costs and "destroy value." While this might be beneficial for the partnership, in the paper we focus on symmetric equilibria. In Section 5, we show how our solution extends to the case in which partners are allowed to destroy value, in the direct form of observable unproductive effort. Formally, a *Strongly Symmetric Equilibrium (SSE)* is a PPE in which the strategies $\{a_t^1, a_t^2\}$ satisfy $a_\tau^1 \equiv a_\tau^2$, after every public history in $\mathcal{F}_t$,

3 Solution

In this section, we present the main technical results of the paper; we postpone analyzing the partnership's value, its dynamic, and its incentivizing mechanisms until Section 4.

In equilibrium, partners share public beliefs about the fundamentals, and we let

$$\overline{\mu}_\tau = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}} [\mu_\tau]$$

denote the expected level of fundamentals at time τ, given the pair of strategies $\{a_t^1, a_t^2\}$ and the history of public signals. With no observational noise, $\sigma_Y = 0$, partners observe the fundamentals, and so $\overline{\mu}_\tau = \mu_\tau$, always. Whenever observational noise σ_Y is strictly positive, a simple application of the Kalman-Bucy filter yields that for a fixed pair of strategies $\{a_t^1, a_t^2\}$, the publicly expected level of fundamentals $\overline{\mu}_t$ follows

$$d\overline{\mu}_t = (r + \alpha)(a_t^1 + a_t^2)dt - \alpha\overline{\mu}_t dt + \gamma_t[dY_t - \overline{\mu}_t dt], \tag{4}$$

for an appropriate gain parameter γ_t, and

$$dY_t = \overline{\mu}_t dt + \sigma_Y dB_t,$$

where $\{B_t\}$ is a Brownian Motion with respect to the public filtration $\{\mathcal{F}_t\}$. For simplicity, we will assume that at time zero, partners believe that μ_0 is Normally distributed with steady-state variance σ^2. This implies that both the posterior variance σ_t^2 of their

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estimate and the gain parameter γ_t remain constant throughout the game, $\sigma_t^2 = \sigma^2$ and $\gamma_t = \gamma$, and equal (see Liptser and Shiryaev [2013])

$$\gamma = \sqrt{\alpha^2 + \frac{\sigma_\mu^2}{\sigma_Y^2}} - \alpha, \tag{5}$$

$$\sigma^2 = \gamma \times \sigma_Y^2.$$

Exploiting the exponential decay of fundamentals, we may rewrite the continuation payoffs as

$$W_\tau^i = \mathbb{E}_\tau^{\{a_t^1, a_t^2\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(\frac{\mu_\tau}{2} e^{-\alpha(t-\tau)} + \int_\tau^t (\alpha + r) \frac{a_s^1 + a_s^2}{2} e^{-a(t-s)} ds - c(a_t^i) \right) dt \right] \tag{6}$$

$$= \frac{\overline{\mu}_\tau}{2(r+\alpha)} + \mathbb{E}_\tau^{\{a_t^1, a_t^2\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(\frac{a_t^1 + a_t^2}{2} - c(a_t^i) \right) dt \right].$$

For a fixed pair of symmetric strategies, define relational capital at time τ, w_τ, as the continuation value net of the expected value of the fundamentals,

$$w_\tau = W_\tau - \frac{\overline{\mu}_\tau}{2(r+\alpha)} = \mathbb{E}_\tau^{\{a_t, a_t\}} \left[\int_\tau^\infty e^{-r(t-\tau)} (a_t - c(a_t)) dt \right]. \tag{7}$$

Even if the relationship unravels at some time τ and partners stop exerting effort (play repeated static Nash), they will keep collecting partnership profits. The "inherited" (expected) fundamentals $\overline{\mu}_\tau$ may be positive, due to past effort or luck, and will only slowly revert to zero, yieding expected profits all along. Their value to a partner is $\frac{\overline{\mu}_\tau}{2(r+\alpha)}$.

In our setting, value of partnership (continuation value) may be factored additively into this backward looking value of inherited fundamentals, as well as the forward looking value of efforts undertaken in the future – relational capital. For one, maximizing the value of partnership, for a given level of inherited fundamentals, is equivalent to maximizing relational capital. In the rest of the paper we thus focus on the problem of maximizing relational capital. Moreover, factorization of continuation value implies similar factorization of incentives. They consist of Markov incentives, driven by the desire to shape fundamentals, and so profit flows, as well as relational incentives, driven by the

desire to shape relational capital, and so future efforts.

Since the level of inherited fundamentals does not interact with productivity or cost of effort, direct Markov incentives are very simple. Partners do not work to invest in a better production technology: marginal value fundamentals is constant, equal to $\frac{1}{2(r+\alpha)}$. Higher effort bumps up fundamentals by a factor of $(r + \alpha)\, dt$, and so Markov incentives – marginal effect of effort on the value of fundamentals (scaled down by dt) – are constant, equal to $\frac{1}{2}$ (see (2)).

Relational incentives resemble the incentives players have in repeated games. In symmetric equilibria, they are provided by "burning value:" partners coordinate on relatively efficient effort after "good" histories, indicating high past effort, and on relatively inefficient effort after "bad" histories. In our construction of equilibria, relational capital plays the role of an accounting device that tracks good and bad histories. In a sense, partners exert effort to build up relational capital, and so boost future efforts. Relational incentives are the marginal effect of effort on relational capital (scaled down by dt); they are an endogenous, equilibrium object.

For a given level of relational incentives x, let $a(x)$ be the locally optimal choice of effort,

$$a(x) = \arg\max_a \left\{ (x + 1/2) \times a - c(a) \right\}. \tag{8}$$

Perfectly Monitored Fundamentals   The following benchmark result focuses on the case of observable fundamentals, and is closely related to the results in Sannikov and Skrzypacz [2007], Faingold and Sannikov [2011] and Bohren [2018].   When partners observe fundamentals (i.e., $\sigma_Y = 0$), they play a stochastic game with perfect state monitoring. Doing away with observation noise improves monitoring of partners' effort. Intuitively, this should help partners incentivize and sustain high effort. However, we have the following result.

Proposition 1 *Suppose that $\sigma_\mu > 0$ and there is no observational noise, $\sigma_Y = 0$.*

i) A symmetric strategy profile $\{a_t, a_t\}$ is a SSE with associated relational capital

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process $\{w_t\}$ *if and only if there is a* L^2 *processes* $\{I_t\}$ *such that*

$$dw_t = (rw_t - (a_t - c(a_t))) \, dt + I_t \times (d\mu_t - [(r + \alpha) \, 2a_t - \alpha\mu_t] \, dt) + dM_t^w, \qquad (9)$$

where $a_t = a((r + \alpha) \, I_t)$ *and* $\{M_t^w\}$ *is a martingale orthogonal to* $\{Y_t\}$, *and the transversality condition* $\mathbb{E}\left[e^{-rt}w_t\right] \to_{t \to \infty} 0$ *holds.*

ii) The unique SSE is the repeated static Nash equilibrium.

The representation of relational capital as a solution to the differential equation (9) in part i) is standard (see Sannikov [2007], Sannikov [2008]). Roughly speaking, the additional public signal at time t is the innovation to the process of fundamentals, $d\mu_t - [(r + \alpha) \, 2a_t - \alpha\mu_t]$. While in discrete-time models the continuation value can respond to the signal in an arbitrary, nonlinear way, it follows from the Martingale Representation Theorem that in continuous-time the signal can affect continuation value only linearly, with sensitivity I_t^i.[12] This result underlies much of the tractability of Brownian continuous-time models. As the expected value of signal is zero, the promise keeping constraint on the relational capital pins down it's drift.

Consider now relational incentives for exerting effort. Extra effort bumps up the current increment of fundamentals $d\mu_t$ by a factor of $(r + \alpha)dt$. Since higher public signal $d\mu_t$ increases relational capital by a constant factor of I_t, relational incentives are constant at $(r + \alpha) \, I_t$. Consequently, effort at any point in time is chosen to maximize $((r + \alpha) \, I_t + 1/2) \, a - c(a)$.

Part ii) relies on relational capital changing linearly in the new signal realization. [13] In a strongly symmetric equilibrium, incentivizing effort greater than that in the repeated

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[12]More precisely, the Martingale Representation Theorem implies the result only in the case when the filtration $\mathcal{F}_t$ is generated by the public signals, in which case the martingale $M_t^{i,w}$ is zero; the more general case follows from, e.g., Proposition 3.4.14 in Karatzas [1991].

[13]See Sannikov and Skrzypacz [2007] for an excellent justification, using a sequence of discrete time models, of why the relational incentives must involve both large rewards and large punishments. Roughly speaking, in a Brownian continuous-time, or near-Brownian discrete-time agency model, principal observes an extremely noisy signal of agent's effort. Providing adequate relational incentives thus requires extremely large changes in continuation value–large stakes. Large changes in continuation value can be balanced out – to prevent large on-path losses – only if they include both large rewards and large punishments.

static Nash (normalized to zero) requires strictly positive sensitivity of relational capital to signals. Thus, for example, at the bliss point of maximal relational capital, strictly positive relational incentives would result in an escape to the right, due to volatility. On the other hand, no incentives and so no effort would also result in an escape to the right, due to positive drift – in order to satisfy promise keeping, as long as the relation capital is strictly positive.[14] In Section 5 we provide conditions for the impossibility of nontrivial asymmetric PPE (following Sannikov and Skrzypacz [2007]).

Imperfectly Monitored Fundamentals In the rest of the paper we assume that fundamentals are imperfectly observed, $\sigma_Y > 0$.

Let us begin our analyzis by revisiting the accounting of relational incentives. Increasing effort by ε at time τ bumps up expected fundamentals to $\overline{\mu}_\tau^{dev} = \overline{\mu}_\tau + \varepsilon \left(r + \alpha \right) dt$. For fixed strategies of the partners, this changes the probability distribution of effort paths in the future. Relational incentives F_τ capture the effect effort today has over the discounted value of future efforts,

$$F_\tau := \frac{\partial}{\partial \varepsilon} \mathbb{E}_\tau^{\{a_t, a_t\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \left(a_t - c(a_t) \right) dt \right],$$

for a given revenue process $dY_t = \overline{\mu}_t^{dev} dt + \sigma_Y dB_t$, where the publicly expected level of fundamentals evolve according to $d\overline{\mu}_t^{dev} = (r + \alpha) 2a_t dt - \alpha \overline{\mu}_t^{dev} dt + \gamma [dY_t - \overline{\mu}_t^{dev} dt]$, and $\overline{\mu}_\tau^{dev} = \overline{\mu}_\tau + \varepsilon \left(r + \alpha \right)$. A *local Strongly Symmetric Equilibrium (local SSE)* is a profile of symmetric strategies such that at any point of time actions are locally optimal, $a_t = a(F_t)$, for the function a defined in (8).

Proposition 2 *A symmetric strategy profile $\{a_t, a_t\}$ with relational capital and relational incentives processes $\{w_t\}$ and $\{F_t\}$ is a local SSE if and only if there are L^2 processes*

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[14]The argument immediately generalizes to the case when the the set of relational capitals is open on the right, and so the supremum is not achievable; see Appendix.

$\{I_t\}$, $\{J_t\}$ such that

$$dw_t = (rw_t - (a_t - c(a_t))) \, dt + I_t \times (dY_t - \overline{\mu}_t dt) + dM_t^w, \tag{10}$$

$$dF_t = (r + \alpha + \gamma) F_t dt - (r + \alpha) I_t dt + J_t \times (dY_t - \overline{\mu}_t dt) + dM_t^F,$$

and actions a_t satisfy $a_t = a(F_t)$, where $\{M_t^w\}$ and $\{M_t^w\}$ are martingales orthogonal to $\{Y_t\}$, and the transversality conditions $\mathbb{E}\left[e^{-rt} w_t\right], \mathbb{E}\left[e^{-(r+\alpha)t} F_t\right] \to_{t\to\infty} 0$ hold. Moreover, every SSE is a local SSE.

The interpretation of the result is as follows. The first equation is analogous to the one in Proposition 1, but now with imperfect state monitoring. Relational capital responds linearly to the public signal innovation, which is now $dY_t - \overline{\mu}_t dt$, with sensitivity I_t. The second equation characterizes relational incentives, F_τ. Whereas under full observability they are equal to $(r + \alpha) I_\tau$, now F_τ is proportional to the expected discounted integral of future $(r + \alpha) I_t$,

$$F_\tau = \mathbb{E}_\tau^{\{a_t, a_t\}} \left[\int_\tau^\infty e^{-(r+\alpha+\gamma)(t-\tau)} (r + \alpha) I_t dt \right]. \tag{11}$$

The process $\{J_t\}$ measures how sensitive relational incentives are to the public profit signals. Because of this representation, we may think of $\{I_t\}$ as the flow, and $\{F_t\}$ as the stock of relational incentives.

Let us provide an intuition for why the integral in (11) captures the relational incentives. As in the case of observable fundamentals, a deviating increased effort at time τ increases the wedge between the correct beliefs and the equilibrium beliefs about the fundamentals, $\overline{\mu}_t^{dev} - \overline{\mu}_t$, by $(r + \alpha)dt$. This increases expected relational capital today by $(r + \alpha)I_\tau$. Moreover, given persistence, it follows from the Kalman formula (4) that the wedge reverts to the mean at rate $\alpha + \gamma$, and so in the next instant $\tau + \Delta$ the wedge will scale down by $e^{-(\alpha+\gamma)\Delta}$. The term α is the mean reversion of the fundamentals. The second term γ follows from a ratchet effect: if the wedge is positive, and, thus, the equilibrium beliefs $\overline{\mu}_t$ are relatively low, the new profit realization will be surprisingly high, and so $\overline{\mu}_t$ will move up faster than the correct beliefs $\overline{\mu}_t^{dev}$, by a factor of γ. Overall, the

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discounted effect of the wedge on relational capital at time $\tau + \Delta$ is $e^{-(r+\alpha+\gamma)\Delta}(r+\alpha)I_{\tau+\Delta}$. Thus, the integral in (11) captures the expected discounted marginal effect of extra effort on relational capital.

The main step in our solution relies on the following novel parametrization. Our problem does not yield a natural choice of objective function, such as one player's continuation value (in the Principal-Agent problems, or asymmetric equilibria in dynamic games). In order to construct local SSE we (i) use the relational capital as a state variable, and (ii) parametrize maximal relational incentives as a function of it. To the best of our knowledge, this approach of maximizing incentives as a way to characterize the optimal equilibrium is novel.[15]

To highlight the role of the new parametrization, we split the problem of characterizing the optimal local SSE into two steps. First, the following proposition suggests a strategy of constructing any, not necessarily optimal, equilibria.

