

# SUPPLY NETWORK FORMATION AND FRAGILITY

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ABSTRACT. We model the production of complex goods in a large supply network. Firms source several essential inputs through relationships with other firms. Relationships may fail, and given this idiosyncratic risk, firms multisource inputs and invest to make relationships with suppliers stronger (less likely to fail). We find that aggregate production is discontinuous in the strength of these relationships. This has stark implications for equilibrium outcomes. We give conditions under which the supply network is endogenously fragile, so that arbitrarily small negative shocks to relationship strength lead to a large, discontinuous drop in aggregate output.

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## 1. INTRODUCTION

Modern production is complex. Consider a product such as an airliner. It consists of multiple parts, each of which is essential for its production, and many of which are sourced from suppliers. The parts themselves are produced using multiple inputs, and so on.<sup>1</sup> The induced complementarities can severely amplify disruptions in supply. The failure of one firm to obtain an input crucial for producing one subpart (e.g., a circuit needed for an electronic control in the engine of an airliner) can lead to the idleness of a great many other productive resources.

The situation is exacerbated by the presence of *specific* sourcing relationships, which have come to play a very important role, particularly in the production of complex goods. For instance, there is not an off-the-shelf engine for the Airbus A380. Instead, the part is customized to meet particular specifications and there are only a small number of potential suppliers. The same is true at the subpart level, and so on. Thus, production of the A380 is not exposed just to market level shocks in the availability of each needed part, but rather to idiosyncratic shocks in the operation of the few particular supply relationships that can deliver the part.

We give a few examples. A fire at a Philips Semiconductor plant in March 2000 halted production, preventing Ericsson from sourcing critical inputs, causing its production to also stop (*The Economist*, 2006). Ericsson is estimated to have lost hundreds of millions of dollars in sales as a result, and it subsequently exited the mobile phone business (Norrman and Jansson, 2004). In another example, two strikes at General Motors parts plants in 1998 led 100 other parts plants, and then 26 assembly plants, to shut down, reducing GM's earnings by \$2.83 billion (Snyder et al., 2016). Carvalho et al. (2016) quantify the cascading disruptions in supply chains following the 2011 tsunami in Japan, and the large associated costs. Though these cases are particularly dramatic and well-documented, disruptions to supply networks occur frequently. In a survey of studies on this subject in operations and management, Snyder et al. (2016) write, "It is tempting to think of supply chain disruptions as rare events. However, although a given type of disruption (earthquake, fire, strike) may occur very infrequently, the large number of possible disruption causes, coupled with the vast scale of modern supply chains, makes the likelihood that some disruption will strike a given supply chain in a given year quite high." An industry study found 1,069 supply chain disruption events globally during a six-month period in 2018 (Supply Chain Quarterly, 2018). Moreover, typical disruptions seem to cause lasting damage. Hendricks and Singhal (2003, 2005a,b) examine hundreds of supply chain problems reported in the business press. Even minor disruptions are associated with significant and long-lasting declines in sales growth and stock returns.<sup>2</sup>

To summarize, relationship-specific sourcing is practically important and exposes producers of complex goods to considerable risk. In view of this, firms make costly efforts to increase the probability of sourcing their inputs successfully. For example, they insure themselves by multisourcing key inputs—maintaining several relationships that can substitute for one another. They also invest in relationships to make it likelier that a given one operates as intended.<sup>3</sup>

<sup>1</sup>An Airbus A380 has millions of parts produced by more than a thousand companies (Slutsken, 2018).

<sup>2</sup>Craighead et al. (2007) find a positive association between supply chain complexity and disruptions.

<sup>3</sup>A recent survey found that more than 80% of companies are concerned about the resilience of their supply chains (Bhatia et al., 2013). On the importance of relational contracts as a way to manage non-contractable shocks in supply chains, and the associated frictions, see Antràs (2005).

Our model is designed to examine these phenomena and their aggregate implications. There is a large supply network consisting of many producers. Each one sources multiple essential inputs through specific relationships with potential suppliers. Any given relationship may fail, which prevents its use (at least at a given time) for sourcing a needed input. In view of this, firms may maintain multiple sourcing relationships and invest to make them *stronger*—likelier to operate. In other words, firms invest to reduce the probability of idiosyncratic supply disruptions. We study the endogenous determination of the aggregate robustness of the supply network, and the implications for the stability of the economy that it is a part of.

We show that this environment features *precipices*: aggregate production is discontinuous in relationship strength. We first explain how the technological structure of complex supply networks gives rise to precipices even with exogenous relationship strength. We then examine the endogenous determination of relationship strength as a strategic network formation problem, where firms invest in their specific supply relationships. Our full model has both this endogenous investment, and endogenous entry of firms into the network formation process. Indeed, a tractable model with these features is our main methodological contribution.

The modeling framework allows us to derive several qualitative insights about fragility. Though the possibility of endogenous investment in relationships allows each firm to insure against idiosyncratic disruptions, it can lead supply networks to configure themselves into an equilibrium that is extremely fragile at the aggregate level. Small shocks to institutional quality that slightly reduce the strength of relationships (e.g., a reduction in the quality of courts handling commercial disputes) can undergo extreme amplification and cause a supply network to fail. We describe the parameters that put a supply network at risk, and show that they are not knife-edge cases. Moreover, once a fragile supply network is damaged by a shock, linkages between different supply networks can propagate the damage and lead to widespread distress.

The novelty of the framework comes from the combination of two features often studied separately: (i) complexity, specifically that each product requires multiple other products as inputs; and (ii) firms' exposure to failures in specific supply relationships. Jointly, these phenomena create the potential for very stark amplification effects, whose equilibrium implications we explore.

We then examine the welfare properties and regulation of complex supply networks. Markets in which too many firms can profitably enter are more vulnerable to the extreme externalities created by the fragility we have analyzed. Indeed, the problem of endogenous underinvestment in the reliability of sourcing is exacerbated in congested markets. We also discuss the consequences of the results for economies consisting of multiple complex supply networks, and describe domino effects that follow from the basic fragility phenomenon.

We now describe the model and results in a bit more detail. There are many products and many firms specialized in each product; each firm is small relative to the overall supply network. These firms form supply relationships with suppliers to secure the inputs they require. There may be multiple relationships with different producers of a given input. Each of these relationships may fail, independently, with some probability. Relationships are said to be *stronger* if they are more likely to operate successfully. For a firm to source an input from a given supplier, it must be that (i) the relationship with that supplier operates successfully and (ii) the supplier is able to produce the desired input. Key parameters of the supply network at this stage are: the number of distinct

inputs required in each production process (a measure of complexity); the number of potential suppliers of each input (a measure of the availability of multisourcing); and the strength of each relationship. Our first set of results examines the mechanics of such a system when the strength of the relationships is exogenous. It gives conditions under which there is a discontinuity in aggregate productivity: when relationship strength falls below a certain threshold, which we call a *precipice*, production drops discontinuously to zero. This raises the prospect of fragility: a small negative shock to relationship strengths (e.g., a public event that makes contracts less likely to be enforced) can lead to considerable economic damage.

We next show that a social planner will always choose costly relationship strengths so that the supply network is away from a precipice. A natural question is then whether a supply network will be near a precipice when relationship strengths are determined by equilibrium choices rather than by a planner. To analyze this, we model the incentives of firms that are attempting to produce. Firms that succeed in producing sell their (differentiated) output to consumers to make a gross profit.<sup>4</sup> They choose a level of costly investment toward making relationships stronger—i.e., likelier to operate.<sup>5</sup> These investment decisions, which depend on the gross profit that firms expect to make, determine the strength of relationships in equilibrium. The gross profit is itself endogenous: it is determined by the number of firms competing to sell a given product—i.e. on the crowdedness of the market. When more firms enter, markups are lower and each earns a smaller gross profit.

The basic force that can push the supply network toward a precipice is as follows. When the supply network is reliable and gross profits are high, firms want to enter. Competition drives down gross profits and makes it less appealing to pay costs to make relationships strong. So relationships get weaker. (Indeed, because firms do not internalize the full benefit of their reliability for other firms, there is inefficient underinvestment.) The question is where this dynamic stops. The precipice can be a natural stopping point, because near it reliability suffers so much that entry is no longer appealing. An interesting effect of the precipice in equilibrium outcomes is that it can ration entry while those entering obtain positive profits. This causes economies to get stuck at the precipice for a range of parameters.

We study how the outcome depends on an aggregate parameter of the supply network, which can be either a productivity shifter (which multiplies the value of production) or a cost shifter (a parameter that facilitates relationship quality, such as institutional quality). The supply network in equilibrium can end up in one of three configurations: (i) a *noncritical equilibrium* where the equilibrium investment by entering firms is enough to keep relationship strength away from the precipice; (ii) a *critical equilibrium* where the number of firms is just on the edge of overcrowding and the equilibrium relationship strength is just on the precipice; and (iii) an unproductive equilibrium where positive investment cannot be sustained. These regimes are ordered. As the productivity of the supply network (or, equivalently, the quality of institutions) decreases from a high to a low level, the regimes occur in the order just given. Each regime occurs for a positive interval of values of the parameter.

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<sup>4</sup>Gross of fixed entry costs, that is.

<sup>5</sup>This can be interpreted in two ways: (1) investment on the intensive margin, e.g. to anticipate and counteract exogenous risks or improve contracts; (2) on the extensive margin, to find more partners out of a set of potential ones.

The critical equilibrium is fragile in the following sense. Suppose a small shock occurs that affects the strengths of all relationships. Then output can fall drastically: indeed, production falls off the precipice. We discuss some implications concerning welfare, and the distinctive considerations that the precipice raises. A policy implication of our results is that rationing entry could be justified by macroprudential considerations.

**Related work.** A large literature has examined the economic implications of complementarities in production. Jovanovic (1987) shows how strategic interdependencies or complementarities can produce aggregate volatility in endogenous variables despite no such volatility in exogenous variables. Kremer (1993) and subsequent work have argued that complementarities can help provide a unified account of many economic phenomena. These include very large cross-country differences in production technology and aggregate productivity; rapid output increases during periods of industrialization; the macroeconomic propagation of idiosyncratic shocks; and the structure of production networks and international trade flows; see, among many others, Ciccone (2002), Acemoglu et al. (2007), Levchenko (2007), Jones (2011) and Levine (2012).

There is also a vibrant literature in macroeconomics on production networks. Carvalho (2014) provides a comprehensive survey covering much of this literature. Two recent developments in the literature are particularly relevant to our work: (i) the modeling of the endogenous determination of the input-output structure; and (ii) a firm-level approach as opposed considering inter-industry linkages at a more aggregated level. Some of the most relevant work on these issues includes Atalay et al. (2011), Oberfield (2012), Carvalho and Voigtländer (2014), Acemoglu and Azar (2019), Taschereau-Dumouchel (2017), Boehm and Oberfield (2018), Tintelnot et al. (2018), Baqaee and Farhi (2019), Baqaee and Farhi (2017) and König et al. (2019).

There has been much recent interest in markets with non-anonymous trade mediated through relationships.<sup>6</sup> The work most closely related to ours in this area also studies network formation in the presence of shocks. This includes work in the context of production (e.g., Levine (2012), Brummitt et al. (2017), Bimpikis et al. (2019), Yang et al. (2019), Amelkin and Vohra (2019)), work on financial networks (e.g., Cabrales et al. (2017), Elliott et al. (2018), Erol (2018), Erol and Vohra (2018), Jackson and Pernoud (2019)), and work in varied other contexts (e.g., Blume et al. (2011), Jackson et al. (2012), Talamàs and Vohra (2018)). More broadly, the aggregate implications of non-anonymous trade have been studied across a variety of settings. For work on thin financial markets see, for example, Rostek and Weretka (2015), for buyer-seller networks see, e.g., Kranton and Minehart (2001), and for intermediation see, e.g., Gale and Kariv (2009).

Our model combines complementarities in production and strategic network formation choices. Agents make a continuous choice that determines the probability of their supply relationships operating successfully. The links that form may, however, fail in a “discrete” (i.e., non-marginal) way. The first feature makes the model tractable, while the second one yields discontinuities in the aggregate production function and distinguishes the predictions from models where the aggregate production function is differentiable. It might be thought that aggregating over many supply chains, these discontinuities would be smoothed out at the level of the macroeconomy; we show they are not.

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<sup>6</sup>A literature in sociology emphasizes the importance of business relationships, see for example Granovetter (1973) and Granovetter (1985). For a survey of related work in economics see Goyal (2017).

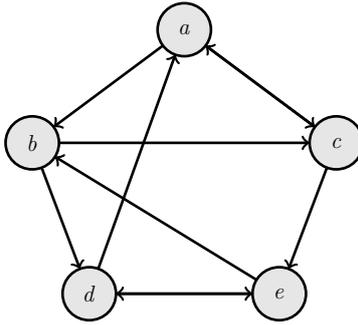


FIGURE 1. An example of a production network, describing the interdependencies between products. An arrow from node  $i$  to  $j$  represents that to produce  $i$ , it is necessary to procure product  $j$ .

Mathematically, our work is related to a recent applied mathematics literature on so-called *multilayer networks* and their phase transitions (Buldyrev et al., 2010). Discontinuities such as the one we study are termed first-order phase transitions in this literature.<sup>7</sup> Buldyrev et al. (2010), and subsequent papers in this area such as Tang et al. (2016) and Yang et al. (2019), study quite different network processes, typically with exogenous networks. We point out that discontinuities of this kind arise even in the most standard models of networked production (e.g., that of Acemoglu et al. (2012)), once specific sourcing relationships are taken into account, and, more importantly, that endogenous forces can put equilibria on a precipice. Predating the recent literature on multilayer networks, Scheinkman and Woodford (1994) used insights from physics models on self-organized criticality to provide a “sandpile” model of the macroeconomy in which idiosyncratic shocks have large aggregate effects.<sup>8</sup> The setup and behavior of the model are rather different from ours: the main point of commonality is in the concern with endogenous fragility. In our work, the supply network is robust to idiosyncratic shocks but very sensitive to arbitrarily small aggregate shocks to relationship strength.

## 2. COMPLEX SUPPLY NETWORKS AND DISCONTINUITIES IN RELIABILITY

In this section, we describe the structure of the supply network. We examine how the aggregate production function depends on a parameter measuring the strength of supply relationships. This gives the first and simplest manifestation of precipices. We describe other features of the economy needed to fill out the economic model only at a high level for now. Readers preferring a more complete presentation of the entire economic model can skip directly to Section 3.

**2.1. The supply network.** We first define a *production network* that describes technological relationships in the economy. Its nodes are a finite set  $\mathcal{I}$  of *products*. Product  $i$ 's production function requires as inputs a set of other products, which we denote by  $\mathcal{I}_i \subseteq \mathcal{I}$ . If a firm producing product  $i$  procures none of a required input product, it is unable to produce any output. For a concrete production function with these properties, take any CES production function with an elasticity of substitution equal to or greater than 1.

<sup>7</sup>These can be contrasted with second-order phase transitions such as the emergence of a giant component in a communication network, which have been more familiar in economics—see Jackson (2010).

<sup>8</sup>Endogenously, inventories reach a state analogous to a sandpile with a critical slope, where any additional shock (grain dropped on the sandpile) has a positive probability of leading to an avalanche.

*A regular-network assumption.* For simplicity, here and in much of our exposition, we take the production network to be *regular*: the number of different inputs to produce a given product is the same number  $m$  across products (i.e.,  $|\mathcal{I}_i| = m$  for each  $i$ ). We call  $m$  the *complexity* of the network. The advantage of this assumption is that it allows us to study the mechanics and equilibrium outcomes of the system using intuitive one-dimensional fixed-point equations amenable to exact analysis. The substantive conclusions continue to hold in a much richer model, where different industries require different numbers of inputs, have different (and endogenous) levels of investment in robustness, etc. These extensions are discussed in Section 6.

Figure 1 provides a very simple illustrative example of a regular production network involving five products. Product  $a$  requires products  $b$  and  $c$  as inputs; product  $b$  requires products  $d$  and  $c$  as inputs; and so on.

The *supply network* describes relationships among firms rather than products. For each product  $i$ , there is a continuum of firms  $\mathcal{F}_i$ , with the same production function. We denote a typical firm  $if$ , where  $i$  is the product and  $f \in \mathbb{R}$  is the label of the firm. A firm making product  $\mathcal{I}_i$  requires the products in  $\mathcal{I}_i$  as inputs. Thus a firm  $if$  has  $n$  *potential supply relationships* for each product in  $\mathcal{I}_i$ . Only some of these relationships will turn out to be operational. (The firm multisources for its inputs precisely because link operation is uncertain, as we will see.) The firms that  $if$  has supply relationships with for input  $j \in \mathcal{I}$ , are drawn independently, according to an atomless distribution, from the continuum of firms  $\mathcal{F}_j$  who produce the required product.

Figure 2 depicts a small subpart of the supply network—a firm, its suppliers, and their suppliers. Consider a firm  $a1$ . According to the production network, making product  $a$  requires two input products— $b$  and  $c$ . The firm sources these via supply relationships with particular firms specialized in these products. For each of  $b$  and  $c$ , the firm has potential supply relationships with two producers of each input it requires. The suppliers of our focal firm,  $a1$  are in a position similar to that of  $a1$  itself.<sup>9</sup> These supply relationships are directed edges, from the firm that is doing the procuring or ordering to its supplier. Thus, orders flow upstream, while products flow downstream.

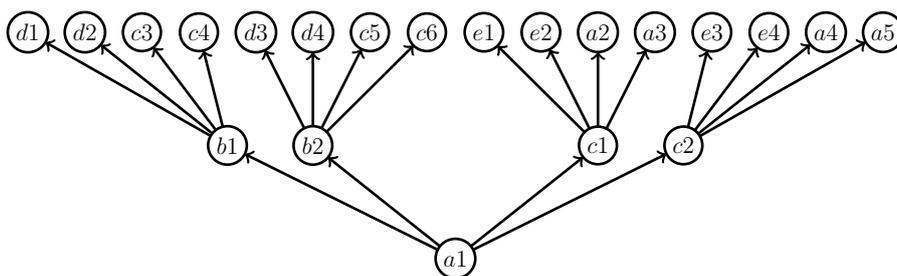


FIGURE 2. This figure illustrates the first three layers of the supply network for producer  $a1$ , corresponding to the production network shown in Figure 1. Firms higher up are upstream of  $a1$ , and a directed edge from one firm to another reflects that the first may try to source from the second.

**2.2. Uncertainty in sourcing and firm functionality.** We now introduce the uncertainty of sourcing inputs, coming from the stochastic failure of some supply relationships. Each of a firm’s supply relationships may be *operational* or not—a binary random

<sup>9</sup>Because there is a continuum of firms and only finitely many (randomly sampled) firms appear upstream of any given firm, the probability of any firm appearing multiple times upstream of a given firm is 0.

outcome. There is a parameter  $x$ , called *relationship strength* (for now exogenous and homogeneous across the supply network) which is the probability that any relationship is operational. All these realizations are independent.<sup>10</sup> We can think of such a realization as describing whether the “pipe” meant to ship inputs from the particular (upstream) supplier to its (downstream) customer is working. The actual supply network is then obtained by keeping each potential supply link independently with probability  $x$ . We illustrate this in Figure 3.

For a firm to be *functional*, it must be able to source *each* input. This means that, for each input, at least one of the suppliers of that input must itself be functional. (Conditional on at least one supplier being able to supply, it can fulfill the needed quantity, so the possibility of failure is the only reason to multisource.) Thus, functionality is interdependent—a point we will return to soon.

Continuing with our running example, take the firm  $a1$  for illustration. For this firm to be functional, it must be that for any input product (say,  $b$ ), there is at least one of  $a$ ’s potential suppliers ( $b1$  or  $b2$ ) of that product such that (i) the link that  $a1$  has to source from this supplier is operational (the pipe works) and (ii) this supplier is functional itself (able to source its required inputs and produce, so there is something to ship down the pipe). Whether (ii) holds depends on other functionality realizations throughout the supply network.

**2.3. Which firms can function?** We are interested in the physical production possibilities frontier. Which firms *can* produce, given the supply link realizations, and what is the size of this set?

For a given realization of which supply links are operational we need to determine which firms are functional. As the functionality of firms is interdependent, it can be consistent for different sets of firms to be functional. We focus throughout the paper on the maximal set of firms that can be consistently functional.

A simple algorithm determines the maximal functional set for a given realization of which supply links are operational. Consider all firms, i.e., the union of all the  $\mathcal{F}_i$ . (1) Initialize the set of (putatively) functional firms,  $\widehat{\mathcal{F}}(0)$ , to be all firms. (2) At any stage of the algorithm  $s = 1, 2, \dots$  let  $\widehat{\mathcal{F}}(s)$  be the set of firms in  $\widehat{\mathcal{F}}(s - 1)$  that, for each input, have an operational link for that input to some firm in  $\widehat{\mathcal{F}}(s - 1)$ .

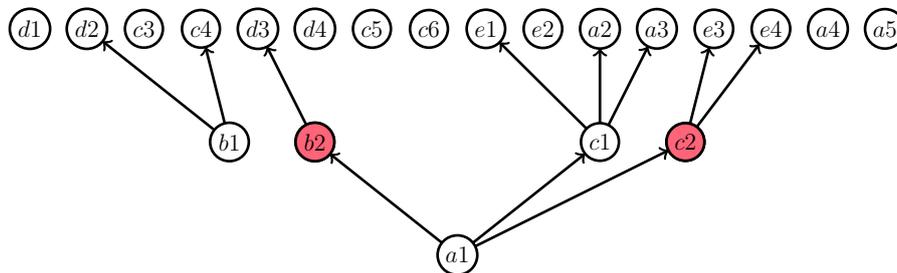
Intuitively, this algorithm begins by optimistically assuming that all firms are functional, and removing firms only when they have no operational links supplying some needed input.

This procedure gives a decreasing sequence of sets. Define  $\overline{\mathcal{F}}$  to be the intersection (i.e., limit) of these sets. By a standard application of the Tarski fixed-point theorem, this turns out to be the maximal set of firms such that it is consistent for all firms in the set to produce.<sup>11</sup> The algorithm is simply a convenient and intuitive procedure for finding

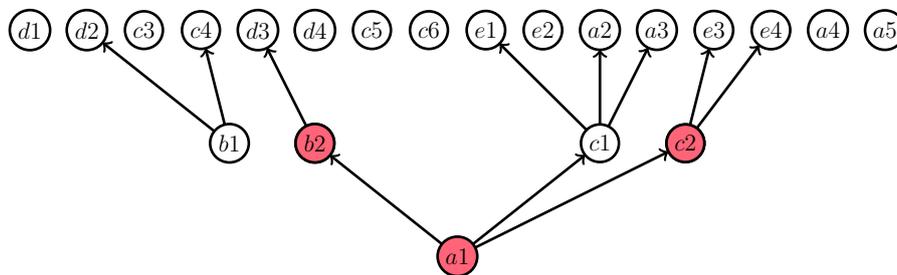
<sup>10</sup>For a formal construction of the potential supply network and the functionality realizations of the links, see Appendix A.

<sup>11</sup>Formally, given a realization of which supply links are operational, it is the maximal set  $\widehat{\mathcal{F}}$  of firms such that: for every firm  $if$  in  $\widehat{\mathcal{F}}$ , and each input  $j$  necessary to produce product  $i$ , the firm has at least one operational supply relationship to a supplier of product  $j$ , which is also in  $\widehat{\mathcal{F}}$ . In Appendix B we discuss more formally the production that takes place conditional on the set of functional firms.

this set. This determines the physical production possibilities of the supply network.<sup>12</sup> For an illustration of this process, see Figure 3.



(A) Stage 1: firms  $b2$  and  $c2$  are removed from  $\widehat{\mathcal{F}}(0)$ . They can't produce because they can't get all the essential inputs they need. This gives  $\widehat{\mathcal{F}}(1)$ .



(B) Stage 2: now firm  $a1$  is removed from  $\widehat{\mathcal{F}}(1)$ . As  $b2 \notin \widehat{\mathcal{F}}(1)$ ,  $a1$  is unable to source a  $b$ -product from a functional supplier and so cannot produce. This gives  $\widehat{\mathcal{F}}(2)$ .

FIGURE 3. An illustration of the algorithm for determining the maximal set of functional firms. At stage  $s$ , only those firms with at least one operational supply relationship to a firm in the set of still-functional firms ( $\widehat{\mathcal{F}}(s-1)$ ) for each input required remain functional. Firms that are not functional are shaded red. The algorithm terminates after stage 2. The set of functional firms is  $\widehat{\mathcal{F}}(2)$ .

2.3.1. *The reliability function.* Next we are interested in the mass of firms in  $\overline{\mathcal{F}}$  producing any given product; by symmetry it does not depend on the product. We denote this quantity by  $\rho(x)$  and call it the *reliability* of the supply network. This is the mass of firms that are able to produce, or equivalently the probability that a randomly selected firm can produce. The reliability depends on relationship strength  $x$ , which is the probability of any given supply link being operational.

Recall that each firm requires  $m$  different inputs and has  $n$  potential suppliers of each of these inputs. In Figure 4 we plot reliability as a function of the probability  $x$  of each relationship being operational; we use the  $m = n = 2$  case here, as in our illustrations above. The key fact about this plot is that reliability is *discontinuous* in relationship strength  $x$ , jumping at a value that we will call  $x_{\text{crit}}$ . The probability of successful production is 0 when  $x < x_{\text{crit}}$ , but then increases discontinuously to more than 70% at this threshold.

<sup>12</sup>It is worth noting that the maximal fixed point is *the* fixed point selected in the limit of finite versions of our model (see Appendix G).

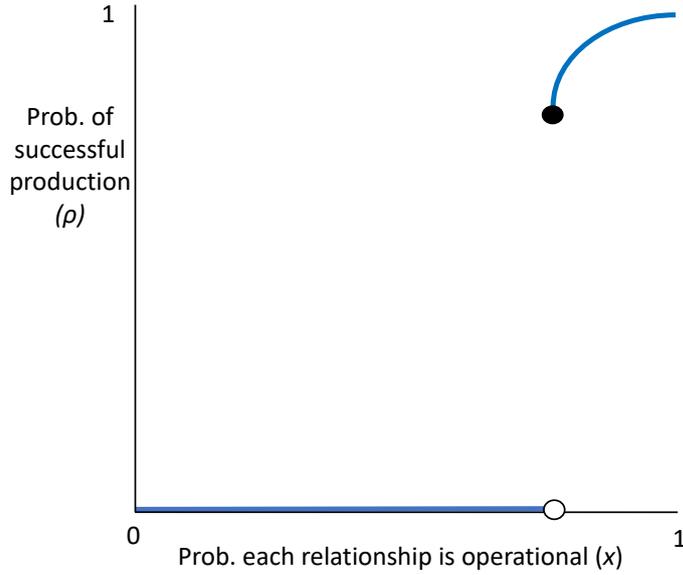


FIGURE 4. The reliability of a supply network,  $\rho(x)$ , which is the probability that a random firm can successfully produce, plotted as the probability any given relationship is operational,  $x$ , varies.

This property of  $\rho(x)$  is general—for any number  $m \geq 2$  of required inputs and any potential multisourcing level  $n \geq 1$ ,  $\rho(x)$  has a unique point of discontinuity at  $x_{\text{crit}}$ , and for all lower levels of  $x$ , production always fails ( $\rho(x) = 0$ ). This is summarized in the following result:

**Proposition 1.** Let the complexity of production  $m$  be at least 2 and the number  $n$  of potential suppliers for each input be at least 1. The measure of the set of functional firms  $\overline{\mathcal{F}}$ , denoted by  $\rho(x)$ , is a nondecreasing function with the following properties.

- (1) There are numbers  $x_{\text{crit}}, r_{\text{crit}} > 0$  such that  $\rho$  has a discontinuity at a critical level of relationship strength  $x_{\text{crit}}$ , where it jumps from 0 to  $r_{\text{crit}}$  and is strictly increasing after that.
- (2) If  $n = 1$ , we have that  $x_{\text{crit}} = 1$ ; otherwise  $x_{\text{crit}} < 1$ .
- (3) If  $x_{\text{crit}} < 1$ , then as the relationship strength  $x$  approaches  $x_{\text{crit}}$  from above, the derivative  $\rho'(x)$  tends to  $\infty$ .

The result is proved in Appendix E. We explain the intuition behind the result in the next subsection.

