ECON 897 - Waiver Exam
August 19, 2019

Important: This is a closed-book test. No books or lecture notes are permitted. You have **180** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Please write parts I, II, and III in separate blue books. Write legibly. Good luck!

Part I: Real Analysis

1. (6 points) Let (a_n) be a sequence of real numbers. Define

$$b_n = \sum_{j=1}^{n} a_j, \qquad c_n = \sum_{j=1}^{n} |a_j|.$$

Show that, if (c_n) converges, then so does b_n.

2. (14 points) Let $C([0,1])$ denote the space of all continuous real-valued functions on $[0,1]$.

   (a) (7 points) Prove that $C([0,1])$ is complete under the uniform norm $||f||_u : \sup_{x\in[0,1]} |f(x)|$.

   (b) (7 points) Prove that $C([0,1])$ is not complete under the L^1-norm: $||f||_1 = \int_0^1 |f(x)|dx$.

3. (20 points) Let $\mathbb{R}^\infty$ be the space of sequences (or the space of infinite vectors). For instance, one element $x \in \mathbb{R}^\infty$ can be denoted as $x = (x_1, x_2, x_3, ...)$. Consider the function $||\cdot|| : \mathbb{R}^\infty \to \mathbb{R}$:

$$||x|| = \sup_{i\in\mathbb{N}} |x_i|.$$

   (a) (5 points) Show that $||\cdot||$ is a norm of $\mathbb{R}^\infty$.

   (b) (5 points) Show that the function $d : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}$ such that $d(x,y) = ||x - y||$ is a distance in $\mathbb{R}^\infty$.

   (c) (5 points) Define the unit closed ball in the sequence space $B^\infty = \{x \in \mathbb{R}^\infty : ||x|| \leq 1\}$. Is B^∞ closed? Is it bounded?

   (d) (5 points) Is B^∞ compact?

Part II: Linear Algebra and Differentiation

1. (20 points) Let $U : \mathbb{R} \to \mathbb{R}$ be a third-order differentiable function such that $U'(x) > 0$, $U''(x) \leq 0$ for all $x \in \mathbb{R}$. Suppose that there are S states with probabilities $\pi_s \in [0, 1]$, $\sum_{s=1}^{S} \pi_s = 1$. Each state has a constant payoff $a \in \mathbb{R}$ and an additional state-specific payoff $\epsilon_s \in \mathbb{R}$, with the property that $\sum_{s=1}^{S} \pi_s \epsilon_s = 0$. Define the certainty equivalent function $c : \mathbb{R} \to \mathbb{R}$, implictly, as:

$$U(c(a)) \equiv \sum_{s=1}^{S} \pi_s U(a + \epsilon_s)$$

   (a) (5 points) Write second-order Taylor expansions of $U(a + \epsilon_s)$ around $a \in \mathbb{R}$ to show that:

$$U(a) \geq \sum_{s=1}^{S} \pi_s \; U(a + \epsilon_s), \quad \forall a \in \mathbb{R}$$

   [Hint: Compute separate taylor-series for each s and then combine them].

   (b) (3 points) Use the result in the previous question to show that $c(a) \leq a$, for all $a \in \mathbb{R}$.

   (c) (3 points) Use the implicit function theorem to show that

$$c'(a) = \frac{\sum_{s=1}^{S} \pi_s U'(a + \epsilon_s)}{U'(c(a))}$$

   For Questions (d) and (e) assume that $A(x) := -\frac{U''(x)}{U'(x)}$ is weakly decreasing.

   (d) (5 points) Define $g(z) := U'(U^{-1}(z))$. Use the chain rule and the inverse function theorem to show that $g'(z) = \frac{U''(U^{-1}(z))}{U'(U^{-1}(z))}$. Furthermore, show that g is convex.
   [Hint: Use the second derivative characterization of convexity. To simplify the derivation you can assume that $g''(z)$ exists.]

   (e) (4 points) Show that $c'(a) = \frac{\sum_{s=1}^{S} \pi_s g(U(a+\epsilon_s))}{g(U(c(a)))}$ and use this result to show that $c'(a) \geq 1$.
   [Hint: Use the the definitions of g and $c(a)$]

2. Suppose that $\{K_1, \dots, K_H\}$ is a collection of non-empty convex sets, such that $K_h \subseteq \mathbb{R}^n_+$. Define the set K, as

$$K := \sum_{h=1}^{H} K_h := \Big\{ x \in \mathbb{R}^n : x = \sum_{h=1}^{H} x_h, \quad x_h \in K_h, h \in \{1, \dots H\} \Big\}$$

   (a) (3 points) Show that $K \subseteq \mathbb{R}^n_+$.

   (b) (5 points) Show that K is non-empty and convex.

   (c) (2 points) Suppose that $e \in \mathbb{R}^n_+ \backslash K$. Show that there exists a non-zero vector $p \in \mathbb{R}^n$ such that $p^t x \geq w$ for all $x \in K$, where $w := p^t e$.

   (d) (4 points) Further assume that if $x \in K$ and $x^* \geq x$ (each coordinate weakly larger) then $x^* \in K$ (monotonicity). Show that in question (c), p cannot have strictly negative coordinates, i.e. $p \in \mathbb{R}^n_+$.

3. (5 points) Let X be an $n \times k$ matrix. Show that if X is full rank then $X^t X$ is full rank.

