Econ 703

August Prelim

Instructions:

- This is a *closed* book examination.
- The total points are 120.
- You have 2.5 hours to answer questions.
- Please give a complete and intelligible solution.
- All the best.

1. (40 points) Consider the following method of dissolving a partnership. Two partners each own one share of a firm. They have a dispute and each partner wishes to either buy-out, or be bought-out by the other partner. Partner i values both his own and her partner's share at θ_i per share. These values θ_1 and θ_2 are independently and uniformly distributed on $[0,1]$. Each partner i knows her own θ_i but not θ_{-i}.

(1) (10 points) Suppose that each partner i chooses a bid $b_i \geq 0$ for the other's share. The highest bidder wins and obtains her partner's share for the amount of the opponent's (losing) bid. So if i is the winner, i's payoff is $2\theta_i - b_j$ and the loser's payoff is b_j, where $j \neq i$. In the event of a tie, each partner wins with equal probability.

Compute the interim expected payoff of bidder i with type θ_i and bid b_i, when the opponent uses a linear symmetric strategy $b_j(\theta_j) = \alpha + \beta\theta_j$ with $\beta > 0$.

Solution. Suppose that $b_i \leq \alpha + \beta$. Then bidder i wins if and only if $\theta_j < \frac{b_i - \alpha}{\beta}$. So type θ_i's expected payoff with bid b_i is

$$\begin{aligned}
&\int_0^{\frac{b_i-\alpha}{\beta}} (2\theta_i - \alpha - \beta\theta_j)d\theta_j + \int_{\frac{b_i-\alpha}{\beta}}^1 b_i d\theta_j \\
&= \frac{b_i-\alpha}{\beta}(2\theta_i - \alpha) - \frac{\beta}{2}\left(\frac{b_i-\alpha}{\beta}\right)^2 + \left(1 - \frac{b_i-\alpha}{\beta}\right) b_i \\
&= \frac{b_i-\alpha}{\beta}(2\theta_i - \alpha) - \frac{1}{2\beta}(b_i-\alpha)^2 + \left(1 - \frac{b_i-\alpha}{\beta}\right) b_i.
\end{aligned}$$

Suppose now that $b_i > \alpha + \beta$. Then bidder i always wins. So type θ_i's expected payoff with bid b_i is

$$\int_0^1 (2\theta_i - \alpha - \beta\theta_j)d\theta_j = 2\theta_i - \alpha - \frac{\beta}{2}.$$

(2) (10 points) Find a linear symmetric equilibrium.

Solution. A standard argument shows that the optimal bid is $b_i \leq \alpha + \beta$. From part (1), type θ_i's expected payoff with such a bid b_i is

$$\frac{b_i - \alpha}{\beta}(2\theta_i - \alpha) - \frac{1}{2\beta}(b_i - \alpha)^2 + \left(1 - \frac{b_i - \alpha}{\beta}\right) b_i.$$

Taking FOC,

$$\frac{1}{\beta}(2\theta_i - \alpha) - \frac{b_i - \alpha}{\beta} + \left(1 - \frac{b_i - \alpha}{\beta}\right) - \frac{b_i}{\beta} = 0.$$

Arranging,

$$b_i = \frac{\alpha + \beta}{3} + \frac{2}{3}\theta_i.$$

At the same time, we must have $b_i = \alpha + \beta\theta_i$. Hence we have $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$.

(3) (10 points) Suppose now that the winner pays the amount of her winning bid. Find a unique linear symmetric equilibrium.

Solution. Again, the optimal bid must be $b_i \leq \alpha + \beta$. Pick such a b_i. Then bidder i wins if and only if $\theta_j < \frac{b_i - \alpha}{\beta}$. So type θ_i's expected payoff with such a bid b_i is

$$\begin{aligned}
&\int_0^{\frac{b_i-\alpha}{\beta}} (2\theta_i - b_i)d\theta_j + \int_{\frac{b_i-\alpha}{\beta}}^1 (\alpha + \beta\theta_j)d\theta_j \\
&= \frac{b_i - \alpha}{\beta}(2\theta_i - b_i) + \alpha + \frac{\beta}{2} - \frac{b_i - \alpha}{\beta}\alpha - \frac{\beta}{2}\left(\frac{b_i - \alpha}{\beta}\right)^2 \\
&= \frac{b_i - \alpha}{\beta}\left(2\theta_i - \frac{3}{2}b_i - \frac{1}{2}\alpha\right) + \alpha + \frac{\beta}{2}.
\end{aligned}$$

Taking FOC,

$$\frac{1}{\beta}\left(2\theta_i - \frac{3}{2}b_i - \frac{1}{2}\alpha\right) - \frac{3}{2} \cdot \frac{b_i - \alpha}{\beta} = 0.$$

Equivalently,

$$b_i = \frac{1}{3}\alpha + \frac{2}{3}\theta_i.$$

At the same time, we must have $b_i = \alpha + \beta\theta_i$. Hence we have $\alpha = 0$ and $\beta = \frac{2}{3}$.