Proposition 3 *Consider a continuous* $I : [\underline{w}, \overline{w}] \to \mathbb{R}_+$ *and a* C^2 *strictly concave function* $F : [\underline{w}, \overline{w}] \to R$ *that satisfy the differential equation*

$$(r+\alpha+\gamma)F(w) = (r+\alpha)I(w) + F'(w) \times (rw - (a(F(w)) - c(a(F(w))))) + \frac{F''(w)}{2}\sigma_Y^2 I^2(w),$$
(12)

where a *is defined in (8), such that each boundary point* $w^\partial \in \{\underline{w}, \overline{w}\}$ *together with* $F\left(w^\partial\right)$ *is either achievable by a local SSE, or satisfies*

$$(r + \alpha + \gamma)F(w^\partial) = F'(w^\partial) \times \left(rw^\partial - \left(a(F(w^\partial)) - c(a(F(w^\partial)))\right)\right), \qquad (13)$$

$$sgn\left(\frac{\overline{w} + \underline{w}}{2} - w^\partial\right) = sgn\left(rw^\partial - \left(a(F(w^\partial)) - c(a(F(w^\partial)))\right)\right).$$

Then, for every $w_0 \in [\underline{w}, \overline{w}]$, *there is a local SSE* $\{a_t, a_t\}$ *achieving* $(w_0, F(w_0))$.

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[15]In contrast, the use of an additional *state variable* that relates to marginal incentives is well established; see e.g. Werning [2001] or Kapička [2013], who introduce expected marginal utility of consumption, Williams [2011] and Prat and Jovanovic [2014], who introduce expected marginal utility of a state, and Sannikov [2014] who introduces marginal incentives, in models of contracting with persistence.

The proposition provides a tractable tool, in the form of an Ordinary Differential Equation, for constructing local SSE. It is used below, in Proposition 4 to establish existence of nontrivial equilibria. Function I parametrizes the flow of relational incentives, $I_t = I(w_t)$ as a function of internal capital w_t, $w_t \in [\underline{w}, \overline{w}]$. Ito formula implies that if w_t is the relational capital that follows (10) and F is a solution to (12), then the process $\{F(w_t)\}_{t \geq 0}$ captures the associated relational incentives. More precisely, process $\{F(w_t)\}_{t \geq 0}$ satisfies the differential equation (10) in Proposition 2 with sensitivities $J_t = F'(w_t) \times I(w_t)$. In equation (12), the left hand side is the average flow of relational incentives that is needed to generate the stock of relational incentives $F(w)$, given the exponential discounting, mean reversion, and ratchet effect (r, α and γ, respectively). The first term on the right captures the flow of relational incentives, the second term is the change in the relational incentives resulting from the change in relational capital, and the last term is the loss (since $F'' < 0$) resulting from the second-order variation in relational capital.

When the boundary point is a relational capital known to be achievable by a local SSE, upon reaching this point, the game simply follows this local SSE. The alternative boundary conditions (13) are more complicated. The first clause requires that at the boundary point the flow of relational incentives $I(w^\partial)$ can be set to zero.[16] The second clause requires that at the boundary point the relational capital drifts inside of the set. Consequently, at the boundaries, relational incentives $F(w^\partial)$ are made up only of the discounted relational incentives in the next instant, due to persistence.

The following result is one of the main results of this paper.

Theorem 1 *Let w^* be the supremum of the relational capital achievable in a local SSE. Then, there exists a strictly concave function F on $[0, w^*)$ that satisfies*

$$(r + \alpha + \gamma)F(w) = \max_{I \geq 0} \left\{ (r + \alpha)I + F'(w)\left(rw - (a(F(w)) - c(a(F(w))))\right) + \frac{F''(w)\sigma_Y^2}{2}I^2 \right\} \quad (14)$$

$$= F'(w)\left(rw - (a(F(w)) - c(a(F(w))))\right) - \frac{(r + \alpha)^2}{2\sigma_Y^2 F''(w)},$$

---

[16]More precisely, the stock of incentives $F(w^\partial)$ at the boundary can be generated by having either $I(w^\partial) = 0$, or $I(w^\partial) = -2\frac{r + \alpha}{\sigma_Y^2 F''(w)} > 0$. In the proof in the Appendix, we show how to reduce the second case to the first one, by extending the functions F and I beyond $[\underline{w}, \overline{w}]$, with $I = 0$.

as well as the boundary conditions

$$F(0) = 0, \qquad (15)$$

$$\lim_{w \uparrow w^*} \left\{ (r + \alpha + \gamma)F(w) - F'(w) \times (rw - (a(F(w)) - c(a(F(w))))) \right\} = 0,$$

$$\lim_{w \uparrow w^*} \left\{ rw^* - (a(F(w^*)) - c(a(F(w^*)))) \right\} = 0.$$

Moreover, w^* is not attained by any local SSE.

Function F in the theorem parametrizes the upper boundary of relational incentives achievable in local SSE, for any level of relational capital. The theorem establishes that the boundary starts at $(0,0)$, and satisfies equation (14), as well as the right-hand boundary conditions in (15). The differential equation differs from the one in Proposition 3 in that the incentive flow I is set optimally, given F,

$$I^*(w) = -\frac{r + \alpha}{\sigma_Y^2 F''(w)}. \qquad (16)$$

Moreover, conditions (15) require that not only $I(w)$, but also the drift of the relational capital disappears close to the right boundary.

Crucially, the result provides a procedure for finding the supremum w^* of relational capitals achievable in local SSE. It is based on the fact that w^* is, by definition, the argument at the right end of the upper boundary of relational capital-relational incentive pairs achievable in local SSE. Thus, the solution of equation (14) with boundary conditions (15) that reaches furthest to the right characterizes w^*.

Despite similarities, equation (14) is not the Hamilton-Belmann-Jacobi (HJB) equation for the solution of a stochastic control problem (maximizing stock of incentives, with incentive flows as the instrument, given relational capital as the state variable). The reason is that, in our problem, the law of motion of relational capital depends, via actions chosen $a(F(w))$, on the value function F (see (10)). This is not allowed in a stochastic control problem, and so we may not rely on the existing verification theorems. Nevertheless, our proof establishes that an HJB-like characterization (14) of the solution

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is also available for a problem in which the value function doubles as a state variable.

The theorem also shows that the supremum w^* is not attainable, and so the optimal local SSE does not exist. This follows from the last two equations in (15), which imply that $(w^*, F(w^*))$ would have to be generated by a repeated static Nash, with zero drift and volatility of relational capital. Intuitively, positive drift or volatility at w^* would lead to an escape beyond w^*. If the drift was strictly negative, one could generate relational capital above w^*, simply by letting it drift down (formally: by extending F together with $I = 0$ to the right of w^*, and applying Proposition 3).

Moreover, since the policy $I^*(w)$ in (16) is positive for any positive w, the pair I^* and F does not satisfy the conditions of Proposition 3 on any strict closed subset of $[0, w^*)$. In sum, we cannot invoke Proposition 3 to construct local equilibria that achieve relational capital-incentive pairs $(w, F(w))$, for w below w^*. Thus, while the above theorem characterizes the supremum value w^*, we need a separate result to construct near-optimal equilibria.

This is achieved in the following result. In the rest of this section, we assume that the cost of effort is quadratic:[17]

$$\text{(Quadratic Cost)} \qquad c(a) = \frac{1}{2}a + \frac{C}{2}a^2. \tag{17}$$

Theorem 2 *For $\varepsilon > 0$, let w_ε^* be the upper bound on relational capital achievable in a local SSE with policies I_t constrained to be either zero or above ε. Then, there exists a strictly concave function F_ε on $[0, w_\varepsilon^*)$ that satisfies*

$$
\begin{aligned}
(r + \alpha + \gamma)F_\varepsilon(w) &= \max_{I_\varepsilon \in \{0\} \cup [\varepsilon, \infty)} \left\{ (r+\alpha)I_\varepsilon + F_\varepsilon'(w)\left(rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))\right) + \frac{F_\varepsilon''(w)\sigma_Y^2}{2}I_\varepsilon^2 \right\} \\
&= F_\varepsilon'(w)\left(rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F_\varepsilon''(w)}\mathbf{1}_{F_\varepsilon''(w) \geq -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}} \\
&\quad + \varepsilon\left(r + \alpha + \varepsilon\frac{F_\varepsilon''(w)\sigma_Y^2}{2}\right)\mathbf{1}_{F_\varepsilon''(w) \in \left(-2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}, -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}\right)},
\end{aligned}
\tag{18}
$$

as well as the boundary conditions (15).

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[17]Quadratic costs greatly simplify deriving the bounds in Proposition 4, and in Theorems 2 and 3, but we are confident that the result can be extended to more general cost functions, with appropriate bounds on third derivatives.

Moreover, there exists $\overline{w}_\varepsilon < w_\varepsilon^*$, with $w_\varepsilon^* - \overline{w}_\varepsilon = O(\varepsilon^{1/3})$, such that function F_ε together with the optimal policy function I_ε^* restricted to $[0, \overline{w}_\varepsilon)$ satisfy the conditions of Proposition 1, and so define a local SSE. Finally, on $(0, \overline{w}_\varepsilon)$ F_ε satisfies

$$F_\varepsilon' \leq \frac{r + \alpha}{4r\sigma_Y^2 C\varepsilon}, \ \ and \ F_\varepsilon'' \geq -2\frac{r + \alpha}{\sigma_Y^2 \varepsilon}. \tag{19}$$

In what follows, the local SSE that achieves $(\overline{w}_\varepsilon, F_\varepsilon(\overline{w}_\varepsilon))$, with $\overline{w}_\varepsilon$ and F_ε as in Theorem 2, is called ε-*optimal*, for any $\varepsilon > 0$. The *near-optimal* equilibria are $\varepsilon-$optimal equilibria, for small ε.

An additional benefit of working with approximately optimal equilibria, highlighted in the Theorem, is that the differential equation (18) is better suited for computational problems. The bounds on the derivatives guarantee stability of numerical iteration.[18] Moreover, the bound on the first derivative at the initial condition $w = 0$, at which $F_\varepsilon(0) = 0$, narrows down the search for $F_\varepsilon'(0)$ to a compact set.

So far we have characterized only local equilibria. The following result shows conditions on the primitives, under which local equilibria satisfy full incentive compatibility constraints.[19]

Theorem 3 *Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$. Then $\{a_t, a_t\}$ is a SSE when $C\sigma_Y$ is sufficiently high, where C is the second derivative of cost function and σ_Y is observational noise.*

The problem in establishing global incentive compatibility consists in showing that, after any history, the effort choice is concave. Given that effort cost function is strictly convex, with second derivative C, this boils down to establishing bounds on how convex the expected benefit of effort is. Crucially, in a dynamic environment with persistence, like ours, a deviation affects the strength of incentives that the agent faces in the future. This knock-on effect makes accounting for the benefits of deviations much more involved than in a static setting, or without persistence.

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[18]In contrast, equation (14) is not uniformly elliptic, since $I^*(w)$ can be arbitrarily small.

[19]Formula 37 in the Appendix provides a precise sufficient condition on the parameters that guarantees global incentive compatibility.

Following up on this intuition, in order to bound how convex the benefit of effort is, it is sufficient to establish a uniform bound on how sensitive the relational incentives $F_\varepsilon(w)$ are with respect to public signals. The first part of the proof is related to the results in the literature and shows that there are no global deviations from a local SSE if this sensitivity of relational incentives $F_\varepsilon(w)$ is uniformly bounded (see Williams [2011], Sannikov [2014], and Cisternas [2017]).

In the second part of the proof, we bound this endogenous sensitivity of relational incentives by a function of the primitives of the model. This part of the proof relies heavily on the analytical tractability of our solution. The sensitivity equals $F'_\varepsilon(w) \times I^*_\varepsilon(w)$ (see discussion under Proposition 3 and Theorem 1), with $I^*_\varepsilon(w) \leq -\frac{r+\alpha}{\sigma_Y^2 F''(w)}$. Thus, the result follows from the bound on F'_ε, from Theorem 2, together with an upper bound on F''_ε, established in the proof. Intuitively, C measures convexity of costs, whereas large noise σ_Y makes incentives costly, resulting in their low sensitivity, and so relatively linear benefit of effort.

To conclude this section, we show that the characterizations are not vacuous, and nontrivial SSE exist.[20]

Proposition 4 *The supremum w^* of relational capitals achievable in local SSE is strictly positive when $C^2\sigma_Y^2(r + \alpha + \gamma)$ is sufficiently small and the bound A large. This is also the supremum of relation capitals achievable in SSE if $C\sigma_Y$ is sufficiently large.*

Relational incentives in partnership are discounted at a rate $r + \alpha + \gamma$ (see discussion below Proposition 2). The proposition establishes that if this discount rate is sufficiently low, then nontrivial local SSE exist – in analogy with Folk Theorems, in repeated games literature. (We further discuss the effect of r, α and γ on the solution in Section 4.2.) Full incentive compatibility follows then from the proof of Theorem 3.

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[20]Formulas 40 and 42 in the Appendix provide precise sufficient conditions on the parameters that guarantee existence and global incentive compatibility.

4 Value of Partnership and Equilibrium Dynamics

The results in the previous section provide a convenient tool with which to explore the comparative statics of the value of the partnership. They also provide a complete characterization of the dynamics of the fundamentals, or the profitability of the partnership, together with the level of effort that the partnership sustains. In this section, we explore some of their properties. Given nonexistence of the optimal local SSE (see Theorem 1), in this section we refer to the near-optimal equilibria characterized in Theorem 2.

4.1 Dynamics of Effort

Figure 1 illustrates near-optimal SSE. Given quadratic costs as in (17), the horizontal parabola is the locus of the relational capital-incentives pairs (w, F) that can be achieved by symmetric play in a stage game, absent any incentive constraints.[21] This is an analogue of feasible stage game payoffs. Specifically, define the lower and upper arms of the parabola, $\underline{F}$ and $\overline{F}$, such that for any $w \leq w_{EF}$, we have $\underline{F}(w) < \overline{F}(w)$ and

$$rw = a(\underline{F}(w)) - c(a(\underline{F}(w))) = \frac{\underline{F}(w)}{2C}\left(1 - \underline{F}(w)\right) \tag{20}$$
$$= a(\overline{F}(w)) - c\left(a(\overline{F}(w))\right) = \frac{\overline{F}(w)}{2C}\left(1 - \overline{F}(w)\right).$$

Relative to the first best, in which the efficient stock of relational incentives F_{EF} equals $1/2$ (see (2)), $\underline{F}$ traces out the pairs at which the partners have too little incentives, whereas at $\overline{F}$, they have too much incentives.

