Frictional Capital Reallocation with Ex Post Heterogeneity*

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Abstract

This project studies economies with markets for capital reallocation, where gains from trade are driven by firm-specific productivity shocks, but are hindered by search frictions and liquidity considerations. Results are provided on existence, uniqueness and efficiency. The model is tractable enough to analyze monetary and fiscal policy using simple graphs. Additionally, we calibrate it to investigate quantitatively the effects of changes in productivity and credit conditions. The framework can capture several facts deemed interesting in the literature – e.g., capital misallocation is countercyclical, while its price and reallocation are procyclical. We also discuss how well productivity dispersion measures inefficiencies or frictions.

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1 Introduction

This paper studies economies where capital is accumulated in primary markets and reallocated in frictional secondary markets driven by firm-specific productivity shocks. As motivation, note that efficient economic performance requires getting the right amount of investment over time, plus getting existing capital into the hands of those best able to use it at a point in time, and of course these are intimately related: the ease with which used capital can be retraded affects incentives for the accumulation of new capital, just like the attributes of secondary markets for houses, cars and other assets influence primary markets. Also note that reallocation is sizable: purchases of used capital constitute 25% to 30% of total investment, even ignoring mergers, acquisitions and rentals, and only looking at big, publicly-traded firms, and so it looks to be important at the macro level.[1]

A reason to study capital markets is to see how the outcomes depend on fiscal and monetary policy, and one version of our formulation is tractable enough to analyze these policies using simple graphs.[2] A reason to focus on frictional reallocation is that many people argue real-world capital markets are far from the perfectly competitive ideal.[3] Imperfections include adverse selection, financial constraints, the difficulty of finding an appropriate counterparty, and holdup problems due to bargaining. We downplay adverse selection (on that, see Eisfeldt

[1] See Eisfeldt and Rampini (2006), Cao and Shi (2016), Dong et al. (2016), Cui (2017), and Eisfeldt and Shi (2018). Also, this only concerns reallocation across firms, but one could also consider movement of capital within firms (Giroud and Mueller 2015), across sectors (Ramey and Shapiro 1998) or between countries (Caselli and Feyrer 2007).

[2] On fiscal policy, papers showing taxation is crucial for capital formation include Cooley and Hansen (1992), Chari et al. (1994), McGrattan et al. (1997) and McGrattan (2012). On monetary policy, papers studying investment and inflation include Tobin (1965), Sydrauski (1967), Stockman (1981) and Cooley and Hansen (1989), although they use reduced-form monetary models, while we use the microfoundations in the literature surveyed by Lagos et al. (2017). Examples of that literature focusing on capital include Aruoba and Wright (2003), Lagos and Rocheteau (2008), Aruoba et al. (2011), Andolfatto et al. (2016), Shi (1998,1999*a,b*), Shi and Wang (2006), Menner (2006), Molico and Zhang (2006) and Berentsen et al. (2011).

[3] See Gavazza (2010,2011*a,b*), Kurman (2014), Ottonello (2015), Cao and Shi (2016), Dong et al. (2016), Kurman and Rabinovitz (2018) and Horner (2018). Note that these are all nonmonetary analyses, however, and hence cannot address some key issues in this paper.

and Rampini 2008 or Li and Whited 2014) to concentrate on other issues: our secondary capital market features bilateral exchange and bargaining, as in search theory, and the use of assets in facilitating payment, as in monetary economics.

As additional motivation, consider Ottonello (2015), who compares models of capital with and without search, and argues the former fit the facts better and generate more interesting propagation. Horner (2018) shows vacancy rates for commercial real estate resemble unemployment data, suggesting that search may be as important for capital as it is for labor, and argues that rents on these properties vary considerably, inconsistent with Walrasian theory. In a particular market, the one for aircraft, Pulvino (1998), Gilligan (2004) and Gavazza (2011*a,b*) find that used sales are thrice new sales, that prices vary inversely with search time, and that market thickness affects trading frequency, average utilization, utilization dispersion, average price and price dispersion. This work emphasizes the importance of specificity, making it hard for firms to trade certain types of customized capital, further suggesting a search-based approach.

Reallocation moves capital from lower- to higher-productivity firms (Maksimovic and Phillips 2001; Andrade et al. 2001; Schoar 2002). In the specification studied below, productivity differences come from idiosyncratic shocks.[4] Conditional on investment, reallocation is efficient iff the nominal interest rate is $\iota = 0$ – the Friedman rule – but investment can be too high or low depending on the tax rate τ and bargaining power θ. We prove that $\iota = 0$ implies optimal fiscal policy is $\tau > 0$ for θ too low and $\tau < 0$ for θ too high; and that $\tau = 0$ implies optimal monetary policy is $\iota > 0$ both for θ too low and too high. This is interesting because monetary and fiscal policy are not symmetric, and because it is quite difficult to get $\iota > 0$ optimal in most monetary models.

[4] This contrasts with our previous work on ex ante heterogeneity (Wright et al. 2018), where by assumption some firms cannot trade in the primary market, and hence must get k in the secondary market. Here all firms can get k in the primary market, but might retrade it in the secondary market. This seems more natural and lets us dispense with a few awkward features of the earlier model – e.g., if an agent can only get k in a frictional market, if an opportunity arises in this market he should want to get a lot, but for tractability we had to preclude that by assuming they can store it for just one period.

We show that the model is able to match some observations deemed important in the literature: reallocation is procyclical but mismatch countercyclical (Eisfeldt and Rampini 2006; Cao and Shi 2016); the price of used capital is procyclical (Lanteri 2016); and the ratio of spending on used capital to total investment is procyclical (Cui 2016). However, to match these facts we need to have both increases in productivity and decreases in financial constraints during good times. Another quantitative findings is that the welfare cost of inflation is quite high due to the way it discourages capital reallocation. We also discuss quantitatively how well productivity dispersion measures misallocation, frictions, or welfare, related to some interesting empirical research.[5] Also related is much recent work on OTC (over-the-counter) asset markets.[6] Also related is real business cycle theory, where the textbook model is a special case of our model.[7] Finally, the paper contributes to research on money demand by firms.[8]

As in any good monetary model, a fundamental property of the theory presented below is that (some) economic activity decreases with the cost of liquidity, as measured by either inflation or nominal interest rates. In particular, the model

[5] See Hsieh and Klenow (2009), Buera et al. (2011), Midrigan and Xu (2014), Cooper and Schott (2016), Ai et al. (2015) and David and Venkateswaran (2017). Our results are broadly consistent with their findings – e.g., Buera et al. (2011) conclude the data is consistent with financial frictions distorting capital across firms, even if self-financing mitigates the problem, which is what our emphasis on liquidity is meant to capture.

[6] In spirit this includes work following Duffie et al. (2005), but there are major differences: First, our asset is neoclassical capital (instead of a Lucas tree) that togther with labor produces goods as in standard growth theory. Second, our agents face genuine credit problems that they can address by holding liquid assets. And we do not restrict asset holdings to $\{0, 1\}$, making our setup closer to extensions of Duffie et al. by Lagos and Rocheteau (2009), Geromichalos and Herrenbrueck (2016) or Lagos and Zhang (2018).

[7] In particular, by shutting down the idiosyncratic productivity shocks, we get exactly the model in Hansen (1985), right down to functional forms. A more recent business cycle model with productivity dispersion is Asker et al. (2014), but that setup has neither a secondary capital market nor liquidity considerations. So, one can say that we extend modern monetary theory to incorporate capital in more detail, or that we extend mainstream macro to include secondary markets with liquidity, search and bargaining frictions.

[8] For older studies of firms' cash holdings, see Mulligan (1997), Bates et al. (2009) and references therein. More recently, and more closely related to this paper, Rocheteau et al. (2018) develop a model in the New Monetarist tradition, except focusing on money demand by firms rather than the usual practice of focusing on households (see the survey by Lagos et al. 2017). He and Zhang (2019) have a related model where firms and households both use money; we consider a similar setup in Section 6.2.