(4) (10 points) Compare the equilibrium payoffs in parts (2) and (3) for each type θ_i. Discuss.

Solution. In both cases, the equilibrium payoff of type θ_i is $\theta^2 + \frac{1}{3}$. This result follows from a revenue equivalence theorem, which ensures that any symmetric and increasing equilibrium of any mechanism yields the same equilibrium payoff to the bidder, up to a constant. In this setup, the equilibrium payoff of type $\theta_i = 0$ is $\frac{1}{3}$ in both parts (2) and (3), so the constant term is exactly the same in these two auction formats. Accordingly, they yield the same equilibrium payoff for each type θ_i.

2. (40 points) There are two players, and infinitely many periods $t = 1, 2, \cdots$. Player 1 moves in odd periods, and chooses "Stop" or "Go." Similarly, player 2 moves in even periods, and chooses "Stop" or "Go." Once someone chooses "Stop," the game ends. If the game ends in some odd period t, player 1's payoff is $\delta^{t-1}a$ and player 2's payoff is $\delta^{t-1}b$. If the game ends in some even period t, player 1's payoff is $\delta^{t-1}b$ and player 2's payoff is $\delta^{t-1}a$. If no one chooses "Stop," then the payoff is $(0,0)$. Assume that $\delta \in (0,1)$.

Find all *pure-strategy* subgame-perfect equilibrium for the following regions of the parameter space. To have a full credit, prove that you indeed find *all* equilibria.

(1) (10 points) $a > 0$ and $a > \delta b$.

Solution. Note that for each period t, there is only one information set. Indeed, to reach period t, all the past actions must be "Go." So a strategy profile in the infinite-horizon game is represented by a sequence $(s_1, s_2, \cdots)$, where $s_t \in \{\text{Stop}, \text{Go}\}$ is the action in period t.

When $a > 0$ and $a > \delta b$, "Stop" is a dominant action in each information set. Indeed, given a current period t, choosing "Stop" yields a payoff of a, while choosing "Go" yields at most $\max\{\delta^2 a, \delta b\} < a$ regardless of the opponent's strategy. Accordingly, in this case, "Always stop" is the unique SPE. That is, in the unique SPE, $s_t = \text{Stop}$ for all t.

(2) (10 points) $a < 0$ and $a < \delta b$.

Solution. As in part (1), we can show that "Go" is a dominant action in each information set. Accordingly, "Always go" is the unique SPE.

(3) (10 points) $a > 0$ and $a < \delta b$.

Solution. In this case, there are two SPEs. In one SPE, player 1 always chooses "Go," while player 2 always chooses "Stop." In another SPE, player 1 always chooses "Stop," while player 2 always chooses "Go." It is straightforward to show that these two profiles are indeed SPE.

The following two lemmas imply that there is no other equilibrium.

Lemma 1. *In any subgame-perfect equilibrium, if $s_t = Stop$ for some t, then $s_{t+1} = Go$.*

Proof. Suppose not, so that there is a subgame-perfect equilibrium in which $s_t = s_{t+1} = \text{Stop}$ for some t.

Consider the subgame which begins from period t. Let i denote the player who moves in period t. If player i follows the equilibrium strategy and chooses "Stop," her payoff is a. On the other hand, by deviating to "Go" in this period, she can obtain a payoff of δb, because $s_{t+1} = \text{Stop}$. By the assumption, this payoff is greater than the equilibrium payoff a, which means that player i wants to deviate in period t. This is a contradiction. *Q.E.D.*

Lemma 2. *In any subgame-perfect equilibrium, if $s_t = Go$ for some t, then $s_{t+1} = Stop$.*

Proof. Suppose not, so that there is a subgame-perfect equilibrium in which $s_t = s_{t+1} = \text{Go}$ for some t.

Consider the subgame which begins from period t. We consider the following two cases:

Case 1: $s_\tau = \text{Go}$ for all $\tau \geq t+2$. In this case, the equilibrium payoff in the subgame is zero. But the player who moves in period t can better off by stopping the game in that period, which is a contradiction.