This result already has some stark implications. As relationship strength increases, a threshold is passed at which production becomes possible. Moreover, reliability (the probability of successfully producing the good) jumps abruptly from 0 to a positive number (in the  $n = m = 2$  case, approximately 0.8) as  $x$  improves around the threshold  $x_{\text{crit}}$ . This implies that small improvements in relationship strength  $x$ , for example through the improvement of institutions, can have large payoffs for an economy, and the net marginal returns on investment in  $x$  can change sharply from being negative to being positive and very large.

As supply networks become large and decentralized, one might think that the impact of uncertainty on the probability of successful production would be smoothed somehow by averaging. We find the opposite: in aggregating up the uncertainty through the

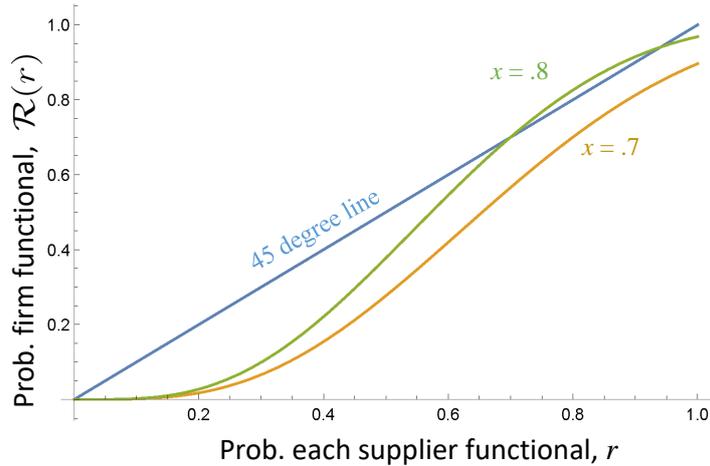


FIGURE 5. The probability,  $\mathcal{R}(r)$ , that a focal firm is functional as a function of  $r$ , the probability that a random supplier is functional. Here we use the parameters  $n = 4$  and  $m = 3$ .

interdependencies of the supply network, we get a very sharp sensitivity of aggregate productivity to relationship strength. This is in contrast to standard production network models (e.g., Baqaee and Farhi (2019)), where the aggregate production function is differentiable at any point. The difference comes from the fact that we model the failure of nodes to produce when they do not receive (enough) inputs.

Another tempting but false conjecture is that the regularity of the network structure, and symmetry more generally, play an important role in generating the discontinuity in the probability of successful production. We discuss in Section 6 our results showing that the discontinuity persists in the presence of asymmetric networks where different products require different numbers of inputs, there are different numbers of suppliers for each input, and relationship strengths vary by input.

2.3.2. *Deriving the reliability function: Analyzing the mass of functional firms.* We now explain the reasoning behind Proposition 1.

Recall the algorithm given earlier for determining the mass of functional firms. Suppose that at some stage of this process, the fraction of firms in  $\widehat{\mathcal{F}}(s-1)$  is  $r$ . Consider an arbitrary firm; let us examine the probability that it is in  $\widehat{\mathcal{F}}(s)$ . We claim that this number is given by

$$\mathcal{R}(r) = (1 - (1 - xr)^n)^m. \quad (1)$$

This equation is straightforward to derive. By definition, being in  $\widehat{\mathcal{F}}(s)$  requires that for each input, our focal firm has an operational link to some firm in  $\widehat{\mathcal{F}}(s)$  producing that input. Consider the first input of our focal firm. For a given one of its suppliers, the probability that supplier is in the set  $\widehat{\mathcal{F}}(s-1)$  is  $r$ , and the probability that the link the supplier is operational is  $x$ . The probability that both events happen is  $xr$ . The probability that this combination of events happens for at least one of the  $n$  potential suppliers of the first input is therefore  $1 - (1 - xr)^n$ . Finally, the probability that for all  $m$  inputs our focal firm has an operational link to some firm in  $\widehat{\mathcal{F}}(s)$  producing that input is  $(1 - (1 - xr)^n)^m$ .

Let  $R(s)$  be the mass of  $\widehat{\mathcal{F}}(s)$  in the market producing any product  $i$ . The above discussion shows that  $R(s) = \mathcal{R}(R(s-1))$ . To initialize this sequence, we observe that

$R(0) = 1$ , since we initialize the process by supposing no firms fail. The sequence  $(R_s)_{s=1}^{\infty}$  converges to the largest fixed point of equation (1), which is, by definition  $\rho(x)$ .

We can interpret the fixed-point condition  $r = \mathcal{R}(r)$  in an intuitive way. When the firms in  $\overline{\mathcal{F}}$  are functioning, we can write

$$r = \left( 1 - \underbrace{(1 - xr)^n}_{\text{probability a given input cannot be acquired}} \right)^m.$$

By the same reasoning we have given above, when all firms in the supply network are functioning with probability  $r$ , the probability of failing to source some given input is  $(1 - xr)^n$ , and therefore the probability of sourcing all inputs successfully—i.e., functioning—is the right-hand side.

This discussion allows us to describe  $x_{\text{crit}}$  in Proposition 1: it is the smallest  $x$  for which the equation has a nonzero solution  $r$ , and  $r_{\text{crit}}$  is that solution.

The last step is to understand why the largest fixed point  $\rho(x)$  jumps to a positive level discontinuously as we vary  $x$ . Consider Figure 5, which shows the shape of  $\mathcal{R}$  as  $x$  is varied. We see that beyond a certain value of  $x$ , this curve has a nontrivial intersection with the 45-degree line, but not before. The essential feature of the curve that leads to this shape is its convexity for low values of  $r$ , which occurs whenever the exponent  $m$  is at least 2. In that case, the curve  $\mathcal{R}$  is initially bounded above by a quadratic function in  $x$ . This means that it cannot have an intersection with the 45-degree line close to zero.<sup>13</sup> Thus, any positive fixed point must emerge discontinuously as  $x$  increases. The fact that  $m \geq 2$ —that is, that the supply network is complex—is crucial, as we will see again in Section 2.4.2.

**2.4. Discussion.** The key features of our model are that (i) each firm’s production relies on multiple non-commodity inputs and (ii) that these are sourced through failure-prone relationships with particular suppliers. In this section we first discuss the motivation and interpretation of these assumptions. We then compare the model to two benchmarks by relaxing each assumption in turn. In the first benchmark, we consider the production of simple goods, where each firm requires only one non-commodity input. In the second benchmark, we consider market-based, as opposed to relationship-based, sourcing of all inputs. Finally, we discuss some first implications of Proposition 1.

**2.4.1. Comments on key concepts and assumptions.** The fact that key inputs are supplied via a limited number of supply relationships is a key assumption in our model. Firms are often constrained in the number of supply relationships they can maintain—for example, by technological compatibility, geography, trust, understanding, etc. Supplier relationships have been found to play important roles in many parts of the economy—for relationship lending between banks and firms see Petersen and Rajan (1994, 1995); for traders in Madagascar see Fafchamps and Minten (1999); for the New York apparel market see Uzzi (1997), for food supply chains see Murdoch et al. (2000), for the diamond industry see Bernstein (1992), for Japanese electronics manufacturers see Nishiguchi (1994), and so on. Indeed, even in fish markets, a setting where we might expect relationships to play a minor role, they seem to be important (Kirman and Vriend, 2000; Graddy, 2006). The parameter  $x$  that we have called relationship strength, which is the probability that a supply relationship is operational, can capture a variety of considerations: uncertainty regarding compatibility, whether delivery can happen on time, possible misunderstanding about the required input, etc. It will depend on the context or environment in which

<sup>13</sup>Since  $\mathcal{R}(r) < Cr^2$  for some  $C$ , and the right-hand side is much smaller than  $r$  for small  $r$ .

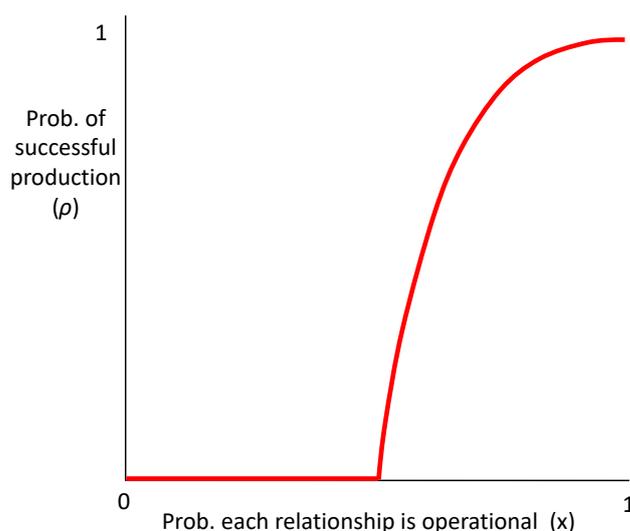


FIGURE 6. The probability of a complete supply tree for a simple product, which facilitates successful production of the final good, as relationship strength varies.

production occurs, and also (as we explicitly model below) on the investments agents make.

We assume that a firm can produce as long as it has an operational link to at least one functioning producer of each input. More realistically, it may be that to produce, a sufficient quantity or quality of each input is needed, and the shock is to whether this quantity can be delivered. The shock need not destroy all the output, but may destroy or reduce the value of the output by some amount (say, a random fraction of the output). We consider the starkest case for simplicity; the key force is robust to these sorts of extensions.

*2.4.2. Contrast with sourcing for simple production.* To contrast with the case of complex products, we consider a supply network where each firm requires only a single relationship-sourced input ( $m = 1, n = 2$ ). We call such production simpler in that each firm requires only one type of risky input relationship to work.<sup>14</sup> We plot how the probability of successful production varies with relationship strength in Figure 6. In comparison to the case of complex production illustrated in Figure 4, there is a stark difference. For values of  $x < 0.5$  the probability of successful production is 0 and for values of  $x > 0.5$  the probability of successful production is strictly positive, but the change about this point is continuous.

The change at  $x = 0.5$  is abrupt with the derivative changing discontinuously. The intuition for this change is one that is familiar from the networks literature and in particular from studies of contagion (see, for example, Elliott et al. (2014) in the context of financial contagion). For a given producer, production will be successful if the supply network doesn't die out after a finite number of steps. This depends on whether the rate at which new branches in the network are created is higher or lower than the rate at which existing branches die out due to failure. It turns out that when  $x > 0.5$ , a supply

<sup>14</sup>As a matter of interpretation, there may be more than one physical input at each stage. The key assumption is that all but one are sourced as commodities rather than through relationships, and so are not subject to disruption.

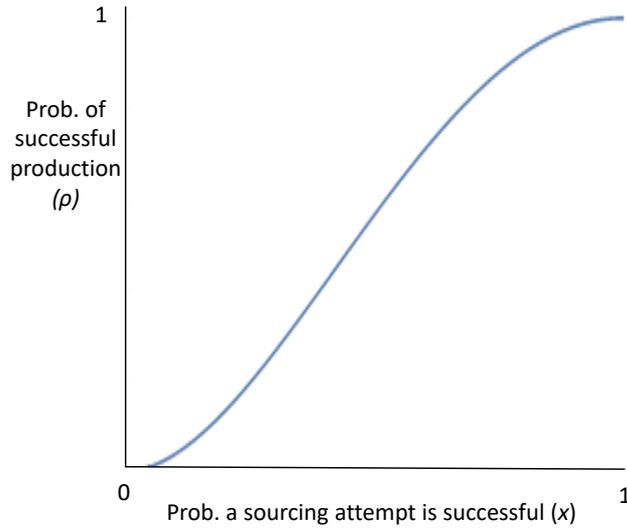


FIGURE 7. The probability of successful production for a firm as market-based sourcing attempts become more likely to succeed.

tree grows without bound in expectation, while when  $x < 0.5$  it dies out.<sup>15</sup> The kink in the probability of successful production around the key threshold of 0.5 is related to the emergence of a giant component in an Erdős–Rényi random graph. That is not the case, as we have seen, for complex production. That is a different sort of phase transition, reliant on the need for multiple inputs at each stage. As we’ll see, this difference has stark economic implications once investments in reliability are endogenous.

2.4.3. *Contrast with market-based sourcing.* We let each firm attempt to source a given input type it requires not through pre-established relationships, but through a market. The market is populated by those potential suppliers that are able to successfully produce the required input. However, upon approaching a supplier there is still a chance that sourcing fails for one reason or another. (A shipment might be lost or defective, or a misunderstanding could lead the wrong part to be supplied.) In the metaphor we introduced earlier, we now assume each firm extends pipes *only* to functional suppliers, but we still keep the randomness in whether the pipes work.<sup>16</sup>

Let the probability a given attempt at sourcing an input succeeds be  $x$ , independently. As before, in view of this idiosyncratic risk, each firm *if* multisources by contracting with two potential suppliers of each input. The probability that both potential suppliers of a given input type fail to provide the required input is  $(1 - x)^2$ , and the probability that at least one succeeds is  $1 - (1 - x)^2$ . As the firm needs access to all its required inputs

<sup>15</sup>Given that each producer has two potential suppliers for the input, and each of these branches is operational with probability  $x$ , the expected number of successful relationships a given firm in this supply network has is  $2x$ . When  $x < 1/2$ , each firm links to on average less than 1 supplier, and so the rate at which branches in the supply tree fail is faster than the rate at which new branches are created. The probability that a path in the supply network reaches beyond a given layer  $l$  then goes to 0 as  $l$  gets large and production fails with probability 1. On the other hand, when  $x > 1/2$ , the average number of children each node has is greater than 1 and so new branches appear in the supply tree at a faster rate than they die out, leading production to be successful with strictly positive probability.

<sup>16</sup>Formally, in the construction of the random graph, we allow each firm’s potential supply links to be formed conditional on the realization of its supplier’s functionality; in particular, these links are extended only to functional suppliers.

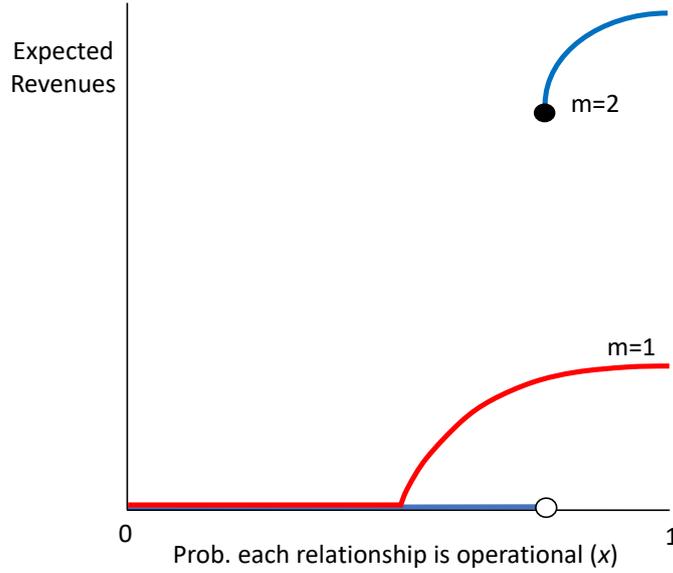


FIGURE 8. A contrast of the  $m = 1$  and  $m = 2$  cases with the degree of multisourcing being held at  $n = 2$ . Expected revenues are on the vertical axis. This is a product of the probability of successful production and price of goods. The case in which the complex good retails for a price of 1 while the simple good retails for a price of  $1/4$  is illustrated.

to be able to produce, and it requires 2 different input types, the probability the firm is able to produce is  $(1 - (1 - x)^2)^2$ . In Figure 7 we plot how the probability that a given firm is able to produce varies with the probability their individual sourcing attempts are successful. This probability increases smoothly as  $x$  increases.

2.4.4. *An implication for industrial development.* We can use the comparison between the production of complex and simple products to sketch a rudimentary theory of industrial development. Suppose a complex good ( $m = 2$ ) can be sold for more in the market than a simple one ( $m = 1$ ), and (just for simplicity) the level of multisourcing is the same in both cases ( $n = 2$ ). We then vary  $x$  exogenously, which can be thought of as varying quality of commercial institutions. As illustrated in Figure 8, as  $x$  increases there is a first threshold at which it will become possible to produce a simple good with  $m = 1$ , and the probability of such production being successful then increases in a continuous way in  $x$ . At some higher threshold value of  $x$ , it will become possible to produce a complex good. Hence, at the point where its institutions become good enough to support complex production, the value-weighted productivity of the economy can jump discontinuously.

2.5. **A planner's problem.** We consider now a planner who chooses the value of  $x$ , which can be thought of as the quality of institutions. The planner's problem is

$$\max_{x \in [0,1]} \kappa \rho(x) - c(x), \quad (2)$$

where  $c(x)$  is a convex function representing the cost of maintaining institutions of quality  $x$ . We assume that  $c(0) = 0$ ,  $c'(0) = 0$ , and  $\lim_{x \rightarrow 1} c'(x) = \infty$ . Here  $\kappa > 0$  is a parameter we use to shift the value of production. Thus the planner seeks to maximize expected social surplus, which is the total surplus produced by the firms that are functional minus the cost of maintaining institutions.

Define the correspondence  $x^{SP}(\kappa) = \operatorname{argmax}_{x \in [0,1]} \kappa(1 - (1 - x\rho(x))^n)^m - c(x)$ . This gives the values of  $x$  that solve the social planner's problem for a given  $\kappa$ .

**Proposition 2.** Fix any  $n \geq 2$  and  $m \geq 2$ . Then there exists a  $\kappa_{\text{crit}} > 0$  such that

- (i) for all  $\kappa < \kappa_{\text{crit}}$ ,  $x^{SP}(\kappa) = 0$ .
- (ii) for all  $\kappa > \kappa_{\text{crit}}$ , all values of  $x^{SP}(\kappa)$  are strictly greater than  $x_{\text{crit}}$ .
- (iii) for  $\kappa = \kappa_{\text{crit}}$ , all values of  $x^{SP}(\kappa)$  are either strictly greater than  $x_{\text{crit}}$  or else equal to 0.

The first part of Proposition 2 says that when  $\kappa$  is sufficiently low, it is too costly for the social planner to invest anything in the quality of institutions. As  $\kappa$  increases, a threshold  $\kappa_{\text{crit}}$  is reached and at this value of  $\kappa$  it first becomes optimal to invest in institutional quality. At this threshold, the social planner's investment increases discontinuously. Moreover, it immediately increases to a level strictly *above*  $x_{\text{crit}}$ , and for all larger  $\kappa$  all solutions stay above  $x_{\text{crit}}$ .

It is worth emphasizing that the planner never chooses to invest at the critical level  $x_{\text{crit}}$ . An efficient supply network never operates at the point of discontinuity. This is intuitive: at  $x = x_{\text{crit}}$  the marginal social benefits of investing are infinite, as can be seen in Figure 4, while marginal costs at  $x_{\text{crit}}$  are finite, and so the social planner can always do better by increasing investment a little. In contrast, we show next that individual investment choices *will* sometimes put the supply network on the precipice in equilibrium.

### 3. SUPPLY NETWORKS WITH ENDOGENOUS RELATIONSHIP STRENGTH AND ENTRY

We now introduce our full model. This builds on the structure of physical production described in the previous section. However, relationship strength, as well as decisions to enter, are now both endogenous choices.

**3.1. Entry decisions and the supply network.** There is a finite set  $\mathcal{I}$  of *products*. For each  $i$ , the set  $\mathcal{I}_i \subseteq \mathcal{I}$  is the set of input products necessary for the production of  $i$ .

There is a continuum of firms  $\tilde{\mathcal{F}}_i$  in each product  $i \in \mathcal{I}$  that are *potential entrants*. Each  $\tilde{\mathcal{F}}_i$  is a copy of the interval  $[0, 1]$ ; its firms are labeled  $if$ , where  $i$  is the product label and  $f \in [0, 1]$  is the firm label. Each firm  $if$  may enter or not; this is captured by  $e_{if} \in \{0, 1\}$ . Firm  $if$  has a cost  $\Phi_i(f)$  of entering, where  $\Phi_i : [0, 1] \rightarrow \mathbb{R}_+$  is an increasing function.<sup>17</sup>

**Assumption 1.** For each  $i$ ,  $\Phi_i(f)$  is a strictly increasing function with  $\Phi(0) = 0$ .

First, firms decide whether to pay the fixed cost to enter<sup>18</sup> or not, and if a firm does choose to enter, it also chooses its investment in relationship strength. To produce one unit of its good, firm  $if$  must source inputs from some supplier of product  $j$  for each  $j \in \mathcal{I}_i$ . A firm obtains the inputs it needs to produce by forming relationships with suppliers of those inputs. We assume that for each  $if \in \mathcal{F}_i$ , and each input  $j \in \mathcal{I}_i$ , there is a finite set  $\mathcal{F}_{if,j}$  of firms that, exogenously, are *potential suppliers* of good  $j$  to firm  $if$ . We suppose the identities (i.e., indices  $f$ ) of these firms are drawn uniformly at random

<sup>17</sup>The set of firms producing each product is endowed with the Lebesgue measure and so the c.d.f. of the fixed costs of entry is  $\Phi^{-1}$ .

<sup>18</sup>The assumption that the lowest entry cost is zero is not necessary for our results, as we discuss in Section 4.

from among the producers of product  $j$  that enter.<sup>19</sup> The supply relationships that can be used by  $if$  to source inputs will come from this set.

A (random) *potential supply network*  $\mathcal{G}$  is a directed graph whose nodes are  $\mathcal{F} = \{if : e_{if} = 1\}$ , consisting of all firms that enter, and where each node has directed links to all of its suppliers  $\mathcal{F}_{if,j}$ .

**3.2. Investment decisions.** The key choice each firm makes conditional on entry is how much to invest to increase the probability with which each of its potential supply relationships is *operational*, delivering any goods produced by the supplier. For the simplest description of the investment decisions, we make some strong symmetry assumptions. First, we assume  $|\mathcal{I}_i| = m$  for each product so that  $m$  different inputs are required to produce each good. Second, we assume that  $|\mathcal{F}_{if,j}| = n$  for all firms  $if$  and all required inputs  $j \in \mathcal{I}_i$ . Third, we take  $\Phi_i = \Phi$  for all  $i$ . As was the case for the basic analysis of discontinuities, these assumptions facilitate a simple analysis of key forces via one-dimensional fixed-point equations, but they are not crucial, and are relaxed in Section 6.

With the symmetry assumptions made, we can formulate a simple model of the formation of production links. The firm chooses an investment level  $y_{if} \geq 0$ , which has a private cost  $c(y_{if})$  and results in a relationship strength

$$x_{if} = \underline{x} + y_{if}.$$

The intercept  $\underline{x} \geq 0$  is a baseline probability of success that occurs absent any costly investment, which might, for example, reflect the quality of institutions. The main purpose of this baseline level is as a simple channel to shock relationship strength (and other specifications, such as a multiplicative one, could be used).<sup>20</sup>

Recalling the definition of  $x_{\text{crit}}$  from Proposition 1, we make the following assumptions:

**Assumption 2.**

- (i)  $\underline{x} < x_{\text{crit}}$ ;
- (ii)  $c'$  is increasing and weakly convex, with  $c(0) = 0$ ;
- (iii) the Inada conditions hold:  $\lim_{y \downarrow 0} c'(y) = 0$  and  $\lim_{y \uparrow 1 - \underline{x}} c'(y) = \infty$ .

The first part of this assumption ensures that baseline (free) relationship strength is not so high that the supply network is guaranteed to be productive even without any investment. The second assumption imposes assumptions on investment costs that help guarantee agents' optimization problems are well-behaved; the Inada conditions, as usual, guarantee interior investments.

**Remark 1.** There are two interpretations of  $x_{if}$ . The first interpretation is that the set of possible suppliers is fixed, and the investment works on the intensive margin to improve the quality of these relationships (by reducing misunderstandings and so on). The second interpretation is that the investment works on the extensive margin—i.e., firms work to find a supplier capable of and willing to supply a given required input type, but their success is stochastic. In this interpretation relationships never fail and there is a fixed set of  $n$  potential suppliers capable of supplying the required input to be found, and each one of them is found independently with probability  $x_{if}$ . In Appendix C

<sup>19</sup>Without losing tractability we could also dispense with the conditioning on who enters and allow a firm's potential relationships to include some firms that don't end up in the market.

<sup>20</sup>Equivalently, we can think of firms directly choosing the strength of their relationships,  $x_{if} \geq \underline{x}$ , and paying the corresponding investment cost  $c(x_{if} - \underline{x})$ .

we discuss a richer extensive-margin interpretation, and also one that permits separate efforts to be directed to the extensive and intensive margins simultaneously.

Given the chosen relationship strengths  $x_{if}$ , we can define a random *realized supply network*  $\mathcal{G}' \subseteq \mathcal{G}$  that consists of those links in  $\mathcal{G}$  that are operational, as introduced in Section 2. For this construction, let a link from  $if$  to a supplier  $jf'$  be operational with probability  $x_{if}$ , which is determined by the investment of the sourcing firm. These realizations are effectively<sup>21</sup> independent across links.

Given the realization of  $\mathcal{G}'$ , Section 2.3 describes which firms are capable of producing—*functional*.<sup>22</sup> We denote by  $F_{if} \in \{0, 1\}$  the random ( $\mathcal{G}'$ -determined) realization of whether firm  $if$  is functional. Finally, we denote by  $r_i$  the probability that a producer of product  $i$ , selected uniformly at random among the entering producers of product  $i$ , is functional, which we call the *reliability* of that product.

**Remark 2.** This formulation assumes that a firm cannot produce anything if it fails to source any one of its inputs. This is a feature of many standard production functions. For example, all CES production functions with an elasticity of substitution less than or equal to 1 have this feature (e.g., the Leontief and Cobb-Douglas production functions).

**3.3. Payoffs and equilibrium.** We now turn to a firm's payoffs, which we will specify only for symmetric behavior of other firms. Indeed, for our study of symmetric equilibria, it will suffice to examine the case where firms  $[0, \bar{f}]$  enter for each product  $i$ . Recall that  $r_i$  is the probability that a firm, selected uniformly at random from producers of product  $i$  that enter, is functional. In a symmetric equilibrium all producers of all products have the same reliability. Denote this reliability by  $r$ . Then the mass of functional firms in the supply network is  $\bar{f}r|\mathcal{Z}|$ . Conditional on being functional, we assume a firm earns a profit—gross of fixed entry costs and the cost of investment—of  $G(\bar{f}r) = \kappa g(\bar{f}r)$ ; where  $g : [0, 1] \rightarrow \mathbb{R}_+$  is a decreasing function. That is, functional firms each earn greater gross profits when the market of functional firms is less crowded. We interpret the multiplier  $\kappa > 0$  as a quantity that shifts total factor productivity, and use it to explore the comparative statics of the supply network as productivity is varied. For some microfoundations giving rise to such profits, see Appendix B.

Thus, conditional on entering and making an investment  $y_{if}$ , the net expected profit of firm  $if$  is

$$\Pi_{if} = \underbrace{\mathbf{E}[F_{if}]}_{\text{prob. functional}} \underbrace{G(\bar{f}r)}_{\text{gross profit}} - \underbrace{c(y_{if})}_{\text{cost of investment}} - \underbrace{\Phi(f)}_{\text{entry cost}}. \quad (3)$$

If a firm does not enter, its net profit is 0. Note that the distribution of  $F_{if}$  depends on others' relationship strengths (and hence investment decisions) as well as one's own.

To summarize, the timing is:

1. Firms make their entry decisions.
2. Firms simultaneously choose their investment levels to maximize  $\Pi_{if}$ .
3. The realized supply network  $\mathcal{G}'$  is drawn, and payoffs are enjoyed.

A *firm outcome* is given by entry decisions  $e_{if}$  and relationship strengths  $x_{if}$  for all firms  $if$ .<sup>23</sup>

<sup>21</sup>There are some technical subtleties here arising from the continuum of agents in our model. See Appendix A.

<sup>22</sup>We also give a formal definition in Appendix A.

<sup>23</sup>

**Definition 1.** A firm outcome is an *equilibrium* if the following conditions hold:

- Optimal investment: conditional on all entry decisions and all others' investment decisions, each firm sets its  $y_{if}$  to maximize the expectation of its net profit  $\Pi_{if}$ .
- Optimal firm entry: Correctly anticipating subsequent investment decisions, and conditional on others' entry decisions, no positive mass of firms can increase its profit by making a different entry decision.<sup>24</sup>

#### 4. EQUILIBRIUM SUPPLY NETWORKS AND THEIR FRAGILITY

We now study the equilibrium of our model. In order to do so we break the equilibrium problem down into two steps. First we study firms' equilibrium investment decisions conditional on a given level of entry. This lets us describe equilibrium relationship strengths. Then we study equilibrium entry decisions using that analysis of equilibrium relationship strengths.