Part III: Optimization and Probability

1. (25 points) Consider functions $f_1, f_2 : \mathbb{R} \times \Theta \to \mathbb{R}$ and correspondence $D_1, D_2 : \Theta \to P(\mathbb{R})$[1] specified as follows:

$$f_1(q;\theta) = p(\overline{q} + q) - cq, \quad f_2(q;\theta) = p(\overline{q} + \frac{1}{2}q) - \frac{1}{2}cq$$

$$D_1(\theta) = \{q \in \mathbb{R} | q \leq 2\overline{q}, q \geq 0\}, \quad D_2(\theta) = \{q \in \mathbb{R} | q \leq 4\overline{q}, q \geq 0\}$$

The optimization problem (P_θ) is as follows:

$$(P_\theta) \quad \min\{\max_{q \in D_1(\theta)} f_1(q;\theta), \max_{q \in D_2(\theta)} f_2(q;\theta)\}$$

For $D_i^*(\theta) = \arg\max_{q \in D_i(\theta)} f_i(q;\theta)$, $i \in \{1, 2\}$, the solution set $D^*(\theta)$ is defined as

$$D^*(\theta) := \begin{cases} D_1^*(\theta) & \text{if } \max_{q \in D_1(\theta)} f_1(q;\theta) < \max_{q \in D_2(\theta)} f_2(q;\theta) \\ D_2^*(\theta) & \text{if } \max_{q \in D_1(\theta)} f_1(q;\theta) > \max_{q \in D_2(\theta)} f_2(q;\theta) \\ D_1^*(\theta) \cup D_2^*(\theta) & \text{if } \max_{q \in D_1(\theta)} f_1(q;\theta) = \max_{q \in D_2(\theta)} f_2(q;\theta) \end{cases}$$

Suppose all the given parameters are strictly positive.

(a) (2 points) How many parameters are used in (P_θ)?

(b) (8 points) Solve (P_θ), for $\forall \theta \in \Theta$ (Characterize $D^*(\theta)$)

Consider a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(x) = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

And consider another optimization problem

$$(P'_\theta) \quad \min\{\max_{q \in D_1(\theta)} \varphi(f_1(q;\theta)), \max_{q \in D_2(\theta)} f_2(q;\theta)\}$$

Denote the solution set for (P'_θ) by $D^{*'}(\theta)$.

(c) (5 points) Prove or disprove by counterexample the following statement:

$$D^*(\theta) \neq D^{*'}(\theta) \text{ for } \forall \theta \in \Theta$$

Now, fix $p = 2$, and $\overline{q} = 1$. And consider a solution set $D^*(c)$ that changes over c.

(d) (5 points) Is $D^*(c)$ upper hemicontinuous correspondence? If yes, prove it. If not, find all $c \in \mathbb{R}_{++}$ such that $D^*(c)$ is not upper hemicontinuous.

(e) (5 points) Is $D^*(c)$ lower hemicontinuous correspondence? If yes, prove it. If not, find all $c \in \mathbb{R}_{++}$ such that $D^*(c)$ is not lower hemicontinuous.

[1] P(S) is a power set of a set S.

2. (15 points) Let X and Y be random variables with joint probability density function f_{XY} as follows:

$$f_{XY}(x,y) = \begin{cases} ce^{-(x+y)} & \text{if } 0 \le x < \infty,\ 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

for some $c \in \mathbb{R}$

(a) (3 points) Calculate c that validates f_{XY} as a joint probability density function.

(b) (3 points) Are X and Y independent? If so, prove it. If not, explain why independence does not hold.

(c) (4 points) Consider a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ F_Y(y) \end{pmatrix}$$

where F_Y is the marginal cumulative distribution function of the random variable Y. Define $\begin{pmatrix} U \\ V \end{pmatrix} := g\begin{pmatrix} X \\ Y \end{pmatrix}$. Are U, V random variables? If so, 1) explain the reason, and 2) calculate the joint cumulative distribution $F_{U,V}$ of U and V. If not, please explain the reason.

(d) (2 points) Consider a function $Z : (\mathbb{R}, \mathcal{B}_0, \mathbb{P}) \to (\mathbb{R}_+, \mathcal{B}_1)$ such that

$$Z(\omega) = \begin{cases} |\omega| & \text{if } \omega > 0 \\ 0 & \text{if } \omega = 0 \\ 1 & \text{otherwise} \end{cases}$$

where $\mathbb{R}_+ = \{a \in \mathbb{R} | a \ge 0\}$ and $\mathcal{B}_i,\ i \in \{0,1\}$ is a Borel σ-algebra associated with the corresponding set. Is Z a random variable? Please explain the reason.

(e) (3 points) Consider a random variable $Q : (\mathbb{R}, \mathcal{B}_0, \mathbb{P}) \to (\mathbb{R}_+, \mathcal{B}_1)$ such that

$$Q(\omega) = |\omega|$$

and the induced probability $\mathbb{P}_{\mathbb{Q}}$'s cumulative distribution function satisfies $F_Q(\omega) = F_X(\omega)$ for $\forall \omega \in \mathbb{R}_+$, where F_X is the marginal cumulative distribution function of the random variable X given in the problem. Using $\mathbb{P}_{\mathbb{Q}}$ (induced probability function), can we fully characterize $\mathbb{P}$ (original probability function)? If yes, please characterize $\mathbb{P}$. If not, please specify the cases where the probability $\mathbb{P}$ is not determined.