Function F describes the partners' relational incentives in a near-optimal SSE and fully characterizes the equilibrium dynamics. Given the quadratic cost of effort, the level of incentives, $F + 1/2$, translates linearly into the level of efforts taken by the partners. The effort, together with the level of relational capital w determine the drift of the relational capital (see Proposition 2), whereas its sensitivity with respect to profit flows,

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[21]We are also assuming that the upper bound A on efforts does not bind along the parabola, and so $a(\overline{F}(w)) < A$, for all $w \leq w_{EF}$, which requires $A > 1$.
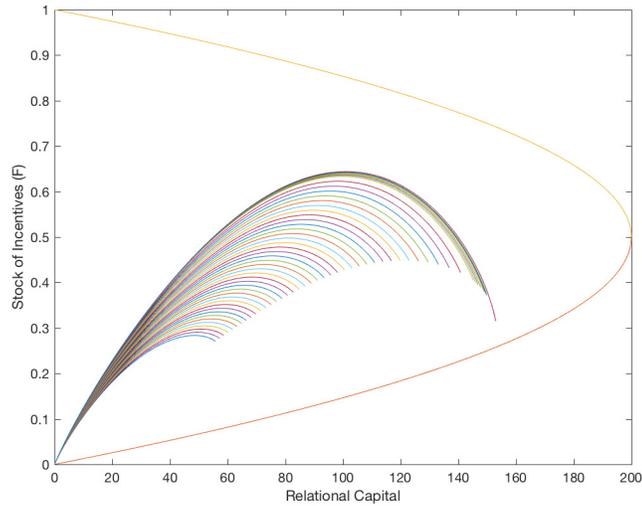
This figure displays the relational incentives as a function of the relational capital of the partnership. We fix a parametrization of the model,[22] and find local SSE as described in Proposition 3. Each curve describes an SSE, and the highest one a near-optimal one. The parametrization used satisfies the second-order condition in Theorem 3.
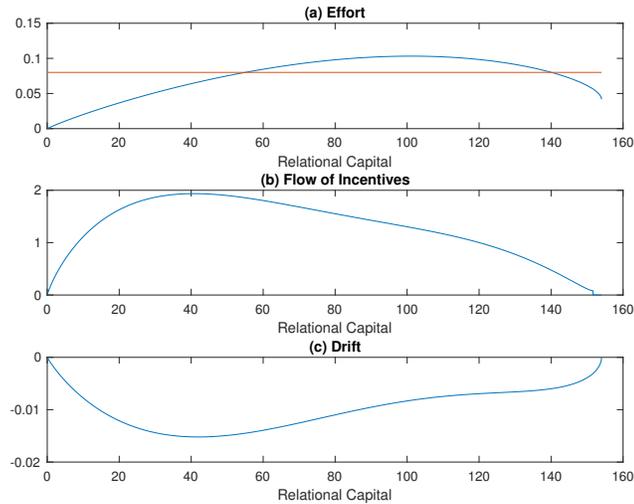
Figure 1: Relational Incentives in Near-optimal SSE

or flow of relational incentives $I(w)$, is proportional to the inverse of F''.[23]

The graph of F starts on the left at the stationary repeated Nash equilibrium point $(0,0)$. In the Appendix we establish that the graph never dips below $\underline{F}$, and when $C\sigma_Y$ is sufficiently large it also never reaches above $\overline{F}$ (Lemmas 1 and 6), as displayed in Figure 1. Proposition 2 implies that at any internal point, where the graph lies within the parabola, the drift of the relational capital is negative. When agents exert effort, the relational capital has no drift if the discounted value of effort is equal to the return on the relational capital, rw, as given by (20). Negative drift means that agents exert a too efficient level of effort, given their relational capital. In a sense, they gradually consume it away.

Relational capital responds linearly to unexpected profit outcomes (see equation (10)). As shown in Figure 2, the flow of relational incentives must vanish at the extremes so that the relational capital does not cross them. When relational capital is close to zero,

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[23]This is true as long as $F_{\varepsilon}''(w) \geq -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, as discussed in Theorem 2. Formula 28 in the Appendix gives us an exact characterization of it.

This figure displays in panel (a) the effort of a partner in the near-optimal SSE, as a function of the relational capital. The horizontal line represents the efficient level of effort. Panels (b) and (c) display the flow of relational incentives, and the drift as a function of the relational capital of the partnership, in the near-optimal SSE.

Figure 2: Effort, Flow of Relational Incentives, and Drift in Near-optimal SSE

the threat of unraveling starts binding; similarly, close to the bliss point partners efforts cannot be rewarded anymore.

Our results in Theorems 1 and 2 show that the (stock of) relational incentives, F, has three properties. The corollary and discussion below provides intuition and their economic implications. First, relational incentives are strictly concave in relational capital. Second, they are zero when relational capital is zero, and are increasing for low w. Third, relational incentives are decreasing when the relational capital is high. This implies that relational incentives have a unique internal maximizer. Since effort is linear in F, we have the following corollary regarding effort as a function of the relational capital of the partnership.

Corollary 1 (Rallying, Over-Working and Coasting) *In near-optimal local SSE effort $a(w)$ is increasing to the left, and decreasing to the right of $w^{\#}$, for $w^{\#} < w^*_{\varepsilon}$. For some parameter values, there is a neighborhood of $w^{\#}$ in which $a(w) > a_{EF}$.*

Note that in the near-optimal local SSE, relational capital always increases after high

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profit flows–positive $dY_t - \overline{\mu}_t dt$–since $I(w)$ is positive (see Figure 2). Thus, the corollary says that when relational capital is low, the partners' effort increases after good outcomes. Intuitively, a good profit outcome rallies the partnership further away from the brink of dissolution. Longer expected life of the partnership means more time when today's effort yields relational benefits, which encourages higher effort. A direct implication of rallying is that the equilibrium effort prescribed for today and for the immediate future are complements: Higher effort today increases relational capital, which, in turn, increases equilibrium effort.

Rallying and complementarity of efforts across time is related to the encouragement effect defined in Bolton and Harris [1999] (see also, for instance, Georgiadis [2014], Cetemen et al. [2017]). The difference is that in the literature on experimentation in teams, the effect comes from good signals making partners believe that the exogenous success of the project is more likely. In our case, good signals make partners believe that the endogenous failure (the dissolution of the partnership) is less imminent.

On the other hand, when relational capital is sufficiently high, the corollary implies that the partners' effort decreases after good outcomes. In other words, partners coast on their past good performance. The reason is that as the relational capital approaches the highest achievable value, the flow of relational incentives $I(w)$ must die out. At these high values, positive effort is driven almost entirely by deferred incentives: relational incentives discounted from the future, when the relational capital drifts down. Coasting also implies that when relational capital is sufficiently high, the equilibrium effort prescribed for today is a direct substitute of the equilibrium effort in the immediate future: Higher effort today increases relational capital, which makes partners subsequently coast more.

Deferred incentives would drive effort in any environment, as long as an agent can substitute effort across time to smooth its marginal cost. Intuitively, it pays off to exert effort and invest in relational capital now, when the marginal cost of effort is low, so that it bears fruit later, in the cut-throat phase, when partners work much and the marginal cost of effort is high. In our partnership model, deferred incentives rely on persistence and the imperfect state monitoring, which imply that incentive benefits of effort accrue

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gradually over time (compared with Proposition 1). Moreover, as they support effort even close to the bliss point, deferred incentives are precisely the force that prevents the triviality of equilibria result, as in Sannikov and Skrzypacz [2007] and Proposition 1.

(a) A short-lived partnership   (b) A long-lived partnership
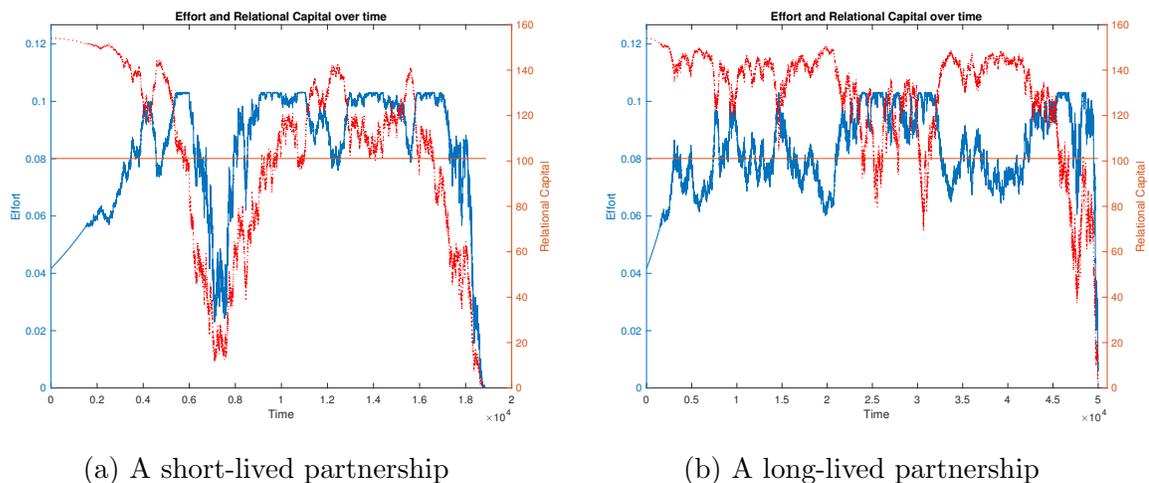
This figure displays two sample paths of effort (on the left axis) and relational capital (on the right axis) over time. Panel (a) displays an eventful sample path, where the relationship reached the brink of dissolution and was rallied back by partners' efforts. Panel (b) displays a long-lived relationship, in which the partners exerted effort for a long time. The horizontal line represents the efficient effort (on the left axis), and the relational capital where effort is maximized (on the right axis).

Figure 3: Effort and Relational Capital over Time

Figure 3 displays two sample paths for the evolution of effort and relational capital over time. In both sample paths, the relationship starts at the maximal relational capital. In the beginning, players coast, and as I is very small, the relational capital drifts down, undisturbed by shocks. When relational capital is above the horizontal line, profit outcomes that increase relational capital lead players to exert less effort. In these coasting phases, changes in effort and relational capital are negatively correlated. For instance, the sample path in Panel (b) shows, almost everywhere, that when relational capital goes up, effort goes down. Also note that on both sample paths, players exert effort higher than the stationary efficient quite frequently. When relational capital is below the horizontal line, changes in effort and relational capital are positively correlated. For instance, in Panel (a) at around time 0.7, the relational capital reached a very low point,

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with a very low effort as well, and the partnership was on the brink of dissolution. Good profit realizations raised the relational capital and partners rallied their effort, giving the partnership more time.

Another implication of our model is that in the near-optimal local SSE, partners may exert inefficiently high effort. This is illustrated in Figure 1, where the stock of relational incentives F reaches above the efficient level of $1/2$, and in both sample paths of Figure 3. The paradox is, of course, that the incentive problem is to find ways to provide nontrivial incentives, rather than to curb excessive incentives. In a model without persistence (and discrete time), such over-working may not happen.[24] In our case, high incentives for moderate levels of relational capital lead to inefficiently high effort, but this is outweighed by the benefit of sustaining nontrivial incentives at higher levels of relational capital, due to deferred incentives.

We believe that there are two ways to interpret the empirical implications of those results, depending on how one thinks about relational capital. In the model, relational capital is an endogenous variable that is not directly observable; however, it has a clear interpretation, facilitating the search for observable instruments. Those range from informal expressions of partners' optimism or "bad blood" to the goodwill of the joint venture, which is recorded in financial statements. Alternatively, one can calculate relational capital directly from the observable profit flows. Under this interpretation, for example, coasting means that after a string of good outcomes, the partners' effort starts decreasing. Although their effort is not observable, the joint effort can be identified from the changes in profitability of the partnership.

4.2 Sand in the Wheels

The first result below characterizes parameter values such that provision of relational incentives is impossible.

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[24]While we are not aware of a reference for this result, the intuition is clear: as long as the set of equilibrium payoff vectors is convex, scaling down the incentives is feasible in the Bellman problem.

Proposition 5 *The upper bound w^* on relational capital achievable in a local SSE converges to 0 as $r + \alpha + \gamma \to \infty$.*

Let us look at the role of each of these three parameters in our model. The first implication of Proposition 5 is that impatience (high r) hinders relational incentives. This confirms the folk intuition present in all dynamic models, as long as the cost of effort is born today, but the benefits are delayed.

The impossibility in the case of high α and high γ provide a continuity result relative to the models with no persistence, and with no observational noise (Sannikov and Skrzypacz [2007]; Proposition 1 above).

First, high α means that fundamentals revert quickly to the mean, and so, the effect of effort is distributed over only the immediate future. Intuitively, models with high mean reversion approximate the i.i.d. models, when the effect of effort on fundamentals is instantaneous.

In the limiting model without persistence, impossibility of relational incentives follows from the fact that they cannot be provided at (or near) the optimal level of continuation value: Nontrivial incentives, equivalent to strictly positive sensitivity of continuation value to profit flows, $I_t > 0$, would result in continuation value escaping to the right. In our case, the proof requires an additional step. Just as in i.i.d. models, the flow of relational incentives must be vanishing close to the bliss point w^*, but this still leaves the possibility of nontrivial deferred incentives. Given little persistence, however, this would require unbounded incentives delivered for moderate levels of relational capital, which is not possible.[25]

Second, the gain parameter γ is high when the observation noise is much smaller than the production noise, $\sigma_Y << \sigma_\mu$ (see formula (5)). In this sense, models with high gain parameter γ approximate models with perfect state monitoring.

The impossibility of relational incentives under near-perfect monitoring relies on the following new mechanism. High gain parameter γ means that partners' estimate of

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[25]Relational incentives is an integral of discounted volatilities of the relational capital process, belonging to a bounded interval.

fundamentals is very sensitive to the observed profits. This is because in our Brownian model the standard deviation of the change in fundamentals is large – which results in very volatile estimate, when signals are precise. High γ, therefore, gives rise to a very strong ratchet effect: When a partner increases effort, and so profit flows increase, the main effect will be a steep increase in the estimate of fundamentals – the benchmark against which the relational incentives are provided – and partners will be held to higher standards in the future. Put simply, high profit outcomes are credited not to high effort, but mostly to a lucky change in fundamentals. Thus, it becomes impossible to incentivize effort.

A direct consequence of the result is that improved monitoring of the state may hurt incentive provision. Formally, starting with a set of parameters that result in nontrivial relational capital (see Proposition 4), decreasing observational noise (increasing γ) leads eventually to a decrease in relational capital achievable in a local SSE. The result is related to the celebrated result in Abreu et al. [1991]. The intuition for the result in our setting— a consequence of extreme ratcheting—is different (in particular, it relies on the incomplete information about the state, which is absent in Abreu et al. [1991]) and to the best of our knowledge new.

The effect of small observational noise σ_Y is entirely different, however, in the case when fundamentals are near deterministic, $\sigma_\mu \approx 0$. Recall from Proposition 4 that nontrivial relational incentives exist when the noise σ_Y is small, as long as $\sigma_\mu << \sigma_Y$. Moreover, when the fundamentals are fully deterministic, we have the following:

Proposition 6 *Suppose that there is no production noise, $\sigma_\mu = 0$. The upper bound w^* on relational capital achievable in a local SSE is increasing in the precision of the monitoring technology σ_Y^{-1}.*

Unlike in the case of stochastic fundamentals, if fundamentals are deterministic, then better monitoring always improves efficiency. Indeed, very precise monitoring guarantees nontrivial relational incentives. This is because with deterministic fundamentals, profit flows are used only to provide incentives, and not to estimate the fundamentals.