**4.1. Symmetric investment equilibria for a given level of entry.** Suppose a measure  $\bar{f}$  of the producers of each product  $i \in \mathcal{I}$  enter; such entry decisions will be the relevant ones for our further analysis. We will now analyze investment decisions in this situation for any  $\bar{f}$ : essentially the subgame at stage 2 in our timing described in Section 3.3. We focus on *symmetric investment equilibria*.

To define these, it is helpful to write a more explicit profit function: suppose all entering firms choose relationship strength  $x$ . We often refer to firms choosing strength  $x_{if} \geq \underline{x}$  directly and paying the corresponding investment cost  $c(x_{if} - \underline{x})$ . Then firm  $if$ 's expected profit conditional on entering the market and choosing strength  $x_{if}$  is:

$$\Pi_{if}(x_{if}; x, \bar{f}) = \underbrace{\kappa g(\bar{f}\rho(x))}_{G(\bar{f}\rho(x))} \underbrace{(1 - (1 - x_{if}\rho(x))^n)^m}_{\mathbf{E}[F_{if}]} - c(x_{if} - \underline{x}) - \Phi(f), \quad (4)$$

where  $\rho(x)$  is the reliability of any of a firm's suppliers when all firms in the supply network have relationship strength  $x$ .<sup>25</sup> This comes from applying (3) and using the same reasoning as in Section 2 to give an explicit formula for  $\mathbf{E}[F_{if}]$ . We write  $P(x_{if}; x)$  for  $\mathbf{E}[F_{if}]$ .

**Definition 2.** We say  $x \geq \underline{x}$  is a *symmetric investment equilibrium* for  $\bar{f}$  if  $x_{if} = x$  maximizes  $\Pi_{if}(x_{if}; x, \bar{f})$ .<sup>26</sup>

Note that a symmetric investment equilibrium is defined by the level of relationship strength  $x$  realized in it, rather than the level of investment. This turns out to be more convenient.

Because firms symmetrically choosing investments  $y_{if} = 0$  results in a reliability of  $\rho(\underline{x}) = 0$  (using, first, the fact that they then obtain relationship strengths  $\underline{x} < x_{\text{crit}}$  by Assumption 2, and second, the characterization of  $\rho$  in Proposition 1), there is always a

<sup>24</sup>Our entry condition is formulated in terms of a positive measure of firms. In Appendix F we tighten our equilibrium definition to require that no firms can profitably enter the market in equilibrium, and show that all our results are robust when we also introduce a competitive banking sector to finance (and hence ration) entry.

<sup>25</sup>Note that because there is a continuum of firms the probability that a firm appears in its potential supply network upstream of itself is 0. Thus the reliability of  $if$ 's suppliers does not depend on  $x_{if}$ .

<sup>26</sup>Formally, the maximization problem that a firm  $if$  is solving depends on  $f$  through  $\Phi(f)$ . However, by this time entry costs have been sunk and the value of  $f$  does not matter.

symmetric investment equilibrium (in fact a zero-investment equilibrium) with relationship strengths  $\underline{x}$ . Our next proposition says that either (i) this is the only possibility for a symmetric investment equilibrium, or (ii) there is, in addition to  $\underline{x}$ , at most one other symmetric investment equilibrium with relationship strengths  $x > \underline{x}$ . The proposition will also describe how the equilibrium relationship strength depends on  $\bar{f}$ .

Before stating the result, we deal with some technicalities. First note that when production is complex ( $m \geq 2$ ), if there is no scope for multisourcing ( $n = 1$ ) then  $x_{\text{crit}} = 1$ . By the Inada condition on the investment cost function (Assumption 2, part (iii)) there cannot then be a symmetric investment equilibrium in which there is a positive probability of successful production. We thus focus on the case in which  $n \geq 2$ .

In analyzing positive symmetric investment equilibria, it is helpful to make an assumption on the environment that ensures that local optimality implies global optimality.

**Assumption 3.** For any  $x \geq x_{\text{crit}}$  and  $\bar{f}$ , the function  $\Pi_{if}(y_{if}; x, \bar{f})$  has a unique interior local maximum.

Assumptions 1–3 will be maintained in the sequel. We now give a simple condition on primitives that is sufficient for Assumption 3 to hold.

**Lemma 1.** For any  $m \geq 2$ ,  $n \geq 2$ , there is a number<sup>27</sup>  $\hat{x}$ , depending only on  $m$  and  $n$ , such that (i)  $\hat{x} < x_{\text{crit}}$ ; and (ii) if  $\underline{x} \geq \hat{x}$ , then Assumption 3 is satisfied.

Consider any environment where  $\underline{x} \in [\hat{x}, x_{\text{crit}})$ . Part (i) of the lemma guarantees that the interval  $[\hat{x}, x_{\text{crit}})$  is nonempty, and part (ii) guarantees that Assumption 3 is satisfied. For any such  $\underline{x}$ , a symmetric zero-investment equilibrium continues to exist. However, as we have said, there may also be a positive symmetric investment equilibrium; in this case, Assumption 3 guarantees that any such equilibrium must satisfy a simple first-order condition.<sup>28</sup>

We now characterize the behavior of positive investment equilibria under the assumption.

**Proposition 3.** Fix any  $n \geq 2$  and  $m \geq 3$ , and any  $\kappa$  and  $g$  consistent with the maintained assumptions. Then there is a unique function  $x^*(\bar{f})$  such that:

- (1) if  $x^*(\bar{f}) > \underline{x}$ , then this value is the unique symmetric investment equilibrium with positive investment;
- (2) if  $x^*(\bar{f}) = \underline{x}$ , the only symmetric investment equilibrium is  $\underline{x}$ .

There is a value  $f_{\text{crit}} \in [0, 1]$  (depending on  $g$ ,  $\kappa$ ,  $n$ , and  $m$ ) such that  $x^*(\bar{f}) > x_{\text{crit}}$  and is weakly decreasing for all  $\bar{f} > f_{\text{crit}}$  and is equal to  $\underline{x}$  for all  $\bar{f} \leq f_{\text{crit}}$ .

Proposition 3 shows that either there is no symmetric positive investment equilibrium, or else such an equilibrium is unique. Further, the function  $x^*(\bar{f})$  gives the highest possible value of relationship strength that can be realized in a symmetric investment equilibrium (for a given entry level  $\bar{f}$ ). The function  $x^*(\bar{f})$  is decreasing in the mass of firms that enter ( $\bar{f}$ ) and drops discontinuously to  $\underline{x}$  (i.e., zero investment for all firms) when entry is above a threshold  $f_{\text{crit}}$  as illustrated in Figure 9.

Establishing this result involves several challenges, which we explain before outlining the argument. First, the reliability of supply relationships at a symmetric investment level

<sup>27</sup>In the proof, we give an explicit description of  $\hat{x}$  in terms of the shape of the function  $P(x_{if}; x)$ .

<sup>28</sup>While conditions we identify in Lemma 1 are sufficient for satisfying Assumption 3, as we will see later (for example, in Figure 12) they are not necessary.

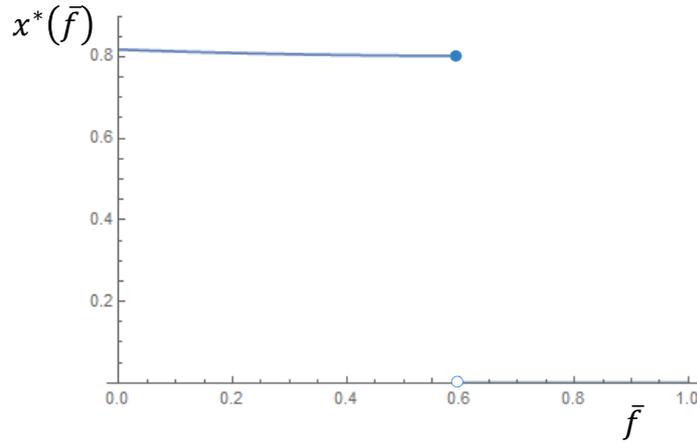


FIGURE 9. The relationship strength that occurs in a symmetric investment equilibrium is plotted as entry  $\bar{f}$  is varied. This is illustrated for  $m = 5$ ,  $n = 3$ ,  $\underline{x} = 0$ ,  $c(x_{if} - \underline{x}) = x_{if}^2$ , and  $G(\bar{f}\rho(x)) = 2(1 - \frac{\bar{f}\rho}{2})$ .

is only known to satisfy a certain fixed point condition. Second, investment incentives are nuanced, and non-monotonic. When a firm’s suppliers are very unreliable there is little incentive to invest in stronger relationships with them—there is no point in having a working pipe when the suppliers have nothing to put in it. On the other hand, when a firm’s suppliers are extremely reliable, a firm can free-ride on this reliability and invest relatively little in its pipes, knowing that as long as it has one working pipe for each input it requires, it is very likely to be able to source the inputs it requires. Investments in relationships strength are strategic complements in some regions of the parameter space, and strategic substitutes in others. Proposition 3 shows that, nevertheless, equilibrium investment has nice uniqueness and monotonicity properties.

The complete proof of Proposition 3 is in the appendix. We give the main ideas here. First, if the relationship strengths of all firms are symmetric (whether in or out of equilibrium), the investment problem each firm faces is identical, because—generalizing the example of Section 2—the supply network upstream of any firm looks identical (up to product labels that do not matter for reliability).

A “law of physics” generalizing the fixed-point equation (1) from our example characterizes reliability for any symmetric level of investment. To make an outcome a symmetric investment equilibrium, a second condition must hold: firms’ choices of relationship strength,  $x_{if}$ , must be optimal given the reliability of each of their suppliers. As there are a continuum of firms, firm  $if$  appears upstream in its own supply chain with probability 0, and so  $i$ ’s investment choice  $x_{if}$  has no impact on the reliability of its suppliers. Given this, and that the same investment choice  $x$  is made by the other firms, the reliability of each of  $if$ ’s suppliers is  $\rho(x)$  which firm  $if$  takes as given. These conditions—the law of physics and a best-response equation for choosing one’s own relationship strength—together determine symmetric investment equilibria, and allow us to characterize when there is one with positive investment. We now expand on each of these.

4.1.1. *A law of physics.* Suppose the strength of all relationships is  $x$ . From Proposition 1 and the argument we gave accompanying it, the *physical consistency* of the relationship

strengths with the reliability  $r$  requires  $r$  to satisfy the equation

$$r = \left( 1 - \underbrace{(1 - x r)^n}_{\text{Probability a given input cannot be acquired}} \right)^m, \quad (\text{PC})$$

and, moreover, to be the *largest* solution to this equation for the given  $x$  (recall Section 2.3). We denote this solution by  $\rho(x)$ .

For  $r > 0$ , the pair  $(x, r)$  satisfies (PC) if and only if it satisfies

$$x = \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \quad (\text{PC}')$$

For a sketch of the right-hand side of equation (PC') as a function of  $r$ , see Figure 10(a). Since the *largest*  $r$  satisfying (PC) is the one that determines reliability  $\rho(x)$ , it follows that the increasing part of the function (PC'), where  $r \in [r_{\text{crit}}, 1]$ , is the part relevant for determining equilibrium reliability. This can also be seen by comparing panels (a) and (c) of Figure 10.

4.1.2. *Best-response investment.* For a positive symmetric investment equilibrium to be played, an entering firm must choose a relationship strength  $x_{if}$  such that its marginal benefits and marginal costs of investment are equalized, given the relationship strength  $x$  chosen by other firms. Recall from (4) the function  $\Pi_{if}(x_{if}; x, \bar{f})$ . Setting its derivative in  $x_{if}$  to zero gives:

$$\underbrace{G(r\bar{f})}_{\text{Profits given successful production}} \underbrace{\frac{\partial \mathbf{E}[F_{if}]}{\partial x_{if}}}_{\text{Marginal increase in probability } if \text{ can produce}} = \underbrace{c'(x_{if} - \underline{x})}_{\text{Marginal cost}}. \quad (\text{OI})$$

It is easy to get an analytical expression<sup>29</sup> for  $\partial \mathbf{E}[F_{if}]/\partial x_{if}$ , the derivative of the probability that firm  $if$  successfully produces in  $x_{if}$ . The (gross) profits that  $if$  receives conditional on successful production,  $G(r\bar{f})$ , do not vary with  $if$ 's investment. Finally,  $c'(x_{if} - \underline{x})$  is the strictly increasing marginal cost that firm  $if$  faces to increase its relationship strength  $x_{if}$ .

The following definitions will be helpful.

$$MB(x_{if}; \bar{f}, r, \kappa) = G(\rho\bar{f}) \frac{\partial \mathbf{E}[F_{if}]}{\partial x_{if}} \quad (5)$$

$$MC(x_{if}) = c'(x_{if} - \underline{x}) \quad (6)$$

4.1.3. *Combining physical consistency and optimal investment.* We want to show that there is at most one solution to (OI) and (PC) simultaneously for  $r \in [r_{\text{crit}}, 1]$ . A first observation is that for  $r \in [r_{\text{crit}}, 1]$  equation (PC') pins down the value that  $x$  must take as a function of  $r$ . We can therefore substitute  $x$  out of equation (OI) using equation (PC'). This gives us the following condition:

$$\underbrace{G(r\bar{f})mnr^{2-\frac{1}{m}}(1-r^{1/m})^{1-\frac{1}{n}}}_{\mathfrak{b}(r)} = c' \left( \underbrace{\frac{1 - \left(1 - r^{\frac{1}{m}} - \underline{x}\right)^{\frac{1}{n}}}{r}}_x \right) \quad (7)$$

<sup>29</sup>Specifically,  $\frac{\partial \mathbf{E}[F_{if}]}{\partial x_{if}} = rn(1-x_{if}r)^{n-1}m(1-(1-x_{if}r)^n)^{m-1}$ .

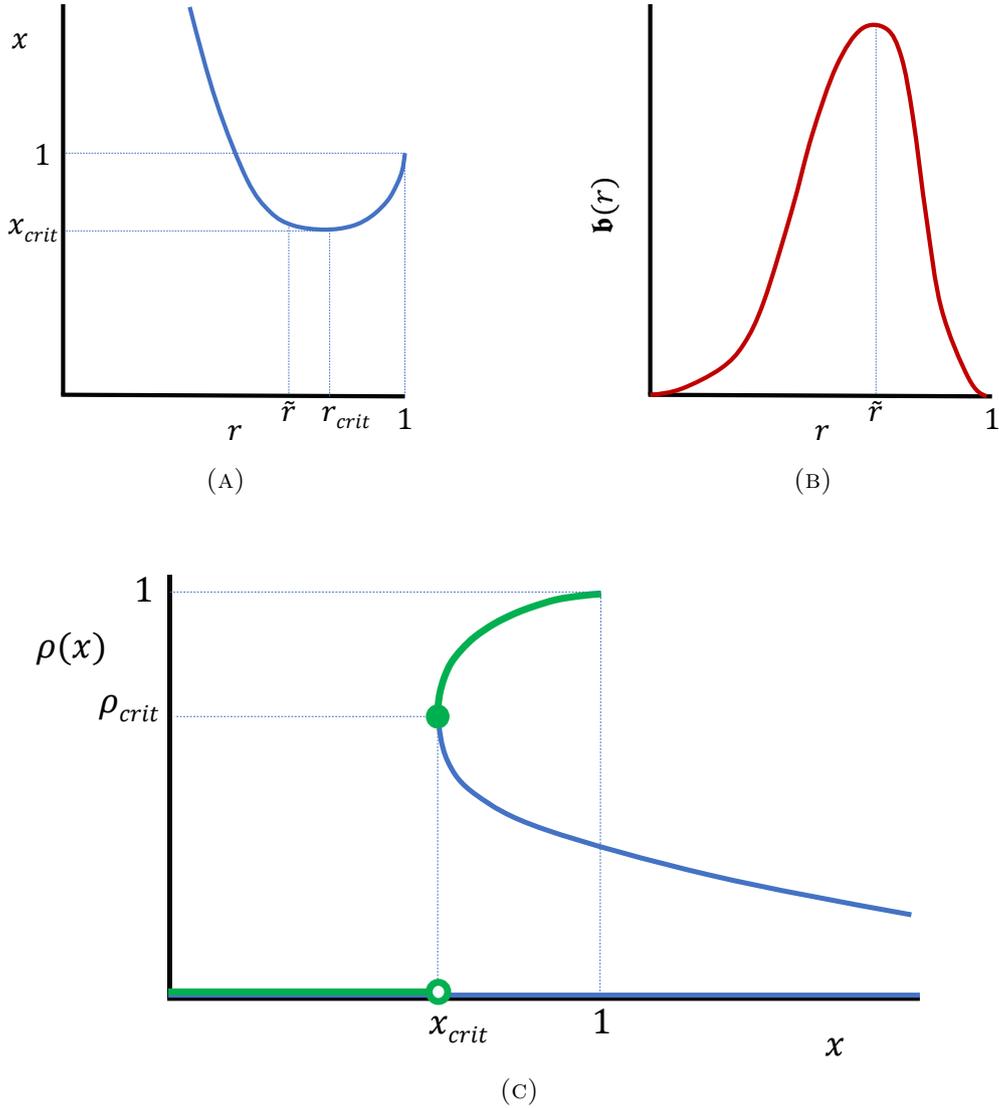


FIGURE 10. Panel (a) shows the relationship between  $r$  and  $x$  implied by physical consistency. Panel (b) plots the marginal benefits. Panel (c) shows how the relationship between  $r$  and  $x$  plotted in panel (a) generates the discontinuity in the functionality probability when the axes are swapped.

We need to show that there is at most one value of  $r \in [r_{crit}, 1]$  that solves this equation. The right-hand side of the equation is increasing in  $r$ .<sup>30</sup> If we could establish that the left-hand side, which we call  $\mathbf{b}(r)$ , is decreasing in  $r$ , it would follow that there is a unique  $r$  solving (7). Unfortunately,  $\mathbf{b}(r)$  is not decreasing in  $r$ . However, we can show that it *is* decreasing in  $r$  for  $r \geq r_{crit}$ , which is sufficient.<sup>31</sup> Panel (b) of Figure 10 gives a representative depiction of  $\mathbf{b}(r)$ , reflecting that it is decreasing to the right of  $r_{crit}$ .

<sup>30</sup>This follows directly: The cost function is convex in  $x$ , so marginal costs are increasing in  $x$ , while  $x$  is increasing in  $r$  for all  $r \geq r_{crit}$  (see Panel (a) of Figure 10).

<sup>31</sup>We do this by showing that the global maximum of  $\mathbf{b}(r)$  is achieved at a number  $\tilde{r}$  that we can prove is smaller than  $r_{crit}$ .

**4.2. Two possible types of positive symmetric equilibria.** We now close the model by making the entry cutoff endogenous. To study firm behavior, first note that optimal behavior by firms entails monotonicity of entry strategies: if firm  $if$  finds it profitable to enter given others' strategies, any firm  $f' < f$  finds it strictly profitable (note that as firms are zero measure a firm's entry decision has no impact on the aggregate environment). Thus, without loss, in any equilibrium entry decisions can be summarized by a cutoff  $\bar{f}_i^*$  for each product  $i$ . A symmetric equilibrium is associated with a pair  $(\bar{f}^*, x^*(\bar{f}^*))$ , where  $\bar{f}^*$  is the entry cutoff for producers of every product and  $x^*(\bar{f}^*)$  is the equilibrium relationship strength. Indeed, recall by Proposition 3, when a measure  $\bar{f}$  of the producers of each product enter, there is at most one positive symmetric investment equilibrium  $x^*(\bar{f}^*)$ . We call such an equilibrium positive if  $\bar{f}^* > 0$  and  $x^*(\bar{f}^*) > \underline{x}$ . We will show that there is at most one positive symmetric equilibrium, and that it can take one of two forms.

To simplify the analysis we make a further assumption, guaranteeing that there is not a corner solution to the entry problem in which all firms enter:

**Assumption 4.**  $\Phi(1) > G(\rho(x^*(1)))\rho(x^*(1)) - c(x^*(1) - \underline{x})$ .

Assumption 4 guarantees that if all firms enter and play the investment equilibrium  $x^*(1)$  (corresponding to all firms entering), then the highest-cost firm of those that enter makes a loss.

**Proposition 4.** Fix any  $n \geq 2$  and  $m \geq 3$ , and any  $g$  consistent with the maintained assumptions. Then:

- (i) For any  $\kappa$ , there is at most one positive symmetric equilibrium.
- (ii) For generic  $\kappa$ , a unique positive investment equilibrium takes one of two forms:
  - $x^*(\bar{f}^*) = x_{crit}$  and  $\Pi_{if} > 0$  for all entering firms; we call this a *critical equilibrium*.
  - $x^*(\bar{f}^*) > x_{crit}$  and marginal firms  $i\bar{f}^*$  make zero profits ( $\Pi_{i\bar{f}^*} = 0$ ), for all  $i \in \mathcal{I}$ ; we call this a *noncritical equilibrium*.

Part (i) of Proposition 4 shows the uniqueness of a positive symmetric equilibrium when one exists. Part (ii) of Proposition 4 shows that when a positive symmetric equilibrium exists, generically we are in one of two regimes—either equilibrium relationship strength is at the critical level and entry is restricted such that all firms make positive profits, or equilibrium relationship strength is above the critical level and all firms that could make a positive profit by entering do.<sup>32</sup>

Proposition 4 is silent on existence of the positive symmetric equilibrium. We show when such equilibria exist, and show which regime we will be in and when, in Proposition 5. The critical regime turns out *not* to be a knife-edge case.

We defer the proof of Proposition 4 to the appendix. To gain some intuition suppose we start from an (out of equilibrium) entry level  $\bar{f}$  in each market, such that all entering firms make strictly positive profits and relationship strength is above the critical level (i.e.,  $x^*(\bar{f}) > x_{crit}$ ). Some firms not entering the market are then able to enter the market and make strictly positive profits. As additional firms enter the market the incentives of firms to invest in reliability diminish ( $x^*(\bar{f})$  is decreasing in  $\bar{f}$ ). There are then two possibilities. First, entry may increase until a zero profit condition is satisfied. In this case

<sup>32</sup>There is an additional case that occurs for a non-generic  $\kappa$ . At a specific level of  $\kappa$  we transition between the two regimes and the marginal firm makes zero profits *and* relationship strength is at the critical level.

we end up with an equilibrium  $x^*(\bar{f}^*) > x_{\text{crit}}$ —what we term non-critical. Alternatively, before a zero profit condition is reached, we may increase entry to the point where firms would choose a relationship strength  $x^*(f^*) = x_{\text{crit}}$ . This is then an equilibrium. Even though the marginal entering firm makes a strictly positive profit, there is no positive mass of non-entering firms that can enter because they tip the system over the precipice, causing production to fail and rendering these entry choices unprofitable.<sup>33</sup>

The proposition says that only one of these two situations can occur for a given set of parameter values, and thus the environment uniquely determines the type of positive equilibrium that occurs, when one exists.

**4.3. Dependence on productivity parameter  $\kappa$ : an ordering of regimes.** Critical equilibria are important because they create the possibility of fragility: small shocks to relationship strengths via a reduction in  $\underline{x}$  can result in a collapse of production. We show now, that when a productivity parameter  $\kappa$  is below a certain threshold, there is no equilibrium with positive production. However, for  $\kappa$  above this threshold and up to another strictly higher threshold, there is a critical positive equilibrium as described in Proposition 4. In such an equilibrium, all firms make positive profits and relationship strength is at the critical level. Finally, for  $\kappa$  above the second threshold, there is a non-critical positive equilibrium, where the marginal firm entering the market makes zero profits.

**Proposition 5.** There exist thresholds  $\underline{\kappa}, \bar{\kappa}$  satisfying  $0 < \underline{\kappa} < \bar{\kappa} < \infty$  such that

- (i) For  $\kappa \leq \underline{\kappa}$ , no firms enter in equilibrium:  $\bar{f}^* = 0$ .
- (ii) For  $\kappa \in (\underline{\kappa}, \bar{\kappa}]$ : a positive fraction of firms enter, so that  $\bar{f}^* > 0$ . The level of investment is critical, i.e.,  $x^*(\bar{f}^*) = x_{\text{crit}}$ , and, for  $\kappa \neq \bar{\kappa}$ , the profit  $\Pi_{if} > 0$  for all entering firms.
- (iii) For  $\kappa > \bar{\kappa}$ : a positive fraction of firms enter, so that  $f^* > 0$ . The level of investment is above the critical level, i.e.  $x^*(\bar{f}^*) > x_{\text{crit}}$ , and  $\Pi_{i\bar{f}^*} = 0$  for all  $i \in \mathcal{I}$ .

Proposition 5 shows that there exists an equilibrium with positive production if and only if  $\kappa$  is above a key threshold. It also describes how the unique productive equilibrium changes as  $\kappa$  increases beyond this threshold. In Figure 11 we illustrate these changes in more detail with an example.

While the proof of Proposition 5 is relegated to the appendix, we now provide some intuition. When  $\kappa$  is very low the value of production does not cover the investment costs of forming a supply network capable of successful production.

As  $\kappa$  increases to a certain level, which we call  $\underline{\kappa}$ , the private gains from production become just sufficient to support equilibrium investments by entering firms that create a supply network capable of successful production for the first time. For all values of  $\kappa < \underline{\kappa}$ , the markup  $\kappa g(\rho_{\text{crit}} f)$  is sufficiently small for all  $f$  that even if all other entering firms have reliability  $\rho_{\text{crit}}$ , an entering firm  $if$  would want to free-ride on others' investments and its profit function would be maximized at an investment level  $x_{if} < x_{\text{crit}}$ . In contrast, when  $\kappa = \underline{\kappa}$  this changes. For the first time, an entering firm  $if$  can best-respond to others choosing investments  $x_{\text{crit}}$  by also choosing  $x_{if} = x_{\text{crit}}$ . This is shown in Figure 12 which plots the gross profits of firm  $if$  for different values of  $x_{if}$  and a

<sup>33</sup>Recall that our equilibrium definition just requires that a positive mass of firms cannot enter profitably. We show in Appendix F that we can tighten our equilibrium definition to require that no firms can profitably enter the market in equilibrium without affecting our results, by adding a competitive banking sector to the model.

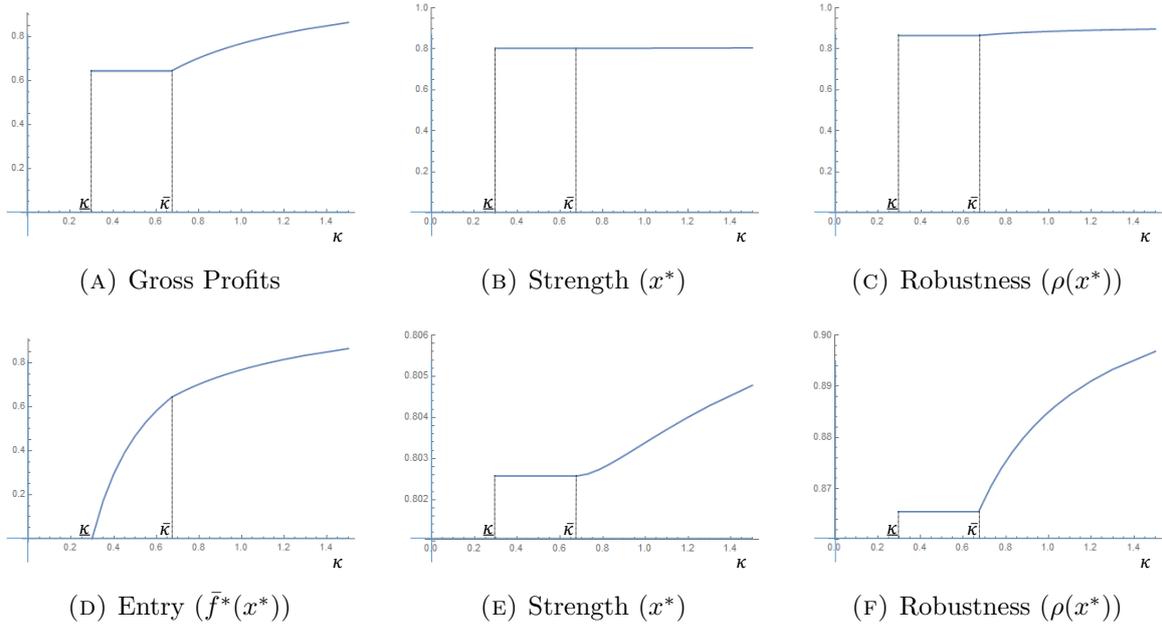


FIGURE 11. Equilibrium values of different variables as  $\kappa$  changes for the case of  $m = 5$ ,  $n = 3$ ,  $g(\bar{f}) = 5(1 - \bar{f})$ ,  $c(x) = x^2$  and firms of each type distributed uniformly over the unit interval. Panel (A) considers the gross profits that entering firms obtain. Panels (B) and (E) shows how equilibrium relationship strength changes (but with different scales for the  $y$ -axis). Panel (C) and (F) show how the equilibrium probability of successful production changes (also with different scales for the  $y$ -axis). Finally, Panel (D) reports how equilibrium entry changes.

markup  $\underline{\kappa}g(0) = \max_f \underline{\kappa}g(\rho_{\text{crit}}f)$ . Given these investments production is successful with probability  $\rho_{\text{crit}}$  and the entering firms receive strictly positive gross profits. Intuitively these profits are strictly positive despite entering firms' only just being willing to make investments  $x_{\text{crit}}$  because the entering firms benefit from the positive externalities other firms' supply chain investments confer on them.