Case 2: $s_\tau = \text{Stop}$ for some $\tau \geq t+2$ Let τ^* denote the smallest $\tau \geq t+2$ such that $s_\tau = \text{Stop}$. Let i denote the player who moves in this period τ^*. Then this player i can better off by stopping the game earlier (either in period t or $t+1$). This is a contradiction. *Q.E.D.*

(4) (10 points) $a < 0$ and $a > \delta b$.

Solution. In this case, there are two SPEs. One SPE is "Always Go." The other SPE is "Always Stop." Again, it is straightforward to show that these two are SPE.

The following lemmas imply that there is no other equilibrium.

Lemma 3. *In any subgame-perfect equilibrium, if $s_t =$ Stop for some t, then $s_{t+1} =$ Stop.*

Proof. Suppose not, so that there is a subgame-perfect equilibrium in which $s_t =$ Stop and $s_{t+1} =$ Go for some t.

Consider the subgame which begins from period t. Let i denote the player who moves in period t. If player i follows the equilibrium strategy and chooses "Stop," her payoff is a. On the other hand, by deviating to "Go" in this period and "Stop" in period $t+2$, she can obtain a payoff of $\delta^2 a$, because $s_{t+1} =$ Go. This improves her payoff, which is a contradiction. *Q.E.D.*

Lemma 4. *In any subgame-perfect equilibrium, if $s_t =$ Go for some t, then $s_{t+1} =$ Go.*

Proof. Suppose not, so that there is a subgame-perfect equilibrium in which $s_t =$ Go and $s_{t+1} =$ Stop for some t.

Consider the subgame which begins from period t. Let i denote the player who moves in period t. If player i follows the equilibrium strategy and chooses "Go," her payoff is δb. On the other hand, by deviating to "Stop" in this period, she can obtain a payoff of a, which is better than δb. This is a contradiction. *Q.E.D.*

3. (40 points) Consider the following signaling game. Player 1 is an entrepreneur, who owns all of the stock in her company. She wants to start a new project, but to do so she must have an investment of $I = 1$ from player 2, a venture capitalist. The only way player 1 can do that is by selling player 2 an equity stake in the company.

The profitability π of the company is either 0 or 1, which is private information for player 1. Player 2 believes that the probability of $\pi = 1$ is $\frac{1}{2}$. Player 1 offers player 2 a stake $s \in [0,1]$ of the company. Player 2 observes s but not π, and then either accepts or rejects the offer. If player 2 rejects, player 1's payoff is π while player 2's payoff is zero. If player 2 accepts, player 1's payoff is $(1-s)(\pi+V)$ while player 2's payoff is $s(\pi+V)-I$, where $V = 2$ is the value of the new project.

(1) (5 points) Show that player 2 accepts any offer $s > \frac{1}{2}$, regardless of her posterior belief.

Solution. If $s > \frac{1}{2}$, then regardless of her belief about π, her payoff by accepting the offer is at least

$$sV - I > 0.$$

This implies the result.

(2) (10 points) Is there any pure-strategy PBE in which both types of player 1 make the same offer $s = 0$ and player 2 rejects?

Solution. Suppose that such an equilibrium exists. Then the equilibrium payoff of type $\pi = 0$ is zero. However, if she deviates by offering $s \in (\frac{1}{2}, 1)$, then the offer will be accepted and she can earn a positive payoff. Hence such an equilibrium does not exist.

(3) (15 points) Is there any pure-strategy PBE in which both types of player 1 offer the same share $s > 0$ and player 2 accepts?

Solution. Yes. One of such an equilibrium is as follows: Both types of player 1 offer $s = \frac{2}{5}$. Player 2 puts equal probability on $\pi = 0, 1$ if $s = \frac{2}{5}$, but believes that $\pi = 0$ for sure in other cases. Player 2 accepts the offer if and only if $s = \frac{2}{5}$ or $s > \frac{1}{2}$. It is easy to check that this equilibrium indeed satisfies sequential rationality and Bayes' rule.

(4) (10 points) Is there a pure-strategy PBE in which two types of player 1 make different offers and at least one of them is accepted?

Solution. Such an equilibrium does not exist. This result directly follows from the following lemmas:

Lemma 5. *There is no equilibrium in which player* 2 *rejects some offer on the equilibrium path.*

Proof. Suppose that such an equilibrium exists. Let π^* denote the type whose offer is rejected in equilibrium. Then this type π^* has an incentive to deviate to $s = \frac{3}{5}$; this offer will be accepted and improves her payoff. *Q.E.D.*

Lemma 6. *There is no equilibrium in which the two types make different offers and both of them are accepted.*

Proof. The type who chooses a higher s in equilibrium has an incentive to mimic the other type; the offer is still accepted and it improves her payoff. *Q.E.D.*