The result is stated for the solutions of the relaxed problem, assuming only local incentive-compatibility constraints. Given Theorem 3, however, this implies that for any fixed cost function, for the range of observational noises σ_Y for which the local equilibrium is fully incentive-compatible, the same comparative statics holds for the supremum of relational capital achievable in an SSE.[26]

4.3 Beatles' Break-up

The only state that determines the flow ("stage game") payoffs in the game is the mean fundamentals, $\overline{\mu}_t$. It seems natural to restrict attention to Markov equilibria, in which this is also the only variable that determines the equilibrium actions, and serves as an accounting device for the good and bad histories – with high fundamentals indicative of past high effort. However, aligned with the literature,[27] our results demonstrate that the near-optimal equilibria are not Markovian. While stage game payoffs are driven by mean fundamentals, it is optimal to have actions driven by the endogenous relational capital.

Fundamentals and relational capital have different dynamics, and this has several empirical implications. The sensitivity of fundamentals with respect to the profit flows is constant, $\frac{d\overline{\mu}_t}{dY_t} = \gamma$. The sensitivity of relational capital, which determines effort, with respect to profits is variable and equals $I(w)$. Thus, unlike in Markov equilibria the relationship between the mean profitability and effort is not deterministic: two equally profitable partnerships can differ in the level of relational capital and effort, as Figure 4 shows.

For example, consider the following two partnerships. The first, on the brink of unraveling, experiences large positive shocks, while the second, close to the bliss point w^*, experiences large negative shocks. Equation (4) gives us that the fundamentals of the first (second) partnership grow (fall) proportionally to the shock. However, their relational capitals hardly budge, as relational capital's sensitivity and drift vanishes close

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[26]The tractable analytical characterization of the parameters for which the First Order Approach is valid seems unlikely. Consequently, the comparative statics of the SSE payoffs for a fixed cost function and the entire range of observational noise seems analytically intractable.

[27]See, for instance, the literature on contracting with persistence in footnote 18.

to the boundaries.

In contrast, if the production noise σ_μ is low or absent, fundamentals barely respond to profit outcomes (as γ is close to zero) and, thus, are much more sluggish than relational capital. In this case, a short string of sharp, low profit realizations will unravel the partnership, with hardly any effect on profitability. In other words, even a very profitable partnership may unravel, when its goodwill is tested by a series of adverse outcomes, even if they have a negligible effect on the partnership's profitability. This intuition is formalized in the next proposition.
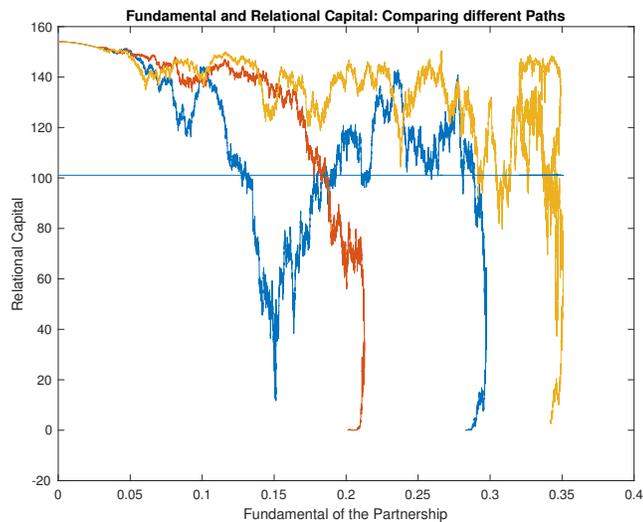

This figure displays three different sample paths of the relational capital of a partnership, as a function of the fundamentals of the relationship. The three different paths highlight that the relationship between fundamentals and relational capital is not a one-to-one relationship. Furthermore, the three partnerships unravel, when relational capital drops to zero, at different levels of fundamentals. The drops are sharp, with small effect on fundamentals. The horizontal line marks the relational capital where effort is maximized.

Figure 4: Relational Capital and Fundamentals of a Partnership

Figure 4 displays the differences in the dynamics of the two capitals. It shows three different sample paths, highlighting that partnership's relational capital is not determined by it's profitability. Furthermore, even at dissolution, different partnerships have different levels of productivity.

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Proposition 7 *In near-optimal local SSE, at any time $t > 0$, the distribution of mean fundamentals $\overline{\mu}_t$ and relational capital w_t has full support. Starting at any level of mean fundamentals $\overline{\mu}_t$, a partnership may unravel in an arbitrarily short period of time, with an arbitrarily small change in $\overline{\mu}_t$, when production noise σ_μ is sufficiently small.*

Finally, Figure 5 displays the relationship between the longevity of the partnership and its fundamentals at the moment of unraveling. Longer partnerships have, in general, better fundamentals when they unravel, as can be seen by the positive relationship displayed in Figure 5. This suggests that the partnerships are relatively unstable. They last if relational capital stays at intermediate levels, when partners work hard and keep fundamentals at high levels. Partnerships lingering on with low relational capital and little effort are rare.
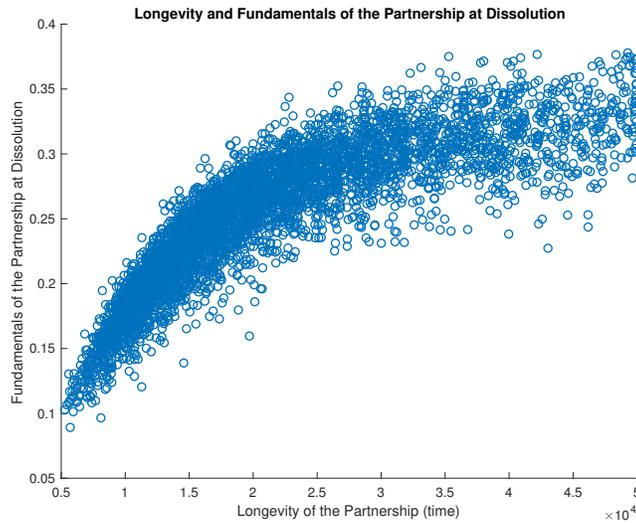
This figure displays the relationship between the longevity of the partnership and the partnership's fundamentals at dissolution. It displays the time to dissolution-fundamentals at dissolution pairs of five thousand simulated paths of the near-optimum SSE.

Figure 5: Longevity and Fundamentals of a Partnership

5   Concluding Remarks and Extensions

In this paper, we present a dynamic model of partnership whose two central features are persistent effect of effort and imperfect state monitoring. We develop a method that allows us to characterize near-optimal strongly symmetric equilibria of the game with a simple differential equation. Its solution describes the supremum of relational incentives achievable in an SSE for a given level of relational capital, and fully characterizes equilibrium dynamics in near-optimal equilibria. Persistence and imperfect state monitoring allows to sustain nontrivial relational incentives and equilibrium effort; we also identify a new way in which precise state monitoring, via stark ratcheting effect, may prevent incentive provision.

Relational capital, which captures the goodwill, or mutual trust in the partnership, is the only state variable, which evolves differently than the persistent "hard" fundamentals of the venture. The model generates novel predictions about the dynamics of effort, fundamentals and profits in a partnership. We also identify new channels for motivating partners in the setting with persistence and imperfect state monitoring.

Both the model of partnership and the solution method can be generalized. Below, we briefly discuss few extensions.

Negative effort. In the paper we assumed that the Nash equilibrium effort of 0 is also the lowest available effort to the partners. Just as in the standard repeated game analysis, this implies that the lowest equilibrium relational capital is zero, providing one of the boundary conditions in Theorems 1 and 2. The possibility of equilibrium "burning value" through negative effort might help partners and enlarge the set of SSE.[28] Regarding the solution, while the differential equation (14) and the right boundary conditions in Theorems 1 and 2 do not change, now not only the derivative $F'(w)$ but also the starting point $\underline{w} \leq 0$ is free.[29]

Observable effort. We may allow agents to also exert observable effort. It may

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[28]We note that the local SSE constructed in Theorem 2 remain local SSE even if negative efforts are allowed; they remain SSE when the condition of Theorem 3 is satisfied.

[29]It is easy to establish that just as $(0,0)$, the point $(\underline{w}, F(\underline{w}))$ must belong to the "parabola" of pairs (w, F) with zero drift.

be productive and drive profits up, or it may be unproductive, with the only effect of "burning value". We conjecture that the only difference in the resulting differential equation (14) is the extra term in the drift of the relational capital. When observable effort is parametrized by the effect o it has on the value of the partnership, in the case when $F' < 0$, partners exert the efficient level of observable effort $\bar{o} > 0$, which drives relational capital down. When relational capital is low and $F' > 0$, partners exert the most unproductive effort $\underline{o} < 0$. This "conspicuous toiling" is best viewed as an investment in relational capital, which moves up quickly in response.[30]

Trivial PPE with observable fundamentals. While Proposition 1 established only the impossibility of nontrivial strongly symmetric equilibria when fundamentals are observable, the result can be extended to asymmetric PPE, under appropriate conditions. Namely, suppose that $a(I) + a(-I) \leq 0$, for every $I > 0$.[31] The condition implies that when the sensitivity of the sum of private relational capitals $w_t^1 + w_t^2$ with respect to profits is nonpositive – as when the two sensitivities have opposite signs – then the total effort is negative. The proof is almost analogous to the one in symmetric case (see Sannikov and Skrzypacz [2007] and Bernard [2018]).

Selling partnership. An alternative interpretation of a partnership unravelling is that partners sell it. If the venture's market value is the value of its fundamentals, partners part with it when relational capital and, thus, their value added dries up. Realistically, the market value of a venture may well exceed its fundamental value. In this case, if partners cannot commit not to sell the partnership, the scope for incentives diminishes. For example, when market offers a fixed markup above the fundamental value, the relational capital may not decrease below this level. Formally, as in the case of negative effort, the left boundary condition in the equation (14) changes. When the market's markup is a fixed fraction of the fundamental value, the boundary condition includes both the fundamental value and relational capital. This requires adding the fundamental

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[30]Similar investment in the value of a partnership has been documented in the equilibrum setting by Fujiwara-Greve and Okuno-Fujiwara [2009] and verified in the lab setting by Lee [2018].

[31]For example, the condition is satisfied when efforts are chosen from the set $\mathcal{A} = [-A, A]$ and the cost of effort satisfies $c''' \leq 0$.

value as the second state variable in equation (14).[32]

Nonlinear Production. In our model, efforts of the partners contribute to the fundamentals linearly. One might argue, however, that an important feature of a successful team is complementarity of its members, suggesting a supermodular stage game production function.[33] Generalizing production function in this way would change the optimal levels of effort and fundamentals, but not change the nature of the incentive problem, or the solution method.

Similarly, in our model the effect of effort on fundamentals does not depend on the level of fundamentals. Allowing for such interdependence would create additional Markov incentives, as partners exert effort to affect the production technology in the future, much like in dynamic decision problems or in Markov equilibria.[34] For example, when marginal productivity of effort is decreasing in fundamentals, incentives are dampened, since higher effort today makes effort less productive tomorrow. In the opposite case, one exerts extra effort to invest in a better production technology tomorrow.

Formally, when production function depends on the level of fundamentals, equation (14) needs to include fundamentals as the additional state variable, just as with the selling of the partnership. Analysis of such partnerships, and, in particular, how Markov incentives interact with relational incentives, is a subject for future research.

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[32]When there is no production noise, and so fundamentals change deterministically, the new differential equation does not have additional second-order derivatives.

[33]We implicitly assume a different form of complementarity, namely that the project cannot be split in two, run separately by each partner.

[34]A different analogy is with a Stackelberg game with complementarities: the leader has direct incentives to take high action, so that the follower finds it optimal to take high action as well. See the analysis by Board and Meyer-ter Vehn [2013] of a Product Choice game with persistence.

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6   Appendix: Proofs

6.1   Proofs of Propositions 1, 2 and 3.

Proof of Proposition 1. i) The proof can be slipt in two parts: first, establishing that for an arbitrary pair of strategies relational capital $\{w_t\}$ follows a process (9) for some $\{I_t\}$, and second, establishing the relationship between processes $\{a_t\}$ and $\{I_t\}$. The proof follows similar steps as Propositions 1 and 2 in Sannikov [2007].

Specifically, in order to establish that the relational capital has the representation (9), observe that the process $\left\{\mu_t - \int_0^t \left[(r+\alpha)\, 2a_s - \alpha\mu_s\right] ds\right\}$, scaled by σ_μ, is a Brownian Motion, and that the process $\widetilde{w}_t = \int_0^t e^{-rs}\left(a_s - c(a_s)\right) ds + e^{-rt}w_t$ is a martingale. Since efforts, and so $\widetilde{w}_t$ are bounded, it follows directly from Proposition 3.4.14 in Karatzas [1991] (of which the Martingale Representation Theorem is a special case, when the filtration $\mathcal{F}_t$ is generated only by the process of fundamentals) that $\widetilde{w}_t$ equals $\int_0^t e^{-rs}I_s^i\left(d\mu_s - \left[(r+\alpha)\, 2a_s - \alpha\mu_s\right] ds\right) + M_t^w$, for appropriate $\{I_t\}$ and martingale $\{M_t^w\}$. Differentiating and equating both expressions for $\widetilde{w}_t$ yields the representation.

Conversely, for a bounded process $\{v_t\}$ that satisfies (9), define the process $\widetilde{v}_t = \int_0^t e^{-rs}\left(a_s - c(a_s)\right) ds + e^{-rt}v_t$, together with $\widetilde{w}_t$ as above. Both $\{\widetilde{v}_t\}$ and $\{\widetilde{w}_t\}$ are bounded martingales and so, as their values agree at infinity, they agree after every history. It follows that the processes $\{v_t\}$ and $\{w_t\}$ are the same.

As regards incentive compatibility, fix an alternative strategy $\{\widetilde{a}_t\}$ for player i and note that the relational capital satisfies

$$
\mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right) dt\right]
$$
$$
= \mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right) dt + w_\tau + \int_\tau^\infty d\left(e^{-rt}w_t\right)\right]
$$
$$
= w_\tau + \mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right) dt + \int_\tau^\infty e^{-rt}\left(dw_t - rw_t dt\right)\right]
$$
$$
= w_\tau + \mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(\frac{\widetilde{a}_t - a_t}{2} - c(\widetilde{a}_t) + c(a_t) + I_t\,(r+\alpha)\,(\widetilde{a}_t - a_t)\right) dt\right],
$$

where the first equality follows from $\mathbb{E}_\tau^{\left\{\widetilde{a}_t^i, a_t^{-i}\right\}}\left[e^{-r(t-\tau)}w_t\right] \to 0$, as $t \to \infty$ (given that

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efforts are bounded), and the last one follows from $\mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}} [d\mu_t - [(r + \alpha) 2a_t - \alpha\mu_t] dt] = (r + \alpha) \mathbb{E}_\tau^{\{\widetilde{a}_t, a_t\}} [\widetilde{a}_t - a_t]$. Since continuation value and relational capital differ by a constant, it follows from this representation and convexity of costs that there exists no profitable deviating strategy for partner i if and only if his effort process satisfies $a_t = a((r + \alpha) I_t)$.

ii) From representation (9) it follows that when $w_t \geq \varepsilon > 0$, then either the volatility satisfies $I_t \sigma_\mu \geq \delta > 0$, in order to incentivize a strictly positive, more efficient effort, or the drift satisfies $\mathbb{E}_\tau^{\{a_t^1, a_t^2\}} [dw_t] \geq \delta > 0$, to satisfy promise keeping (where δ depends on ε). It follows that if $w_0 > 0$ then the process $\{w_t\}$ escapes to infinity with positive probability, which, given bounded efforts, yields contradiction.