As firms located at  $f = 0$  have no entry costs they find it strictly profitable to enter. But then so will a positive mass of other firms with sufficiently low entry costs holding investment fixed. However, were a positive mass of such firms to also enter, markups would decline (recall that  $g(\bar{f}\rho(x))$  is strictly decreasing in  $\bar{f}$ ). This in turn would reduce the incentives for firms to invest in their supply network robustness and now the entering firms would no longer be willing to make the investments required to form a functioning supply network. (The discontinuity is key to this: it is not possible to sustain production with a slightly smaller investment, so this small diminution in incentives results in a large drop in production.) As a result, it is actually not profitable for any positive mass of firms to enter, and the only equilibrium involves an entry threshold at  $f = 0$ . Thus at  $\underline{\kappa}$  we have an equilibrium in which those firms located at 0 enter, and those firms that do enter receive strictly positive profits while choosing relationship strength  $x_{\text{crit}}$ , resulting in a probability of successful production equal to  $\rho_{\text{crit}}$ .

As  $\kappa$  increases above  $\underline{\kappa}$ , the incentives to invest in supply chain robustness improve. If entry were held fixed, we would then have an investment equilibrium  $x^* > x_{\text{crit}}$ . However, the increase in  $\kappa$  allows more firms to enter without tipping the supply network over the

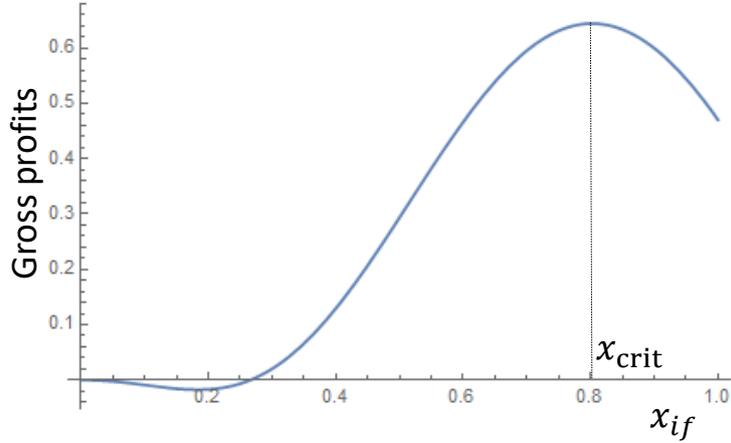


FIGURE 12. The gross profits that firm  $if$  can achieve by choosing different relationship strength  $x_{if}$ , when all other firms have relationship strength  $x_{crit}$  and  $\kappa = \underline{\kappa}$ .

edge of the precipice. This continues until firms have relationship strengths  $x_{crit}$ . An illustration of this is shown in Panels (B) and (E) of Figure 11: for the values of  $\kappa$  between  $\underline{\kappa}$  and  $\bar{\kappa}$ , equilibrium relationship strength remains constant at  $x_{crit}$ . Likewise, as shown in Panels (C) and (F), the probability of successful production remains fixed at  $\rho_{crit}$  for these values of  $\kappa$ .

On the one hand, as  $\kappa$  increases the profits of those firms already in the market directly increase. On the other hand, increased entry reduces profits. These two effects must exactly offset each other. Entry must occur up until the point where firms have exactly the same investment incentives as before (and so again choose relationship strength  $x_{crit}$ ), and, at this point, these firms must receive exactly the same mark-up as before. Note that although all entering firms receive this constant level of gross profits as  $\kappa$  increases, the marginal entering firm receives ever lower *net* profits (because they have ever higher entry costs). This can be seen by comparing panels (A) and (D) in Figure 11.

As  $\kappa$  increases further it reaches  $\bar{\kappa}$ . At this point the marginal entering firm receives zero profits. From this point forward the equilibrium regime changes. As  $\kappa$  increases further, entry increases, but only until the zero profit condition is satisfied. This additional entry is now insufficient to keep gross profits constant, and gross profits increase. This increases investment incentives for the entering firms and investment now increases above  $x_{crit}$ .

**4.4. Criticality and fragility.** We now formalize the idea that supply networks in the critical regime will be fragile. We do so by explicitly examining how the supply network responds to shocks, which for simplicity are taken to have zero probability (though the analysis is robust to anticipated shocks that happen with sufficiently small probability).

**Definition 3** (Equilibrium fragility).

- A productive equilibrium with investments  $y^*$  is *fragile* if *any* negative institutional shock, such that  $\underline{x}$  decreases to  $\underline{x} - \epsilon$  for  $\epsilon > 0$ , results in output falling to 0 (i.e.,  $\rho(\underline{x} - \epsilon + y^*) = 0$ ).
- A productive equilibrium is *robust*, if it is not *fragile*.

In a fragile equilibrium we hold firms' investment decisions and entry choices fixed. Implicitly, we are assuming that investments in supply relationships and entry decisions are made over a sufficiently long time frame that firms cannot change the quality of their supply relationships or their entry decisions in response to a shock.

**Proposition 6.** If  $\kappa \leq \bar{\kappa}$ , then any productive equilibrium is fragile and if  $\kappa > \bar{\kappa}$ , then any productive equilibrium is robust.

Proposition 6 follows immediately from the definition of a fragile equilibrium.<sup>34</sup>

**4.5. Robustness of results.** A number of remarks about the robustness of our results are in order:

**4.5.1. Infinite supply networks.** We have assumed that supply trees are infinite. This plays an important role generating the discontinuity in the probability that production is successful, which is central to our paper. However, were we to limit the length of supply trees to  $T$ , the probability of production has a similar shape which converges to the discontinuous function we study in the limit as  $T$  gets large. For example, when  $m = 5$  and  $n = 4$  if  $T = 7$  then a decrease in the investment level from  $x = 0.66$  to  $x = 0.61$  causes the probability of successful production to drop from more than 80% to less than 10%. For higher  $m$ , this convergence is faster. Thus, relatively small shocks can still cause production to collapse to near zero for relatively low values of  $T$ . See Appendix G.

In Appendix G we also consider a version of our model where each firm requires no inputs with probability  $\tau$ . This nests our current model when  $\tau = 0$ . We show in this appendix that as  $\tau \rightarrow 0$  equilibrium investments  $x^*(\tau)$  converge to the equilibrium investment levels we study by setting  $\tau = 0$ . We conclude that our main economic insights will continue to hold with complex but finite supply trees.

**4.5.2. Heterogeneity.** We show in Section 6, supported by Appendix D, that our key economic insights also continue to hold when we introduce heterogeneity in many dimensions. In this section we: (i) permit different products to require different numbers of inputs; (ii) allow the number of potential suppliers to differ by the input being sourced; (iii) accommodate product specific mark-ups and entry costs; (iv) allow the cost of achieving a given strength of supply relationship to differ depending on the product being sourced; and (v) allow firms to make different investments into sourcing different inputs. We find that production exhibits the same discontinuities and solve two examples numerically in which the production of some but not all products occurs in the critical regime. Finally, using insights from these examples, we show that there is a *weakest link* phenomenon that holds between certain products that are intertwined in the supply network. Such products are either all in the critical regime or else none of them are.

## 5. WELFARE IMPLICATIONS

We now briefly discuss inefficiencies and welfare considerations within our model. We begin with some thought experiments about the planner intervening on only one margin (investment given entry, or entry only) and then consider richer policies.

A basic point is that there may be underinvestment in robustness because of positive externalities: each firm's functionality contributes to the profits of many other firms.

<sup>34</sup>In Appendix I we show that even if investments can adjust in response to a shock, an equilibrium is fragile if  $\kappa \leq \bar{\kappa}$  and robust if  $\kappa > \bar{\kappa}$ .

Any model where firms don't fully appropriate the social surplus of their production will have this property.

To analyze this inefficiency, let us for a moment hold entry fixed to isolate effects arising from underinvestment, and study the investment equilibrium only. There are actually two effects that occur when firms become more reliable: one is the positive externality we have just described, but another effect is that, because each produces with a higher probability, there is more competition in the product market, which can exert a negative pecuniary externality on other firms. However, total surplus and consumer surplus unambiguously increase, since both are increasing in total sales. So, on the whole, there is too little investment in reliability. Indeed, holding entry fixed, because of the discontinuity in  $\rho$ , the consequences of firms' failing to internalize the positive externalities can be extremely stark, yielding no production in equilibrium. Another sense in which there is too little investment is that lower  $x$  makes the supply network more susceptible to shocks, and thus more fragile.

Now we consider a separate question, about entry: starting at an equilibrium, would the planner want to drive more firms into the market, knowing that reliability will be determined in an investment equilibrium subsequently? If reliability is held *fixed*, more entry increases total surplus after entry (because more goods are produced). Moreover, the resulting social gain covers the entry cost of the new firms. This is because at the status quo, the marginal firm was just indifferent to paying its entry cost, but it was appropriating only a part of its contribution to social surplus. So far we have held reliability fixed. But of course reliability is not fixed: new entrants reduce all firms' gross profits and therefore reduce investment in reliability. This effect could overpower the first. Indeed, in parallel to what we have said above, our most distinctive observation here is that a small amount of new entry, as long as it pushes up production by a positive amount (which would be needed for the positive effect on welfare we have just discussed) can destroy the positive investment equilibrium completely, or at least damage it severely if the status quo is on the steep part of the  $\rho$  curve. So unless existing firms' gross profits can somehow be protected (which would prevent them from wanting to decrease investment), subsidizing entry can be dangerous.

We turn now to some other policy options. Given what we have said, an obvious goal for the planner to focus on is increasing investment in robustness. This is particularly valuable at a critical equilibrium, where the supply network is fragile. But, as we will now explain, the fragile regime is one in which it is hard to effect beneficial changes on this margin.

A tempting solution is to subsidize effort to increase robustness, for example making the cost of a given investment  $1 - \theta$  times its original cost. More precisely, we make the gross (i.e., after-entry) profit equal to

$$\Pi_{if} = \underbrace{\mathbf{E}[F_{if}]}_{\text{prob. functional}} \underbrace{G(\bar{f}r)}_{\text{gross profit}} - \underbrace{(1 - \theta)c(y_{if})}_{\text{cost of effort}}.$$

Suppose the status quo was a critical equilibrium. It can readily be seen that a subsidy of this form cannot change the equilibrium reliability as long as the equilibrium remains critical—which it will for an open set of  $\theta$ . The reason is that at a critical equilibrium, the relationship strength and reliability are at  $(x_{\text{crit}}, r_{\text{crit}})$ . The subsidy will increase the profitability of the marginal firm and allow the market to bear more entrants while

providing sufficient incentives to keep reliability at  $r_{\text{crit}}$ . So, in the fragile regime, the reliability subsidy is entirely dissipated by increased entry.<sup>35</sup>

Given the limitations of subsidies, a potentially appealing alternative, especially at a fragile equilibrium, is to impose a minimum required level of reliability for any firm that enters. Whether this is practical will depend heavily on the setting. If a reliability requirement above  $r_{\text{crit}}$  can be imposed, then holding entry fixed, it will move the outcome away from the precipice. As argued above when we examined whether there is too little investment, this leads to an unambiguous improvement in total and consumer surplus. However, what if entry is now allowed to respond in a world with this new regulation? If *more* firms enter in equilibrium, then consumer and total surplus go up relative to the status quo, since both factors in the expression  $r\bar{f}$  (total output) increase in equilibrium:  $r$  increases because of the new regulation, and  $\bar{f}$  increases by hypothesis. Could equilibrium entry decrease? This will not be a concern in the fragile regime: all entering firms make strictly positive profits at the status quo (by the properties of the fragile regime we have catalogued). Thus, when the equilibrium is critical, quality regulation is an unambiguous improvement both for consumers and for total surplus.

Regulating reliability directly may not be feasible: especially in complex markets with many supply relationships, most of the relevant outcomes might not be contractible or observable to an outsider. In this case, within our model, the only policy lever that remains is to regulate entry. The costs of collapse or volatility are not explicitly in our model; but if these are in the planner's objective, the planner will be willing to sacrifice some of the surplus from the market to avoid these costs. Thus, limiting entry only to firms that can earn considerable rents could be justified by macroprudential considerations.

## 6. HETEROGENEITY

We now study a version of our supply network with heterogeneity. We take the model as described in Section 3.1, and enrich it to accommodate asymmetries. Our analysis so far has relied on symmetry. Even in formulating the analytical expression for the reliability function, symmetry was important. This might raise concerns that it is the symmetry that is driving the discontinuities central to our analysis. In this section we show this is not the case. Paralleling the analysis of the homogeneous case, we show that the reliability of the supply network as a function of link strength is discontinuous. We then consider an analogue of the model of Section 3, with firms investing in the strength of their relationships with their suppliers. Through numerical examples, we then show that heterogeneous economies behave in a similar way to the homogeneous ones we have studied. Moreover, we show that supply networks in the presence of heterogeneity feature a *weakest link* property. When the production of one product is fragile, this makes the production of all products that rely directly or indirectly on it also fragile.

We now formalize the different kinds of heterogeneity that we introduce. (1) The set of producers of product  $i$ ,  $\mathcal{I}_i$ , is an arbitrary set with some cardinality  $m_i$ . Thus, we no longer require the input requirements to be symmetric. (2) For each product  $i$  and input product  $j \in \mathcal{I}_i$ , there is a number  $n_{ij}$  of potential suppliers of product  $j$  that each firm has; thus  $n_{ij}$  replaces the single multisourcing parameter  $n$ . (3) Investment and link strength are input-specific: For a firm  $if$  and an input  $j \in \mathcal{I}_i$ , there is a relationship

<sup>35</sup>At a non-fragile equilibrium, this strong non-responsiveness will not hold. Holding entry fixed, a reliability subsidy will increase reliability and make everyone better off. However, this will be partly undone by increased entry, which will drive down gross profits and incentives to invest.

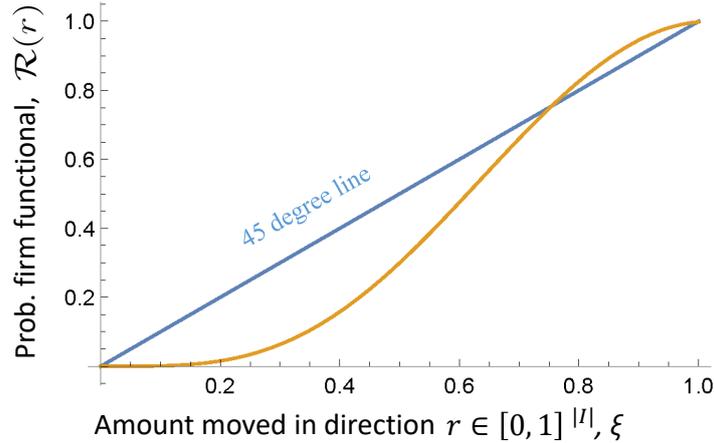


FIGURE 13. The probability,  $\mathcal{R}(r)$ , that a focal firm is functional as we increase the probabilities that other types of firms are functional by moving in an arbitrary direction  $\mathbf{r} \in [0, 1]^{|Z|}$ ,  $\xi$ .

strength  $x_{if,j}$  which replaces the single number  $x_{if}$ . The cost of effort is  $c_{ij}(x_{ij} - \underline{x}_{ij})$ . (4) The gross profit conditional on producing product  $i$  is  $G_i(\bar{f}_i r_i)$ . Here  $G_i$  is a product-specific function (which can capture many different features of different product markets that affect their profitabilities),  $\bar{f}_i$  is the fraction of producers of product  $i$  that enter, and  $r_i$  is the reliability of producers of product  $i$ . (5) The distribution of entry costs is product-specific: there is a function  $\Phi_i$  such that the cost of entry for firm  $if$  is  $\Phi_i(f)$ .

**6.1. Discontinuities in reliability with exogenous link strength.** Our first result shows that the basic physical implications of supply chain complexity are robust to heterogeneity. In the general environment we have described, there is an analogue of Proposition 1.

To formalize this we introduce just for simplicity a single parameter  $\xi$  reflecting institutional quality. We posit that  $x_{if,j} = x_{ij}(\xi)$ , where  $x_{ij}$  are strictly increasing, differentiable, surjective functions  $[0, 1] \rightarrow [0, 1]$ .

**Proposition 7.** Suppose that for all products  $i$ , the complexity  $m_i$  is at least 2. Moreover, suppose whenever  $j \in \mathcal{I}_i$ , the number  $n_{ij}$  of potential suppliers for each firm is at least 1. For any product  $i$ , the measure of the set of functional firms  $\bar{\mathcal{F}}_i$ , denoted by  $\rho_i(\xi)$ , is a nondecreasing function with the following properties.

- (1) There is a number  $\xi_{\text{crit}}$  and a vector  $\mathbf{r}_{\text{crit}} > 0$  such that  $\rho$  has a discontinuity at  $\xi_{\text{crit}}$ , where it jumps from 0 to  $\mathbf{r}_{\text{crit}}$  and is strictly increasing in each component after that.
- (2) If  $n_{ij} = 1$  for all  $i$  and  $j$ , we have that  $\xi_{\text{crit}} = 1$ ; otherwise  $\xi_{\text{crit}} < 1$ .
- (3) If  $\xi_{\text{crit}} < 1$ , then as  $\xi$  approaches  $\xi_{\text{crit}}$  from above, the derivative  $\rho'_i(x)$  tends to  $\infty$  in some component.

The idea of this result is simple, and generalizes the graphical intuition of Figure 13. For any  $\mathbf{r} \in [0, 1]^{|Z|}$ , define  $\mathcal{R}^\xi(\mathbf{r})$  to be the probability, under the parameter  $\xi$ , that a producer of product  $i$  is functional given that the reliability vector for producers of other products is given by  $\mathbf{r}$ . This can be written explicitly:

$$[\mathcal{R}^\xi(\mathbf{r})]_i = \prod_{j \in \mathcal{I}_i} [1 - (1 - r_j x_{ij}(\xi))^{n_{ij}}].$$

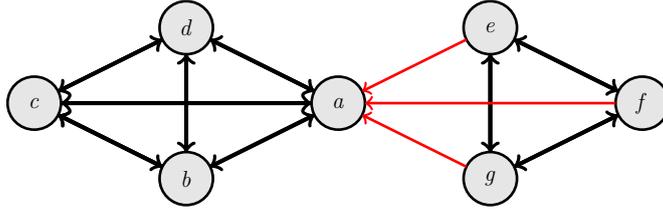


FIGURE 14. Supply dependencies: Black bold arrows represent reciprocated supply dependencies in which both products require inputs from each other. A red arrow from one product to another means that the product at the origin of the arrow uses as an input the product at the end of the arrow (e.g. product  $e$  requires product  $a$  as an input, but not the other way around). Product  $a$  also depends on itself, but the corresponding self-link is not shown.

By the same reasoning as in Section 2, there is a pointwise largest fixed point of this function and this reflects the shares of firms that can produce each product.

Near 0, the map  $\mathcal{R}^\xi : [0, 1]^n \rightarrow [0, 1]^n$  is bounded above by a quadratic function (as a consequence of  $m_i \geq 2$  for all  $i$ ). Therefore it cannot have any fixed points near 0. Thus, analogously to Figure 5, as illustrated in Figure 13, fixed points disappear abruptly as  $\xi$  is reduced past a critical value  $\xi_{\text{crit}}$ .

**6.2. Numerical examples.** We now provide two numerical examples. Details regarding how we numerically solve these examples are in Appendix D.

**Example 1.** There are seven products. Only product  $a$  is used as an input into its own production. Products  $a$  to  $d$  all use inputs from each other, products  $e$  to  $g$  all use inputs from each other, and products  $e$  to  $g$  also require product  $a$  as an input. Figure 14 shows the input dependencies between these products. We let products  $a$  and  $b$  have three potential suppliers for each of their required inputs, while products  $c$  to  $f$  have only two potential suppliers for each of their required inputs. We let the profitability of the seven products differ systematically. All else equal, product  $a$  is associated with the highest margins, then  $b$ , then  $c$  and so on. Specifically, we let  $G_i(r_i \bar{f}_i) = \alpha_i(1 - r_i \bar{f}_i)$  and set

$$\alpha = [40, 30, 15, 10, 3.5, 3, 2.8].$$

We let the cost of a producer of product  $i$  from investing in supplier relationships with producers of product  $j$  be  $\frac{1}{2}\gamma_{ij}x_{ij}^2$ , and set, for now,  $\gamma_{ij} = 1$  for all product pairs  $ij$ . Finally, we use an entry cost function  $\Phi_i(f) = \beta_i f$ , and set

$$\beta = [40.44, 39.85, 2.30, 2.28, 0.30, 0.40, 0.50].$$

The equilibrium investments that firms must make are pinned down by equating the marginal costs and benefits of each investment for each firm. Equilibrium entry levels are then pinned down by either a zero profit condition, or else by the requirement that investments are at the critical level. For this configuration, production of products  $e$ ,  $f$  and  $g$  is critical, while production is non-critical for products  $a$ ,  $b$ ,  $c$  and  $d$  and entry is pinned down by a zero profit condition. We report the equilibrium levels of entry and investment, along with gross and net profits, in Appendix D.

We then model a small unanticipated shock to the cost of firms producing product  $i$  investing in their relationship strength with suppliers of product  $j$ , by increasing  $\gamma_{ij}$  from 1 to  $1 + \epsilon$ . After the shock to  $\gamma_{ij}$ , producers of product  $i$  choose to exert less effort

sourcing input  $j$ . For  $\gamma_{ij}$  such that  $i = a, b, c$  or  $d$ , the impact of this is minor. The probability of successful production for those products ( $r_i$ ) only drops continuously and for a small shock the change will be small. Nevertheless, this is still sufficient for the output of the firms producing products  $e, f$  and  $g$  to collapse to 0 since they are ‘critical’ and they source (directly or indirectly) products  $a, b, c$  and  $d$ .

Similarly, if the shock occurs to the sourcing efforts of firms producing products  $e, f$  or  $g$ , output of these products will collapse to 0 since they are ‘critical’. On the other hand, output of products  $a, b, c$  and  $d$  will not be affected since these producers do not require inputs  $e, f$  or  $g$ .

**Example 2.** We adjust the configuration of the previous example by letting the vector of product profitabilities be

$$\alpha = [4, 5, 6, 7, 10, 15, 20],$$

and setting

$$\beta = [10.00, 4.00, 0.20, 0.20, 1.36, 1.36, 1.43].$$

Everything else, remains the same as before.

Given these parameters we numerically solve for the equilibrium (see Appendix D). For these parameter values production of products  $a, b, c$  and  $d$  is now critical, while production of products  $e, f$  and  $g$  is not.

Consider now a shock to  $\gamma_{ij}$  for  $i \in \{a, b, c, d\}$ . Output of the shocked product  $i$  then collapses to 0, and thus so will the output of the firms producing products  $\{a, b, c, d\}$ . Output of products  $e, f$  and  $g$  will then also collapse to 0 since those producers all source (directly or indirectly) inputs  $a, b, c$  and  $d$ .

Consider now a shock to  $\gamma_{ij}$  for  $i \in \{e, f, g\}$ . Output of the shocked product will adjust to accommodate the shock and the probability of successful production for the affected product will fall continuously. For a small shock, this decrease in output will be small. Products  $a, b, c$  and  $d$  will be unaffected even though they are critical, since their firms do not source products  $e, f$  and  $g$ .

Examples 1 and 2 illustrate that the relationships between the failures of production across products in the same supply network can be subtle. We now investigate this more systematically. For this section, we use  $x^* = (x_1^*, x_2^*, \dots, x_{|\mathcal{I}|}^*)$  to describe equilibrium investment profiles for the different products. As before we identify the profile with the relationship strength achieved. We let  $x_{-i} = (x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_{|\mathcal{I}|})$ . Note that for this section  $x_i^* = \{x_{i,j}^*\}_{j \in \mathcal{N}_i}$  is a vector, whereas elsewhere it is a scalar. An investment profile  $x_i$  is critical if, fixing the relationship strengths of the producers of other products  $x_{-i}$ , the probability of successful production of product  $j$  would be zero at any profile  $\hat{x}_i < x_i$ , where  $\hat{x}_i < x_i$  means that each entry of  $\hat{x}_i$  is weakly lower than the corresponding entry of  $x_i$  and at least one such entry is strictly lower. Product  $i$  is critical in an equilibrium when the equilibrium investment profile of producers of product  $i$  is critical.

**Proposition 8.**

- (i) Suppose product  $i$  is critical, then any other product  $j$  on a directed path from  $j$  to  $i$  (on the product interdependencies graph) will fail following a shock  $\epsilon > 0$  to  $\gamma_{ik}$  for any  $k \in \mathcal{N}_k$ .
- (ii) Let  $\mathcal{I}_c \subset \mathcal{I}$  be a set of products that are part of a strongly connected component of the product interdependencies graph. Then any equilibrium with positive effort

is such that either all producers of products  $i \in \mathcal{I}_c$  have critical relationship strengths or no producers of products  $i \in \mathcal{I}_c$  have critical relationship strengths.

Proposition 8 shows that supply networks suffer from a weakest link phenomenon. First, if a product is critical, then a shock to it causes the production of other products that use it as an input, directly or indirectly, to also fail. Second, if we take a strongly connected component of products where none of them are critical and, say, reduce entry costs for the producers of one product until it becomes critical, then all products in that component will also become critical at the same time. The component is only as strong as its weakest link.

## 7. CONCLUDING DISCUSSION: AGGREGATE FRAGILITY

So far we have focused on a single complex supply network with particular parameters. The economy may be made up of many such supply networks. In this section, we consider the extent to which these forces can lead to fragility in the aggregate for an economy consisting of many such supply networks, any one of which is small relative to the macroeconomy (we will use “small” in this sense throughout this section).

Suppose there are many small sectors which operate independently of each other, with heterogeneity across sectors but, for simplicity, homogeneity within sector. The parameters of these different supply networks, including their complexities and the extent of multisourcing opportunities, are drawn from a distribution. We know from the above that a small shock can discontinuously reduce the production of some of these supply networks. We now point out that a small shock can have a large macroeconomic effect, and that previous results on the fragile regime, especially Proposition 3 are essential to this.

For simplicity, fix the functions  $c(\cdot)$  and  $\Phi(\cdot)$ .<sup>36</sup> A given *sector* of the economy is described by a tuple  $\mathfrak{s} = (m, n, \kappa)$ . We consider the space of sectors induced by letting the parameters  $m, n$  and  $\kappa$  vary. In particular, we let  $\mathcal{M}$  be the set of possible values of  $m$ , the set of integers between 1 and  $M$ ; we let  $\mathcal{N}$  be the set of possible values of  $n$ , integers between 1 and  $N$ , and we allow  $\kappa \in \mathcal{K} = [0, K]$ . The space of possible sectors is now  $\mathcal{S} = \mathcal{M} \times \mathcal{N} \times \mathcal{K}$ . We let  $\Psi$  be a distribution over this space, and assume that it has full support.

In some sectors, there will not exist a positive equilibrium (for example, when  $\kappa$  is sufficiently low fixing the other parameters). Consider now those sectors for which there is a positive equilibrium. There are now two possibilities. It may be that the only sectors for which there is a positive equilibrium have  $m = 1$ . That is, the only sectors with positive reliability are simple. In this case, there is no aggregate fragility.

But if, in contrast,  $\mathcal{S}$  contains sectors where production is *not* simple, then we will have macroeconomic fragility. Indeed, an immediate consequence of Propositions 5 and 6 is that if there are some complex ( $m \geq 2$ ) sectors with positive equilibria, then some of the lower- $\kappa$ -sectors with the same  $(n, m)$ —which are included in  $\mathcal{S}$ —are in the fragile regime. The measure that  $\Psi$  assigns to sectors in the fragile regime is positive. Thus, a shock to relationship strengths through  $\underline{x}$  will cause a discontinuous drop in expected aggregate output.