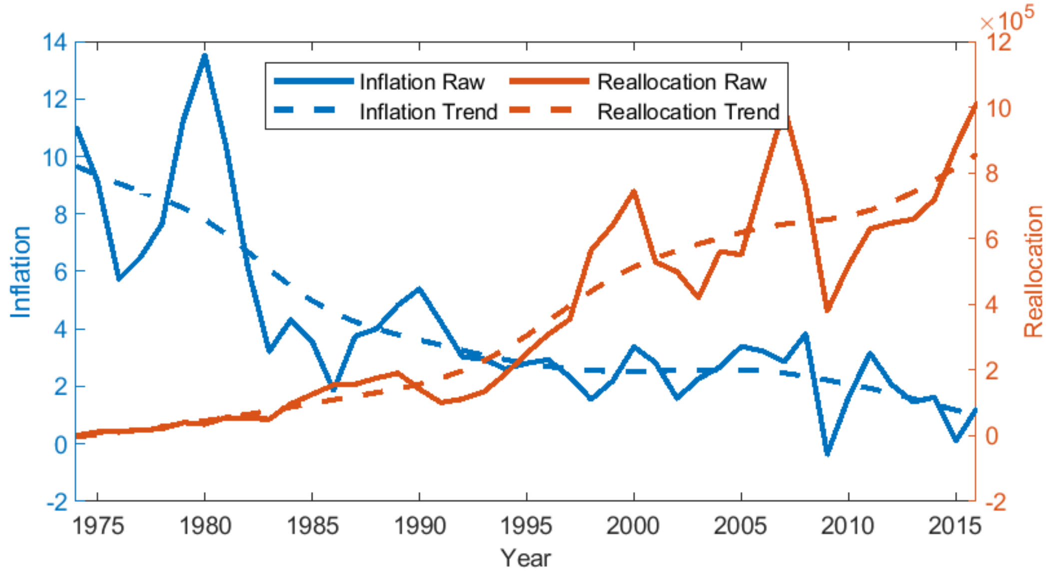


Figure 1: Inflation and Capital Reallocation, 1974-2016, USA

unambiguously predicts that reallocation decreases with inflation or interest rates. As show in Figure 1, this is consistent with the data (where reallocation comes from Computat and inflation from FRED). However, we do not make too much of that because it is mainly due to the trend – i.e., the effect is operative at the medium- to long-run frequency, not the standard business cycle frequency. This is not a problem per se, since we are happy to interpret the mechanism in the formal model as applying mainly in the medium- to long-run, in the spirit of Berentsen et al. (2011), say. Still, there are potentially many reasons why inflation has trended down and reallocation up over the sample, and we do not want to argue here for causation.

In what follows, Section 2 describes the environment. Sections 3 and 4 discuss equilibrium with perfect credit and with money. Sections 5 analyzes a tractable special case. Section 6 presents extensions where, among other things, we use price posting instead of bargaining. Section 7 presents the calibration and quantitative implications. Section 8 concludes.

2 Environment

Time is discrete and continues forever. As sown in Figure 2, at each t two markets convene sequentially: a frictional decentralized market, or DM; and a frictionless centralized market, or CM. This alternating market structure, adapted from Lagos and Wright (2005), is ideal for our purposes because the CM and DM correspond well to primary and secondary capital markets. In the CM, agents consume a numeraire good c, supply labor hours h and accumulate capital k as in standard growth theory. Then in the DM, rather than households trading consumption goods as in most of the related monetary literature, here firms trade capital (even if, for convenience, we sometimes call them households that own firms rather than firms per se). All agents (firm owners) have utility $u(c) - Ah$, where $u'(c) > 0$, $u''(c) < 0$ and $u'(c) \to \infty$ as $c \to 0$, and discount between the CM and the next DM using $\beta \in (0, 1)$.

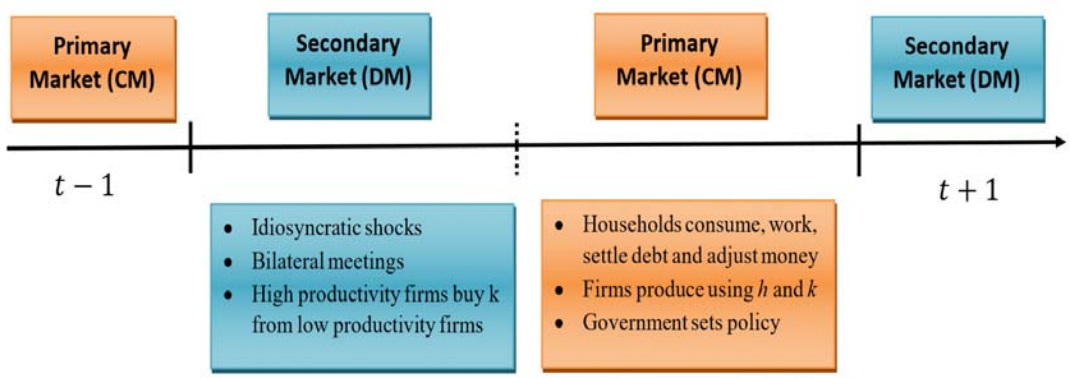


Figure 2: Time Line

We start at $t = 0$ in the DM with all agents holding k_0. Then each firm gets a productivity shock $\varepsilon \in [0, \bar{\varepsilon}]$ with CDF $G(\varepsilon)$. Here the shocks are i.i.d. so that all firms look the same after the CM but before ε is realized (Cui et al. 2019 study a version with persistent shocks). This generates gains from trade in the DM, where agents meet randomly in pairs, with α denoting the meeting probability. Each meeting is characterized by $\mathbf{s} = (k_b, \varepsilon_b, k_s, \varepsilon_s)$, where b and s index the buyer and seller, and clearly the former is the one with higher ε. A firm has CM technology $F(k,h,\varepsilon)$ that is increasing and concave in (k,h) with continuous second derivatives. It is also increasing in ε $\forall\,(k,h)$. While in principle F can display CRS or DRS (constant or decreasing returns to scale), we usually impose the latter to make it more likely that, when high and low ε firms meet, the former gets some but not all the capital.[9]

To discuss efficiency, first, firms in the CM hire labor $h = h^*(k,\varepsilon)$ where

$$F_2[k, h^*(k,\varepsilon), \varepsilon]u'(c) = A, \tag{1}$$

given $h > 0$ under standard assumptions. Aggregating this across firms gives total hours, and the time constraint for households, say $h \leq 1$, is assumed slack unless stated otherwise. Next, for capital reallocation, when two firms meet in the DM let q be the amount the one with lower ε gives to the one with higher ε. If the obvious constraint $q \leq k$ is slack then $q = q^*(\mathbf{s})$ satisfies

$$F_1[k+q, h^*(k+q,\varepsilon_b), \varepsilon_b] = F_1[k-q, h^*(k-q,\varepsilon_s), \varepsilon_s]. \tag{2}$$

With these results on h and q in hand, consider a planner choosing a path for k to maximize utility for the representative agent, subject to search frictions, an initial k_0 and resource feasibility after government takes g_t units of numeraire each period. The problem can be written as

$$\begin{aligned} W^*(k_0) &= \max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - Ah_t\right] \qquad (3) \\ \text{st } c_t &= y_t + (1-\delta)k_t - g_t - k_{t+1} \\ y_t &= (1-\alpha)\int_0^\infty F[k_t, h^*(k_t,\hat{\varepsilon}), \hat{\varepsilon}]dG(\hat{\varepsilon}) \\ &\quad +\alpha \int_{\hat{\varepsilon}>\tilde{\varepsilon}} F\{k_t + q^*(\hat{\mathbf{s}}), h^*[k_t + q^*(\hat{\mathbf{s}})], \hat{\varepsilon}\}dG(\tilde{\varepsilon})\,dG(\hat{\varepsilon}) \\ &\quad +\alpha \int_{\hat{\varepsilon}<\tilde{\varepsilon}} F\{k_t - q^*(\tilde{\mathbf{s}}), h^*[k_t - q^*(\tilde{\mathbf{s}})], \hat{\varepsilon}\}dG(\tilde{\varepsilon})\,dG(\hat{\varepsilon}), \end{aligned}$$

[9] As Jovanovic and Rousseau (2002) say, "Used equipment and structures sometimes trade unbundled in that firm 1 buys a machine or building from firm 2, but firm 2 continues to exist. At other times, firm 1 buys firm 2 and thereby gets to own all of firm 2's capital." We focus here on cases where it is efficient for high ε firms trading with low ε firms to get some but not all the capital; Cui et al. (2019) assume CRS and focus on the other case.

where δ is depreciation, and output y_t includes production by the $1-\alpha$ measure of firms that did not have a DM meeting, the α measure that had a meeting and increased k, plus the α measure that had a meeting and decreased k.