Proof of Proposition 2. Fix a strategy profile $\{a_t, a_t\}$. The first part of the proof is identical to the first part of the proof of Proposition 1: since the process $\left\{ Y_t - \int_0^t \overline{\mu}_s ds \right\}$, scaled by σ_Y, is a Brownian Motion, it follows from Proposition 3.4.14 in Karatzas [1991] that a process $\{w_t\}$ is the relational capital process associated with $\{a_t, a_t\}$, defined in (7), precisely when it can be represented as in (10), for some L^2 process $\{I_t\}$ and a martingale $\{M_t\}$ orthogonal to $\{Y_t\}$.

Let us now evaluate the marginal benefit of effort, and the marginal relational benefit of effort F_τ in particular. Consider the Brownian Motion $\sigma_Y^{-1} \left\{ Y_t - \int_0^t \overline{\mu}_s ds \right\}$. It follows from Girsanov's Theorem that the change in the underlying density measure of the output paths induced by the change in expected fundamentals from $\overline{\mu}_\tau$ to $\overline{\mu}_\tau^{dev} = \overline{\mu}_\tau + \varepsilon (r + \alpha)$ is

$$\Gamma_t^\varepsilon = e^{-\frac{1}{2} \int_\tau^t \frac{\left(\overline{\mu}_s^{dev} - \overline{\mu}_s\right)^2}{\sigma_Y^2} ds + \int_\tau^t \frac{\overline{\mu}_s^{dev} - \overline{\mu}_s}{\sigma_Y} \frac{dY_s - \overline{\mu}_s ds}{\sigma_Y}},$$

for $t > \tau$, where $\{\overline{\mu}_s\}_{s \geq \tau}$ and $\{\overline{\mu}_s^{dev}\}_{s \geq \tau}$ are the associated paths of estimates, defined in (4), with $\overline{\mu}_s^{dev} - \overline{\mu}_s = \varepsilon e^{-(\alpha+\gamma)(s-\tau)}$, $s > \tau$. The relational capital at time τ thus changes to

$$\mathbb{E}_\tau^{\{a_t, a_t\}} \left[\int_\tau^\infty e^{-r(t-\tau)} \Gamma_t^\varepsilon (a_t - c(a_t)) dt \right].$$

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Since

$$\left.\frac{\partial}{\partial\varepsilon}\Gamma_t^\varepsilon\right|_{\varepsilon=0} = (r+\alpha)\int_\tau^t e^{-(\alpha+\gamma)(s-\tau)}\frac{dY_s - \overline{\mu}_s ds}{\sigma_Y},$$

it follows that

$$\begin{aligned}
F_\tau &= \frac{\partial}{\partial\varepsilon}\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\Gamma_t^\varepsilon\left(a_t - c(a_t)\right)dt\right]\\
&= (r+\alpha)\,\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty e^{-r(t-\tau)}\left(a_t - c(a_t)\right)\left(\int_\tau^t e^{-(\alpha+\gamma)(s-\tau)}\frac{dY_s - \overline{\mu}_s ds}{\sigma_Y}\right)dt\right]\\
&= (r+\alpha)\,\mathbb{E}_\tau^{\{a_t,a_t\}}\left[\int_\tau^\infty\left(\int_t^\infty e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right],
\end{aligned}$$

where the last equality follows from the change of integration.

Intuitively, in the last integral above, the inside integral corresponds to the forward looking relational capital, which is then multiplied by a Brownian innovation, scaled by the discounted impact of shifted (expected) fundamentals. The correlation between the relational capital and the Brownian innovation equals I_t, from the representation of the relational capital. This yields F_τ as the expected discounted integral of I_t.

Formally, for $\tau' \geq \tau$

$$\begin{aligned}
\mathbb{E}_{\tau'}^{\{a_t,a_t\}}\left[F_\tau\right] &= (r+\alpha)\,\mathbb{E}_{\tau'}^{\{a_t,a_t\}}\left[\int_\tau^\infty\left(\int_t^\infty e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right]\\
&= (r+\alpha)\left[\int_\tau^{\tau'}\left(\int_t^{\tau'} e^{-r(s-t)}\left(a_s - c(a_s)\right)ds\right)e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right]\\
&\quad + (r+\alpha)\,w_{\tau'}\times\left[\int_\tau^{\tau'} e^{-r(\tau'-t)}e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right] + e^{-(r+\alpha+\gamma)(\tau'-\tau)}F_{\tau'}
\end{aligned}$$

is a martingale, as a function of τ'. Using the representation of the relational capital established above, the drift of this martingale equals

$$\begin{aligned}
(r+\alpha)&\left[e^{-(r+\alpha+\gamma)(\tau'-\tau)}I_{\tau'} + \left((a_{\tau'} - c(a_{\tau'})) - rw_{\tau'}\right)\int_\tau^{\tau'} e^{-r(\tau'-t)}e^{-(r+\alpha+\gamma)(t-\tau)}\frac{dY_t - \overline{\mu}_t dt}{\sigma_Y}\right]\\
&+ \frac{d}{dt}e^{-(r+\alpha+\gamma)(\tau'-\tau)}F_{\tau'},
\end{aligned}$$

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where the first term is the covariance of the Brownian increments of $(r + \alpha)w_{\tau'}$ and of the bracketed stochastic intergral in the last line. Integrating over $[\tau, \infty)$ and taking expectation at time τ yields

$$0 = (r + a) \, \mathbb{E}_\tau^{\{a_t, a_t\}} \left[\int_\tau^\infty e^{-(r+\alpha+\gamma)(t-\tau)} I_t \right] - F_\tau.$$

Using Proposition 3.4.14 from Karatzas [1991] one more time, F_τ satisfies the above equation precisely when it can be represented as in (10).

Finally, since effort increases fundamentals by a factor of $r + \alpha$, and given the decomposition of the continuation value as in (7), the effort process is a local SSE exactly when a_t satisfies $a_t = a(F_t)$ (see e.g. the Verification Theorem in Yong and Zhou [1999] Ch.3.2).

Proof of Proposition 3. Note that the boundary condition (13) can be satisfied in two ways. The first line of (13) is equivalent to $I\left(w^\partial\right) \left(r + \alpha + \frac{F''(w)}{2} \sigma_Y^2 I(w^\partial)\right) = 0$, which can hold either when $I\left(w^\partial\right) = 0$, or $I(w^\partial) = -2\frac{r+\alpha}{\sigma_Y^2 F''(w)} > 0$. Construction of a local SSE that achieves the boundary in the case $I(w^\partial) > 0$, when relational capital "escapes" the interval $[\underline{w}, \overline{w}]$, requires an additional step, as we detail below.

First, we extend the functions F and I beyond the boundary points w^∂, at which condition (13) is satisfied with $I\left(w^\partial\right) > 0$. Consider a boundary point $w^\partial = \overline{w}$ and $rw^\partial - \left(a(F(w^\partial)) - c(a(F(w^\partial)))\right) < 0$. We use the Implicit Function Theorem to extend function F to a point $\overline{\overline{w}} > \overline{w}$, so that conditions (13) and $F''(w) < 0$ hold on $\left[\overline{w}, \overline{\overline{w}}\right]$. We also extend I continuously to the interval $\left[\overline{w}, \overline{\overline{w}}\right]$ with $I(w) = -2\frac{r+\alpha}{\sigma_Y^2 F''(w)} > 0$, so that F and I satisfy the equation (12) on $\left[\overline{w}, \overline{\overline{w}}\right]$. In words, on the interval $\left[\overline{w}, \overline{\overline{w}}\right]$ the relational incentives can be provided in two ways: they can either consist entirely of the discounted future relational incentives, with zero flow, or by providing inefficiently high flow of relational incentives I. The extension to the interval $\left[\underline{\underline{w}}, \underline{w}\right]$ in the case of $w^\partial = \underline{w}$ is analogous.

Fix $w_0 \in [\underline{w}, \overline{w}]$. We first construct a process $\{w_t\}$ of continuation values that satisfies the stochastic equation (10). Let τ^∞ be the stopping time when $\{w_t\}$ reaches a boundary

point that is a local SSE. Moreover, define a sequence of stopping times $(\tau_n)_{n \in \mathbb{N}_+}$ such that $\tau_0 = 0$; for n odd, $\tau_n \geq \tau_{n-1}$ is the stopping time when $\{w_t\}$ reaches either of the new, "outside" boundary points $\{\underline{\underline{w}}, \overline{\overline{w}}\}$; and for $n > 0$ even, $\tau_n \geq \tau_{n-1}$ is the stopping time when $\{w_t\}$ reaches either of the original "inside" boundary points $\{\underline{w}, \overline{w}\}$. For times $t \in [\tau_n, \tau_{n+1})$ with n even and $t < \tau^\infty$ we let $\{w_t\}$ be the weak solution to (10), with $I_t = I(w_t)$ and $\{M_t^w\} = 0$, starting at w_{τ_n}. Existence of a weak solution follows from the continuity if it's drift (which is a consequence of continuity of F and action defined via (8)) and volatility I (see e.g. Karatzas [1991], Theorem 5.4.22). For times $t \in [\tau_n, \tau_{n+1})$ with n odd and $t < \tau^\infty$ we let $\{w_t\}$ be the weak solution to (10), with $I_t = 0$ and $\{M_t^w\} = 0$, starting at w_{τ_n}. In words, the process $\{w_t\}$ has positive volatility until it reaches an "outside" boundary point in $\{\underline{\underline{w}}, \overline{\overline{w}}\}$, after which it drifts "inside" till it reaches the "inside" boundary point in $\{\underline{w}, \overline{w}\}$, when it resumes with the positive volatility, and so on.

It follows from the Ito formula that before τ^∞ the process $F_t = F(w_t)$, satisfies the differential equation in (10), with $\{M_t^F\} = 0$ and $J_t = F'(w_t) \times I(w_t)$. Since both w_t and F_t are bounded, the transversality conditions are satisfied. Finally, we may extend the processes $\{w_t\}, \{I_t\}, \{F_t\}$ and $\{J_t\}$, together with martingales $\{M_t^w\}$ and $\{M_t^F\}$ beyond τ^∞ by letting them follow a local SSE that achieves $(w_{\tau^\infty}, F(w_{\tau^\infty}))$. Then the processes satisfy conditions of Proposition 2.

6.2 Proof of Theorem 1.

Let us define $\mathcal{E}$ to be the set of the pairs of relational capital and relational incentives, (w, F), achievable in local SSE, and let the partial function $E : \mathbb{R} \to \mathbb{R}$ parametrize the upper boundary of this set, as a function of w.

The proof of the Theorem follows in several steps. Lemma 1 below establishes convexity and the bounds on the set $\mathcal{E}$. Lemma 3 establishes the novel "local" escape argument, which is the key step of the proof. It is a version of the escape arguments used in the stochastic control verification theorems, adapted to our setting, in which the law

of motion of a state variable w_t depends also on the value of the problem, $F(w_t)$.

Using those results, together with Proposition 3, Proposition 8 shows that E is differentiable at any interior w, whereas Propositions 9 and 10 show that E locally satisfies the differential equation (14) whenever the rate of change of the tangent is bounded – condition that is verified in Proposition 11.

Let us define the efficient level of relational capital as $w_{EF} = \frac{1}{r}(a_{EF} - c(a_{EF}))$, for a_{EF} as in (2).

Lemma 1 *The set $\mathcal{E}$ is convex and $w^* \leq w_{EF}$. Moreover, for the upper boundary E we have $E(w) \geq \underline{F}(w) > 0$, for all $w \in (0, w^*)$.*

Proof. Convexity is immediate from the possibility of public randomization, and the inequality $w \leq w_{EF}$ follows from the definitions. Finally, suppose by the way of contradiction that there exists w, $0 \leq w < w^*$, such that $E(w) < \underline{F}(w)$. Note that at w the slope of E is smaller than the slope of $\underline{F}$: otherwise, the repeated static Nash point $(0,0)$, belonging to the graph of the convex function $\underline{F}$, and the convex set $\mathcal{E}$ would not overlap. This implies that E is bounded away below $\underline{F}$ to the right of w, and so in any local SSE the relational capital has drift bounded away above zero, as long as $w_t \geq w$ (see (10)). The possibility of escape of relational capital beyond w^* establishes the contradiction. ∎

Lemma 2 *If $(1, E')$ is a tangent vector at $(w_0, E(w_0))$ then in must be that*

$$(r + \alpha + \gamma)E(w_0) \geq E' \times (rw_0 - (a(E(w_0)) - c(a(E(w_0))))). \tag{21}$$

Proof. Suppose that (21) is violated, and "drift" $rw_0 - (a(E(w_0)) - c(a(E(w_0)))) < 0$ (when the inequality is reversed the proof is analogous; when drift equals 0 then both sides of (21) equal zero). Let $\overline{E}' > E'$ be such that (21) holds with equality, with $\overline{E}'$ in place of E'. Consider the function F defined over $[w_0, w']$, where w' is in the right neighborhood of w_0, such that F satisfies (21) with equality, with initial contition $(F(w_0), F'(w_0)) = (E(w_0), \overline{E}')$, $w - (a(F(w)) - c(a(F(w)))) < 0$ for all $w \in [w_0, w']$, and F lies strictly above the upper boundary E. The function F is the solution of the implicit function

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second order ordinary differential equation. Finally, from the continuous dependence on initial parameters[35], let $\widetilde{F}$ be a function defined in the right neighborhood $[w_0, \widetilde{w}']$ of w_0 that satisfies (21) with equality, $\widetilde{F}(w_0) < E(w_0)$, $w - (a(F(w)) - c(a(F(w)))) < 0$ for all $w \in [w_0, \widetilde{w}']$ and $\widetilde{F}(w) > E(w)$ for some $w > w_0$ ($\widetilde{F}$ starts just below, and "cuts above" E).