<sup>36</sup>These could also be drawn from a distribution, but the notation would be more cumbersome.

**7.1. Interdependent supply networks and cascading failures.** So far we have looked at the case in which the different sectors operate independently and all business-to-business transactions occur through supply relationships confined to their respective sectors. We think of these relationships as mediating the supply of inputs that are tailored to the specifications of the business purchasing them. They are not products that can be purchased off-the-shelf. For example, most business use computers, but many do not maintain failure-prone specific relationships with computer manufacturers. So far we have abstracted from any interdependencies between businesses created by such purchases. However, these interdependencies might matter. A failure of one product could reduce the productivity of others that purchase this product off-the-shelf. If this reduction in productivity is long-lasting it can reduce the incentives of firms to invest in their supply relationships. A shock that causes one fragile supply network to fail might then precipitate the failures of others, even though these others are disconnected in our supply network. Indeed, there can then be cascades of failures of the following sort: producers in supply networks that are not *initially* in the fragile regime are put into the fragile regime by others' failures, and then also are disabled by a small shock. We flesh out this idea in Appendix I.1.

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## APPENDIX A. FORMAL CONSTRUCTION OF THE SUPPLY NETWORK

Fix a production network as described in Section 2: a finite set  $\mathcal{I}$  of products and, for each  $i$ , a set  $\mathcal{I}_i$  of products which are inputs necessary to produce this project. Let  $\mathcal{P}$  denote the production network specified by these data.

Next we specify the nodes of the supply network. For each  $i \in \mathcal{I}_i$  there is a set  $\mathcal{F}_i = \{if : f \in [0, \bar{f}_i)\}$  of firms producing product  $i$ .<sup>37</sup> We endow this set with the Borel  $\sigma$ -algebra.

A *supply network with production network  $\mathcal{P}$  and multisourcing number  $n$*  is a random graph  $\mathcal{G} = (\mathcal{F}, \mathcal{E})$  satisfying the following properties.

- Its nodes are the set  $\mathcal{F}$ ;
- Edges are ordered pairs  $(if, jf')$  where  $j \in \mathcal{I}_i$ —the meaning that  $if$  sources from  $jf'$ , and we depict such an edge as an arrow from  $if$  to  $jf'$ .
- For any  $if \in \mathcal{F}$ , for each  $j \in \mathcal{I}_i$  there are  $n$  edges  $(if, jf')$  to  $n$  distinct producers  $f'$  of product  $j$ , i.e. elements of  $\mathcal{F}_j$ . For any firm, define its neighborhood  $N_{if} = \{jf' : (if, jf') \in \mathcal{E}\}$ .
- For any firm  $if$ , the elements of  $N_{if} \cap \mathcal{F}_j$  are independently drawn from an atomless distribution, and these realizations are independent across  $j$ .
- For any countable set of firms  $\mathcal{F}'$ , neighborhoods  $N_{if}$  are independent.

Define  $\mathcal{G}'$  to be a random subgraph of  $\mathcal{G}$  in which each edge is kept independently, with probability  $x$ . More formally, define for every edge  $e$  a random variable  $O_e \in \{0, 1\}$  (whether the edge is operational) such that

- $\mathbf{P}[O_e = 1 \mid \mathcal{G}] = x$  for every  $e \in \mathcal{G}$  and,
- for any countable subset  $E$  of edges in  $\mathcal{G}$ , the random variables  $(O_e)_{e \in E}$  are independent conditional on  $\mathcal{G}$ .

A subset  $\widehat{\mathcal{F}} \subseteq \mathcal{F}$  is defined to be *consistent* if, for each  $if \in \widehat{\mathcal{F}}$ , the following holds: for all products  $j$  that  $if$  requires as inputs ( $\forall j \in \mathcal{I}_i$ ) there is a potential supplier of project  $j$  for firm  $if$  ( $\exists jf' \in \mathcal{F}_{if,j}$ ) so that  $if$  has an operational link to that supplier  $(if, jf')$  (s.t.  $(if, jf') \in \mathcal{G}'$ ) with  $if$  and  $jf' \in \widehat{\mathcal{F}}$ . There may be many consistent sets, but by a standard argument, there will be a maximal one,  $\overline{\mathcal{F}}$ , which is a superset of any other consistent set. This set can be found via the algorithm in Section 2.

## APPENDIX B. SIMPLE MICROFOUNDATIONS

In the main text, we have described the production in the supply network at an abstract level. Here we give a more detailed foundation. We make the simplest choices possible at each stage, but remark where generalizations are straightforward.

**B.1. Production possibilities and allocations.** We take the parameters laid out in Appendix A and the realized supply network  $\mathcal{G}'$  as constructed there, and then define the following environment based on these data. There is a representative household that inelastically supplies  $\bar{L}$  units of labor. Labor is the only factor (unproduced input).

There is one consumption good, produced as a CES aggregate of *consumption components*  $(C_i)_{i \in \mathcal{I}}$ , one for each product. We turn now to the production of these components.

<sup>37</sup>Here we have chosen a particular set of labels, namely numbers in  $[0, 1)$ ; in the illustrations of the main text we have used real numbers outside this interval for convenience.

Letting  $c_{if}$  be the amount of firm  $if$ 's output contributed to that component, we have the CES production of consumption component  $i$ :

$$C_i = \left( \int_0^1 c_{if}^{\frac{\sigma-1}{\sigma}} df \right)^{\frac{\sigma}{\sigma-1}}.$$

An allocation is described by specifying

- (1) firm sourcing of intermediates: for each  $if$  and  $jj'$ , the amount that  $if$  buys from  $jj'$ ; called  $z_{if,jj'}$ ; this is 0 if  $(if, jj') \notin \mathcal{G}'$  and may be 0 or positive otherwise;
- (2) an amount  $\ell_{if} \geq 0$  of labor contributed by the household to each firm;
- (3) firm output: the output  $Y_{if}$  produced by each producer of product  $i$ , which is constrained to be 0 if  $e_{if} = 0$ ;
- (4) an amount  $c_{if} \geq 0$  of firm  $if$ 's output that allocated to consumption.

Each firm  $if \in \mathcal{F}_i^e$  has a production technology that gives

$$\mathcal{Y}_i(\ell_{if}, (z_{if,jj'})_{j \in \mathcal{I}_i, jj' \in \mathcal{F}^e})$$

units of output when its labor and intermediate inputs are the ones given in the arguments. For simplicity only, we will work with a Leontief technology.<sup>38</sup> For fixed constants  $0 < \epsilon, \lambda, \iota < 1$

$$\mathcal{Y}_i(\ell_{if}, (z_{if,jj'})_{jj' \in \mathcal{F}^e}) = \min \left\{ \frac{\ell_{if}}{\lambda}, \left\{ \frac{z_{if,jj'}}{\iota/m} \right\}_{j \in \mathcal{I}_i, jj' \in \mathcal{F}^e} \right\}.$$

In words, to produce a unit of its product, a firm needs  $\lambda$  units of labor and  $\iota$  units of intermediates, the latter needing to come equally from its inputs.

**B.2. Payments.** We now describe how the surplus from production is allocated. The allocation is efficient. Rents accrue when firms sell their goods to produce the consumption components (i.e., effectively to the consumer). Firms sell these differentiated goods at a markup of  $\sigma$ , as implied by a Nash-in-prices solution.

In sales of intermediates among firms, all bargaining power is allocated to the buyer at each stage of production. This avoids any distortions, double marginalization, etc., arising from markups by suppliers.<sup>39</sup> By using this specification we make the point that pricing or bargaining frictions play no role in the sources of fragility we identify.

**B.3. The equilibrium.** The technology implies that the ultimate factor requirement for one unit of production of any good is  $\lambda/(1-\iota)$  units of labor. By the bargaining power assumption, all intermediate inputs are sold at marginal cost, which is ultimately their factor content. We normalize the wage to 1.

Using all the labor produces  $(1-\iota)/\lambda$  units of output. Noting also that, by the technology, producing any unit of output requires  $1/(1-\iota)$  units of some intermediates, we can calculate that the units of output that go to consumption (as opposed to intermediates) is

$$\gamma = \frac{(1-\iota)^2}{\lambda(2-\iota)}.$$

Focusing on symmetric equilibria of the entry game, suppose firms  $f \leq \bar{f}$  have entered for each product. In any efficient outcome, due to the convexity in the production of the

<sup>38</sup>All our calculations would be very similar for any symmetric CES production technology at each step.

<sup>39</sup>In Appendix H we show that our basic conclusions are robust to any distribution of the post-entry surplus generated by efficient production.

consumption good, a share  $\gamma/n$  is output that comes from product  $i$ , and  $c_{if} = \gamma/(n\rho\bar{f})$  is contributed by each firm. (Here  $\rho$  is the fraction of firms that are functional.)

These calculations imply that the total amount of consumption good  $C_i$  that comes out of the corresponding CES aggregator in equilibrium is

$$C_i = \frac{\gamma}{n}(\rho\bar{f})^{\frac{1}{\sigma-1}}.$$

The profits of firm  $i$  come from earning a markup of  $\frac{\sigma}{\sigma-1}$  over their marginal cost. This yields, after some calculation, gross (i.e., post-entry) profits of

$$G = \frac{\gamma}{n\rho\bar{f}} \cdot \frac{1}{1-\sigma}.$$

Thus, we have that consumer surplus is strictly increasing in  $\rho\bar{f}$ , the mass of functional firms that have entered, and the profits to each firm are strictly decreasing in this quantity.

## APPENDIX C. INTERPRETATION OF INVESTMENT

**C.1. A richer extensive margin model.** In Remark 1 we gave an extensive margin search effort interpretation of  $x_{if}$ . In some ways this interpretation was restrictive. Specifically, it required there to be exactly  $n$  suppliers capable for supplying the input and that each such supplier be found independently with probability  $x_{if}$ . This alternative interpretation is a minimal departure from the intensive margin interpretation, which is why we gave it. However, it is also possible, through a change of variables, to see that our model encompasses a more general and standard search interpretation.

Fixing the environment a firms faces, specifically the probability other firms successfully produce  $\rho > 0$  and a parameter  $n$  that will index the ease of search, suppose we let each firm  $if$  choose directly the probability that, through search, it finds an input of given type. When  $\rho = 0$  we suppose that all search is futile and that firms necessarily choose  $\hat{x}_{if} = 0$ . Denote the probability firm  $if$  finds a supplier of a given input type by  $\hat{x}_{if}$ . Conditional on finding an input, we let it be successfully sourced with probability 1 so all frictions occur through the search process. Implicitly, obtaining a probability  $\hat{x}_{if}$  requires search effort, and we suppose that cost of achieving probability  $\hat{x}_{if}$  is  $\hat{c}(\hat{x})$ , where  $\hat{c}$  is a strictly increasing function with  $\hat{c}(0) = 0$ .

We suppose firms choose  $\hat{x}_{if}$  taking the environment as given. In particular, firms take as given the probability that suppliers of the inputs they require successfully produce. When many potential suppliers of an input produce successfully we let it be relatively easy to find one, and if none of these suppliers produce successfully then it is impossible to find one. In addition, the parameter  $n$  shifts how easy it is to find a supplier.

Given this set up we can let the probability of finding a supplier have the functional form  $\hat{x} := 1 - (1 - x_{if}\rho)^n$ , and the cost of achieving this probability be given by  $\hat{c}(\hat{x}) := c\left(\frac{1-(1-\hat{x})^{1/n}}{\rho}\right)$ . Although these functional form assumptions might seem restrictive, we still have freedom to use any function  $c$  satisfying our maintained assumptions. This degree of freedom is enough for the model to be quite general as all that matters is the size of the benefits of search effort relative to its cost, and not the absolute magnitudes. Further, these functional form assumptions satisfy all the desiderata we set out above. As  $1 - (1 - x_{if}\rho)^n$  is the key probability throughout our analysis, all our results then go through with this interpretation.

**C.2. Effort on both the extensive and intensive margins.** This section supports the claims made in Remark 1 that our model is easily extended to allow firms to make separate multi-sourcing effort choices on the intensive margin (quality of relationships) and the extensive margin (finding potential suppliers).

Suppose a firm  $if$  chooses efforts  $\hat{e}_{if} \geq 0$  on the extensive margin and effort  $\tilde{e}_{if} \geq 0$  on the intensive margin, and suppose that  $x_{if} = h(\hat{e}_{if}, \tilde{e}_{if})$ . Let the cost of investment be a function of  $\hat{e}_{if} + \tilde{e}_{if}$  instead of  $y_{if}$ . This firm problem can be broken down into choosing an overall effort level  $e_{if} = \hat{e}_{if} + \tilde{e}_{if}$  and then a share of this effort level allocated to the intensive margin, with the remaining share allocated to the extensive margin. Fixing an effort level  $e$ , a firm will choose  $\hat{e}_{if} \in [0, e]$ , with  $\tilde{e}_{if} = e - \hat{e}_{if}$ , to maximize  $x_{if}$ . Let  $\hat{e}_{if}^*(e)$  and  $\tilde{e}_{if}^*(e) = e - \hat{e}_{if}^*(e)$  denote the allocation of effort across the intensive and extensive margins that maximizes  $x_{if}$  given overall effort  $e$ . Given these choices, define  $h^*(e) := h(\hat{e}_{if}^*(e), \tilde{e}_{if}^*(e))$ . As  $h^*$  is strictly increasing in  $e$ , choosing  $e$  is then equivalent to choosing  $x_{if}$  directly, with a cost of effort equal to  $c(h^{*-1}(e))$ . Thus, as long as the cost function  $\hat{c}(e) := c(h^{*-1}(e))$  continues to satisfy our maintained assumptions on  $c$ , everything goes through unaffected.

#### APPENDIX D. HETEROGENEITY

In this appendix, we first explain how we numerically solve the examples from Section 6, and then report some additional information about these examples.

**D.1. Solving examples with heterogeneities numerically.** To compute the equilibria in Examples 1 and 2, we proceed as follows.

As before we consider symmetric equilibria, such that all entering producers of product  $i$  invest the same amount sourcing a given input  $j$ , and let entry be described by a threshold on entry costs, which is now product specific ( $\bar{f}_i$ ). As we are interested in these symmetric investment choices conditional on entering, and about the entry decision of the marginal producer of product  $i$ , we dispense with the subscript  $f$ . In such an equilibrium the profit of a marginal producer of product  $i$  is given by

$$\Pi_{i, \bar{f}_i} = G_i(\bar{f}_i r_i) r_i - \frac{1}{2} \sum_{j \in \mathcal{N}_i} \gamma_{ij} x_{ij}^2 - \Phi_i(\bar{f}_i), \quad (8)$$

where

$$r_i = \prod_{j \in \mathcal{N}_i} (1 - (1 - x_{ij} r_j)^{n_{ij}}),$$

$\mathcal{N}_i$  is the neighborhood of  $i$  on the product dependency graph, with  $|\mathcal{N}_i| = m_i$  (the complexity of production for product  $i$ ), and  $n_{ij}$  is the number of potential suppliers a producer of product  $i$  has for input  $j$  (i.e. the potential level of multisourcing by producers of product  $i$  for input  $j$ ).

The marginal benefit a producer of product  $i$  receives from investing in its relationships with suppliers of input  $j$  is

$$MB_{ij} = G(\bar{f}_i r_i) \prod_{l \in \mathcal{N}_i, l \neq j} (1 - (1 - x_{il} r_l)^{n_{il}}) n_{ij} (1 - x_{ij} r_j)^{n_{ij}-1} r_j. \quad (9)$$

Letting  $\gamma_{ij} = 1$  (as in the examples), the marginal cost for a producer of product  $i$  investing in a relationship with a supplier of input  $j$  is

$$MC_{ij} = x_{ij}. \quad (10)$$

We look for  $|Z| \times |Z|$  matrix  $X$ , with entries  $x_{ij}$  satisfying  $MB_{ij} = MC_{ij}$ . The value of  $x_{i1}$  that equates the marginal benefits and marginal costs for firm  $i$ 's investment into sourcing product 1 is increasing in  $G_i(\bar{f}_i r_i) = \alpha_i(1 - r_i \bar{f}_i)$ , which an entering producer of product  $i$  takes a given. As we still have the freedom to choose  $\alpha_i$  we can select an arbitrary  $x_{i1} \in (0, 1)$ . However, doing so pins down the value of  $x_{ij}$  for all  $j \neq 1$ . Specifically, we must have

$$\frac{MB_{ij}}{MB_{i1}} = \frac{MC_{ij}}{MC_{i1}}, \quad \forall i, j$$

which can be expressed as

$$\frac{G(\bar{f}_i r_i) \prod_{l \in \mathcal{N}_i, l \neq j} (1 - (1 - x_{il} r_l)^{n_{il}}) n_{ij} (1 - x_{ij} r_j)^{n_{ij}-1} r_j}{G(\bar{f}_i r_i) \prod_{l \in \mathcal{N}_i, l \neq 1} (1 - (1 - x_{il} r_l)^{n_{il}}) n_{i1} (1 - x_{i1} r_1)^{n_{i1}-1} r_1} = \frac{x_{ij}}{x_{i1}},$$

and reduces to

$$\frac{(1 - x_{ij} r_j)^{n_{ij}-1}}{(1 - (1 - x_{ij} r_j)^{n_{ij}})} \frac{n_{ij}}{x_{ij}} = \frac{n_{i1} r_1}{x_{i1} r_j} \frac{(1 - x_{i1} r_1)^{n_{i1}-1}}{(1 - (1 - x_{i1} r_1)^{n_{i1}})}. \quad (11)$$

The left-hand side is decreasing in  $x_{ij}$  while the right-hand side is given, so there can be only one solution  $x_{ij}$  satisfying the above.

We initialize  $x_{i1}$  for all  $i$  to values just smaller than 1. We then decrease these values incrementally by a small amount. After each reduction we calculate the  $X$  matrix using the above procedure and calculate the probability of successful production  $r_i$  for each industry. We continue until the probability of successful production decreases to 0 for one of the products  $i$ . This gives us values of  $X$  such that at least one product is in the critical regime.

The values of  $G_i(r_i \bar{f}_i)$  are then set so that  $MB_{i1} = MC_{i1}$  using equations (9) and (10). This ensures that all firms are choosing profit maximizing investments that result in at least one product being fragile. Recall that  $G_i(r_i \bar{f}_i) = \alpha_i(1 - r_i \bar{f}_i)$ , and so depends on both  $\alpha_i$  and  $\bar{f}_i$ . Thus for a given value of  $G_i(r_i \bar{f}_i)$  we can choose an arbitrary  $\bar{f}_i \in (0, 1)$  and then set  $\alpha_i$  so that  $G_i(r_i \bar{f}_i)$  is the value we require. The values of  $\bar{f}_i$  we pick don't matter because entry costs can be set to ensure that all entering firms make weakly positive profits, and no positive measure of non-entering firms want to enter. Specifically, using the entry cost function,  $\Phi_i(f) = \beta_i f$ , we choose the values of  $\beta_i$  that set the profit of the marginal firm to 0 (in equation (8)) for the products where production is not critical. Likewise, we choose values of  $\beta_i$  that set this profit to a strictly positive number for the products where production is in the critical regime.

Using this procedure there turn out to be essentially two types of equilibria with fragile firms. Either a firm in the set  $\{a, b, c, d\}$  becomes fragile first, in which case all firms become fragile simultaneously, or else a firm in the set  $\{e, f, g\}$  becomes fragile first in which case all firms in this set simultaneously become fragile while firms in the set  $\{a, b, c, d\}$  do not. When a firm in the set  $\{a, b, c, d\}$  becomes fragile first, a shock to any one of  $\{a, b, c, d\}$  that reduces the reliability of sourcing an input (either directly, or indirectly by reducing incentives to invest in reliability) is sufficient for the probability of successful production of all firms to fall to 0. When a firm in the set  $\{e, f, g\}$  becomes fragile first, a similar shock to any one of these firms is sufficient for the probability of

successful production of these firms, but not firms  $\{a, b, c, d\}$ , to fall to 0. The parameters selected in Examples 1 and 2 are chosen to illustrate these two possible cases.

**D.2. Example 1—additional information.** The equilibrium relationship strengths are reported in the matrix  $X$  below, where an entry  $x_{ij}$  represents the strength chosen by a producer of product  $i$  in a relationship sourcing input  $j$ .

By pinning down the first column of  $X$  with arbitrary values and solving for the other entries, we get

$$X = \begin{bmatrix} 0.8873 & 0.8872 & 0.9315 & 0.9385 & 0 & 0 & 0 \\ 0.8773 & 0 & 0.9204 & 0.9272 & 0 & 0 & 0 \\ 0.8673 & 0.8672 & 0 & 0.9084 & 0 & 0 & 0 \\ 0.8573 & 0.8572 & 0.8915 & 0 & 0 & 0 & 0 \\ 0.7573 & 0 & 0 & 0 & 0 & 0.9726 & 0.9783 \\ 0.7473 & 0 & 0 & 0 & 0.9464 & 0 & 0.9572 \\ 0.7373 & 0 & 0 & 0 & 0.9265 & 0.9317 & 0 \end{bmatrix}.$$

Such an  $X$  corresponds to the following product reliabilities:

$$r = [0.9926, 0.9928, 0.9387, 0.9307, 0.5384, 0.5262, 0.5145].$$

Also, for such values of  $x_{ij}$ , products  $a, b, c, d$  are non critical, while products  $e, f, g$  are critical.

We can obtain  $G = [21.0836, 17.7538, 3.2818, 3.0451, 2.6780, 2.5859, 2.4990]$ .

Recall that we let  $G_i(r_i \bar{f}_i) = \alpha_i(1 - r_i \bar{f}_i)$ . Setting

$$\alpha = [40, 30, 15, 10, 3.5, 3, 2.8],$$

we can find the values

$$\bar{f} = [0.4764, 0.4112, 0.8322, 0.7473, 0.4362, 0.2623, 0.2089].$$

Given those entry fractions  $\bar{f}_i$ , we then choose values of  $\beta_i$  that set the profit of the marginal firm to 0 for products  $a, b, c, d$  and values that set this profit to a strictly positive number for products  $e, f, g$ . Such values are

$$\beta = [40.4397, 39.8544, 2.3021, 2.2770, 0.3000, 0.4000, 0.5000].$$

Gross profits *before* the entry costs are

$$\tilde{\Pi} = [19.2666, 16.3872, 1.9159, 1.7016, 0.2036, 0.1756, 0.1506],$$

and the net profits of the marginal entering producer of each product (after entry costs) are

$$\Pi = [0, 0, 0, 0, 0.0728, 0.0707, 0.0461].$$

**D.3. Example 2—additional information.** The equilibrium investment levels are reported in the matrix  $X$  below, where an entry  $x_{ij}$  represents the investment made by a producer of product  $i$  towards sourcing input  $j$ .

By pinning down the first column of  $X$  with arbitrary values and solving for the other entries, we get

$$X = \begin{bmatrix} 0.7965 & 0.7792 & 0.8735 & 0.8663 & 0 & 0 & 0 \\ 0.8065 & 0 & 0.8859 & 0.8785 & 0 & 0 & 0 \\ 0.8165 & 0.8029 & 0 & 0.8681 & 0 & 0 & 0 \\ 0.8265 & 0.8124 & 0.8855 & 0 & 0 & 0 & 0 \\ 0.8965 & 0 & 0 & 0 & 0 & 0.8947 & 0.8894 \\ 0.9065 & 0 & 0 & 0 & 0.9103 & 0 & 0.8992 \\ 0.9165 & 0 & 0 & 0 & 0.9204 & 0.9146 & 0 \end{bmatrix}.$$

Such an  $X$  corresponds to the following product reliabilities:

$$r = [0.8837, 0.9132, 0.7653, 0.7756, 0.8778, 0.8865, 0.8951].$$

Also, for such values of  $x_{ij}$ , the production of products  $a, b, c, d$  is critical, while the production of products  $e, f, g$  are now non critical.

We can obtain  $G = [3.7758, 3.9399, 1.9995, 2.0736, 2.6608, 2.7929, 2.9372]$ .

Recall that we let  $G_i(r_i \bar{f}_i) = \alpha_i(1 - r_i \bar{f}_i)$ . Setting

$$\alpha = [4, 5, 6, 7, 10, 15, 20],$$

we can find the values

$$\bar{f} = [0.0634, 0.2322, 0.8712, 0.9074, 0.8361, 0.9180, 0.9531].$$

Given those entry fractions  $\bar{f}_i$ , we then choose values of  $\beta_i$  that set the profit of the marginal firm to a strictly positive number for products  $a, b, c, d$ , and values that set the profits of the marginal entering firm to 0 for products  $e, f, g$ . Such values are

$$\beta = [10, 4, 0.2, 0.2, 1.3613, 1.3580, 1.4345].$$

Gross profits *before* the entry costs are

$$\tilde{\Pi} = [1.9590, 2.4942, 0.4978, 0.5446, 1.1381, 1.2466, 1.3673],$$

and the net profits of the marginal entering producer of each product (after entry costs) are

$$\Pi = [1.3246, 1.5655, 0.3236, 0.3632, 0, 0, 0].$$

## APPENDIX E. OMITTED PROOFS

**E.1. The shape of the reliability function.** We start with an important lemma about  $\rho$ , which is proved in Section 1.1 of the Supplementary Appendix, and of which Proposition 1 is a corollary.

**Lemma 2.** Suppose the complexity of the economy is  $m \geq 2$  and there are  $n \geq 1$  potential input suppliers of each firm. For  $r \in (0, 1]$  define

$$\chi(r) := \frac{1 - \left(1 - r^{\frac{1}{m}}\right)^{\frac{1}{n}}}{r}. \quad (12)$$

Then there are values  $x_{\text{crit}}, r_{\text{crit}} \in (0, 1]$  such that:

- (i)  $\rho(x) = 0$  for all  $x < x_{\text{crit}}$ ;
- (ii)  $\rho$  has a (unique) point of discontinuity at  $x_{\text{crit}}$ ;
- (iii)  $\rho$  is strictly increasing for  $x \geq x_{\text{crit}}$ ;

- (iv) the inverse of  $\rho$  on the domain  $x \in [x_{\text{crit}}, 1]$ , is given by  $\chi$  on the domain  $[r_{\text{crit}}, 1]$ , where  $r_{\text{crit}} = \rho(x_{\text{crit}})$ ;
- (v)  $\chi$  is positive and quasiconvex on the domain  $(0, 1]$ ;
- (vi)  $\chi'(r_{\text{crit}}) = 0$ .

**E.2. Proof of Proposition 1.** Proposition 1 is a direct corollary of Lemma 2 above.

**E.3. Proof of Lemma 1.** We establish some notation. Recall that  $P(x_{if}, x) = (1 - (1 - x_{if}\rho(x))^n)^m$ . For the extended domain  $x_{if} \in [0, 1/\rho(x)]$ , we define

$$Q(x_{ik}; x) := P'(x_{ik}; x) = \frac{d}{dx_{if}} P(x_{if}; x). \quad (13)$$

We will need two steps to prove Lemma 1. The first step consists of establishing Lemma 3 on the basic shape of  $Q(x_{if}; x)$ .

**Lemma 3.** Fix any  $m \geq 2$ ,  $n \geq 2$ , and  $x \geq x_{\text{crit}}$ . There are uniquely determined real numbers  $x_1, x_2$  (depending on  $m, n$ , and  $x$ ) such  $0 \leq x_1 < x_2 < 1/\rho(x)$  so that:

0.  $Q(0; x) = Q(1/\rho(x); x) = 0$  and  $Q(x_{if}; x) > 0$  for all  $x_{if} \in (0, 1/\rho(x))$ ;
1.  $Q(x_{if}; x)$  is increasing and convex in  $x_{if}$  on an interval  $[0, x_1]$ ;
2.  $Q(x_{if}; x)$  is increasing and concave in  $x_{if}$  on an interval  $(x_1, x_2]$ ;
3.  $Q(x_{if}; x)$  is decreasing in  $x_{if}$  on an interval  $(x_2, 1/\rho(x)]$ .
4.  $x_1 < x_{\text{crit}}$ .

The proof of Lemma 3 is in Section 1.2 of the Supplementary Appendix. Figure 15 illustrates the shape of  $Q(x_{if}, x)$  as implied by Lemma 3.