Routine methods yield the planner's investment Euler equation

$$\begin{aligned} r_t + \delta = {} & (1-\alpha)\int_0^\infty F_1[k_{t+1}, h^*(k_{t+1}, \hat{\varepsilon}), \hat{\varepsilon}]dG(\hat{\varepsilon}) \\ & + \alpha \int_{\hat{\varepsilon}>\tilde{\varepsilon}} F_1\{k_{t+1} + q^*(\hat{\mathbf{s}}), h^*[k_{t+1} + q^*(\hat{\mathbf{s}})], \hat{\varepsilon}\} dG(\tilde{\varepsilon})\, dG(\hat{\varepsilon}) \\ & + \alpha \int_{\hat{\varepsilon}<\tilde{\varepsilon}} F_1\{k_{t+1} - q^*(\tilde{\mathbf{s}}), h^*[k_{t+1} - q^*(\tilde{\mathbf{s}})], \hat{\varepsilon}\} dG(\tilde{\varepsilon})\, dG(\hat{\varepsilon}). \end{aligned} \tag{4}$$

where r_t is given by $1 + r_t = u'(c_t)/\beta u'(c_{t+1})$ (in equilibrium this is an interest rate; here $1 + r_t$ is just notation for the MRS). While (4) may look like a static condition, it is not, since out of steady state k_t affects c_t and r_t. Intuitively, the LHS of (4) is the marginal cost of investment due to discounting and depreciation, while the RHS is the benefit, taking into account shocks and reallocation.

Given h_t, q_t and k_t, consumption c_t is given by $u'(c_t) = A/F_2(\cdot)$, where $F_2(\cdot)$ is the same for all firms since labor is allocated without frictions. One can show the outcome exists uniquely and is summarized as follows:[10]

Proposition 1 *Given k_0 and the time path of g_t, the solution to the planner's problem is characterized by (nonnegative, bounded) paths for $\left\langle \hat{k}_t^*, q_t^*(\cdot), h_t^*(\cdot), c_t^* \right\rangle$, where $k_t^* \in \mathbb{R}$, $q_t^* : \mathbf{s} \to \mathbb{R}$, $h_t^* : (k, \varepsilon) \to \mathbb{R}$, and $c_t^* \in \mathbb{R}$ satisfy (4), (2), (1) and the constraints in (3).*

3 Perfect-Credit Equilibrium

While our main interest is in economies with payment frictions, perfect credit provides a natural benchmark. Thus, when firms meet in the DM, the one with higher ε gets q units of capital in exchange for a promise to deliver d units

[10]Note: when we say paths are bounded, we mean the usual transversality condition for capital, and after adding money money the analog for real balances (e.g., see Rocheteau and Wright 2013); we do not dwell on this in what follows.

of numeraire in the next CM, where without loss of generality we can restrict attention to one-period debt as long as the constraint $h \in [0,1]$ is slack. Let the CM and DM value functions be $W(a,k,\varepsilon)$ and $V(k,\varepsilon)$, where the CM state includes one's financial asset position, capital holdings and productivity, while the DM state includes just capital and productivity. In general, $a = z - d - T$ where z is real money balances introduced below, d is debt from the previous DM, and T is a lump sum tax.

The CM problem is then

$$\begin{aligned} W(a,k,\varepsilon) &= \max_{c,h,\hat{k}} \{u(c) - Ah + \beta \mathbb{E}_{\hat{\varepsilon}} V_{+1}(\hat{k},\hat{\varepsilon})\} \\ \text{st } c + \hat{k} &= wh + a + \Pi(k,\varepsilon) + (1-\delta)k \\ \Pi(k,\varepsilon) &= \max_{\tilde{h}} \{F(k,\tilde{h},\varepsilon) - w\tilde{h}\}, \end{aligned} \tag{5}$$

where w is the real wage, $\Pi(k,\varepsilon)$ is profit income, and we omit t subscripts when the timing is obvious. From profit maximization, labor demand is

$$\tilde{h}(k,\varepsilon) = \arg\max_{\tilde{h}} \{F(k,\tilde{h},\varepsilon) - w\tilde{h}\}. \tag{6}$$

Of course $\tilde{h}$ also depends on w, but that is subsumed in the notation, to highlight the dependence on (k,ε).[11]

Using the constraints, we reduce (5) to

$$\begin{aligned} W(a,k,\varepsilon) &= \frac{A}{w}\left[\Pi(k,\varepsilon) + a + (1-\delta)k\right] + \max_c \left\{u(c) - \frac{A}{w}c\right\} \\ &\quad + \max_{\hat{k}} \left\{-\frac{A}{w}\hat{k} + \beta \mathbb{E}_{\hat{\varepsilon}} V(\hat{k},\hat{\varepsilon})\right\}. \end{aligned}$$

When nonnegativity constraints are slack, the FOC's are

$$c : \frac{A}{w} = u'(c) \tag{7}$$

$$\hat{k} : \frac{A}{w} = \beta \mathbb{E}_{\hat{\varepsilon}} V_1(\hat{k},\hat{\varepsilon}) \tag{8}$$

[11] Note that labor demand $\tilde{h}$ by a firm does not generally coincide with the supply h of its owner – indeed, with hours traded in the frictionless CM, we do not pin down who works for whom. An interesting extension would be to incorporate frictional labor markets (as Berentsen et al. 2010 or Dong and Xiao 2018 do in a similar model but without capital).

plus the budget equation. The envelope conditions are

$$
\begin{aligned}
W_1(a,k,\varepsilon) &= \frac{A}{w} \qquad (9)\\
W_2(a,k,\varepsilon) &= \frac{A}{w}\left[F_1(k,\tilde{h},\varepsilon)+1-\delta\right]. \qquad (10)
\end{aligned}
$$

Assuming an interior solution for $h \in (0,1)$, the following is immediate:[12]

Lemma 1 *(i) the CM choice $\hat{k}$ is independent of (a,k,ε); (ii) W is linear in a.*

As in the planner problem, DM meetings are characterized by $\mathbf{s} = (k_b, \varepsilon_b, k_s, \varepsilon_s)$, and the buyer is the agent with higher ε. The trading surpluses are

$$
\begin{aligned}
S_b(\mathbf{s}) &= W[-d(\mathbf{s}), k_b + q(\mathbf{s}), \varepsilon_b] - W(k_b, \varepsilon_b)\\
S_s(\mathbf{s}) &= W[d(\mathbf{s}), k_s - q(\mathbf{s}), \varepsilon_s] - W(k_s, \varepsilon_s),
\end{aligned}
$$

where the buyer gets $q(\mathbf{s}) \le k_s$ in exchange for debt $d(\mathbf{s})$. Simplification yields

$$
\begin{aligned}
S_b(\mathbf{s}) &= \frac{A}{w}\left\{\Pi[k_b + q(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) + (1-\delta)\, q(\mathbf{s}) - d(\mathbf{s})\right\}\\
S_s(\mathbf{s}) &= \frac{A}{w}\left\{\Pi[k_s - q(\mathbf{s}), \varepsilon_s] - \Pi(k_s, \varepsilon_s) - (1-\delta)\, q(\mathbf{s}) + d(\mathbf{s})\right\}.
\end{aligned}
$$

Assuming both parties observe $\mathbf{s}$, one can use a variety of mechanisms to determine the terms of trade; we adopt Kalai's (1977) proportional bargaining solution (Aruoba et al. 2007 argue that this is has advantages over Nash bargaining in models with liquidity considerations, although here, with perfect credit, they are the same). If θ is buyers' bargaining power, Kalai's solution sets (p,q) to maximize S_b subject to feasibility and $(1-\theta)\, S_b(\mathbf{s}) = \theta S_s(\mathbf{s})$. Given perfect credit, this leads to $q = q^*(\mathbf{s})$, the same as the planner problem, and

$$
\begin{aligned}
d^*(\mathbf{s}) &= \theta\left\{\Pi(k_s, \varepsilon_s) - \Pi[k_s - q^*(\mathbf{s}), \varepsilon_s]\right\} + (1-\delta)\, q^*(\mathbf{s}) \qquad (11)\\
&\quad + (1-\theta)\left\{\Pi[k_b + q^*(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b)\right\}.
\end{aligned}
$$

[12] Part (i) follows from (8) and part (ii) from (9). As in Lagos and Wright (2005), these use quasi-linear utility, but the same results hold for any utility function that is CRS (Wong 2016), or for any utility function if we assume indivisible labor (Rocheteau et al. 2008).