Note that $(w_0, \widetilde{F}(w_0))$ is achieved by a local SSE and the boundary condition (15) holds at $\widetilde{w}'$. Thus, the function $\widetilde{F}$ satisfies conditions of Proposition 3, together with $I \equiv 0$, and so there are local SSE that achieve it. Since part of $\widetilde{F}$ lies strictly above E, this yields a contradiction. ∎

For $\varepsilon > 0$ consider a differential equation related to the one in Theorem 1, but with an extra "slack" of ε,

$$(r + \alpha + \gamma)G_\varepsilon(w) = G'_\varepsilon(w)\left(rw - (a(G_\varepsilon(w)) - c(a(G_\varepsilon(w))))\right) - \frac{(r+\alpha)^2}{2\sigma_Y^2 G''_\varepsilon(w)} + \varepsilon, \quad (22)$$

where function a is defined in (8), as before. Intuitively, for given initial conditions $(w, G_\varepsilon, G'_\varepsilon)$ the solution G_ε of (22), is more concave (has more curvature) than the solution of (14).

Lemma 3 *For every $\varepsilon, M > 0$ there is $\delta(\varepsilon, M) > 0$ such that for any concave solution G_ε of the differential equation (22) on an interval $[\underline{w}, \overline{w}]$, with $|G'_\varepsilon| \leq M$, the following conditions may not be satisfied:*

$$i)\ G_\varepsilon(\underline{w}) = E(\underline{w})\ and\ G_\varepsilon(\overline{w}) = E(\overline{w}),$$
$$ii)\ 0 < E(w) - G_\varepsilon(w) \leq \delta(\varepsilon, M).\ for\ w \in (\underline{w}, \overline{w})$$

Roughly speaking, a standard escape argument would have G_ε solve the original differential equation (14) and no upper bound on the distance from $E(w)$ to $G_\varepsilon(w)$ in condition ii). The idea is that the only way to generate a value strictly above the solution of the differential equation, which captures the maximum, given the state w_t and shadow

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[35]Recall the definition of function a in (8), and that cost function c is differentiable.

values F' and F'', is for the value to drift higher. This (together with no escape via the endpoints, guaranteed by i)) establishes that the value must be able to grow without bound, with positive probability, establishing the contradiction. In our case, however, the law of motion of w_t also depends on the value F. This implies that, given the state w_t and shadow values F' and F'', the differential equation captures only a *local* maximum, for a given value of F. The idea of the modified result is to show that with F in the vicinity of the solution to the differential equation, the law of motion of w_t does not change much. Thus, in the vicinity of F the value must be close to the solution of the original differential equation (14), and so below the solution of the differential equation (22) with an appropriate slack. This implies that the value of the problem that starts above but sufficiently close to G_ε must drift up, out of the neighborhood (violating the second inequality in ii)).

Proof. Fix (w_0, F_0) with $w_0 \in (\underline{w}, \overline{w})$ and $G_\varepsilon(w_0) < F_0 < E(w_0)$, together with a local SSE that achieves it, and let $\{w_t\}$ and $\{F_t\}$ be the processes of relational capital and relational incentives it gives rise to. Define $D(w_t, F_t)$ as the distance of F_t from the solution G_ε of the differential equation (22),

$$D(w_t, F_t) = F_t - G_\varepsilon(w_t).$$

Using Ito's lemma together with the Proposition 2, at any time t when $D(w_t, F_t) \in [0, \delta(\varepsilon, M)]$, the drift of the process $D(w_t, F_t)$ equals, for appropriate process $\{I_t\}$,

$$
\begin{aligned}
\frac{\mathbb{E}\left[dD(w_t, F_t)\right]}{dt} &= (r + \alpha + \gamma) F_t - (r + \alpha) I_t - G'_\varepsilon(w_t) \times (r w_t - (a(F_t) - c(a(F_t)))) \qquad (23) \\
&\quad - \frac{G''_\varepsilon(w) \left[\sigma_Y^2 I_t^2 + d\langle M_t^w \rangle\right]}{2} \\
&\geq (r + \alpha + \gamma) F_t - (r + \alpha) I_t - G'_\varepsilon(w_t) \times (r w_t - (a(G_\varepsilon(w_t)) - c(a(G_\varepsilon(w_t))))) \\
&\quad - \frac{G''_\varepsilon(w) \left[\sigma_Y^2 I_t^2 + d\langle M_t^w \rangle\right]}{2} - \frac{\varepsilon}{2} \\
&\geq (r + \alpha + \gamma)(F_t - G_\varepsilon(w_t)) + \varepsilon - \frac{\varepsilon}{2} > (r + \alpha + \gamma) \times D(w_t, F_t),
\end{aligned}
$$

The first inequality holds because $|G'_\varepsilon(w_t)| \leq M$, functions a and c are Lipschitz con-

tinuous and $D(w_t, F_t) \in [0, \delta(\varepsilon, M)]$, where $\delta(\varepsilon, M)$ is assumed to be sufficiently small. The second inequality follows because G_ε satisfies

$$(r+\alpha+\gamma)G_\varepsilon(w) = \max_I \left\{ (r+\alpha)I + G'_\varepsilon(w)(rw - (a(G_\varepsilon(w)) - c(a(G_\varepsilon(w))))) + \frac{G''_\varepsilon(w)\sigma_Y^2}{2}I^2 \right\} + \varepsilon,$$

G_ε is concave, and $d\langle M_t^w \rangle$ is positive. Let τ be the stopping time of the process $D(w_t, F_t)$ hitting zero. Due to $D(w_0, F_0) > 0$ and inequality (23), it follows that there is a finite time T such that $E[D(w_T, F_T)|\tau \geq T] > \delta(\varepsilon, M)$. On the other hand, since

$$E\left[D(w_{\min\{T,\tau\}}, F_{\min\{T,\tau\}})\right] = P(\tau \geq T) \times E[D(w_T, F_T)|\tau \geq T]$$
$$+ P(\tau < T) \times E[D(w_\tau, F_\tau)|\tau < T]$$
$$= P(\tau \geq T) \times E[D(w_T, F_T)|\tau \geq T],$$

and the expectation is positive, it follows that $P(\tau \geq T) > 0$. This establishes that $D(w_T, F_T)$ exceeds $\delta(\varepsilon, M)$ with positive probability, contradiction. ∎

Proposition 8 *The upper boundary E of the set of relational capital and relational incentives achievable in a local SSE is differentiable in $(0, w^*)$.*

Proof. Suppose to the contrary that $(w_0, E(w_0))$ is a kink. If follows from Lemma 2 that there exists an interior tangent vector $(1, E')$ at $(w_0, E(w_0))$ such that

$$(r + \alpha + \gamma)E(w_0) > E' \times (rw_0 - (a(E(w_0)) - c(a(E(w_0))))).$$

In this case the differential equation (14), written as $F''(w) = \mathcal{F}(w, F, F')$, has the right hand side Lipschitz continuous in the neighborhood of the point $(w_0, E(w_0), E')$, with $F'' < 0$. Continuous dependence on the initial parameters implies that there exists $\varepsilon > 0$ such that G_ε^* solving (22) with the same initial conditions is strictly above curve E in a neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time shifting the initial condition $(w_0, E(w_0), E')$ down to $(w_0, E(w_0) - \delta, E')$,

for $0 < \delta << \varepsilon$, we construct a function G_ε that satisfies the conditions of Lemma 3, yielding the contradiction. ∎

Let us distinguish points on the boundary E at which the solution to the differential equation (14) would require F'' infinite. Specifically, we will say that $(w_0, E(w_0), E'(w_0)) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ is *nondegenerate* if

$$(r + \alpha + \gamma)E(w_0) > E'(w_0) \times (rw_0 - (a(E(w_0)) - c(a(E(w_0))))). \qquad (24)$$

Proposition 9 *Suppose $(w_0, E(w_0), E'(w_0))$ is nondegenerate. Then the solution F to the differential equation (14) with this initial condition is weakly above curve E in the neighborhood of w_0.*

Proof. Suppose to the contrary that $F < E$ in, say, the right neighborhood of w_0 (the case of the left neighborhood is analogous). Given that $(w_0, E(w_0), E'(w_0))$ is nondegenerate, and so the solution F is continuous in the initial conditions in the neighborhood of $(w_0, E(w_0), E'(w_0))$, there exists $\widetilde{E}' > E'(w_0)$ such that the solution $\widetilde{F}$ of (14) with initial conditions $(w_0, E(w_0), \widetilde{E}')$ "comes back to " E, meaning $\widetilde{F}(\overline{w}) = E(\overline{w})$ for some $\overline{w} > w_0$. But then the function $\widetilde{F}$ defined on $[w_0, \overline{w}]$ satisfies the conditions of Proposition 3, and so its graph is achievable by local SSE. However, since $\widetilde{E}' > E'(w_0)$ it follows that $\widetilde{F}$ is strictly above E in the right neighborhood of w_0, yielding the contradiction. ∎

For the next Proposition we need the following technical Lemma.

Lemma 4 *Let $F, E : [\underline{w}, \overline{w}) \to \mathbb{R}$ be two concave functions such that*

$$i) \ E \leq F,$$

$$ii) \ E(\underline{w}) = F(\underline{w}) \ and \ E'_+(\underline{w}) = F'_+(\underline{w}),$$

$$iii) \ F''_+(\underline{w}) \ exists$$

Then either $E''_+(\underline{w})$ exists and equals $F''_+(\underline{w})$ or there is G with $G(\underline{w}) = E(\underline{w})$, $G'_+(\underline{w}) = E'_+(\underline{w})$ and $G''_+(\underline{w}) < F''_+(\underline{w})$ such that $E \leq G$ in a right neighborhood of $\underline{w}$.

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Proof. Suppose that $E''_+(\underline{w})$ does not exist or is not equal to $F''_+(\underline{w})$. From i), this means that there is a $\varepsilon > 0$ and a decreasing sequence $\{w_n\} \to \underline{w}$ such that

$$E(w_n) \leq F(\underline{w}) + F'_+(\underline{w}) \times (w_n - \underline{w}) + \left(F''_+(\underline{w}) - \varepsilon\right) \times (w_n - \underline{w})^2.$$

However, concavity of E implies that the above inequality holds not only for the sequence $\{w_n\}$ but in a right neighborhood of $\underline{w}$. This implies the result, with $G(w) = F(w) - \varepsilon(w - \underline{w})^2$ in a neighborhood of $\underline{w}$. ∎

Proposition 10 *Suppose $(w_0, E(w_0), E'(w_0))$ is nondegenerate. Then $E''(w_0)$ exists and E satisfies the differential equation (14) at w_0.*

Proof. Let F satisfy (14) with initial conditions $(w_0, E(w_0), E'(w_0))$ and suppose that either $E''_+(w_0)$ does not exist, or $E''_+(w_0) \neq F''_+(w_0)$ (the case of left second derivative is analogous). Propositions 8 and 9 establish that the conditions of Lemma 4 are satisfied at w_0, and so in the right neighborhood of w_0 E is bounded above by $F(w) - \overline{\varepsilon}(w - w_0)^2$, for appropriate $\overline{\varepsilon} > 0$. Continuous dependence on the initial parameters implies that there exists $\varepsilon > 0$ such that G^*_ε solving (22) with the same initial conditions $(w_0, E(w_0), E'(w_0))$ as F has second derivative at w_0 strictly larger than $F''(w_0) - \overline{\varepsilon}$ and is strictly above curve E in a right neighborhood of w_0 (excluding point w_0). Invoking the continuous dependence once again, this time turning the initial condition $(w_0, E(w_0), E'(w_0))$ right to $(w_0, E(w_0), E'(w_0) - \delta)$, for $0 < \delta << \varepsilon$, we construct a function G_ε that satisfies the conditions of Lemma 3, yielding contradiction. ∎

Together with Proposition 10, the following proposition establishes that the upper boundary E satisfies the differential equation (14), for positive relational capitals.

Proposition 11 *For every $w_0 \in (0, w^*)$ the point $(w_0, E(w_0), E'(w_0))$ is nondegenerate.*

Proof. Note that a degenerate point may not satisfy $rw_0 - (a(E(w_0)) - c(a(E(w_0)))) = 0$, since the left hand side of (21) is strictly positive (Lemma 1). Suppose that the degenerate point $(w_0, E(w_0), E'(w_0))$ satisfies $rw_0 - (a(E(w_0)) - c(a(E(w_0)))) < 0$ (the proof for

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the case of reverse inequality is analogous). As in the proof of Lemma 2, the C^2 function F defined in a right neighborhood of w_0, which satisfies (21) with equality and satisfies initial contition $(F(w_0), F'(w_0)) = (E(w_0), E'(w_0))$ must lie below E. Consider now a strictly concave quadratic function $G^* < F$, with $(G^*(w_0), G^{*\prime}(w_0)) = (E(w_0), E'(w_0))$. The function satisfies

$$(r + \alpha + \gamma) G^* (w) < G^{*\prime}(w) (rw - (a(G^*(w)) - c(a(G^*(w))))) - \frac{(r+\alpha)^2}{2\sigma_Y^2 G^{*\prime\prime}}, \qquad (25)$$

in a right neighborhood of w_0. But then, by increasing slightly $G^{*\prime}(w_0)$, we may construct a quadratic function G over an inteval $[w_0, \overline{w}]$ that also satisfies (25), together with $G(w_0) = E(w_0)$, $G'(w_0) > E'(w_0)$, and $G(\overline{w}) = E(\overline{w})$. There exists then a function $I : [w_0, \overline{w}] \to \mathbb{R}$, with $I(w) > -\frac{(r+\alpha)^2}{\sigma_Y^2 G''}$, such that

$$(r + \alpha + \gamma) G (w) = I(w) + G'(w) (rw - (a(G(w)) - c(a(G(w))))) + \frac{G'' \sigma_Y^2}{2} I^2. \ w \in [w_0, \overline{w}]$$

Applying Proposition 2, each point $(w, G(w))$, for $w \in [w_0, \overline{w}]$, can be achieved by a local SSE. Since $G'(w_0) > E'(w_0)$, this yields the desired contradiction. ∎

To conclude the proof of the theorem, it remains to establish the boundary conditions (15).

1. $E(0) = 0$. Strictly positive relational incentives at zero in an SSE would imply that the expected discounted efforts by each agent are strictly positive; consequently, a deviation to zero effort always would yield a nonzero relational capital to a partner, contradiction.

2. $\lim_{w \uparrow w^*} E(w) = \underline{F}(w^*)$. i) Lemma 1 shows that $\lim_{w \uparrow w^*} E(w) < \underline{F}(w^*)$ is impossible. ii) If $\lim_{w \uparrow w^*} E(w) \in \left(\underline{F}(w_0), \overline{F}(w_0)\right)$, then, using Proposition 2, it would be possible to extend the solution to the right, with $I(w) = 0$ for $w > w^*$, contradiction. iii) If $\lim_{w \uparrow w^*} E(w) = \overline{F}(w_0)$ then, whether E approaches $\overline{F}$ from above or below, the differential equation (14) would be violated in the left neighborhood of w_*. iv) If $\lim_{w \uparrow w^*} E(w) > \overline{F}(w_0)$, then relational capital in any local SSE achieving points close to

$(w^*, \lim_{w \uparrow w^*} E(w))$ has strictly positive drift, bounded away from zero. This would lead to the escape of w to the right of w^*, with positive probability.