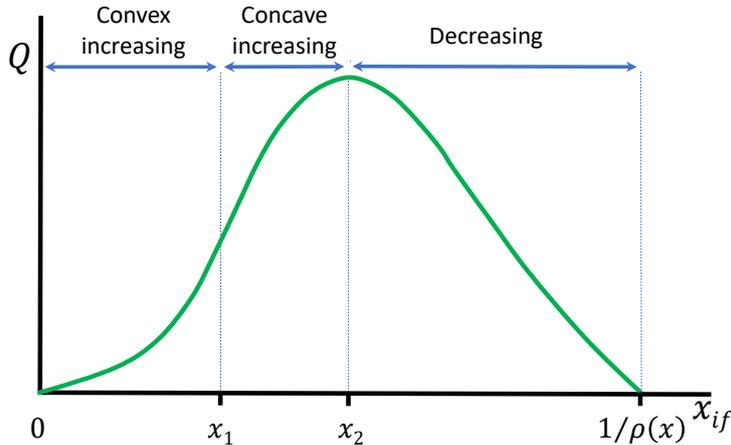


FIGURE 15. The shape of  $Q(x_{if}, x)$ .

We now complete the proof of Lemma 1 by setting  $\hat{x} = x_1$ . By Lemma 3 property 4, the interval  $(\hat{x}, x_{\text{crit}})$  is non-empty. Thus we just need show to that Assumption 3 is satisfied when  $\underline{x} \in (\hat{x}, x_{\text{crit}})$ . Note that by Assumption 2  $c'(0)$  is 0 and increasing and weakly convex otherwise. Since, by Lemma 3,  $P'(x_{if}; x)$  is first concave and increasing (possibly for the empty interval) and then decreasing (possibly for the empty interval) over the range  $x_{if} \in [\underline{x}, 1]$ , it follows that there is at most a single crossing point between the curves  $P'(x_{if}; x)$  and  $c'(x_{if} - \underline{x})$ . This crossing point corresponds to the first-order condition  $P'(x_{if}; x) - c'(x_{if} - \underline{x})$ , yielding the unique maximizer of  $\Pi(x_{if}; x)$ , as illustrated

in Fig. 16. If such a crossing does not exist,  $y_{if} = 0$  is a local and global maximizer of the profit function.

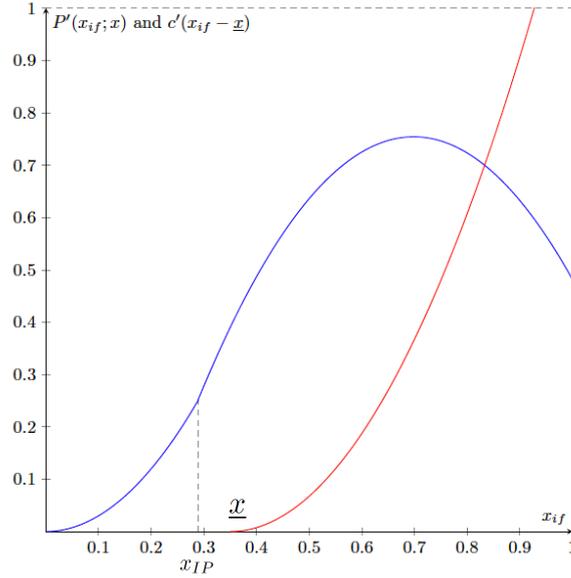


FIGURE 16.  $P'(x_{if}; x)$  is in blue and  $c'(x_{if} - x)$  is in red. There can be only one crossing point between the two curves, which corresponds to the maximizer of  $\Pi(x_{if}; x)$ .

**E.4. Proof of Proposition 2.** By Lemma 2,  $x_{\text{crit}} > 0$  and for all  $x < x_{\text{crit}}$ ,  $\rho(x) = 0$ . Thus no value of  $x \in (0, x_{\text{crit}})$  can be a solution to the social planner's problem. Further, for all values of  $\kappa$  sufficiently small, and for all values of  $x \in [0, 1]$ ,

$$c(x_{\text{crit}}) > \kappa \geq \kappa \rho(x).$$

Thus for sufficiently small values of  $\kappa$  the unique solution to the social planner's problem is to choose  $x = 0$  and so  $x^{SP}(\kappa) = 0$ .

Next note that for all values of  $\kappa$  sufficiently large,

$$c(x_{\text{crit}}) < \kappa \rho_{\text{crit}},$$

and the social planner strictly prefers choosing  $x = x_{\text{crit}}$  to  $x = 0$ . Thus, when  $\kappa$  is sufficiently high all values of  $x^{SP}(\kappa)$  will be weakly greater than  $x_{\text{crit}}$ .

Define  $\kappa_{\text{crit}} := \sup_{\kappa: 0 \in x^{SP}(\kappa)} \kappa$ . As the social planner's unique solution is  $x = 0$  for values of  $\kappa$  sufficiently low, and  $x = 0$  is not a solution for values of  $\kappa$  sufficiently high,  $\kappa_{\text{crit}}$  is a finite, strictly positive number. This establishes part (i).

We now show that  $1 \notin x^{SP}(\kappa)$  for all value of  $\kappa$ . As  $\kappa$  is bounded,  $\lim_{x \rightarrow 1} \frac{1}{\kappa} c'(x) = \infty$ . Moreover, as  $n \geq 2$ ,  $x_{\text{crit}} < 1$ , so  $\rho'(x)$  is bounded. Hence, at  $x = 1$  the social planner can always do a little better by reducing investment.

Thus for values of  $\kappa > \kappa_{\text{crit}}$  all solutions to the social planner's problem must be interior and so the following first order condition must hold

$$\kappa \rho'(x) = c'(x).$$

By Lemma 2, part (vi),  $\lim_{x \downarrow x_{\text{crit}}} \rho'(x) = \infty$ . Thus, as by assumption  $c'(x)$  is bounded for interior value of  $x$ , there cannot be an interior equilibrium at  $x = x_{\text{crit}}$ . This implies that all solutions to the social planner's problem for  $\kappa > \kappa_{\text{crit}}$  are at values of  $x$  strictly greater than  $x_{\text{crit}}$ , establishing part (ii) and part (iii).

### E.5. Proof of Proposition 3.

*Proof.* In the following argument, we defer technical steps to lemmas, which are proved in the Supplementary Appendix.

We begin by sketching the idea of the argument. Recall

$$r = (1 - \underbrace{(1 - xr)^n}_{\text{Probability a given input cannot be acquired}})^m, \quad (\text{PC})$$

Consider any positive symmetric equilibrium<sup>40</sup>  $x_\star$  with reliability  $r_\star = \rho(x_\star)$ . By Lemma 2(i), if  $x_\star < x_{\text{crit}}$ , then  $r = 0$  and marginal benefits from investing are 0, an impossibility by the optimal investment condition in Definition 1. Therefore,  $x_\star \geq x_{\text{crit}}$  and so by Lemma 2(iv) we have  $x_\star = \chi(r_\star)$ ; finally, by Lemma 2(iii)

$$r_\star \geq r_{\text{crit}} := \rho(x_{\text{crit}}). \quad (14)$$

Now, given Assumption 3, the optimal investment condition OI says that we have

$$MB(x_\star; r_\star) = MC(x_\star),$$

where

$$MB(x_{if}; r) = \kappa g(r\bar{f})rn(1 - xr)^{n-1}m(1 - (1 - xr)^n)^{m-1} \quad (15)$$

and  $MC(x) = c'(x)$ . Since we recently deduced  $x_\star = \chi(r_\star)$ , we can substitute out  $x_\star$  to find that the following equation holds:

$$MB(\chi(r_\star); r_\star) = MC(\chi(r_\star)).$$

Define two auxiliary functions of  $r$ :

$$\begin{aligned} \widetilde{MB}(r) &= MB(\chi(r); r) \\ \widetilde{MC}(r) &= MC(\chi(r)). \end{aligned}$$

We now know that the equation

$$\widetilde{MB}(r) = \widetilde{MC}(r) \quad (16)$$

is satisfied for any positive equilibrium reliability  $r = r_\star$  and moreover that (recall inequality (14))  $r_\star \geq r_{\text{crit}}$ . Thus, to prove the proposition it suffices to show that there is at most one solution of equation (16) on the domain  $r \in [r_{\text{crit}}, 1]$ .

We do this by showing that on this domain,  $\widetilde{MB}$  is strictly decreasing, while  $\widetilde{MC}$  is strictly increasing, so they intersect at most once.<sup>41</sup> The claim for  $\widetilde{MC}$  is straightforward:  $\chi$  is increasing on the relevant domain and  $c'$  is increasing by assumption.

Turning now to  $\widetilde{MB}$ : By plugging in  $x = \chi(r)$  into (15) and differentiating in  $r$ , we have

$$\widetilde{MB}(r) = \kappa g(r\bar{f})mnr^{2-\frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1-\frac{1}{n}}. \quad (17)$$

The following Lemma completes the proof.

**Lemma 4.**  $\widetilde{MB}$  is strictly decreasing on the domain  $[r_{\text{crit}}, 1]$ .

<sup>40</sup>Note that we are using a notation distinct from the  $x^*$  notation we later introduce for the *function*  $x^* : \bar{f} \mapsto x_\star$  that gives the positive investment equilibrium (if there is one) for a given level of firm entry.

<sup>41</sup>We note that the functions on both sides of the equation are merely constructs for the proof. In particular, when we sign their derivatives, these derivatives do not have an obvious economic meaning.

To prove lemma 4 we write  $\widetilde{MB}(r)$  as a product of two pieces,  $\alpha(r) := \kappa g(r\bar{f})$  and

$$\beta(r) := mn r^{2 - \frac{1}{m}} \left(1 - r^{\frac{1}{m}}\right)^{1 - \frac{1}{n}}.$$

Note that the function  $\beta(r)$  is positive for  $r \in (0, 1)$ . We will show that it is also strictly decreasing on  $[r_{\text{crit}}, 1)$ . By assumption,  $g(r\bar{f})$  is positive and strictly decreasing in its argument, so  $\alpha(r)$  is also positive and decreasing in  $r$ . Thus, because  $\widetilde{MB}$  is the product of two positive, strictly decreasing functions on  $[r_{\text{crit}}, 1)$ , it is also strictly decreasing on  $[r_{\text{crit}}, 1)$ .

It remains only to establish that  $\beta(r)$  is strictly decreasing on the relevant domain. Two additional lemmas are helpful.

**Lemma 5.** The function  $\beta(r)$  is quasiconcave and has a maximum at  $\hat{r} := \left(\frac{(2m-1)n}{2mn-1}\right)^m$ .

**Lemma 6.** For all  $n \geq 2$  and  $m \geq 3$ ,  $\hat{r} < r_{\text{crit}}$ .

Lemmas 5 and 6 are proved in Sections 1.3 and 1.4 of the Supplementary Appendix. Together show that  $\beta(r)$  is strictly increasing and then strictly decreasing in  $r$  for  $r \in (0, 1)$ , with a turning point in the interval  $(0, r_{\text{crit}})$ . Thus  $\beta(r)$  is strictly decreasing on the domain  $[r_{\text{crit}}, 1)$ , the final piece required to prove Lemma 4.  $\square$

**E.6. A lemma on how investment equilibrium depends on entry.** Consider the class of outcomes where the cutoff for entry is  $\bar{f}$  for each product. Here we study the shape of the investment equilibrium function,  $x^*(\bar{f})$ .

**Lemma 7.** The function  $x^*(\bar{f})$  is decreasing in  $\bar{f}$  on any interval of values of  $\bar{f}$  where  $x^*(\bar{f})$  is positive. Moreover,  $G(r\bar{f})$  is also decreasing in  $\bar{f}$  on an interval of values of  $\bar{f}$  where  $r$  is positive.

The proof of Lemma 7 is in Section 1.5 of the Supplementary Appendix.

### E.7. Proof of Proposition 4.

*Proof.* We begin with an overview of the proof.

*Overview:*

The first step is to construct a function  $H(\bar{f}) : [0, 1] \rightarrow [0, 1]$ , defined as follows. For any (symmetric) cutoff  $\bar{f}$  for entry that may be in effect, supposing that all firms choose relationship strength  $x^*(\bar{f})$  conditional on entering, what mass of firms find it profitable to enter and invest optimally? This is the value  $H(\bar{f})$ . Step 1 establishes some basic facts about the shape of  $H$  that are useful in the sequel. It also notes that in any symmetric equilibrium (and therefore, any SPFE), all firms *if* with locations  $f$  less than a certain cutoff enter, justifying our focus on cutoff entry rules. A value  $\bar{f}$  is defined to be an *entry fixed point* if  $H(\bar{f}) = \bar{f}$ .

Then we consider two possibilities. The first is that there is an entry fixed point (so that free entry determines the number of entering producers of each product). In this case, using what we have established about  $H$ , we show that fixing the parameters of the model ( $\kappa$  being the key one) there is a *unique* entry fixed point  $\bar{f}^*$ , and that  $x^*(\bar{f}^*) \geq x_{\text{crit}}$ . From these facts we deduce there is a unique SPFE; this unique SPFE has the entry cutoff  $\bar{f}^*$  and relationship strength  $x^*(\bar{f}^*)$ . The free-entry equilibrium is critical if and only if  $x^*(\bar{f}^*) = x_{\text{crit}}$ , which we show can happen for exactly one value of  $\kappa$ .

The second possibility is that there is no entry fixed point. Then we show any SPFE must have an entry cutoff  $\bar{f}^*$  such that  $H(\bar{f}^*) > \bar{f}^*$  and  $x^*(\bar{f}^*) = x_{\text{crit}}$ . We again show uniqueness of SPFE and note that it is critical. These arguments use the facts about the shape of  $H$  established in Step 1.

This concludes the overview. We now proceed with the proof.

For notational simplicity, we let  $\underline{x} = 0$  throughout this proof, so that investment and relationship strength are identified. The proof is essentially identical for  $\underline{x} > 0$ .

**Step 1.** To define  $H$ , we suppose that a mass of firms  $\bar{f}$  is entering the market and consider which entry levels are consistent with equilibrium. Since equilibrium entails investments  $x = x^*(\bar{f})$ , firm  $if$  will want to enter the market if  $\Pi_{if}(x; x, \bar{f}) \geq 0$ , i.e.,

$$G(\bar{f}\rho(x^*(\bar{f})))\rho(x^*(\bar{f})) - c(x^*(\bar{f})) - \Phi(f) \geq 0. \quad (18)$$

Rearranging equation (18), a firm  $if$  located at  $f$  will find it profitable to enter if and only if

$$f \leq \Phi^{-1}(G(\bar{f}\rho(x^*(\bar{f})))\rho(x^*(\bar{f})) - c(x^*(\bar{f}))).$$

As the firm locations are uniform on the unit interval,

$$H(\bar{f}) := \Phi^{-1}(\max\{G(\bar{f}\rho(x^*(\bar{f})))\rho(x^*(\bar{f})) - c(x^*(\bar{f})), 0\}). \quad (19)$$

Lemma 8 catalogs some useful properties of  $H(\bar{f})$ .

**Lemma 8.**  $H$  has the following properties:

- $H(0) > 0$ ;
- $H(1) < 1$ ;
- $H(\bar{f})$  is strictly decreasing for all  $\bar{f}$  such that  $x^*(\bar{f}) > 0$ , and  $H(\bar{f}) = 0$  for all  $\bar{f}$  such that  $x^*(\bar{f}) = 0$ .

The proof of Lemma 8 is in Section 1.6 of the Supplementary Appendix.

An entry fixed point is defined by

$$\bar{f} = H(\bar{f}). \quad (20)$$

It will be helpful to define an entry level  $f_{\text{crit}}$  above which production fails for sure, if one exists. If  $x^*(1) = 0$ , then

$$f_{\text{crit}} = \sup\{f : x^*(f) \geq x_{\text{crit}}\}.$$

On the other hand, if  $x^*(1) > 0$  then there is no feasible entry level that results in investments  $x_{\text{crit}}$  and so no feasible entry level at which investment is critical. So that  $f_{\text{crit}}$  is always well defined, in this case we let  $f_{\text{crit}} = 2$ , an infeasibly high entry level (and any infeasibly high entry level would do).

There are two cases to consider.

- Case 1 (noncritical equilibrium): There exists a solution  $\bar{f}^*$  to equation (20) such that  $\bar{f}^* < f_{\text{crit}}$ .
- Case 2 (critical equilibrium): There does not exist a solution  $\bar{f}^*$  to equation (20) such that  $\bar{f}^* < f_{\text{crit}}$ .

**Step 2:** In this step of the proof we consider Case 1. In this case, there exists a  $\bar{f}^*$  such that

$$\Phi(\bar{f}^*) = G(\bar{f}^*\rho(x^*(\bar{f}^*)))\rho(x^*(\bar{f}^*)) - c(x^*(\bar{f}^*)).$$

Hence inequality (18) binds and the equilibrium expected profits is zero for any firm  $i\bar{f}^*$  located at  $\bar{f}^*$  (for any product  $i$ ). Moreover, since  $\Phi$  is increasing in its argument, it follows that any firm  $if$  located at  $f < \bar{f}^*$  will be able to obtain strictly positive

equilibrium expected profits by entering, while any firm located at  $f > \bar{f}^*$  will find it unprofitable to enter.

By the properties of  $H(\bar{f})$  documented in Lemma 8, there can exist at most one solution  $\bar{f}^*$  to equation (20). (See Fig. 17 for an illustration.) We have shown that at this solution the equilibrium profits of firm  $i$   $\bar{f}^*$  are zero. Moreover, since  $\bar{f}^* < f_{\text{crit}}$ , it follows from Lemma 7 that  $x^*(\bar{f}^*) > x^*(f_{\text{crit}}) = x_{\text{crit}}$ .

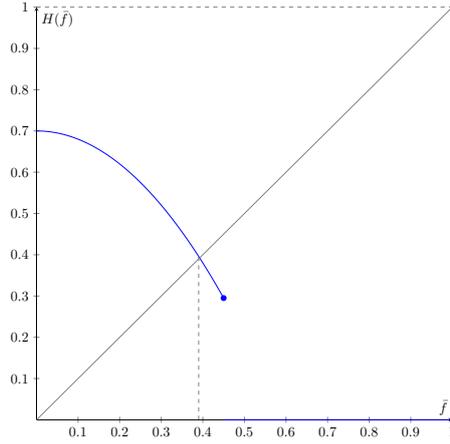


FIGURE 17. Case 1 (noncritical) equilibrium

Finally, note that there is no positive mass of firms that profitably change their entry decisions. Those firms entering make positive profits, while if a positive mass of firms not entering were to enter, investment would be lower (as  $x^*(\bar{f})$  is strictly decreasing in  $\bar{f}$ ) and the mass of firms successfully producing would drop to 0—and hence those firms entering would make losses.

**Step 3:** We now consider Case 2, that of the critical equilibrium. This case can be broken down into two subcases. The first subcase is trivial: There could exist a solution  $\bar{f}^*$  to equation (20), at which  $x^*(\bar{f}^*) = x_{\text{crit}}$ , such that the equilibrium is fragile (as illustrated in Figure 18).

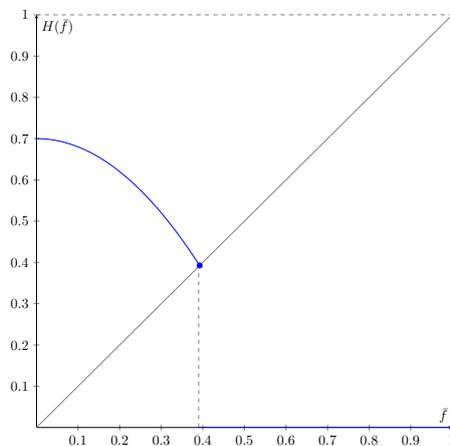


FIGURE 18. Case 2 (critical) equilibrium (first subcase)

The second subcase is that there may not exist a solution to equation (20). Then by Lemma 8 the only way for  $H(\bar{f})$  to not intersect the 45-degree line is for there to be a discontinuity before it crosses it. However, as  $G(\cdot)$  is a continuous function and  $c(\cdot)$  is a continuous function the only discontinuities in  $H(\bar{f})$  must be where  $\rho(\cdot)$  is discontinuous, and by Lemma 2 there is only one discontinuity in  $\rho(x)$ , and it is at  $x_{\text{crit}}$ . Thus, given that investment is optimal, the only possible discontinuity in  $H(\bar{f})$  is at  $f_{\text{crit}}$ . We must therefore have that  $f_{\text{crit}} \in (0, 1)$  and  $H(f_{\text{crit}}) > f_{\text{crit}}$  (as illustrated in Figure 19).

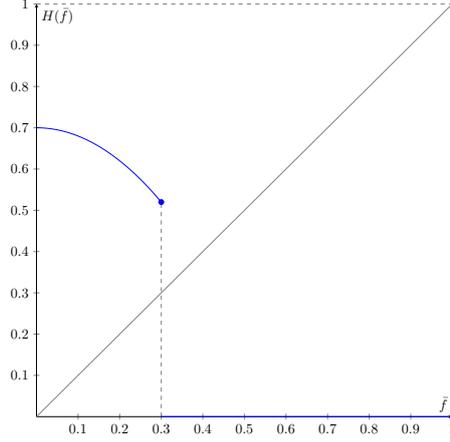


FIGURE 19. Case 2 (critical) equilibrium (second subcase)

We show now that in these cases there is an equilibrium with  $\bar{f}^* = f_{\text{crit}}$  such that  $x^*(\bar{f}^*) = x_{\text{crit}}$ . As

$$H(f_{\text{crit}}) = \Phi^{-1}(G(f_{\text{crit}}\rho(x^*(f_{\text{crit}})))\rho(x^*(f_{\text{crit}}))) - c(x^*(f_{\text{crit}})) > f_{\text{crit}},$$

it follows that

$$G(f_{\text{crit}}\rho(x^*(f_{\text{crit}})))\rho(x^*(f_{\text{crit}})) - c(x^*(f_{\text{crit}})) > \Phi(f_{\text{crit}})$$

and any firm  $f \leq f_{\text{crit}}$  achieves positive expected profits from entering of

$$G(f_{\text{crit}}\rho(x^*(f_{\text{crit}})))\rho(x^*(f_{\text{crit}})) - c(x^*(f_{\text{crit}})) - \Phi(f) > 0.$$

To show that the pair  $(x_{\text{crit}}, f_{\text{crit}})$  is an equilibrium in this case, note that were a positive mass of non-entering firms to deviate and enter there would no longer be a positive production equilibrium and thus the deviation would be unprofitable. Therefore, in equilibrium,  $\bar{f}^* = f_{\text{crit}}$  and  $x^*(\bar{f}^*) = x^*(f_{\text{crit}}) = x_{\text{crit}}$ . □

### E.8. Proof of Proposition 5.

*Proof.* For notational simplicity, we let  $\underline{x} = 0$  here. The proof is essentially identical for  $\underline{x} > 0$ .

We will prove the different parts of the proposition statement in reverse order. By Proposition 4 there is at most one equilibrium with positive entry. We will first show that when  $\kappa$  is sufficiently high (i.e.,  $\kappa > \bar{\kappa}$ ) there is an equilibrium with positive entry and equilibrium investment is increasing in  $\kappa$ , while  $\rho(x^*) > \rho_{\text{crit}}$ . We'll then consider lower values of  $\kappa$  (i.e.,  $\kappa \in (\underline{\kappa}, \bar{\kappa}]$ ) and show that for this range of  $\kappa$  there continues to be an equilibrium with positive entry and production, but with  $\rho(x^*) = \rho_{\text{crit}}$ . Finally, we'll show that when  $\kappa \leq \underline{\kappa}$  there is no equilibrium with positive entry (or production).

**Part (iii):** We begin by considering the cases in which  $\kappa > \bar{\kappa}$ . First we will find the threshold  $\bar{\kappa}$ , and construct the unique positive entry equilibrium for this value of  $\kappa$ . We will then show what happens in equilibrium as  $\kappa$  increases.

Fixing the mass of entering firms at  $\bar{f}$ , Assumption 3 along with the Inada condition implies that equation (OI) identifies a necessary and sufficient condition for a positive investment equilibrium. Away from the precipice a no-profit condition must also be satisfied. Thus, firms are best-responding to each other in their entry ( $\bar{f}$ ) and investment decisions ( $x$ ) when the following two equations are satisfied:

$$MB(x; \bar{f}, \rho(x), \kappa) = MC(x) \quad (21)$$

$$\kappa g(\bar{f}\rho(x))\rho(x) - c(x) - \Phi(\bar{f}) = 0 \quad (22)$$

The first equation equates firms' marginal benefits and marginal costs of investment given entry  $\bar{f}$ . The second equation requires that the marginal entering firm makes 0 profits when entry is  $\bar{f}$  and investments are  $x$ .

We look for a entry level ( $\bar{f}$ ) and  $\kappa$  pair such that when  $\bar{f}$  firms enter, all entering firms choose investments  $x_{\text{crit}}$  and the marginal entering firm receives zero profits. We denote this pair by  $f_{\text{crit}}$  and  $\bar{\kappa}$ , and they satisfy the following two equations

$$MB(x_{\text{crit}}; f_{\text{crit}}, \rho_{\text{crit}}, \bar{\kappa}) = MC(x_{\text{crit}}),$$

$$\bar{\kappa} g(f_{\text{crit}}\rho_{\text{crit}})\rho_{\text{crit}} - c(x_{\text{crit}}) - \Phi(f_{\text{crit}}) = 0,$$

To see that there is a unique pair  $(\bar{\kappa}, f_{\text{crit}})$  solving these equations, and these solutions are valid (i.e.  $\bar{\kappa} > 0$  and  $f_{\text{crit}} \in [0, 1]$ ), note that the first equation implies that

$$\underbrace{\bar{\kappa} g(f_{\text{crit}}\rho_{\text{crit}})mn\rho_{\text{crit}}(1 - x_{\text{crit}}\rho_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}\rho_{\text{crit}})^n)^{m-1}}_{MB(x_{\text{crit}}; f_{\text{crit}}, \rho_{\text{crit}}, \bar{\kappa})} = \underbrace{c'(x_{\text{crit}})}_{MC(x_{\text{crit}})}, \quad (23)$$

and so

$$\bar{\kappa} = \frac{c'(x_{\text{crit}})}{g(f_{\text{crit}}\rho_{\text{crit}})mn\rho_{\text{crit}}(1 - x_{\text{crit}}\rho_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}\rho_{\text{crit}})^n)^{m-1}}.$$

As long as  $f_{\text{crit}} > 0$  (as we show below),  $g(f_{\text{crit}}\rho_{\text{crit}}) > 0$ . Thus, as we also have  $c'(x_{\text{crit}}) > 0$ , we must have  $\bar{\kappa} > 0$ .

The second equation implies that

$$\begin{aligned} f_{\text{crit}} &= \Phi^{-1} \left( \underbrace{\bar{\kappa} g(f_{\text{crit}}\rho(x_{\text{crit}}))\rho_{\text{crit}} - c(x_{\text{crit}})}_{\text{gross profits}} \right) \\ &= \Phi^{-1} \left( \frac{c'(x_{\text{crit}})\rho_{\text{crit}}}{mn\rho_{\text{crit}}(1 - x_{\text{crit}}\rho_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}\rho_{\text{crit}})^n)^{m-1}} - c(x_{\text{crit}}) \right). \end{aligned} \quad (24)$$

Note that this expression for  $f_{\text{crit}}$  is *not* a fixed point condition— $f_{\text{crit}}$  does not appear on the right hand side.<sup>42</sup> As  $\Phi$  is a strictly increasing function, this inverse is well defined and by Assumption 4,  $f_{\text{crit}} < 1$ . Moreover, by Assumption 3, when  $\kappa = \bar{\kappa}$  and  $\bar{f} = f_{\text{crit}}$  if other entering firms choose  $x = x_{\text{crit}}$ , an entering firm *if* strictly prefers also choosing investment level  $x_{if} = x_{\text{crit}}$  to choosing  $x_{if} = 0$ . Thus, by choosing  $x = x_{\text{crit}}$  firm *if* must be making strictly positive *gross* profits (as choosing  $x_{if} = 0$  guarantees 0 gross profits). As  $\Phi(0) = 0$ , this implies that  $f_{\text{crit}} > 0$ .

By construction,  $(x_{\text{crit}}, f_{\text{crit}})$  is a solution to the system consisting of equations (21) and (22) when  $\kappa = \bar{\kappa}$ . Thus, these investment and entry levels constitute an equilibrium.