Before meetings occur in the DM the expected payoff is

$$V(\hat{k},\hat{\varepsilon}) = W(0,\hat{k},\hat{\varepsilon}) + \alpha \int_{\hat{\varepsilon}>\tilde{\varepsilon}} S_b\left(\hat{\mathbf{s}}\right) dG(\tilde{\varepsilon}) + \alpha \int_{\hat{\varepsilon}<\tilde{\varepsilon}} S_s\left(\tilde{\mathbf{s}}\right) dG(\tilde{\varepsilon}). \tag{12}$$

The first term on the RHS is the continuation value from not trading. The second is the surplus from buying capital, where in equilibrium $\hat{\mathbf{s}} = (\hat{k},\hat{\varepsilon},\hat{k},\tilde{\varepsilon})$ and $\tilde{\mathbf{s}} = (\hat{k},\tilde{\varepsilon},\hat{k},\hat{\varepsilon})$, because a firm buys when it realizes $\hat{\varepsilon}$ and meets a firm with $\tilde{\varepsilon} < \hat{\varepsilon}$. Similarly, the last term is the surplus from selling in the DM. After reallocation, output y is given by the condition from the planner problem except $\tilde{h}$ replaces h^*. Goods market clearing requires $c + g + \hat{k} = y + (1-\delta)k$, and then labor market clearing is automatic by Walras' Law.

Conserving notation by not carrying around firm-specific labor demand, only aggregate h, we define equilibrium as follows:

Definition 1 *Given k_0 and time paths for $\langle g,T\rangle$, a perfect-credit equilibrium is a list of time paths for $\left\langle \hat{k}, q(\cdot), p(\cdot), h, c, w \right\rangle$ such that $\forall t$: (i) $(c,h,\hat{k})$ solves the CM maximization problem; (ii) $p(\cdot)$ and $q(\cdot)$ solve the DM bargaining problem; and (iii) markets clear.*

Definition 2 *Given constant $\langle g,T\rangle$, a perfect-credit steady state is a time invariant $\left\langle \hat{k}, q(\cdot), p(\cdot), h, c, w \right\rangle$ that satisfies the definition of equilibrium except for initial conditions.*

With perfect credit, reallocation $q = q^*(\mathbf{s})$ is efficient, in all meetings, taking as given the parties' capital holdings. What about aggregate investment? To answer that we need the capital Euler equation, which after a little algebra reduces to

$$\begin{aligned} r+\delta \;=\; & (1-\alpha)\int_0^\infty \Pi_1(\hat{k},\hat{\varepsilon})dG\left(\hat{\varepsilon}\right) \\ & +\alpha \int_{\hat{\varepsilon}>\tilde{\varepsilon}} \left[\theta \Pi_1[\hat{k}+q\left(\hat{\mathbf{s}}\right),\tilde{\varepsilon}] + (1-\theta)\,\Pi_1(\hat{k},\hat{\varepsilon})\right] dG(\tilde{\varepsilon})dG(\hat{\varepsilon}) \\ & +\alpha \int_{\hat{\varepsilon}<\tilde{\varepsilon}} \left[(1-\theta)\,\Pi_1[\hat{k}-q\left(\tilde{\mathbf{s}}\right),\hat{\varepsilon}] + \theta \Pi_1(\hat{k},\hat{\varepsilon})\right] dG(\tilde{\varepsilon})dG(\hat{\varepsilon}). \end{aligned} \tag{13}$$

By the envelope theorem $\Pi_1(\cdot) = F_1(\cdot)$, making (13) look somewhat more like (4) from the planner problem. In fact, the first line is the same, while the second is the same iff $\theta = 1$ and the third is the same iff $\theta = 0$.

Heuristically, this can be understood in terms of holdup problems: Bargaining in the DM with $\theta < 1$ increases demand for k in the CM relative to the efficient benchmark, because buying in the secondary market is less attractive when sellers extract part of the surplus. Similarly, $\theta > 0$ decreases demand for k in the primary market, because selling it in the secondary market is less attractive when buyers extract part of the surplus. As in many (not all) bargaining models, there is a $\theta^* \in (0,1)$ such that these effects net out to deliver efficiency, a version of Hosios (1990). We summarize as follows, with the proof is omitted, because it is a special case of results derived below for monetary economies:

Proposition 2 *In perfect-credit equilibrium, consumption, hours and reallocation are efficient conditional on investment, while investment is too high if $\theta < \theta^*$, too low if $\theta > \theta^*$ and efficient if $\theta = \theta^*$, for some $\theta^* \in (0,1)$.*

4 Monetary Equilibrium

Money is only essential if credit is imperfect (Kocherlakota 1998). Hence we need limited commitment, plus imperfect information so that it is hard to punish those who renege on their obligations by taking away future credit (as in Kehoe and Levine 1983). One way to formalize the relevant information frictions that has proved useful elsewhere (Gu et al. 2013*a*,*b*) is this: opportunistic behavior can be observed and communicated to others, and hence the culprit can be punished, with probability μ. This gives rise to an endogenous limit on unsecured debt, say $\bar{d}$, and if $\bar{d}$ is low there emerges a role for assets in facilitating intertemporal trade; in particular, money is never essential if $\mu = 1$ but can be if $\mu < 1$ (Gu et al. 2016). For simplicity, we set $\mu = 0$, which implies $\bar{d} = 0$ and hence no unsecured credit, although collateralized credit may work.

In particular, one can in principle punish defaulters by taking away future profit as in Holmstrom and Tirole (1998), and taking away existing or newly-acquired assets as in Kiyotaki and Moore (1997). Following the literature, in general, suppose only fractions χ_{Π} of profit Π, χ_k of existing k, and χ_q of newly-acquired q are pledgeable, which means these are the amounts that can be seized after default. For simplicity, we set $\chi_{\Pi} = \chi_k = 0$, but with a nod to realism set $\chi_q = 1$.[13] Given this, and given that only $(1-\delta)\,q$, can be pledged since δq depreciates, secured credit is equivalent to a rental agreement: the firm we call the buyer uses q and returns $(1-\delta)\,q$ to the one we call the seller. It is equivalent because returning $(1-\delta)\,q$ in the CM is equivalent to keeping it and paying off debt with the same CM value. Importantly, note that a cash down payment is always necessary for DM trade: since a seller's opportunity cost of trade is $(1-\delta)\,q$ plus the output q generates, if a buyer can only get credit up to $(1-\delta)\,q$ he cannot cover that cost without using some cash.