3. $\lim_{w \uparrow w^*} E''(w) = -\infty$. When the condition is violated, then $I^*(w)$ is continuous and strictly positive close to w^*. The proof of the theorem so far establishes that E is C^2 and satisfies the differential equation (14). Given this regularity, standard verification theorem techniques establish that the equilibria achieving $(w, E(w))$, $w < w^*$, must use the optimal flow of relational incentives $I^*(w)$ a.e. (see Yong and Zhou [1999]), ; when $(w, E(w))$ is unattainable, the same is true for (w, F) in the limit, with F approaching $E(w)$. This, however, leads to the relational capital escaping to the right of w^*, with positive probability.

6.3 Proof of Theorem 2

In the following proofs we will need the following result.

Lemma 5 *For any $\varepsilon > 0$ and the function F_ε from Theorem 2*

$$F_\varepsilon \le \frac{(r + \alpha)^2}{256 \sigma_Y^2 (r + \alpha + \gamma) r^2 C^2} + 1. \tag{26}$$

Proof. Let $w_0 \in [0, \overline{w}_\varepsilon]$ be the point at which F_ε is maximized and $F'_\varepsilon(w_0) = 0$. For $w \ge w_0$ such that $F_\varepsilon(w) \ge \overline{F}(0) = 1 \ge \overline{F}(w)$, so that the drift of the relational capital $rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))$ is positive, we have

$$(r + \alpha + \gamma) F_\varepsilon(w) = F'_\varepsilon(w) (rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))) - \frac{(r + \alpha)^2}{2\sigma_Y^2 F''_\varepsilon(w)} \le -\frac{(r + \alpha)^2}{2\sigma_Y^2 F''_\varepsilon(w)}, \tag{27}$$

$$-F''_\varepsilon(w) \le \frac{(r + \alpha)^2}{2\sigma_Y^2 (r + \alpha + \gamma)},$$

where the equality follows from the fact that $F''_\varepsilon(w) \ge -\frac{r + \alpha}{\sigma_Y^2 \varepsilon}$ (otherwse the right hand side would fall short of 1, and so the left hand side). Since $\overline{w}_\varepsilon \le w_{EF} = 1/8rC$, it

therefore follows that

$$F_\varepsilon(w_0) \leq F_\varepsilon(w_0) - F_\varepsilon(\overline{w}_\varepsilon) + 1 \leq \frac{1}{2} \frac{(r+\alpha)^2}{2\sigma_Y^2 (r+\alpha+\gamma)} \left(\frac{1}{8rC}\right)^2 + 1$$

$$= \frac{(r+\alpha)^2}{256\sigma_Y^2 (r+\alpha+\gamma) r^2 C^2} + 1.$$

∎

The proof of the first part of the Theorem is analogous to the proof of Theorem 1. The optimal policy function implied by 18 is given by

$$I_\varepsilon^*(w) = -\frac{r+\alpha}{\sigma_Y^2 F''(w)}, \quad \text{if } F_\varepsilon''(w) \geq -\frac{r+\alpha}{\sigma_Y^2 \varepsilon} \tag{28}$$

$$I_\varepsilon^*(w) = \varepsilon, \quad \text{if } -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon} < F_\varepsilon''(w) < -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$$

$$I_\varepsilon^*(w) = 0. \quad \text{if } F_\varepsilon''(w) \leq -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$$

It is easy to establish that $F_\varepsilon''(0) = -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, since with any other value, the equation (18) would be violated around zero. In what follows we establish that if there is w such that $F_\varepsilon''(w) < -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, then $F_\varepsilon'(w) << 0$ and $rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))) \approx 0$. Given concavity of F_ε, $w - w_\varepsilon^*$ is small, and so this will establish the proof with F_ε restricted to $[0, \overline{w}_\varepsilon]$, where $\overline{w}_\varepsilon$ is the first point such that $F_\varepsilon''(\overline{w}_\varepsilon) = -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ and $F_\varepsilon''(w) < -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ in the right neighborhood of $\overline{w}_\varepsilon$.

Consider w such that $F_\varepsilon''(w) < -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$. Given quadratic costs, we have

$$(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' = \frac{1}{C} (1/2 - F_\varepsilon(w)) F_\varepsilon'(w).$$

Thus, differentiating (18), we get[36]

$$F_{\varepsilon}''(w) = \frac{F_{\varepsilon}'(w)\left(\alpha + \gamma + (a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w))))'\right)}{rw - a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w)))} \tag{29}$$

$$= \frac{F_{\varepsilon}'(w)\left(\alpha + \gamma + \frac{1}{C}\left(1/2 - F_{\varepsilon}(w)\right)F_{\varepsilon}'(w)\right)}{rw - a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w)))}$$

$$\geq -\frac{F_{\varepsilon}'^2(w)}{2C|rw - a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w)))|}, \quad \text{when } F_{\varepsilon}'(w) \leq 0$$

$$\geq -\frac{C_1 F_{\varepsilon}'^2(w)}{C(rw - a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w))))}, \quad \text{when } F_{\varepsilon}'(w) \geq 0$$

where C_1 is the bound on F_ε from Lemma 5. For an appropriate $C_2 > 0$ this yields

$$\frac{F_{\varepsilon}'^2(w)}{|rw - a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w)))|} \geq \frac{C_2}{\varepsilon}. \tag{30}$$

On the other hand, equation (18) implies that

$$F_{\varepsilon}'(w)\left(rw - (a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w))))\right) = (r + \alpha + \gamma)F_{\varepsilon}(w) \leq (r + \alpha + \gamma)C_1. \tag{31}$$

Inequalities (30) and (31) imply that $|rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))| \leq C_3 \varepsilon^{1/3}$, with $C_3 > 0$. Since $F_\varepsilon(w) \geq 1/2$, in the case when $F_\varepsilon'(w) \geq 0$ (so that the drift of relational capital is positive), whereas $F_\varepsilon(w) \geq \lim_{w \to w_\varepsilon^*} F_\varepsilon(w) = \underline{F}(w_\varepsilon^*) \geq C_4 > 0$, in the case when $F_\varepsilon'(w) \leq 0$ (the equality follows from the boundary condition (15)) equation (18) yields

$$|F_{\varepsilon}'(w)| = \frac{(r + \alpha + \gamma)F_{\varepsilon}(w)}{|rw - a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w)))|} \geq C_5 \varepsilon^{-1/3}. \tag{32}$$

Since F_ε is concave and bounded in $[0, C_1]$, inequality (27) implies

$$w_\varepsilon^* - w = O(\varepsilon^{1/3}), \quad \text{when } F_\varepsilon' < 0$$

$$w = O\left(\varepsilon^{1/3}\right). \quad \text{when } F_\varepsilon' > 0$$

It is enough now to show that the case $F_\varepsilon'(w) \geq 0$ is not possible. Note that since

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[36]Note that (18) implies $F_{\varepsilon}'(w) \times [rw - a(F_{\varepsilon}(w)) - c(a(F_{\varepsilon}(w)))] \geq 0$.

w is small and $rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))$ positive, we have $F_\varepsilon(w) \approx \overline{F}(0) = 1$. By differentiating (29),

$$F_\varepsilon'''(w) = \left(\frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \right)' \left(\alpha + \gamma + (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' \right) \quad (33)$$
$$+ \frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))) \right)''$$
$$> \frac{F_\varepsilon'(w)}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))) \right)''$$
$$=_{sgn} \left(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))) \right)'',$$

where the inequality follows from the fact that $F_\varepsilon''(w) < 0$ and

$$(rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' = r - \frac{1}{C} \left(1/2 - F_\varepsilon(w) \right) F_\varepsilon'(w)$$
$$\approx r + \frac{1}{2C} F_\varepsilon'(w) > 0,$$
$$\alpha + \gamma + (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))' = \alpha + \gamma + \frac{1}{C} \left(1/2 - F_\varepsilon(w) \right) F_\varepsilon'(w)$$
$$\approx \alpha + \gamma - \frac{1}{2C} F_\varepsilon'(w) < 0,$$

when ε is small enough. Finally,

$$(a(F_\varepsilon(w)) - c(a(F_\varepsilon(w))))'' = \left(\frac{1}{C} \left(1/2 - F_\varepsilon(w) \right) F_\varepsilon'(w) \right)' \quad (34)$$
$$=_{sgn} \left(1/2 - F_\varepsilon(w) \right) F_\varepsilon''(w) - (F_\varepsilon'(w))^2$$
$$\approx -\frac{1}{2} F_\varepsilon''(w) - (F_\varepsilon'(w))^2$$
$$\approx \frac{1}{4C} \frac{(F_\varepsilon'(w))^2}{rw - a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))} - (F_\varepsilon'(w))^2 > 0,$$

when ε is small enough, where the last line follows from (29). This establishes that $F_\varepsilon''(w^0) \leq -2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$ implies $F_\varepsilon'''(w^0) > 0$, and so case $F_\varepsilon'(w^0) \geq 0$ is not possible.

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To conclude the proof, since $w_{EF} = 1/8rC$, we have

$$F_\varepsilon'(w) \le F_\varepsilon'(0) \le F_\varepsilon'(0) - F_\varepsilon'(w_\varepsilon^*) \le \frac{1}{8rC} \times 2\frac{r+\alpha}{\sigma_Y^2 \varepsilon}.$$

6.4 Proof of Theorem 3

Step 1. Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$, together with the processes $\{w_t\}, \{F_t\}, \{I_t\}$ and $\{J_t\}$ that satisfy equations (10) (Proposition 2 and Theorem 2). In this step we show that as long as

$$J_t \le \frac{C(r + 2(\alpha + \gamma))}{8(r+\alpha)}, \quad \forall t \tag{35}$$

then, for an appropriate $X > 0$ and any deviating strategy $\{\widetilde{a}_t\}$, the relational capital at any time $\tau \ge 0$ to the deviating agent is bounded above by

$$\widetilde{w}_\tau(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau) = w_\tau + \frac{F_\tau}{r+\alpha}(\widetilde{\mu}_\tau - \overline{\mu}_\tau) + X(\widetilde{\mu}_\tau - \overline{\mu}_\tau)^2. \tag{36}$$

In the formula, w_τ is the equilibrium level of relational capital, determined by (10), $\widetilde{\mu}_\tau$ are the correct beliefs, given strategies $\{\widetilde{a}_t\}$ and $\{a_t\}$, and $\overline{\mu}_\tau$ are the equilibrium beliefs, given that both strategies are $\{a_t\}$, both determined by (4). Consequently, using the bound with $\widetilde{\mu}_t = \overline{\mu}_t$, the step establishes that local SSE strategies are globally incentive compatible, as long as the bound (35) holds.

Fix a deviation strategy $\{\widetilde{a}_t\}$ and consider the process

$$v_\tau = \int_0^\tau e^{-rs}\left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt + e_\tau^{-r\tau}\widetilde{w}(\widetilde{\mu}_\tau - \overline{\mu}_\tau, w_\tau),$$

where, from (4), the wedge process $\{\widetilde{\mu}_t - \overline{\mu}_t\}$ follows

$$d(\widetilde{\mu}_t - \overline{\mu}_t) = (r+\alpha)(\widetilde{a}_t - a_t)dt - (\alpha + \gamma)(\widetilde{\mu}_t - \overline{\mu}_t)dt.$$

In order to establish bound (36) it is enough to show that the process $\{v_t\}$ has negative drift. We have

$$e^{-rt}dv_t = \left(\frac{\widetilde{a}_t + a_t}{2} - c(\widetilde{a}_t)\right)dt - r\left(w_t + \frac{F_t}{r+\alpha}(\widetilde{\mu}_t - \overline{\mu}_t) + X(\widetilde{\mu}_t - \overline{\mu}_t)^2\right)$$

$$+ (rW_t - (a_t + c(a_t)))dt + I_t \times (dY_t - \overline{\mu}_t dt)$$

$$+ \frac{\widetilde{\mu}_t - \overline{\mu}_t}{r+\alpha}\left((r+\alpha+\gamma)F_t - (r+\alpha)I_t dt + J_t \times (dY_t - \overline{\mu}_t dt)\right)$$

$$+ \left(\frac{F_t}{r+\alpha} + 2X(\widetilde{\mu}_t - \overline{\mu}_t)\right)\left((r+\alpha)(\widetilde{a}_t - a_t)dt - (\alpha+\gamma)(\widetilde{\mu}_t - \overline{\mu}_t)dt\right).$$

Given that the drift of $dY_t - \overline{\mu}_t dt$ is $(\widetilde{\mu}_t - \overline{\mu}_t)dt$, the drift of the $e^{-rt}dv_t$ process equals

$$\frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + F_t(\widetilde{a}_t - a_t)$$

$$+ (\widetilde{\mu}_t - \overline{\mu}_t)^2\left(\frac{J_t}{r+\alpha} - X(r + 2(\alpha+\gamma))\right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X(r+\alpha)$$

$$\leq \frac{\widetilde{a}_t - a_t}{2} + c(a_t) - c(\widetilde{a}_t) + Ca_t(\widetilde{a}_t - a_t)$$

$$+ (\widetilde{\mu}_t - \overline{\mu}_t)^2\left(\frac{J_t}{r+\alpha} - X(r + 2(\alpha+\gamma))\right) + (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X(r+\alpha)$$

$$= -\frac{C}{2}(a_t - \widetilde{a}_t)^2 + (\widetilde{\mu}_t - \overline{\mu}_t)^2\left(\frac{J_t}{r+\alpha} - X(r + 2(\alpha+\gamma))\right)$$

$$+ (\widetilde{\mu}_t - \overline{\mu}_t)(\widetilde{a}_t - a_t)2X(r+\alpha),$$

where we used that $c(a) = \frac{1}{2}a + \frac{C}{2}a^2$, and $F_t(\widetilde{a}_t - a_t) \leq Ca_t(\widetilde{a}_t - a_t)$, with equality in the case $a_t < A$.

Note that when the matrix

$$\begin{bmatrix} -\frac{C}{2} & X(r+\alpha) \\ X(r+\alpha) & \frac{J_t}{r+\alpha} - X(r + 2(\alpha+\gamma)) \end{bmatrix}$$

has a positive determinant, then the trace is negative, and the matrix is negative semidef-

inite, guaranteing negative drift. Since

$$\max_{X} \left\{ -\frac{C}{2} \times \left(\frac{J_t}{r+\alpha} - X\left(r+2\left(\alpha+\gamma\right)\right) \right) - X^2\left(r+\alpha\right)^2 \right\}$$
$$= \frac{C}{2\left(r+\alpha\right)} \left(\frac{C\left(r+2\left(\alpha+\gamma\right)\right)}{8\left(r+\alpha\right)} - J_t \right),$$

it follows that, indeed, when J_t is bounded as in (35), then the bound (36) holds for X that maximizes the above expression.