<sup>42</sup>Recall that  $\rho_{\text{crit}}$  and  $x_{\text{crit}}$  are defined independently of  $f_{\text{crit}}$ .

We consider now values of  $\kappa > \bar{\kappa}$ . Consider the system of equations comprised of the optimal investment condition  $MB(x; \bar{f}, \rho(x), \kappa) = MC(x)$  and a zero profit condition:

$$\kappa g(\bar{f}\rho(x))mn\rho(x)^{2-\frac{1}{m}}(1-\rho(x)^{1/m})^{1-\frac{1}{n}} = c'(x) \quad (25)$$

$$\Phi^{-1}\left(\max\{\kappa g(\bar{f}\rho(x))\rho(x) - c(x), 0\}\right) = \bar{f}. \quad (26)$$

By Proposition 4 there is at most one pair  $(x, \bar{f}) \in \mathbb{R}_{++} \times (0, 1]$  that solves this system. We show in Lemma 9 that this solution changes in a systematic way with  $\kappa$ .

**Lemma 9.** If for  $\kappa = \hat{\kappa}$ , there is a solution  $(\hat{x}, \hat{f})$  to the system of equations 25 and 26, then for any  $\tilde{\kappa} > \kappa$  there exists a unique solution  $(\tilde{x}, \tilde{f})$  to equations 25 and 26 and it satisfies  $\tilde{x} > \hat{x}$ .

We prove Lemma 9 below.

As  $(x_{\text{crit}}, f_{\text{crit}})$  solves equations (21) and (22) when  $\kappa = \bar{\kappa}$ , by Lemma 9 there is also a solution to this system for  $\kappa > \bar{\kappa}$  and the solution is such that  $x^* > x_{\text{crit}}$ . As before this constitutes an equilibrium. Note that the marginal firm makes zero profits in this equilibrium by equation (22) and so this equilibrium corresponds to a Case 1 (non-critical regime) equilibrium.

**Part (ii):**

There is a unique  $\underline{\kappa} > 0$  that solves  $MB(x_{\text{crit}}; 0, \rho_{\text{crit}}, \underline{\kappa}) = MC(x_{\text{crit}})$ . This value of  $\underline{\kappa}$  is

$$\underline{\kappa} = \frac{c'(x_{\text{crit}})}{g(0)mn\rho_{\text{crit}}(1-x_{\text{crit}}\rho_{\text{crit}})^{n-1}(1-(1-x_{\text{crit}}\rho_{\text{crit}})^n)^{m-1}}.$$

As  $g(0) > 0$  and  $c'(x_{\text{crit}}) > 0$ , we have  $\underline{\kappa} > 0$ .

Thus, in the limit as the mass of entering firms converges to zero, if  $\kappa = \underline{\kappa}$ , then by Assumption 3 and equation (OI), if all firms other than firm  $if$  choose investments  $x_{\text{crit}}$  (resulting in a probability of successful production equal to  $\rho_{\text{crit}}$ ), firm  $if$  has a unique best response to choose  $x_{if} = x_{\text{crit}}$ . In particular, we must have

$$\underbrace{\underline{\kappa}g(0)mn\rho_{\text{crit}}(1-x_{\text{crit}}\rho_{\text{crit}})^{n-1}(1-(1-x_{\text{crit}}\rho_{\text{crit}})^n)^{m-1}}_{MB(x_{\text{crit}}; 0, \rho_{\text{crit}}, \underline{\kappa})} = \underbrace{c'(x_{\text{crit}})}_{MC(x_{\text{crit}})}.$$

Comparing this equation to equation (23) implies that  $\underline{\kappa}g(0) = \bar{\kappa}g(f_{\text{crit}}\rho_{\text{crit}})$ . Thus, as  $f_{\text{crit}} > 0$ , and  $g$  is a strictly decreasing function,  $\bar{\kappa} > \underline{\kappa}$ .

Consider now the case in which  $\kappa \in (\underline{\kappa}, \bar{\kappa})$ . For this range, the equilibrium is such that investment is  $x_{\text{crit}}$ . Firms will enter up until an entry level  $\bar{f}$ , where  $\bar{f}$  is such that  $MB(x_{\text{crit}}; \bar{f}, \rho_{\text{crit}}, \kappa) = MC(x_{\text{crit}})$  for firms to choose investments  $x_{\text{crit}}$ . Entry up to this point is strictly profitable and any further positive mass of entry will result in investments less than  $x_{\text{crit}}$ . Hence, entry adjusts to keep firms' markups just high enough to sustain an investment of  $x_{\text{crit}}$ .

We can rearrange equation (21) to write an explicit expression for the level of entry required to keep marginal benefits equal to marginal costs as  $\kappa$  varies. This gives us the function  $\tilde{f} : (\underline{\kappa}, \bar{\kappa}) \rightarrow [0, 1]$ , where

$$\tilde{f}(\kappa) := \frac{1}{\rho_{\text{crit}}} g^{-1}\left(\frac{c'(x_{\text{crit}})}{\kappa mn\rho(x_{\text{crit}})^{2-\frac{1}{m}}(1-\rho(x_{\text{crit}})^{1/m})^{1-\frac{1}{n}}}\right). \quad (27)$$

When  $\kappa \in (\underline{\kappa}, \bar{\kappa})$  we will show that there is an equilibrium in which firms located in the interval  $[0, \tilde{f}(\kappa)]$  enter and receive positive profits,  $x^* = x_{\text{crit}}$  and  $\rho(x^*) = \rho_{\text{crit}}$ . Note that such an equilibrium is in the critical regime.

We now verify that this is an equilibrium. By construction in such an outcome firms are best-responding with their investment choices to others' investment choices. To see that all firms are making strictly positive profits we compare the profits of the marginal entering firm  $if$  at  $\tilde{\kappa} \in (\underline{\kappa}, \bar{\kappa})$  to the profits of the marginal entering firm  $if_{\text{crit}}$  when  $\kappa = \bar{\kappa}$  (we established above the latter is 0).

$$\begin{aligned}
\tilde{\Pi}_{i\tilde{f}} &= \tilde{\kappa}g(\tilde{f}\rho(x_{\text{crit}}))\rho(x_{\text{crit}}) - c(x_{\text{crit}}) - \Phi(\tilde{f}) && \text{by definition} \\
&> \tilde{\kappa}g(\tilde{f}\rho(x_{\text{crit}}))\rho(x_{\text{crit}}) - c(x_{\text{crit}}) - \Phi(f_{\text{crit}}) && \text{because } f_{\text{crit}} > \tilde{f}. \\
&= \bar{\kappa}g(f_{\text{crit}}\rho(x_{\text{crit}}))\rho(x_{\text{crit}}) - c(x_{\text{crit}}) - \Phi(f_{\text{crit}}) && x_{\text{crit}} \text{ still profit maximizing} \\
&= \Pi_{if_{\text{crit}}} && \text{by definition} \\
&= 0. && \text{as shown above}
\end{aligned}$$

As entering firms are making strictly positive profits, their entry decisions are optimal. No other positive mass of firms can profitably enter as after their entry positive investment could not be sustained.

**Part (i):**

Finally, we consider  $\kappa \leq \underline{\kappa}$ . Recall that for  $\kappa > \underline{\kappa}$  in the regime just considered, as  $\kappa \rightarrow \underline{\kappa}$  from above the mass of firms entering went to 0 while entering firms continued to make investments  $x_{\text{crit}}$  (i.e.,  $MB(x_{\text{crit}}; 0, \rho_{\text{crit}}, \underline{\kappa}) = MC(x_{\text{crit}})$ ) resulting in a probability of successful production equal to  $\rho_{\text{crit}}$ . By Lemma 7,  $x^*(f)$  is strictly decreasing in  $f$ . Thus, with  $\kappa = \underline{\kappa}$ , if a positive mass of firms entered, then equilibrium investments would be below  $x_{\text{crit}}$  and production would fail for sure, and all firms with positive entry costs would make losses.

Consider now  $\kappa < \underline{\kappa}$ . In comparison to  $\kappa = \underline{\kappa}$ , at any fixed entry level and any investment choice, the marginal benefits a firm  $i$  receives from investing are then strictly lower (while the marginal costs remain the same). This can be seen immediately from equation (OI) as  $g(\cdot)$  is a strictly decreasing function and, by assumption,  $\kappa < \underline{\kappa}$ . Thus any intersection of marginal benefits and marginal costs must occur at an investment level  $x < x_{\text{crit}}$ . This implies that  $\rho(x^*) = 0$ , and so the only possible equilibrium is at  $x^* = 0$ . We conclude that there is no positive investment equilibrium for any  $\kappa < \underline{\kappa}$ .  $\square$

## E.9. Proof of Lemma 9.

*Proof.* Let  $\tilde{\rho} = \rho(\tilde{x})$ ,  $\hat{\rho} = \rho(\hat{x})$ , and  $\tilde{\Pi}(if)$  denote the equilibrium profits of firm  $if$  when  $\kappa = \tilde{\kappa}$  and a mass  $f$  of firms enter. Finally, let  $\hat{\Pi}(if)$  denote the equilibrium profits of firm  $if$  when  $\kappa = \hat{\kappa}$  and a mass  $f$  of firms enter.

We will first assume that  $\tilde{x} < \hat{x}$ , and show this yields a contradiction. We'll then assume that  $\tilde{x} = \hat{x}$  and show that this also yields a contradiction, allowing us to conclude that  $\tilde{x} > \hat{x}$  as claimed.

First, towards a contradiction, suppose that  $\tilde{x} < \hat{x}$ . As  $\tilde{x} < \hat{x}$  we have  $\tilde{\kappa}g(\tilde{f}\tilde{\rho}) < \hat{\kappa}g(\hat{f}\hat{\rho})$  by equation (25). Thus, as  $\tilde{\kappa} > \hat{\kappa}$ , it must be that  $g(\tilde{f}\tilde{\rho}) < g(\hat{f}\hat{\rho})$ , which, as  $g(\cdot)$  is a decreasing function implies that  $\tilde{f}\tilde{\rho} > \hat{f}\hat{\rho}$ . Then, as  $\tilde{x} < \hat{x}$ , by part (iii) of Lemma 2,  $\tilde{\rho} < \hat{\rho}$  and so we must have  $\tilde{f} > \hat{f}$ .

When  $\kappa = \tilde{\kappa}$ , as  $\tilde{x}, \tilde{f}$  solve the system of equations (25) and (26), firm  $\tilde{f}$  is the marginal entering firm and so all firms with lower entry costs also enter. Thus all firms located at points  $f < \tilde{f}$  enter. As shown above,  $\tilde{x} < \hat{x}$  implies that  $\hat{f} < \tilde{f}$ . Consider then the equilibrium profits of a firm located at  $\hat{f}$  when  $\kappa = \tilde{\kappa}$ . This firm enters the market and receives profits less than 0. To see this, consider the following sequence of inequalities which we explain below.

$$\begin{aligned}
\tilde{\Pi}_{i\hat{f}} &= \tilde{\kappa}g(\tilde{f}\tilde{\rho})\tilde{\rho} - c(\tilde{x}) - \Phi(\hat{f}) \\
&< \hat{\kappa}g(\hat{f}\hat{\rho})\tilde{\rho} - c(\tilde{x}) - \Phi(\hat{f}) \\
&< \hat{\kappa}g(\hat{f}\hat{\rho})(1 - (1 - \tilde{x}\hat{\rho})^n)^m - c(\tilde{x}) - \Phi(\hat{f}) \\
&< \hat{\kappa}g(\hat{f}\hat{\rho})(1 - (1 - \hat{x}\hat{\rho})^n)^m - c(\hat{x}) - \Phi(\hat{f}) \\
&= \hat{\Pi}_{i\hat{f}} \\
&= 0.
\end{aligned}$$

The first inequality is implied by  $\hat{\kappa}g(\hat{f}\hat{\rho}) > \tilde{\kappa}g(\tilde{f}\tilde{\rho})$ , as argued above. The second inequality is implied by  $\hat{\rho} > \tilde{\rho}$  (also as argued above). The third inequality is implied by the fact that  $\hat{x}$  solves equation (25), when  $\kappa = \hat{\kappa}$  and every other firm chooses investments  $\hat{x}$ . Firm  $i\hat{f}$  must thus strictly prefer choosing  $\hat{x}$  to  $\tilde{x}$ . As, combining these inequalities,  $\tilde{\Pi}_{i\hat{f}} < 0$  the firm located at  $\hat{f}$  has a profitable deviation to not enter the market, which is a contradiction.

The only remaining case to consider is  $\tilde{x} = \hat{x}$ . Towards a contradiction suppose that  $\tilde{x} = \hat{x}$ . From equation (26) note that,

$$\hat{f} = \Phi^{-1}\left(\max\{\hat{\kappa}g(\hat{f}\hat{\rho})\hat{\rho} - c(\hat{x}), 0\}\right) \quad (28)$$

$$\tilde{f} = \Phi^{-1}\left(\max\{\tilde{\kappa}g(\tilde{f}\tilde{\rho})\tilde{\rho} - c(\tilde{x}), 0\}\right). \quad (29)$$

However, as  $\tilde{x} = \hat{x}$  we must have (as essentially argued above) that  $\tilde{\kappa}g(\tilde{f}\tilde{\rho}) = \hat{\kappa}g(\hat{f}\hat{\rho})$ . Also, by the definition of  $\rho(x)$ , it must be that  $\hat{\rho} = \tilde{\rho}$ . Substituting these conditions into equations (28) and (29), and as  $\Phi(\cdot)$  is a strictly increasing function,  $\hat{f} = \tilde{f}$ . But then, as  $\tilde{\kappa}g(\tilde{f}\tilde{\rho}) = \hat{\kappa}g(\hat{f}\hat{\rho})$ , we must have  $\hat{\kappa} = \tilde{\kappa}$  which is a contradiction.  $\square$

**E.10. Proof of Proposition 7.** For any  $\mathbf{r} \in [0, 1]^{|\mathcal{I}|}$ , define  $\mathcal{R}^\xi(\mathbf{r})$  to be the probability, under the parameter  $\xi$ , that a producer of product  $i$  is functional given that the reliability vector for all products is given by  $\mathbf{r}$ . This can be written explicitly:

$$[\mathcal{R}^\xi(\mathbf{r})]_i = \prod_{j \in \mathcal{I}_i} [1 - (1 - r_j x_{ij}(\xi))^{n_{ij}}].$$

Let  $\boldsymbol{\rho}(\xi)$  be the elementwise largest fixed point of  $\mathcal{R}^\xi$ , which exists and corresponds to the mass of functional firms by the same argument as in  $[[\mathbf{x}]]$ .

(1) It is clear that  $\boldsymbol{\rho}(1) = \mathbf{1}$ .

(2) Next, there is an  $\epsilon > 0$  such that if  $\|\mathbf{r}\| < \epsilon$ , then for all  $\xi$ , the function  $\mathcal{R}^\xi(\mathbf{r}) < \mathbf{r}$  elementwise. So there are no fixed points near  $\mathbf{0}$ .

(3) For small enough  $\xi$ , the function  $\mathcal{R}^\xi$  is uniformly small, so  $\boldsymbol{\rho}(\xi) = \mathbf{0}$ .

These facts together imply that  $\boldsymbol{\rho}$  has a discontinuity where it jumps up from  $\mathbf{0}$ . Let  $\xi_{\text{crit}}$  be the infimum of the  $\xi$  where  $\boldsymbol{\rho}(\xi) \neq \mathbf{0}$ .

Define

$$\Gamma(\mathcal{R}^\xi(\mathbf{r})) = \{(\mathbf{r}, \mathcal{R}^\xi(\mathbf{r})) : \mathbf{r} \in [0, 1]^{|Z|}\}$$

to be the graph of the function. What we have just said corresponds to the fact that this graph intersects the diagonal at  $\xi_{\text{crit}}$ , but not for values just below  $\xi_{\text{crit}}$ . What we have just said, along with the Implicit Function Theorem, implies that the derivative map of  $\mathcal{R}^{\xi_{\text{crit}}}(\mathbf{r}) - \mathbf{r}$  must be singular (\*). By smoothness of  $\mathcal{R}^{\xi_{\text{crit}}}$ , this implies that some derivative of  $\rho(\xi)$  for values just above  $\xi_{\text{crit}}$  must diverge—otherwise we could take a limit and contradict (\*).

**E.11. Proof of Proposition 8.** *Part (i):* Let  $\mathcal{P}$  be a directed path of length  $T$  from node  $T$  to node 1 and denote product  $i$  by 1. Since product 1 is critical, then following a shock  $\epsilon > 0$  to  $\gamma_{ij}$ ,  $\rho_1^{*'} = 0$ .

For any product  $t + 1$  that sources input  $t$  and such that  $\rho_t^{*'} = 0$ , we have that

$$\begin{aligned} \rho_{t+1}^{*'} &= \prod_{l \in \mathcal{N}_{t+1}} (1 - (1 - x_{t+1,l}^* \rho_l^{*'})^{n_{t+1,l}}) \\ &= 0 \end{aligned}$$

since  $t \in \mathcal{N}_{t+1}$ .

Since  $\rho_1^{*'} = 0$ , it then follows by induction that the production of all products  $t \in \mathcal{P}$  will fail.

*Part (ii):* Suppose production of some product  $i \in \mathcal{I}_c$  is critical and consider another product  $k \in \mathcal{I}_c$  that is an input for the production of product  $i$  (that is,  $k \in \mathcal{N}_i$ ). As an investment equilibrium  $x^*$  is being played, the investment  $x_{kj}$  of a producer of product  $k$  in the reliability of its relationships with suppliers of input  $j \in \mathcal{N}_k$  must satisfy the following condition

$$MB_{kj} = G(\bar{f}_k \rho_k) \prod_{l \in \mathcal{N}_k, l \neq j} (1 - (1 - x_{kl} \rho_l)^{n_{kl}}) n_{kj} (1 - x_{kj} \rho_j)^{n_{kj}-1} \rho_j = \gamma_{kj} c'(x_{kj}) = MC_{kj}$$

Rearranging this equation yields

$$\frac{G(\bar{f}_k r_k^*)}{\gamma_{kj}} \prod_{l \in \mathcal{N}_k, l \neq j} (1 - (1 - x_{kl}^* \rho_l^*)^{n_{kl}}) n_{kj} r_j^* = \frac{c'(x_{kj}^*)}{(1 - x_{kj}^* r_j^*)^{n_{kj}-1}}$$

The right hand side is strictly increasing in  $x_{kj}^*$ , while the left hand side is constant.

Consider now a shock  $\epsilon > 0$  that changes the value of  $\gamma_{ij}$  to  $\gamma'_{ij} = \gamma_{ij} + \epsilon$ . By Lemma [x], this strictly reduces  $G(\bar{f}_k r_k^*)/\gamma_{ij}$ , and hence the new equilibrium investment level satisfies  $x_{kj}^{*'} < x_{kj}^*$ . This in turn implies that  $\rho_k^{*'} < \rho_k^*$ , and so

$$\rho_i^{*'} = \prod_{l \in \mathcal{N}_i} (1 - (1 - x_{i,l}^* \rho_l^{*'})^{n_{i,l}}) \quad (30)$$

$$< \prod_{l \in \mathcal{N}_i} (1 - (1 - x_{i,l}^* \rho_l^*)^{n_{i,l}}) \quad (31)$$

$$= \rho_i^* \quad (32)$$

Since  $k \in \mathcal{N}_i$ ,  $\rho_i^* = 0$  and the production of product  $i$  fails.

Now since both products  $k$  and  $i$  are part of a strongly connected component, there is also a directed path from  $k$  to  $i$ . From part (i), it follows that every product  $t$  on such a directed path (i.e. every product that uses input  $i$  either directly or indirectly through intermediate products) will also have  $\rho_t^* = 0$ . This is true namely for product  $k$  and thus  $\rho_k^* = 0$ . We therefore conclude that, following a small decrease in its sourcing effort from the initial  $x_k^*$ , production of product  $k$  fails. Product  $k$  was thus necessarily critical and we must have had  $x_k^*$  be critical.

Proceeding similarly for any other product  $e \in \mathcal{N}_k$  such that  $e \in \mathcal{I}_c$ , we get that  $e$  is also critical. By induction it follows that all products in a strongly connected component  $\mathcal{I}_c$  of the product interdependencies graph are critical when one of them is. We conclude that either production of all the products in  $\mathcal{I}_c$  are critical or else production of all the products in  $\mathcal{I}_c$  are noncritical.

## APPENDIX F. ADDING A BANKING SECTOR

Our equilibrium definition requires that a positive mass of firms cannot profitably enter. Arguably, a more reasonable equilibrium definition would be to stipulate that no firm can profitably enter the market. This is problematic. Take a critical equilibrium under the current equilibrium definition. As firms are zero measure, it is possible for an additional firm to enter without changing the mass of firms in the market, and thus without changing investment decisions. Thus, when such firms contemplate entering, they anticipate that their deviation, holding the entry decisions of others fixed, will not affect investments and the economy will not fall off the precipice. Hence, entry for such firms is strictly profitable. So, the critical equilibria we found would not be equilibria under a more stringent equilibrium definition in which no firm can have a profitable deviation from changing its entry choice. Moreover, this problem cannot be resolved by allowing some more firms to enter as there is a positive mass of firms that would want to enter holding fixed the entry decisions of others in any critical equilibrium.

To overcome these technical concerns, in this section we adjust our model by adding a competitive banking sector to it. We now suppose that firms must now pay the fixed entry cost before they produce, and that they do not have cash at this time. Thus they borrow to cover their fixed entry costs. Firms can pledge their profits to repay debt. There is a competitive banking sector consisting of a finite number of banks. Before firms enter, each bank  $b$  simultaneously posts an interest rate  $r_b$  and a debt limit  $\bar{\ell}_b$ : it will not lend to a firm that has more than  $\bar{\ell}_b$  debt in total. Each bank can service any measure of firms at these terms. Banks' cost of capital is normalized to 0.

As before, conditional on entry, firms maximize net profits  $\Pi_{if}$ .<sup>43</sup>

To summarize, the timing is:

0. Banks simultaneously set terms: interest rates and lending limits  $(r_b, \bar{\ell}_b)$ .
1. Firms observe these terms; simultaneously take any loans consistent with the terms; and make their entry decisions.
2. Firms simultaneously choose their effort levels to maximize  $\Pi_{if}$ .
3. The supply tree and payoffs are realized.

<sup>43</sup>This amounts to assuming contracts are good enough to avoid any agency frictions between the firm and bank.

Given a profile of bank terms, a *firm outcome* is now determined by entry decisions  $e_{if}$  and investment decisions  $x_{if}$  for all firms  $if$ , as well as of a specification of how much the firm borrows from each bank (which is purely a firm choice, given bank terms). We do not write it explicitly, but the net interest payments that a firm makes to the banks, if it is nonzero, comes out of its profits.<sup>44</sup>

**Definition 4.** A firm outcome combined with a profile of bank terms is said to be a *full equilibrium* if the firm outcome is an equilibrium and we have two final conditions that are satisfied:

- Feasible entry: a firm  $if$  enters only if it takes a loan of sufficient size to pay the fixed cost of entry  $\Phi(f)$ .
- Optimal bank behavior: no bank can deviate to an interest rate and lending limit  $(r'_b, \bar{\ell}'_b)$  that strictly increases its payoff given that an equilibrium is played conditional on these terms.

We show now that the equilibrium decisions of firms we found before can be embedded in a full equilibrium of this extended model.

Consider a critical equilibrium. Before, in a critical equilibrium a mass  $f_{\text{crit}}$  of firms enter and choose investments  $x_{\text{crit}}$ , resulting in reliability  $\rho_{\text{crit}}$ . We show that there is a full equilibrium in which firms make the same entry and investment choices, while banks set a lending limit of  $\Phi(f_{\text{crit}})$  and charge an interest rate of 0.

Given these lending limits and interest rates, the firm entry and investment problem is unchanged from before so we just need to check that the banks have no incentives to deviate.

First, given all other banks set a lending limit of  $\Phi(f_{\text{crit}})$  and charge an interest rate of 0 it is a best response for a bank  $b$  to also set a lending limit of  $\Phi(f_{\text{crit}})$  and charge an interest rate of 0. A necessary condition for any deviation to be strictly profitable is that a strictly positive interest rate is set. Consider first deviations in which bank  $b$  sets a weakly lower lending limit (tighter lending constraint), and charges an interest rate strictly greater than 0. In this case, no firms would seek credit from  $b$  following the deviation and it would not be profitable.

Consider now the remaining possible profitable deviations—a deviation in which  $b$  sets a strictly higher lending limit and charges a strictly positive interest rate. The only firms willing to pay this higher interest rate must be located at  $f > f_{\text{crit}}$  (as they otherwise have access to lower interest credit). Moreover, if a firm located at  $f > f_{\text{crit}}$  finds it profitable to enter, all firms located at  $f < f_{\text{crit}}$  will also find it profitable to enter. Thus the mass of entering firms must increase above  $f_{\text{crit}}$ . However, in this case, by Lemma 7, equilibrium investment choices after entry would decline below  $x_{\text{crit}}$ . As such the only possible investment equilibrium choice of entering firms is  $x^* = 0$ , and at this investment level, the deviating bank would not be able to recover its investment.

Consider a non-critical equilibrium. Before, in a non-critical equilibrium a mass  $\bar{f}$  of firms enter such that the marginal entering firm makes zero net profits, and firms choose investments  $x > x_{\text{crit}}$ , resulting in reliability  $\rho > \rho_{\text{crit}}$ . We show that there is a full equilibrium in which firms make the same entry and investment choices, while banks set a lending limit of  $\Phi(f_{\text{crit}})$  and charge an interest rate of 0.

Given these lending limit and interest rates, firms entry and investment problem is unchanged from before so, again, we just need to check that the banks have no incentives

<sup>44</sup>As we will see, given the competitive banking sector, interest payments are zero in equilibrium.

to deviate. Note now that, regardless of the lending limit set by a bank, no firm will seek credit from it if it changes an interest rate above 0. As all firms that can profitably enter the market already have access to credit, a bank that deviates by setting an interest rate above 0 will not be able to extend any credit. Thus no bank has a profitable deviation and we have a full equilibrium.

## APPENDIX G. FINITE APPROXIMATION

We have assumed so far that production chains have infinitely many “layers” or stages of production, which is a stylized assumption. We may wonder whether the infinite tree approximates the properties of finite production trees. Our results are largely concerned with the sharp discontinuity in  $\rho(x)$  at  $x_{crit}$  when  $m \geq 2$  and its consequences. In this section we show that when we consider models with a finite number of layers, the key properties of the model extend and, in particular, complex economies are fragile to small shocks in equilibrium.

Typical production chains are not infinite, although they can be quite long. We discuss one example. An Airbus A380 has 4 million parts. The final assembly in Toulouse, France, consists of six large components coming from five different factories across Europe: three fuselage sections, two wings, and the horizontal tailplane. Each of these factories gets parts from about 1500 companies located in 30 countries<sup>45</sup>. Each of those companies itself has multiple suppliers as well as contracts to supply and maintain specialized factory equipment, etc. The stylized facts of this case motivate some of our finite approximation robustness exercises.

We consider two types of robustness exercises. First, we numerically investigate what the reliability function of the economy looks like if we truncate the production tree at a certain depth, as well as allowing for some systematic heterogeneity (for example, more upstream tiers being simpler). Then we consider a theoretical exercise with an analytically convenient form of random truncation.

**G.1. Truncated trees: Numerical calculations.** We consider a  $T$ -tier tree where each firm in tier  $t$  requires  $m_t$  kinds of inputs and has  $n_t$  potential suppliers of each input. We denote by  $\rho_T(x)$  the probability of successful production at the top node of a  $T$ -tier tree with these properties. This is defined as

$$\rho_T(x) = (1 - (1 - \rho_{T-1}(x)x)^{n_T})^{m_T}$$

with  $\rho_1(x) = 1$ , since the bottom-tier nodes do not need to obtain inputs.

We see that the expression is recursive and, if unraveled explicitly, would be unwieldy after a number of tiers. However, we will see in the next subsection that if  $m_t = m$  and  $n_t = n$  for all  $t$ , then for any  $x \in [0, 1]$ , as  $T$  goes to infinity,  $\rho_T(x)$  converges to  $\rho(x)$ , which is defined as the largest fixed point of equation (PC).

We start with some examples where  $m_t$  and  $n_t$  are the same throughout the tree. Figure 20 illustrates the successful production probability  $\rho_T(x)$  for different finite numbers of tier  $T$  and how quickly it converges to the discontinuous curve  $\rho(x)$ .