In terms of notation, let m be nominal and $z = \phi m$ real balances, where ϕ is the inverse of the CM price level, let DM meetings be characterized by $\mathbf{s} = (z_b, k_b, \varepsilon_b, z_s, k_s, \varepsilon_s)$, and let $d(\mathbf{s})$ and $p(\mathbf{s})$ be credit and cash payments. The money supply follows $M_{+1} = (1+\mu)\,M$ and the CM government budget equation is $g = T + \phi(M_{+1} - M)$, where g is consumption of c and T is a lumpsum tax (transfer if negative), and note that for our purposes it does not matter whether changes in M occur via changes in T or g. Inflation is $1 + \pi = \phi/\phi_{+1}$, while $1 + \iota = (1+r)\,(1+\pi)$ is the yield on an illiquid nominal bond and $1 + r = u'(c)\,/\beta u'(c_{+1})$ is the yield on a on an illiquid real bond. Here an illiquid asset is one that cannot be traded in the DM – thus, $1+\iota$ is simply the amount of cash in the next CM that makes agents willing to give up a unit of cash in this CM,

[13] Getting q on credit supported by q is like getting a house with a mortgage, and Gavazza (2011*a*) suggests this is realistic in secondary capital markets (although $\chi_q = 1$ may be too high; then again it may not, as we show below that a cash down payment is necessary even if $\chi_q = 1$). To motivate why q is more pledgeable than k, simply imagine that a seller knows the *quality* of the capital he is selling better than he knows the stuff the buyer holds; see Li et al. (2013) and Lester et al. (2013) for explicit information-theoretic models of pledgeability and acceptability, respectively.

while $1+r$ is the same with numeraire replacing cash, and as always we can price these trades even if the assets do not actually exist. The reason for introducing the notation ι is this: in stationary monetary equilibrium z is constant, so $\pi = \mu$ and $1+\iota = (1+\mu)/\beta$, and therefore it is equivalent to describe monetary policy by μ, π or ι. As usual, we impose $\mu > \beta - 1$, or $\iota > 0$, but also consider $\mu \to \beta - 1$, or $\iota \to 0$, which is the Friedman rule.

The CM problem is similar to (5), except the budget equation is now

$$c + \hat{k} + (1+\pi)\hat{z} = wh + a + \Pi(k,\varepsilon) + (1-\delta)k, \tag{14}$$

where $1+\pi$ is the current price of z for the next DM. The key FOC's are

$$\hat{z} \ : \ \frac{A(1+\pi)}{w} = \beta \mathbb{E}_{\hat{\varepsilon}} V_1(\hat{z}, \hat{k}, \hat{\varepsilon}) \tag{15}$$

$$\hat{k} \ : \ \frac{A}{w} = \beta \mathbb{E}_{\hat{\varepsilon}} V_2(\hat{z}, \hat{k}, \hat{\varepsilon}), \tag{16}$$

while the envelope conditions are still (9)-(10). The extension of Lemma 1, which again assumes an interior solution for $h \in (0,1)$, is

Lemma 2 *(i)* $(\hat{z}, \hat{k})$ *is independent of* (a, k, ε)*; (ii)* W *is linear in* a*.*

The DM trading surpluses are

$$\begin{aligned} S_b(\mathbf{s}) &= \frac{A}{w}\left\{\Pi[k_b + q(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) + (1-\delta)\,q(\mathbf{s}) - p(\mathbf{s}) - d(\mathbf{s})\right\} \\ S_s(\mathbf{s}) &= \frac{A}{w}\left\{\Pi[k_s - q(\mathbf{s}), \varepsilon_s] - \Pi(k_s, \varepsilon_s) - (1-\delta)\,q(\mathbf{s}) + p(\mathbf{s}) + d(\mathbf{s})\right\}. \end{aligned}$$

In addition to $q(\mathbf{s}) \leq k_s$, there are two new constraints, $d(\mathbf{s}) \leq (1-\delta)\,q(\mathbf{s})$ and $p(\mathbf{s}) \leq z_b$. Without loss of generality we can say buyers use all the credit they can get, $d(\mathbf{s}) = (1-\delta)\,q(\mathbf{s})$, and as argued above, they still need some cash.

Lemma 3 $q(\mathbf{s}) > 0 \Rightarrow p(\mathbf{s}) > 0$.

Based on these results, the DM surpluses reduce to

$$S_b(\mathbf{s}) = \frac{A}{w}\left\{\Pi[k_b + q(\hat{\mathbf{s}}), \varepsilon_b] - \Pi(k_b, \varepsilon_b) - \hat{p}(\mathbf{s})\right\} \tag{17}$$

$$S_s(\mathbf{s}) = \frac{A}{w}\left\{\Pi[k_s - \hat{q}(\mathbf{s}), \varepsilon_s] - \Pi(k_s, \varepsilon_s) + \hat{p}(\mathbf{s})\right\}. \tag{18}$$

Notice $(1-\delta)\, q(\mathbf{s})$ and $d(\mathbf{s})$ cancel in (17) and (18), but that does *not* make debt irrelevant – it still allows agents to economize on cash, which they like if $\iota > 0$. As regards $p(\mathbf{s}) \leq z_b$, as usual in these kinds of models, it must bind for at least some $\mathbf{s}$ since, intuitively, $\iota > 0$ makes cash a poor saving vehicle.

With less-than-perfect credit, the mechanism used to determine the terms of trade matters more. Gu and Wright (2016) show that for any mechanism in a reasonable class: (i) if $p \leq z_b$ is slack in a type $\mathbf{s}$ meeting then $q(\mathbf{s}) = q^*(\mathbf{s})$ is the same as perfect credit and the mechanism determines $p(\mathbf{s})$; (ii) if $p \leq z_b$ binds then the mechanism determines $q(\mathbf{s}) < q^*(\mathbf{s})$. Hence there is a set $\mathcal{B}$ such that $p \leq z_b$ binds iff $\mathbf{s} \in \mathcal{B}$, where prob$(\mathbf{s} \in \mathcal{B}) > 0$. In particular, with Kalai bargaining $\mathbf{s} \notin \mathcal{B}$ implies $q(s) = q^*(\mathbf{s})$ and

$$p(\mathbf{s}) = (1-\theta)\left\{\Pi[k_b + q^*(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b)\right\} + \theta\left\{\Pi(k_s, \varepsilon_s) - \Pi[k_s - q^*(\mathbf{s}), \varepsilon_s]\right\},$$

while $\mathbf{s} \in \mathcal{B}$ implies $p(\mathbf{s}) = z_b$ and $q = q(\mathbf{s})$ solves

$$z_b = (1-\theta)\left\{\Pi[k_b + q(\mathbf{s}), \varepsilon_b] - \Pi(k_b, \varepsilon_b)\right\} + \theta\left\{\Pi(k_s, \varepsilon_s) - \Pi[k_s - q(\mathbf{s}), \varepsilon_s]\right\}.$$

To define equilibrium, first note that CM consumption and output (c_t, y_t) satisfy the same conditions as above. Goods market clearing is also the same, while money market clearing is simply $m = M$, and labor market clearing is again ignored by Walras' Law. Therefore, letting all agents start with (z_0, k_0), we have:

Definition 3 *Given (z_0, k_0) and paths for $\langle \mu, g, T \rangle$, monetary equilibrium is a list of paths for $\left\langle \hat{k}, q(\cdot), p(\cdot), h, c, \hat{z}, w \right\rangle$ with $\hat{z} > 0$ such that for $\forall t$: (i) $(c, h, \hat{z}, \hat{k})$ solves the CM maximization problem; (ii) $p(\cdot)$ and $q(\cdot)$ solve the DM bargaining problem; and (iii) markets clear.*

Definition 4 *Given constant $\langle \mu, g, T \rangle$, monetary steady state is a time-invariant $\left\langle \hat{k}, q(\cdot), p(\cdot), h, c, \hat{z}, w \right\rangle$ with $\hat{z} > 0$ that satisfies the definition of equilibrium except for initial conditions.*

We now derive the Euler equations for money and capital. First, take the derivatives of $p(\cdot)$ and $q(\cdot)$ wrt their arguments. Then evaluate these at $\hat{\mathbf{s}} = (\hat{z}, \hat{k}, \hat{\varepsilon}, \hat{z}, \hat{k}, \tilde{\varepsilon})$, insert them into the derivatives of $V(\cdot)$ wrt $(\hat{z}, \hat{k})$, and insert those into the FOC's (15)-(16). For money, the result is

$$\frac{1+\pi}{w} = \frac{\beta}{w_{+1}} \left[1 + \alpha \int_{\hat{\varepsilon} > \tilde{\varepsilon}} \Lambda(\hat{\mathbf{s}})\, dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}) \right], \tag{19}$$

where

$$\Lambda(\hat{\mathbf{s}}) \equiv \theta \frac{\Pi_1[\hat{k} + q(\hat{\mathbf{s}}), \hat{\varepsilon}] - \Pi_1[\hat{k} - q(\hat{\mathbf{s}}), \tilde{\varepsilon}]}{D(\hat{\mathbf{s}})}. \tag{20}$$

Using $w_{+1}/\beta w = u'(c)/\beta u'(c_{+1}) = 1 + r$ and $(1+\pi)(1+r) = 1 + \iota$, we get

$$\iota = \alpha \int_{\hat{\varepsilon} > \tilde{\varepsilon}} \Lambda(\hat{\mathbf{s}})\, dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}). \tag{21}$$

The nominal rate on the LHS can be interpreted as the marginal cost of carrying cash, while the RHS is the benefit, since one can show $\Lambda(\hat{\mathbf{s}})$ is the Lagrange multiplier on $p \leq z_b$, representing a wedge in monetary exchange. Notice: $\iota = 0$ implies reallocation is efficient.