Step 2. Fix $\varepsilon > 0$ and consider an $\varepsilon-$optimal local SSE $\{a_t, a_t\}$. In this step we show that when $C\sigma_Y$ is sufficiently large, then for any w_t the sensitivity J_t of relational incentives is bounded as in (35). Together with step 1, this will establish the proof of Theorem 3.

Recall from Proposition 2 and the discussion below that

$$J_t = J(w_t) = F'_\varepsilon(w) \times I^*_\varepsilon(w).$$

Let us bound $I^*_\varepsilon(w)$, in the case when $F'_\varepsilon(w) \geq 0$. (Since $I^*_\varepsilon \geq 0$, the bound (35) holds in the case when $F'_\varepsilon(w) \geq 0$.) Over the subset $S \subseteq [0, \overline{w}_\varepsilon)$ where $F''_\varepsilon(w) < -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, we simply have $I^*_\varepsilon(w) = \varepsilon$. Over the complement $[0, \overline{w}_\varepsilon) \backslash S$, where $F''_\varepsilon(w) \geq -\frac{r+\alpha}{\sigma_Y^2 \varepsilon}$, we have,

$$I^*_\varepsilon(w) = -\frac{r+\alpha}{\sigma_Y^2 F''_\varepsilon(w)} = \frac{2}{r+\alpha} \left\{ (r+\alpha+\gamma) F_\varepsilon(w) - F'_\varepsilon(w) \left(rw - (a(F_\varepsilon(w)) - c(a(F_\varepsilon(w)))) \right) \right\}$$
$$\leq \frac{2}{r+\alpha} \left\{ \frac{(r+\alpha)^2}{256\sigma_Y^2 r^2 C^2} + r + \gamma + \alpha + \frac{r+\alpha}{4\sigma_Y^2 Cr\varepsilon} \frac{1}{8C} \right\},$$
$$= \frac{r+\alpha}{128\sigma_Y^2 r^2 C^2} + \frac{2(r+\gamma+\alpha)}{r+\alpha} + \frac{1}{16\sigma_Y^2 C^2 r\varepsilon} =: I^{\#}$$

where we use the bound (26) on F_ε, from Lemma 5, the bound (19) on $F'_\varepsilon \leq \frac{r+\alpha}{4\sigma_Y^2 Cr\varepsilon}$, establlished in Theorem 2, and the lower bound of $-\left(a_{EF} - c\left(a_{EF}\right)\right) = -1/8C$ on the drift of relational capital.

Condition (35) thus boils down to

$$J_t = F'_\varepsilon(w) \times I^*_\varepsilon(w) \leq \frac{r+\alpha}{4\sigma_Y^2 Cr\varepsilon} \times (\varepsilon + I^\#) \leq \frac{C(r+2(\alpha+\gamma))}{8(r+\alpha)},$$

or,

$$\varepsilon + \frac{r+\alpha}{128\sigma_Y^2 r^2 C^2} + \frac{2(r+\gamma+\alpha)}{r+\alpha} + \frac{1}{16\sigma_Y^2 C^2 r\varepsilon} \leq C^2 \frac{(r+2(\alpha+\gamma))}{2(r+\alpha)^2}\sigma_Y^2 r\varepsilon, \qquad (37)$$

which is satisfied when $C\sigma_Y$ is large enough. This concludes the proof of the step, and the Theorem.

6.5   Other Proofs

Proof of Proposition 4. The proof strategy is to construct a a C^2 function $F : [0, \overline{w}] \to R$ that satisfies the differential inequality

$$(r+\alpha+\gamma)F(w) \leq F'(w) \times (rw - (a(F(w)) - c(a(F(w))))) - \frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)}, \qquad (38)$$

together with the left boundary condition $F(0) = 0$ (achieveable by the repeated static Nash), and the right boundary condition (13). Given such an F, it is always possible to find an $I(w) \geq -\frac{r+\alpha}{\sigma_Y^2 F''(w)}$ for which the equation (12) in Proposition 3 holds at every $w \in [0, \overline{w}]$.

Given the quadratic cost of effort $c(a) = \frac{a}{2} + \frac{C}{2}a^2$, the flow payoffs (given interior efforts) satisfy

$$a(F) - c(a(F)) = \frac{F(w)}{2C}(1 - F(w)),$$

an also $\underline{F}'(0) = 2Cr$ (see (20)). We will construct a curve F over $[0, \overline{w}]$, with $\overline{w} = \delta/r = \frac{1}{16Cr}$, constant second derivative and with the right boundary condition

$$F(\overline{w}) = \frac{1}{2} > \underline{F}(\overline{w}).$$

We also set $F(0) = 0$, as well as $I(\overline{w}) = 0$ and

$$F'(\overline{w}) = \frac{(r + \alpha + \gamma)F(\overline{w})}{r\overline{w} - \frac{F(\overline{w})}{2C}(1 - F(\overline{w}))} = \frac{\frac{1}{2}(r + \alpha + \gamma)}{\delta - \frac{1}{8C}} = -4C(r + \alpha + \gamma),$$

so that the first equation in (13) is satisfied at $\overline{w}$; the second equation follows from $F(\overline{w}) \in \left(\underline{F}(\overline{w}), \overline{F}(\overline{w})\right)$.

The constant second derivative D is pinned down by

$$F(\overline{w}) = \int_0^{\overline{w}} F'(x)dx = \int_0^{\overline{w}} [F'(\overline{w}) - D(\overline{w} - x)]\, dx$$

$$= F'(\overline{w}) \times \frac{\delta}{r} - \frac{D}{2}\left(\frac{\delta}{r}\right)^2,$$

$$\frac{1}{D} = \frac{1}{2} \frac{1}{F'(\overline{w}) \times \frac{\delta}{r} - F(\overline{w})}\left(\frac{\delta}{r}\right)^2 = \frac{1}{2}\frac{1}{-\frac{1}{4}(r + \alpha + \gamma) \times \frac{1}{r} - \frac{1}{2}}\left(\frac{\delta}{r}\right)^2$$

$$= -\frac{2}{2 + r + \alpha + \gamma}\left(\frac{\delta}{r}\right)^2.$$

It follows that, for all $w \in [0, \overline{w}]$,

$$F(w) \le \frac{1}{2} + 4C(r + \alpha + \gamma) \times \frac{\delta}{r} \le \frac{r + \alpha + \gamma}{r}, \tag{39}$$

$$|F'(w)| \le F'(0) \le |F'(\overline{w})| + \frac{F(\overline{w}) - 0}{|F'(\overline{w})|}|D| = 4C(r + \alpha + \gamma) + \frac{2 + r + \alpha + \gamma}{r + \alpha + \gamma}16Cr^2,$$

$$rw - \frac{F(w)}{2C}(1 - F(w)) \ge -\frac{1}{8C},$$

$$(r + \alpha + \gamma)F(w) - F'(w)\left(rw - \frac{F(w)}{2C}(1 - F(w))\right) \le \frac{(r + \alpha + \gamma)^2}{r}$$

$$+ \frac{r + \alpha + \gamma}{2} + \frac{2 + r + \alpha + \gamma}{r + \alpha + \gamma}2r^2,$$

$$-\frac{(r + \alpha)^2}{2\sigma_Y^2 D} = \frac{(r + \alpha)^2}{2\sigma_Y^2}\frac{2}{2 + r + \alpha + \gamma}\left(\frac{1}{16Cr}\right)^2 \ge \frac{1}{512\sigma_Y^2 C^2},$$

where we also assume that the bound A is high enough so that the efforts are interior,

$$A \ge C\frac{r + \alpha + \gamma}{r} \ge C\max_w F(w) \ge \max_w a\left(F(w)\right).$$

The last two inequalities in (39) establish that inequality (38) is satisfied, and so

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nontrivial local SSE exist, as long as

$$\frac{(r+\alpha+\gamma)^2}{r} + \frac{r+\alpha+\gamma}{2} + \frac{2+r+\alpha+\gamma}{r+\alpha+\gamma}2r^2 \leq \frac{1}{512\sigma_Y^2 C^2}. \tag{40}$$

Moreover, the policy function $I(w)$ equals zero at the extremes, and for any $w \in (0, \overline{w})$ satisfies

$$I(w) \geq -\frac{r+\alpha}{\sigma_Y^2 D} \geq \frac{r+\alpha}{\sigma_Y^2}\frac{2}{2+r+\alpha+\gamma}\left(\frac{1}{16Cr}\right)^2 \geq \frac{1}{256\sigma_Y^2 C^2 r} =: \varepsilon. \tag{41}$$

The above establishes that for ε as in (41) the supremum w_ε^* of relational capitals achievable in $\varepsilon-$optimal local SSE is strictly above zero. Invoking Theorem 3, this local SSE is globally incentive compatible when

$$\frac{1}{256\sigma_Y^2 C^2 r} + \frac{r+\alpha}{128\sigma_Y^2 r^2 C^2} + \frac{2(r+\gamma+\alpha)}{r+\alpha} + \frac{256\sigma_Y^2 C^2 r}{16\sigma_Y^2 C^2 r} \leq \frac{(r+2(\alpha+\gamma))}{512(r+\alpha)^2}. \tag{42}$$

Inequalitiy (40) is satisfied when $\sigma_Y^2 C^2(r+\alpha+\gamma)$ is sufficiently small, and inequality (42) holds when $\sigma_Y^2 C^2$ is sufficiently large. This concludes the proof.

Proof of Proposition 5. Consider a sequence of models parametrized by the persistence parameter α, and with all other parameters fixed. Suppose to the contrary that $\lim_{\alpha\to 0} w_\alpha^* > A > 0$, where w_α^* is the supremum of relational capitals achievable in local SSE, for the model parametrized by α.

We start by pointing out the following three properties. First, note that for every α and w at which the drift of the relational capital is negative, the drift is uniformly bounded by

$$|rw - (a(F_\alpha(w)) - c(a(F_\alpha(w))))| \leq a(F_\alpha(w)) - c(a(F_\alpha(w))) \leq a_{EF} - c_{EF} =: B.$$

Moreover, equation (14) implies that, for all α, at w such that $F_\alpha'(w) \in \left[\frac{A}{2B}(r+\alpha+\gamma), 0\right]$

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we have

$$\frac{(r+\alpha)^2}{-\sigma_Y^2 F_\alpha''(w)} \geq (r+\alpha+\gamma)\left(A - \frac{A}{2B}B\right), \tag{43}$$

$$-F_\alpha''(w) \leq \frac{2(r+\alpha+\gamma)}{A\sigma_Y^2}.$$

Finally, for every α, $D > 0$ and w at which the drift of the relational capital is positive and $F_\alpha'(w) \leq 0$, if $F_\alpha(w) \geq D(r+\alpha+\gamma)$ then

$$\frac{(r+\alpha)^2}{-\sigma_Y^2 F_\alpha''(w)} \geq D(r+\alpha+\gamma)^2, \tag{44}$$

$$-F_\alpha''(w) \leq \frac{1}{D\sigma_Y^2}.$$

We now establish the contradiction as follows. Fix α high (to be determined later), and let $0 < w^0 < w^1 < w^2$ be such that

$$F_\alpha'\left(w^0\right) = 0,$$

$$F_\alpha'\left(w^1\right) = -\frac{A}{4B}(r+\alpha+\gamma),$$

$$F_\alpha'\left(w^2\right) = -\frac{A}{2B}(r+\alpha+\gamma).$$

It follows from concavity of F and (43) that

$$F_\alpha\left(w^1\right) \geq \frac{A}{4B}(r+\alpha+\gamma) \times \left(w^2 - w^1\right)$$

$$\geq \frac{A}{4B}(r+\alpha+\gamma) \times \frac{\frac{A}{4B}(r+\alpha+\gamma)}{\frac{2(r+\alpha+\gamma)}{A\sigma_Y^2}} = \frac{A^3\sigma_Y^2}{32B^2}(r+\alpha+\gamma).$$

Consequently, when α is large enough so that $\frac{A^3\sigma_Y^2}{32B^2}(r+\alpha+\gamma) > 1$ and so the drift of relational capital for $w \in [w^0, w^1]$ is positive, it follows from (44) that

$$w^1 - w^0 \geq \frac{\frac{A}{4B}(r+\alpha+\gamma)}{\frac{32B^2}{A^3\sigma_Y^2}} = \frac{A^4\sigma_Y^2}{128B^3}(r+\alpha+\gamma).$$

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When α is large enough, the last inequality contradicts $w^1 - w^0 < w^1 < w_{EF} = \frac{1}{8Cr}$.

Proof of Proposition 6. Suppose $\gamma = \sigma_\mu = 0$. The existence of local SSE, when σ_Y is sufficiently small, follows from Proposition 4. In order to establish monotonicity, note that decreasing σ_Y changes equation (12) in Proposition 2 only by decreasing the last term. This means that if a pair of functions (F, I) satisfies the conditions of Proposition 2 for some interval $[\underline{w}, \overline{w}]$ and a given σ_Y, then for any σ_Y' with $0 < \sigma_Y' < \sigma_Y$ there is a function $\overline{I} \geq I$ such that the pair $(F, \overline{I})$ satisfies the conditions of Proposition 2 for σ_Y'. Applying the result to the pairs $(F_\varepsilon, I_\varepsilon^*)$ on the intervals $[0, \overline{w}_\varepsilon]$, with ε converging to zero, establishes the proof.

Lemma 6 *When $C\sigma_Y$ is large enough, then the upper boundary F of relational incentives achievable in a local SSE satisfies $F(w) \leq \overline{F}(w)$, for all w.*

Proof. Otherwise, for all $w \in [\underline{w}, \overline{w}]$ with $F(w) \geq \overline{F}(w)$ and $F'(w) \leq 0$ we have

$$-\frac{(r+\alpha)^2}{2\sigma_Y^2 F''(w)} > (r+\alpha+\gamma)F(w) > \frac{r+\alpha+\gamma}{2},$$

which provides a bound $-F''(w) < \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)}$. Let w_0 be the point at which $F(w_0) \geq \overline{F}(w_0)$ and $F'(w_0) = 0$ (note that F' must be strictly positive for low w, and strictly negative close to the right boundary). Since $w^* \leq w_{EF} = \frac{1}{8rC}$, it follows that as long as

$$C\sigma_Y \geq \frac{r+\alpha}{4r\sqrt{r+\alpha+\gamma}}, \tag{45}$$

then for all $w \in (w_0, w^*)$ we have

$$\begin{aligned}
F'(w) &\geq -\frac{1}{8rC} \times \frac{(r+\alpha)^2}{\sigma_Y^2(r+\alpha+\gamma)} = -2rC \times \frac{(r+\alpha)^2}{16r^2C^2\sigma_Y^2(r+\alpha+\gamma)} \\
&\geq -2rC = \overline{F}'(0) \geq \overline{F}'(w).
\end{aligned}$$

This implies $F(w) \geq \overline{F}(w)$, for all $w \in (w_0, w^*)$, and so F violates the boundary condition (13), establishing contradiction. ∎