<sup>45</sup>Source: “FOCUS: The extraordinary A380 supply chain”. *Logistics Middle East*. Retrieved on 28 may, 2019 from <https://www.logisticsmiddleeast.com/article-13803-focus-the-extraordinary-a380-supply-chain>

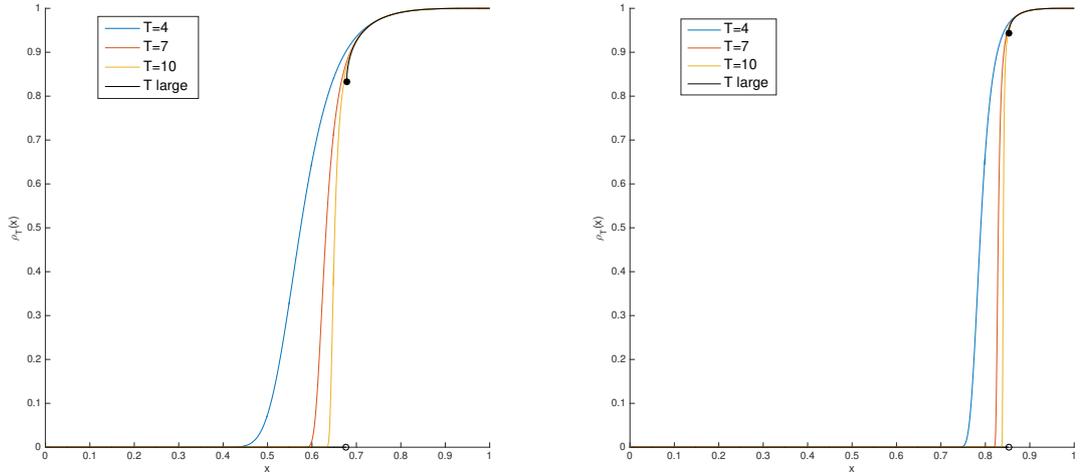


FIGURE 20. Successful production probability  $\rho_T(x)$  for different finite numbers of tiers  $T$ . In panel (a),  $m = 5$  and  $n = 4$ . In panel (b),  $m = 40$  and  $n = 4$ .

In panel (a), we see that  $\rho_T(x)$  exhibits a sharp transition for as few as 4 tiers. The red curve ( $T = 7$ ) shows that when the investment level  $x$  drops from 0.66 to 0.61, or about 7 percent, production probability  $\rho_7(x)$  drops from 0.8 to 0.1. (Thus  $\rho_7$  achieves a slope of at least 14.) In panel (b), we see that increasing product complexity (by increasing  $m$  to 40) causes  $\rho_T$  to lie quite close to  $\rho$ . This illustrates how complementarities between inputs play a key role in driving this sharp transition in the probability of successful production. Note that  $m = 40$  is not an exaggerated number in reality. In the Airbus example described earlier, many components would exhibit such a level of complexity.

However, a complexity number like  $m = 40$  will not occur everywhere throughout the supply network. Indeed, and more generally, one might ask if the regularity in the production tree drives this result. To investigate this possibility, in Figure 21 we plot  $\rho_T(x)$  for a supply tree with irregular complexity, where different tiers may have different values of  $m_t$ . Here we construct 4 trees whose complexity increases with  $T$ . The first tree has  $T = 4$  and  $m_2 = 2$ ,  $m_3 = 6$  and  $m_4 = 10$ . The second tree has  $T = 7$ ,  $m_t = 2$  for  $t = 2, 3$ ,  $m_t = 6$ , for  $t = 4, 5$  and  $m_t = 10$ , for  $t = 6, 7$ . The third tree has  $T = 10$ ,  $m_t = 2$  for  $t = 2, 3, 4$ ,  $m_t = 6$ , for  $t = 5, 6, 7$  and  $m_t = 10$ , for  $t = 8, 9, 10$ . Finally, the fourth tree has large<sup>46</sup> length (here  $T = 1000$ ),  $m_t = 2$  for  $t$  below the first tercile,  $m_t = 6$  for  $t$  between the first and the second terciles and  $m_t = 10$  for  $t$  above the second tercile. We see that trees of moderate length once again exhibit a sharp transition in their probability of successful production. This feature is thus not at all dependent upon the regularity of the trees.

<sup>46</sup>In Figures 20 and 21, all curves are plotted on a grid  $x = [0, 1]$  with increments of 0.001. The black curve represents  $\rho_T(x)$  for  $T = 1000$  and the discontinuity marks the first time  $\rho_T(x)$  is positive.

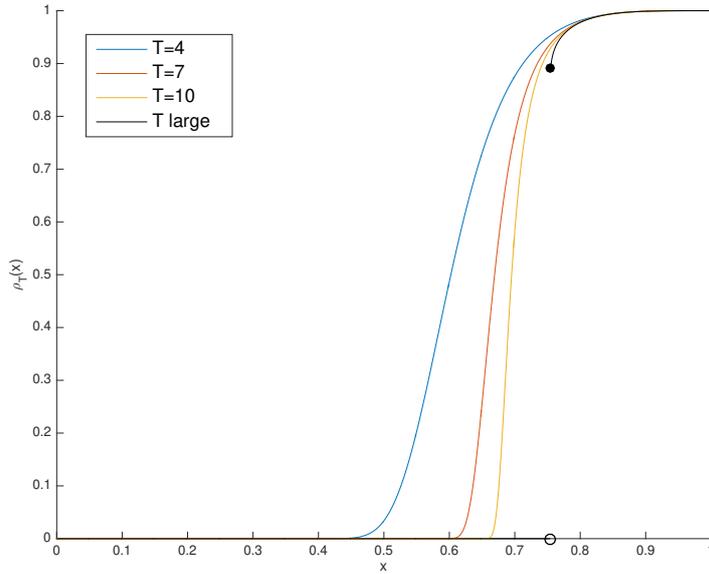


FIGURE 21. Successful production probability  $\rho_T(x)$  for different finite numbers of layers  $T$ , but where different layers may have different complexity  $m$ .

**G.2. Equilibrium in the finite approximation.** The previous section numerically studied a version of our basic production model in a setting where production is “truncated” so that dependency chains don’t continue forever. There, we investigated the physical dependence of reliability on relationship strength. Here we focus on a related model where we can analyze equilibrium investment and entry.

We make production chains in the model finite by assuming that, any stage in the production process, there is a fixed probability  $\tau$  that the firm *if* requires no inputs that are sourced through relationships.<sup>47</sup>

In this extended model, let  $\rho(\tau, x)$  denote the probability of successful production when every firm requiring inputs invests  $x$ . The generalization of equation (1) is

$$r = \tau + (1 - \tau)(1 - (1 - xr)^n)^m.$$

A basic observation is that, analogously to our analysis of the main model, we can define a function  $\chi : [\tau, 1] \rightarrow [0, 1]$  as follows:

$$\chi(\tau, r) = \frac{1}{r} - \frac{1}{r} \left[ 1 - \left( \frac{r - \tau}{1 - \tau} \right)^{1/m} \right]^{1/n}.$$

This function gives the  $x$  that must correspond to a given reliability  $r$ —i.e., the  $x$ , if one exists, such that  $\rho(\tau, x) = r$ . Analyzing this function lets us extend the analysis of our main model. For  $\tau$  near zero, the behavior of  $\rho(\cdot; \tau)$  is very similar to the behavior of  $\rho$  in the non-truncated model.

Consider now the same economic environment as studied in the main text, with the only change being the  $\tau$ -truncated supply network. We take  $\underline{x} = 0$  to save on notation. As before, an equilibrium requires that all entering firms make non-negative profits, no

<sup>47</sup>Practically this could be a raw materials producer that uses only readily sourced commodity technology.

positive mass of firms could make positive profits by entering, and that all entering firms are choosing investments to maximize their profits. The profits of an entering firm  $i_f$  when other firms choose investments  $x$  and all firms located below  $\bar{f}$  enter, are

$$G(\bar{f}\rho(\tau, x))[\tau + (1 - \tau)[1 - (1 - \rho(\tau, x)x_{i_f})^n]^m] - c(x_{i_f}) - \Phi(f).$$

The first order condition, equating marginal benefits and marginal costs of investment, is therefore

$$G(\bar{f}\rho(\tau, x))(1 - \tau)\rho(\tau, x)n(1 - x_{i_f}^*\rho(\tau, x))^{n-1}m(1 - (1 - x_{i_f}^*\rho(\tau, x))^n)^{m-1} = c'(x_{i_f}^*). \quad (33)$$

This is exactly the same condition as before, but with the marginal benefits term multiplied by  $(1 - \tau)$  and with  $\rho(\tau, x)$  replacing  $\rho(x)$ .

For a given entry level  $\bar{\rho}$ , and corresponding symmetric investment equilibrium  $x^*$ , an equilibrium can be either non-critical or critical. At a non-critical one, entry is pinned down by the zero-profit condition.

$$\Phi(\bar{f}) = G(\bar{f}\rho(\tau, x^*))\rho(\tau, x^*) - c(x^*). \quad (34)$$

Thus, at a symmetric non-critical equilibrium, the symmetric investment level and rate of entry satisfy (33) and (34). At a critical equilibrium, entry is such as to make the endogenous investment  $x^*(\bar{f}; \tau)$  equal to  $x_{\text{crit}}(\tau)$ .

Our result is:

**Proposition 9.** Consider the situation of Proposition 5(ii), where in any equilibrium of the  $\tau = 0$  game, we have  $x^* = x_{\text{crit}}$ . In any sequence of positive investment equilibria of the  $\tau$ -truncated model (which exist) investment converges to  $x_{\text{crit}}$  as  $\tau \rightarrow 0$ .

## APPENDIX H. DISTRIBUTION OF PROFITS IN SUPPLY TREES

In this section we relax the assumption that positive profits are only made on final sales to consumers. However, we also make stronger assumptions on the symmetry of technology with respect to different products. For simplicity, we assume that the technology graph is vertex transitive.<sup>48</sup> This means that for any two vertices there is a graph automorphism that maps one vertex to the other.

In this setting we reinterpret  $G(\bar{f}\rho(x))$  to be the overall producer surplus generated by sales of good  $i$  by a given firm, conditional on successful production, aggregated over all firms involved directly or indirectly in the successful production of the good. We let this surplus be distributed among suppliers in the successful (realized) supply tree, which we select uniformly at random from among the possible successful supply trees given the realization of relationship specific shocks, with minimal restriction. For each possible successful supply tree  $S \in \mathcal{S}$  let  $p(S)$  be the different positions in that supply tree, and let  $\phi_{kl} : \mathcal{S} \rightarrow \Delta^{|p(S)|}$  be a measurable function mapping successful supply trees into an allocated share of profits for each position in that tree, where  $\Delta^n$  is the  $n$ -dimensional simplex. For example, following Goyal and Vega-Redondo (2007),  $\phi$  might specify that only critical firms (i.e., those firms in all possible successful supply trees) extract an equal share of profits, while the other firms collectively extract nothing.

Note that by construction a firm  $kl$  can only be allocated a positive share of profits if produces successfully. Consider then the aggregate profits obtained for goods of type  $i \in \mathcal{I}$  and  $j \in \mathcal{I}$ . Suppose that a symmetric investment equilibrium is being played with entry  $\bar{f}$  and investment  $\bar{x}$ . Consider the frequency with which type  $i$  products appear as intermediate goods in a given position in the technology tree. As the technology

<sup>48</sup>We leave the exercise of relaxing this assumption to future work.

graph is vertex transitive, there is graph automorphism that maps type  $i$  into type  $j$ , type  $j$  products must be used as intermediate goods in such a position with exactly the same frequency. Moreover, given that all links fail with the same probability in a type symmetric equilibrium, and as the provider of each input is selected uniformly at random from those producing that good, the probability a given type  $i$  firm appears in any position in a successful supply tree is equal to the probability any type  $j$  firm appears in that position in a successful supply tree.

Let  $H(i, f, \bar{f}, \rho(x))$  be the expected profits of firm  $if$ , for  $f \leq \bar{f}$ , aggregated over all successful supply trees conditional on  $if$ 's successful production. Thus firm  $if$  receives expected overall profits of

$$H(i, f, \bar{f}, \rho(x))\mathbf{E}[F_{if}] - c(x_{if} - \underline{x}) - \Phi(f).$$

However, by the above argument,  $H(j, f', \bar{f}\rho(x)) = H(i, f, \bar{f}\rho(x)) = H(\bar{f}\rho(x))$  for all  $j$  and all  $f' \leq \bar{f}$ . Moreover, as  $\phi$  distributes profits  $G(\bar{f}\rho(x))$  for each successfully produced product, the preservation of these profits implies that  $H(\bar{f}\rho(x)) = G(\bar{f}\rho(x))$ .

## APPENDIX I. ADJUSTING INVESTMENTS IN RESPONSE TO SHOCKS

In this section we consider an alternative definition of a fragile equilibrium in which firms are able to adjust their investment choices in response to a shock. Interestingly, this has no impact on our characterization of fragile equilibria.

**Definition 5** (Equilibrium fragility').

- A productive equilibrium is *fragile'* if, holding the fraction of firms in the market,  $\bar{f}^*$ , fixed, *any* negative shock  $\epsilon > 0$  such that  $\underline{x}$  decreases to  $\underline{x} - \epsilon$  results in equilibrium output falling to 0 (such that  $\rho(x^*(\underline{x} - \epsilon)) = 0$ ).
- A productive equilibrium is *robust'*, if it is not *fragile'*.

In a *fragile'* equilibrium, firms are allowed to adjust their multisourcing efforts following the negative shock, but entry decisions are sunk (i.e. the number of firms  $\bar{f}^*$  in the market cannot change).

**Proposition 10.** If  $\kappa \leq \bar{\kappa}$ , then any productive equilibrium is *fragile'*. If  $\kappa > \bar{\kappa}$ , then any productive equilibrium is *robust'*.

We prove Proposition 10 below in Section I.2. It might be hoped that by adjusting their investment decisions the entering firms could absorb the shock and avoid production failing for sure. Proposition 10 shows that this does not happen. A severe free-riding problem prevents the entering firms from collectively increasing their investment to offset the shock. Of course, if production choices were not allowed to change in response to a shock, then all fragile equilibria would remain fragile. On the other hand, if (perhaps in the longer term) firms can re-optimize their entry decisions, then the probability of successful production need not collapse to 0.

**I.1. Interdependent supply networks and cascading failures.** Making use of Proposition 10, we now posit an interdependence among supply networks wherein each firm's profit depends on the *aggregate* level of output in the economy, in addition to the functionality of the suppliers with whom it has supply relationships. Formally, suppose,  $\kappa_{\mathfrak{s}} = K_{\mathfrak{s}}(Y)$ , where  $K_{\mathfrak{s}}$  is a strictly increasing function and  $Y$  is the integral across all sectors of equilibrium output:

$$Y = \int_{\mathcal{S}} \bar{f}_{\mathfrak{s}}^* \rho(x_{\mathfrak{s}}^*) d\Phi(\mathfrak{s}).$$

Here we denote by  $(\bar{f}_{\mathfrak{s}}^*, x_{\mathfrak{s}}^*)$  the unique positive equilibrium in sector  $\mathfrak{s}$ . The output in the sector is the mass of entering firms,  $\bar{f}_{\mathfrak{s}}^*$  multiplied by the reliability in that sector,  $\rho(x_{\mathfrak{s}}^*)$ .

The interpretation of this is as follows: When a firm depends on a different sector, a specific supply relationship is not required, so the idiosyncratic failures of a given producer in the different sector do not matter—a substitute product can be readily purchased via the market. (Indeed, it is precisely when substitute products are not readily available that the supply relationships we model are important.) However, if some sectors experience a sudden drop in output, then other sectors suffer. They will not be able to purchase inputs, via the market, from these sectors in the same quantities or at the same prices. For example, if financial markets collapse, then the productivity of many real businesses that rely on these markets for credit are likely to see their effective productivity fall. In these situations, dependencies will result in changes to other sectors' profits even if purchases are made via the market. Our specification above takes interdependencies to be highly symmetric, so that only aggregate output matters, but in general these interdependencies would correspond to the structure of an intersectoral input-output matrix, and  $K$  would be a function of sector level outputs, indexed by the identity of the sourcing sector.

This natural interdependence can have very stark consequences. Consider an economy characterized by a distribution  $\Psi$  in which the subset of sectors with  $m \geq 2$  has positive measure, and some of these have positive equilibria. Suppose that there is a small shock to  $\kappa$ . As already argued, this will directly cause a positive measure of sectors to fail. Suppose the shock is sufficiently small that the set of sectors that initially fail is well-approximated by the set of initially fragile supply chains. (In other words, the mass of sectors shifted from non-fragile to fragile by the small shock itself is negligible. We will see that this is a conservative assumption.) The failure of the fragile sectors will cause a reduction in aggregate output. Thus  $\kappa_{\mathfrak{s}} = K_{\mathfrak{s}}(Y)$  will decrease in other sectors *discontinuously*. This will take some other sectors out of the robust regime. Note that this occurs due to the other supply chains failing and not due to the shock itself. As these sectors are no longer robust, they topple too. Continuing this logic, there will be a domino effect that propagates the initial shock. This domino effect could die out quickly, but need not. A full study of such domino effects is well beyond our scope, but the forces in the very simple sketch we have presented would carry over to more realistic heterogeneous interdependencies.

Fig. 22 shows<sup>49</sup> how an economy with 100 interdependent sectors responds to small shocks to the productivity shifter  $\kappa$ . In this example, the technological complexity is set to  $m = 2$  and the number of potential suppliers for each firm is set to  $n = 5$ .

<sup>49</sup>Note that in this example, the cascade dynamics is as follows: At step 1, firms in sectors with a  $\kappa$  in the fragile range fail due to an infinitesimal shock to  $\kappa$ . This is because entry is held fixed at each step while the investments, and hence the relationship strength,  $x_{if}$  are allowed to be readjusted. The initial economy-wide output  $Y_1$  is then decreased to  $Y_2$  and the  $\kappa$ 's are updated using the function  $K$ . Only then, are the firms in the surviving sectors allowed re-adjust both  $\bar{f}$  and  $x_{if}$ . At step 2, infinitesimal shocks hit again and the firms newly found in the fragile regime fail. This process goes on at each step until no further firm fails, at which point the cascade of failures stops. Note that we could also prevent firms from adjusting entry at all throughout all steps of the cascade. This would obviously worsen the number of sectors that fail. This example is thus a conservative estimate of the firms that could fail due to the interdependence of the sectors in an economy.

The cost function for any firm  $if$  is  $c(x_{if}) = 2x_{if}^2$  while the gross profit function is  $g(\bar{f}\rho(x)) = 1 - \bar{f}\rho(x)$  and the entry cost function is  $\Phi(f) = 2f$ . This setup yields values  $\underline{\kappa} = 1.22$  and  $\bar{\kappa} = 1.38$  delimiting the region corresponding to critical (and therefore fragile) equilibria, as per Propositions 5 and 10.

In Fig. 22(a), the productivity shifter of a given sector is distributed uniformly, i.e.  $\kappa_s \sim U(\underline{\kappa}, 4)$ , so that many sectors have high enough productivity to be in a robust equilibrium while a small fraction have low enough productivity to be in a fragile equilibrium. A small shock to the productivity of all sectors thus causes the failure of the fragile sectors (7 in total). This then decreases the output  $Y$  across the whole economy, but only to a small extent (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is thus not enough to bring them into the fragile regime and thus to cause them to fail as well upon a small shock.

In contrast, Fig. 22(b) shows an economy where  $\kappa_s \sim U(\underline{\kappa}, 2.3)$ , so that more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the productivity of all sectors causes the failure of the fragile sectors (now 18). These have a larger effect on decreasing the output  $Y$  across the whole economy (as seen in the right panel). The resulting decrease in the productivities of the robust sectors is now enough to bring some of them into the fragile regime and to cause them to fail too. This initiates a cascade of sector failures, ultimately resulting in 26 sectors ceasing production.

Fig. 22(c) shows an economy where  $\kappa_s \sim U(\underline{\kappa}, 2)$ , so that even more sectors have low enough productivity to be in a fragile equilibrium. A small shock to the productivity of all sectors causes the failure of the fragile sectors (now 20 in total) and this initiates a cascade of sector failures which ultimately brings down all 100 sectors of the economy.

The discontinuous drops in output caused by fragility, combined with the simple macroeconomic interdependence that we have outlined, come together to form an amplification channel reminiscent, e.g., of Elliott et al. (2014) and Baqaee (2018). Thus, the implications of those studies apply here: both the cautions regarding the potential severity of knock-on effects, as well as the importance of preventing first failures before they can cascade.

## 1.2. Proof of Proposition 10.

*Proof.* Consider a reduction in  $\underline{x}$  to  $\underline{x} - \epsilon$  for  $\epsilon > 0$ . From Proposition 5, we know that if  $\kappa \leq \bar{\kappa}(\underline{x})$ , either  $\kappa \leq \underline{\kappa}(\underline{x})$ , in which case there is no productive equilibrium, or  $\underline{\kappa}(\underline{x}) \leq \kappa \leq \bar{\kappa}(\underline{x})$ , in which case, any productive equilibrium is a critical equilibrium.

First observe that  $\bar{\kappa}(\underline{x})$  is strictly increasing in  $\underline{x}$ . Recall that to save on notation we set  $\underline{x} = 0$  for many of our proofs. This is also true in equation 23, but once  $\underline{x}$  is reintroduced we have,

$$\bar{\kappa}(\underline{x}) = \frac{c'(x_{\text{crit}} - \underline{x})}{g(f_{\text{crit}}\rho_{\text{crit}})mn\rho_{\text{crit}}(1 - x_{\text{crit}}\rho_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}\rho_{\text{crit}})^n)^{m-1}},$$

and so  $\bar{\kappa}(\underline{x})$  decreases continuously in  $\underline{x}$ . Similarly, recalling that

$$\underline{\kappa} = \frac{c'(x_{\text{crit}} - \underline{x})}{g(0)mn\rho_{\text{crit}}(1 - x_{\text{crit}}\rho_{\text{crit}})^{n-1}(1 - (1 - x_{\text{crit}}\rho_{\text{crit}})^n)^{m-1}},$$

it is immediate that  $\underline{\kappa}$  decreases continuously in  $\underline{x}$ .

Consider first  $\kappa \leq \underline{\kappa}$  and the corresponding equilibrium  $(\bar{f}^*, x^*(\bar{f}^*))$ , where, by Proposition 5,  $x^*(\bar{f}^*) = 0$ . Fixing the fraction of firms in the market at  $\bar{f}^*$ , as  $\bar{\kappa}(\underline{x})$  is decreasing in  $\underline{x}$ , after the shock to  $\underline{x}$  we must have  $\tilde{x}^*(\bar{f}^*) = 0$

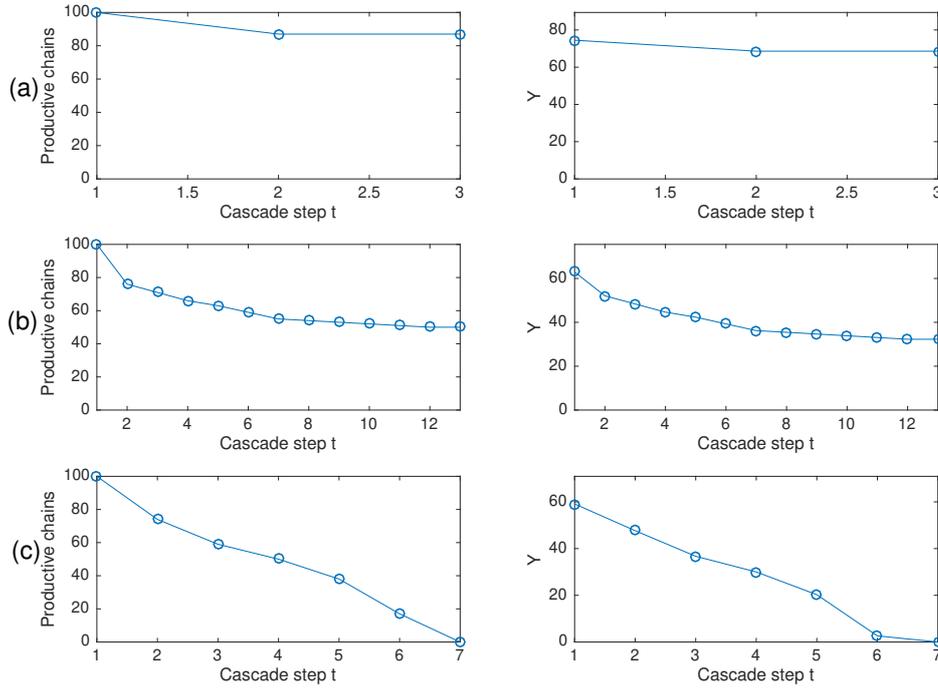


FIGURE 22. Number of sectors that remain productive (left) and economy-wide output  $Y$  (right) for each step of a cascade of failures among 100 interdependent sectors. For all sectors:  $n = 5$ ,  $m = 2$ ,  $c(x_{if}) = 2x_{if}^2$ ,  $g(\bar{f}\rho(x)) = 1 - \bar{f}\rho(x)$  and  $\Phi(f) = 2f$ . This yields  $\underline{\kappa} = 1.22$  and  $\bar{\kappa} = 1.38$ . In row (a), 100 sectors have  $\kappa_s$  initially distributed according to  $U(\underline{\kappa}, 4)$ ; In row (b), 100 sectors have  $\kappa_s$  initially distributed according to  $U(\underline{\kappa}, 2.3)$ ; In row (c), 100 sectors have  $\kappa_s$  initially distributed according to  $U(\underline{\kappa}, 2)$ .

Consider now  $\underline{\kappa} \leq \kappa \leq \bar{\kappa}$  and the corresponding equilibrium  $(\bar{f}^*, x^*(\bar{f}^*))$ , where, by Proposition 5,  $x^*(\bar{f}^*) = x_{\text{crit}}$ . Fixing the fraction of firms in the market at  $\bar{f}^*$ , as  $\bar{\kappa}(\underline{x})$  is decreasing in  $\underline{x}$ ,  $\tilde{x}^*(\bar{f}^*) \leq x_{\text{crit}}$ . By equation (15), the marginal costs from investments evaluated at  $x_{\text{crit}}$  are strictly increasing in  $\underline{x}$ , while the marginal benefits of investment do not depend on  $\underline{x}$ . Thus, following *any* shock  $\epsilon > 0$  to  $\underline{x}$  marginal benefits are the same and marginal costs are strictly higher at the same investment choice  $x_{\text{crit}}$ . As shown in the proof of Proposition 3, marginal benefits are (strictly) decreasing in  $x$  while marginal costs are (strictly) increasing in  $x$ . Thus, as  $\underline{x}$  has decreased the effort level must drop to keep marginal benefits equal to marginal costs. Denote by  $\tilde{x}^*(\bar{f}^*)$ , the optimal effort level (where marginal benefit equals marginal cost) after  $\underline{x}$  as decreased while keeping  $\bar{f}^*$  unchanged:

$$MB(\tilde{x}; \bar{f}^*, \rho(\tilde{x}), \underline{x} - \epsilon) = MC(\tilde{x}).$$

In the new investment equilibrium we therefore have  $\tilde{x} = \tilde{x}^*(\bar{f}^*) < x^*(\bar{f}^*) = x$ . Since in a critical equilibrium,  $x = x_{\text{crit}}$ , it follows that the after-shock, adjusted effort level,  $\tilde{x}$ , is smaller than  $x_{\text{crit}}$  and thus  $\rho(\tilde{x}) = 0$  (the probability of producing collapses to zero). From Definition 5, the equilibrium  $(\bar{f}^*, x^*(\bar{f}^*))$  is thus *fragile*'.

Now consider  $\kappa > \bar{\kappa}$  and the corresponding equilibrium  $(\bar{f}^*, x^*(\bar{f}^*) = x)$ . Then, fixing entry  $\bar{f}^*$ , due to the continuity of the marginal costs in  $x^*(\bar{f}^*) = 0$  and continuity of  $\bar{\kappa}$  in  $\underline{x}$ , there exists a small-enough shock  $\epsilon > 0$  such that  $\bar{\kappa} < \tilde{\kappa}$  and the new investment equilibrium is such that  $x > \tilde{x} > x_{\text{crit}}$ . Thus  $\rho(\tilde{x}) > \rho(x_{\text{crit}}) > 0$  and, by Definition 5, the equilibrium  $(\bar{f}^*, x^*(\bar{f}^*))$  is *robust'*. Only large enough shocks could cause production to fail.  $\square$