Similarly, for capital, the result is

$$\begin{aligned} r + \delta &= (1-\alpha) \int_0^\infty \Pi_1(\hat{k}, \hat{\varepsilon}) dG(\hat{\varepsilon}) \\ &\quad + \alpha \int_{\hat{\varepsilon} > \tilde{\varepsilon}} \Pi_1[\hat{k} + q(\hat{\mathbf{s}}), \hat{\varepsilon}] \Omega(\hat{\mathbf{s}})\, dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}) \\ &\quad + \alpha \int_{\hat{\varepsilon} < \tilde{\varepsilon}} \Pi_1[\hat{k} - q(\tilde{\mathbf{s}}), \hat{\varepsilon}] \Gamma(\tilde{\mathbf{s}})\, dG(\tilde{\varepsilon}) dG(\hat{\varepsilon}), \end{aligned} \tag{22}$$

where we define two other wedges,

$$\Omega(\hat{\mathbf{s}}) \equiv \frac{(1-\theta)\Pi_1(\hat{k}, \hat{\varepsilon}) + \theta \Pi_1[\hat{k} - q(\hat{\mathbf{s}}), \tilde{\varepsilon}]}{D(\hat{\mathbf{s}})} \tag{23}$$

$$\Gamma(\tilde{\mathbf{s}}) \equiv \frac{(1-\theta)\Pi_1[\hat{k} + q(\tilde{\mathbf{s}}), \hat{\varepsilon}] + \theta \Pi_1(\hat{k}, \tilde{\varepsilon})}{D(\tilde{\mathbf{s}})}. \tag{24}$$

Notice: $\theta < 1$ implies $\Omega(\hat{\mathbf{s}}) > 1$, which raises CM demand for k because buying it in the DM is less attractive when sellers extract part of the surplus; $\theta > 0$ implies $\Gamma(\tilde{\mathbf{s}}) < 1$, which lowers CM demand for k because selling it in the DM is less attractive when buyers extract part of the surplus; and at $\iota = 0$ efficiency obtains at $\theta = \theta^*$, as it does with perfect credit.

What happens at $\iota > 0$? First, higher ι increases the demand for k in the CM, because buying it in the DM is less attractive when liquidity is more costly, reminiscent of the Mundell-Tobin effect even if our microfoundations are different. Second, higher ι decreases demand for z, and this reduces CM investment in k, because the option value of selling it in the DM is less attractive when there is less cash in the market, reminiscent of the Keynesian notion that lower nominal rates stimulate real investment even if our microfoundations are again different. Below we discuss which effect dominates. For now, we summarize as follows:[14]

Proposition 3 *(i) Monetary equilibrium with $\iota = 0$ is the same as perfect credit, and thus efficient iff $\theta = \theta^*$. (ii) With $\iota > 0$ it is never efficient.*

5 A Convenient Parameterization

Before studying more general versions, consider a specification that delivers sharp analytic results with clear economic intuition. First suppose $\varepsilon \in \{\varepsilon_H, \varepsilon_L\}$, $\varepsilon_H > \varepsilon_L$, with prob$(\varepsilon_L) = \gamma_L$ and prob$(\varepsilon_H) = \gamma_H = 1 - \gamma_L$. This implies the bargaining solution is a number q, not a function $q(\mathbf{s})$. It also implies $\alpha_H = \alpha\gamma_L$ and $\alpha_L = \alpha\gamma_H$. Moreover, $p \leq z_b$ binds in every DM trade, since it must bind in some trade, and now there is only one kind of meeting with trade – a firm with ε_H meets one with ε_L. Also, suppose $F(k, h, \varepsilon) = \varepsilon f(k) + h$, which is obviously special, but useful because it pins down $w = 1$, independent of other variables.

[14] Apropos the literature, Kurman and Rabinovitz (2018) and the papers cited therein have k holdups problem but no m holdup problem, since they have no money. Papers surveyed by Lagos et al. (2017) have m holdup problems but no k holdup problem, since they have no capital, with exceptions like Aruoba et al. (2011), but there agents trade consumption in the DM, not capital, so our investment channel is missing. Wright et al. (2017) has m and k holdup problems, but the implications are very different since there some agents bring k but not m to the DM while others bring m but not k, and here all agents brings both. Hence, that setup is more like the labor market models of Masters (1998, 2011) or Acemoglu and Shimer (1999), where firms invest in physical capital and workers in human capital. This is not a minor technicality: when some agents bring k and others bring m, there is no θ^* that achieves first best with bargaining (although one can achieve it with posting as in Section 6.1 below).

With this specification output is

$$y = \psi f(k) + \xi \varepsilon_H f(k+q) + \xi \varepsilon_L f(k-q) + h,$$

where $\psi = \gamma_L (1-\alpha_L) \varepsilon_L + \gamma_H (1-\alpha_H) \varepsilon_H$ is average productivity for firms that do not trade in the DM and $\xi = \gamma_H \alpha_H = \gamma_L \alpha_L$ is the volume of DM trades. Then the FOC's from the planner problem are much-simplified versions of the general case:[15]

$$q \; : \; 0 = \varepsilon_H f'(k+q) - \varepsilon_L f'(k-q) \tag{25}$$

$$K \; : \; r + \delta = \psi f'(k) + \xi \varepsilon_H f'(k+q) + \xi \varepsilon_L f'(k-q). \tag{26}$$

Now (25) defines $q = Q(k)$ and (26) defines $k = K(q)$, where both are single-valued. We call $k = K(q)$ the IS curve, the standard name for the investment Euler equation; we call $q = Q(k)$ the CR curve, for capital reallocation; and the planner's solution obtains at their intersection. The slopes of CR and IS are

$$\begin{aligned} \frac{\partial q}{\partial k}_{|CR} &= \frac{\Phi(k,q)}{\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)} \\ \frac{\partial q}{\partial k}_{|IS} &= \frac{\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q) + \psi f''(k)/\xi}{\Phi(k,q)} \end{aligned}$$

where

$$\Phi(k,q) \equiv f'(k+q) f''(k-q) - f'(k-q) f''(k+q).$$

Hence, when they cross, both slope down if $\Phi(k,q) > 0$ and both slope up if $\Phi(k,q) < 0$. Since a few results depend on this, we differentiate cases by[16]

$$\Phi(k,q) < 0. \tag{Condition F}$$

[15]Different from the general case, (26) is a static condition: while we still have $1 + r = u'(c)/\beta u'(c_{+1})$, $w = 1$ implies $u'(c) = A$ in and out of steady state. Thus we can jump to steady state in one period unless $h \in [0,1]$ binds. To see what this entails, consider the case with no DM, a standard growth model with utility and production linear in h. It has a unique steady state $\bar{k} > 0$ where c and h solve $u'(\bar{c}) = A$ and $\bar{h} = \bar{c} + \delta \bar{k} - f(\bar{k})$, and we assume $\bar{h} \in (0,1)$. If k_0 is below $\bar{k}$ but close, we can jump to $\bar{k}$ in one period by setting $h_0 = \bar{c} + \bar{k} - f(k_0) - (1-\delta) k_0$. But if k_0 is so low that $h_0 \leq 1$ binds, the transition has $h_t = 1$ for $t = 1, 2, ...$ until we reach k such that $h = \bar{c} + \bar{k} - f(k) - (1-\delta) k \leq 1$, whence we jump to $\bar{k}$. The situation is symmetric if k_0 is so high that $h_0 \geq 0$ binds.

[16]Both $\Phi < 0$ and $\Phi > 0$ are possible, as can be shown by example, but Condition F holds for common functions like $f(k) = k^\eta$. It implies that when IS shifts left, e.g., due to an increase in $r + \delta$, both k and q fall, as may seem natural but is not necessary in the theory.

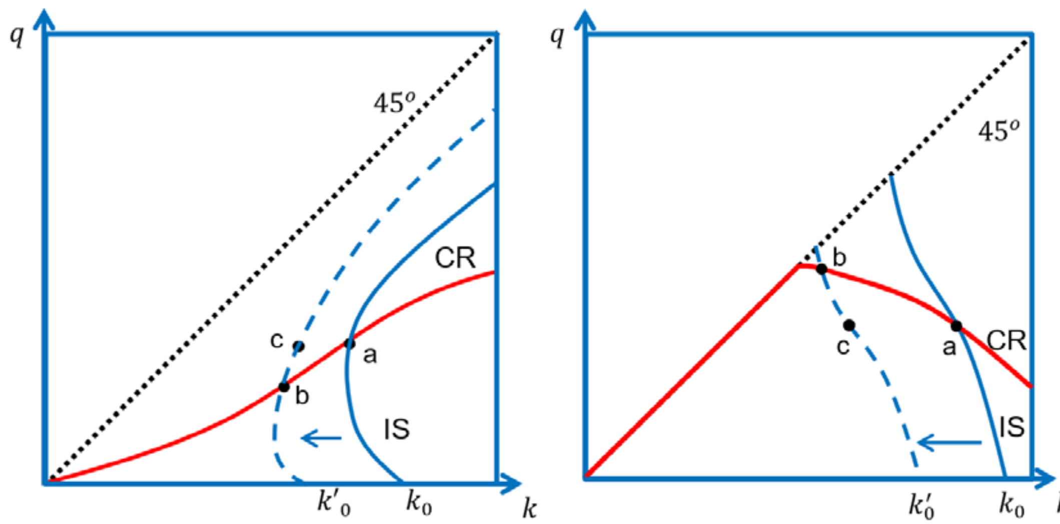


Figure 2: IS and CR for planner with $\Phi < 0$ (left) or $\Phi > 0$ (right)

This is shown Figure 2. Notice in the right panel that the CR curve coincides with the 45^o line when k is low, and the solution can occur on this segment. In this situation $q = k$, so when ε_H firms contact ε_L firms, the former get all the capital – a takeover. However, we can preclude that with $f'(0) = \infty$. This also precludes $k = 0$, and we can rule out $q = 0$ by having ι not too big (see below). Given this, the Appendix proves:

Proposition 4 *For the convenient parameterization, CR and IS cross uniquely at $k^* > 0$ and $q^* > 0$.*

Moving from the planner's problem to equilibrium, let us introduce a proportional tax on capital income τ (we did not do this sooner because the general model is already notationally intense). This changes $\varepsilon f(k)$ to $(1-\tau)\varepsilon f(k)$ in agents' budget equation and changes the bargaining solution to

$$\frac{z}{1-\tau} = (1-\theta)\varepsilon_H[f(k+q) - f(k)] + \theta\varepsilon_L[f(k) - f(k-q)]. \tag{27}$$

Now, jumping right to the Euler equations, we have

$$\iota = \xi\Lambda(k,q) \tag{28}$$

$$\frac{r+\delta}{1-\tau} = \psi f'(k) + \xi\varepsilon_H f'(k+q)\Omega(k,q) + \xi\varepsilon_L f'(k-q)\Gamma(k,q) \tag{29}$$

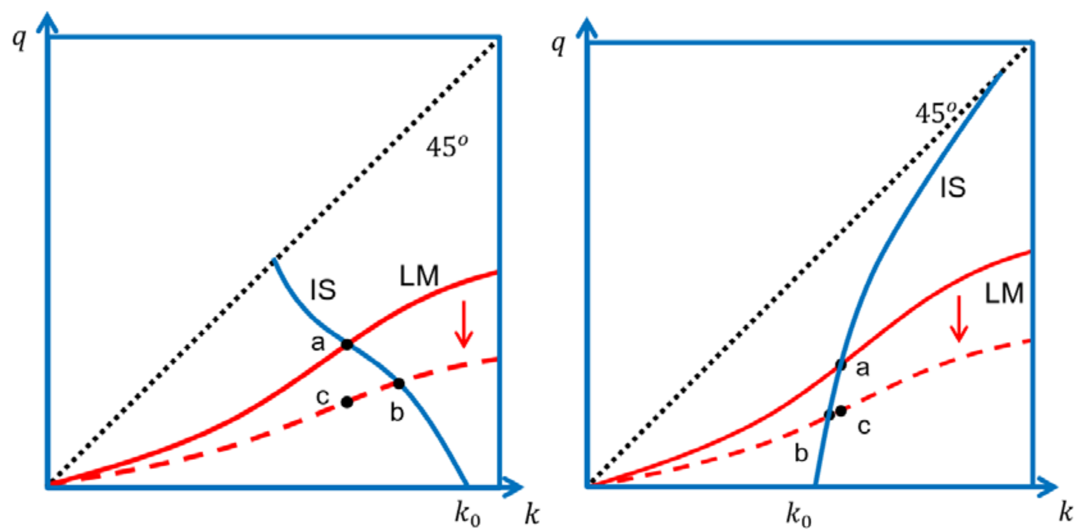
where $\Lambda(k,q)$, $\Omega(k,q)$ and $\Gamma(k,q)$ are the wedges discussed above, written here as functions of (k,q). As in the planner problem, (28)-(29) define two curves, but now (28) is actually the LM curve from (some) undergrad macro classes. So we call it that, even if we continue to plot the curves (k,q) space, because those are our key endogenous variables.

Labels aside, existence of monetary equilibrium requires ι below a threshold $\bar{\iota}$. Uniqueness also requires conditions since there are complementarities at work – i.e., when there is more cash in the market, agents may want to bring more capital, and vice versa. The Appendix proves:

Proposition 5 *For the convenient parameterization, monetary steady state exists iff*

$$\iota < \bar{\iota} \equiv \frac{\alpha_H \gamma_H \theta\left(\varepsilon_H - \varepsilon_L\right)}{\left(1-\theta\right)\varepsilon_H + \theta\varepsilon_L}.$$

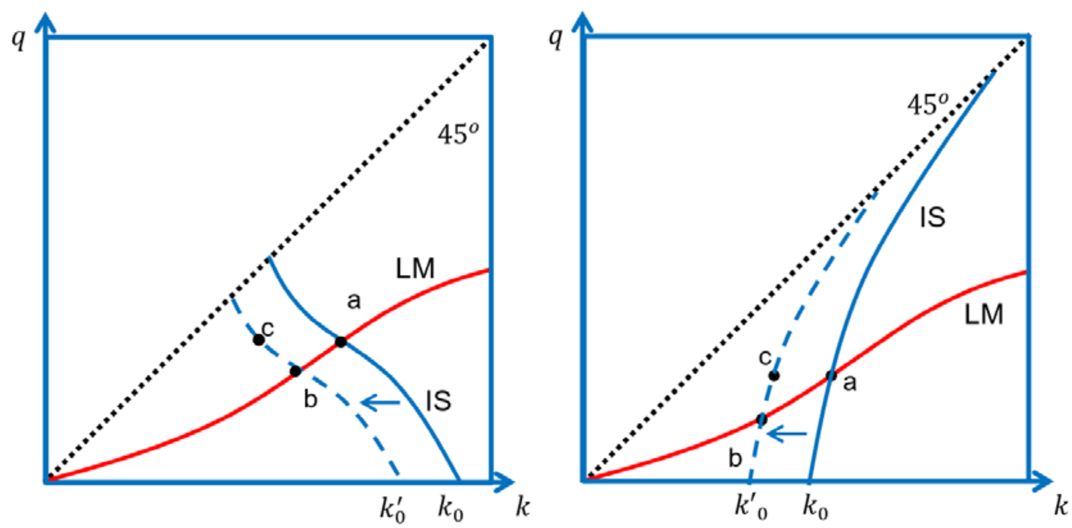
It is unique if either θ is not too small or ι is not too big.



Case 1: High θ Case 2: Low θ

Figure 3: IS and LM, with monetary policy increasing ι

As shown in Figure 3, LM starts at $(0,0)$, lies below the 45^0 line, and while it is not monotone increasing in general it is under Condition F, which we impose for this discussion. Even under Condition F, however, the slope of IS curve depends on θ, as shown in Figure 3.[17] Also shown are the effects of monetary policy. Increasing ι does not affect IS, and rotates LM clockwise until we hit $\bar{\iota}$, at which point it hits the axis and monetary equilibrium breaks down. For $\iota < \bar{\iota}$, increasing ι moves us from a to b, so q decreases, and k decreases or increases as IS slopes up or down. Hence, at least for some θ, lower nominal rates stimulate real investment, consistent with Keynesian doctrine, but the logic is different: here lowering ι reduces the cost of liquidity, which facilitates DM trade, and for low θ this raises k because it improves the option value of selling it in secondary markets. To see the multipliers at work, observe that an increase in ι would move us from a to c if k were fixed, but since k reacts we move to b, attenuating the fall in q in the left panel and accentuating it in the right.



Case 1: High θ Case 2: Low θ

Figure 4: IS and LM, with fiscal policy increasing τ

Figure 4 shows the effects of fiscal policy. Increasing τ shifts IS left but does not affect LM, so q and k both decrease regardless of whether IS slopes up or

[17] Heuristically, there are two effects: first, there is less need to bring k from the CM when q is bigger in the DM, and that tends to make IS decreasing; second, higher q means selling capital in the DM is more lucrative, which tends to raise k, and that makes IS increasing.

down. To see the multipliers at work, observe that an increase in τ would move us from a to c if q were fixed, but since q in fact reacts, we move to b, attenuating the fall in k in the left panel and accentuating it in the right. Summarizing the fiscal implications, higher τ unambiguously reduces both k and q. This is different from monetary policy, where higher ι unambiguously reduces q, but depending on θ can increase or decrease k.

To consider optimal policy, first, note that as in the general model $\iota = \tau = 0$ implies k is efficient iff $\theta = \theta^*$, but now we have a simple expression for

$$\theta^* = \frac{\varepsilon_L \left[f'(k^* - q^*) - f'(k^*) \right]}{(\varepsilon_H - \varepsilon_L) f'(k^*)}. \tag{30}$$

Suppose $\theta \neq \theta^*$ and let us eliminate the monetary wedge by setting $\iota = 0$ to ask about optimal fiscal policy. It is easy to check that full efficiency obtains at $\tau = \tau^*$, where

$$1 - \tau^* = \frac{r + \delta}{\xi \varepsilon_H f'(k^* + q^*) + (1 - \theta \alpha_H) \gamma_H \varepsilon_H f'(k^*) + [1 - (1 - \theta) \alpha_L] \gamma_L \varepsilon_L f'(k^*)}.$$

One can check τ^* is decreasing in θ, and that implies the following:

Proposition 6 *For the convenient parameterization with $\iota = 0$, optimal fiscal policy is $\tau = \tau^*$, where $\tau^* = 0$ if $\theta = \theta^*$, $\tau^* < 0$ if $\theta > \theta^*$, and $\tau^* > 0$ if $\theta < \theta^*$. This achieves full efficiency.*

Similarly, suppose $\theta \neq \theta^*$ and let us eliminate the fiscal wedge by setting $\tau = 0$ to ask about optimal monetary policy. The Appendix proves the next result, showing how monetary and fiscal policy are not symmetric, in the sense that $\iota > 0$ is optimal for both big θ and small θ, and generally full efficiency is illusive.

Proposition 7 *For the convenient parameterization with $\tau = 0$, there exist $\underline{\theta}$ and $\bar{\theta}$, with $0 < \underline{\theta} \leq \theta^* \leq \bar{\theta} < 1$, such that optimal monetary policy is $\iota^* > 0$ for $\theta > \bar{\theta}$ and for $\theta < \underline{\theta}$. In neither case do we achieve full efficiency.*

To understand this, first recall big θ makes agents underinvest in the CM since they get a good deal buying k in the DM. Higher ι counters this by taxing the secondary market, raising primary investment and welfare. Then recall small θ small makes agents overinvest in the CM since they get a good deal selling k in the DM. Higher ι again taxes the secondary market, this time lowering primary investment but raising welfare. Clearly $\iota^* > 0$ is a second-best result: while it mitigates underinvestment for big θ and overinvestment for small θ, in neither case do we get efficient reallocation, because q^* requires $\iota = 0$.

6 Extensions

6.1 Competitive Search

Instead of random search and bargaining, consider directed search and price posting, a combination called competitive search equilibrium (see the survey by Wright et al. 2018). This means agents can communicate before they meet in the DM by posting the terms of trade, to which they commit, as a way to compete for counterparties. As is well known from other applications, this kind of communication and commitment can enhance efficiency, compared to bargaining after agents meet. Heuristically, competitive search can be understood as overcoming holdup problems like those in the benchmark model, which allows us to isolate the effects coming from bargaining and from other features of the environment.

Note that being able to commitment to the DM prices does not mean agents can commit to everything; so we can either use or not use the frictions discussed above precluding perfect credit. Also, since we get the same outcome whether sellers post and buyers choose where to search, or vice versa, for convenience we let buyers post.[18] Then the set of buyers posting the same terms, plus the set of

[18] While it is often equivalent to have buyers post and sellers search or vice versa, there are exceptions (Delacroix and Shi 2017). There is a third approach, where third parties called market makers set up submarkets posting terms to attract both buyers and sellers, that also delivers the same outcome, at least if we get around the complications in Faig and Huangfu (2007) by, say, the method in Rocheteau and Wright (2005).

sellers directing their search toward them, defines a submarket. Although sellers direct their search to a particular submarket, within submarkets there are still bilateral random meetings, where $\alpha_s = \alpha(n)$ is the probability a seller meets a buyer, $\alpha_b = \alpha(n)/n$ is the probability a buyer meets a seller, $n = n_b/n_s$ is submarket tightness, and $\alpha(n)$ satisfies the usual assumptions.

As in Section 5, consider $F(k, h, \varepsilon) = \varepsilon f(k) + h$ and a two-point ε distribution. Then buyers are firms realizing ε_H, after which they post DM terms. When they post, therefore, z and k are predetermined from the CM. Also, anticipating some results, in equilibrium all active submarkets are the same and hence all have $n = \gamma_H/\gamma_L$, but to find equilibrium we first let n be a choice and then equilibrate $n = \gamma_H/\gamma_L$. What gets posted is (p, q, n), meaning this: when buyers and sellers meet in this submarket, they trade q units of capital for a payment p, and tightness in this submarket is n (as usual, it is not important to post n, as agents can figure it out from p and q).

The buyer's posting problem is then

$$\begin{aligned} v_b &= \max_{p,q,n} \frac{\alpha(n)}{n} A \{(1-\tau)\varepsilon_H [f(k_b + q) - f(k_b)] - p\} \\ &\text{st } \alpha(n) A \{p - \varepsilon_L (1-\tau) [f(k_s) - f(k_s - q)]\} = v_s, \end{aligned} \tag{31}$$

where lower case v_j is the per-period version of V_j. The constraint says buyers can get sellers iff they match their market payoff v_s, which is taken as given by individuals but is endogenous in equilibrium. There is another constraint $p \le z$, since buyers cannot hand over cash they do not have, but without loss of generality we can set $z = p$. From (31) we get the terms of trade as functions of (z, k_s). Then solving the CM problem for (z, k_s) we obtain

$$\iota = \frac{\alpha(n)\gamma_H}{n} \frac{\varepsilon_H f'(k+q) - \varepsilon_L f'(k-q)}{\varepsilon_L f'(k-q)} \tag{32}$$

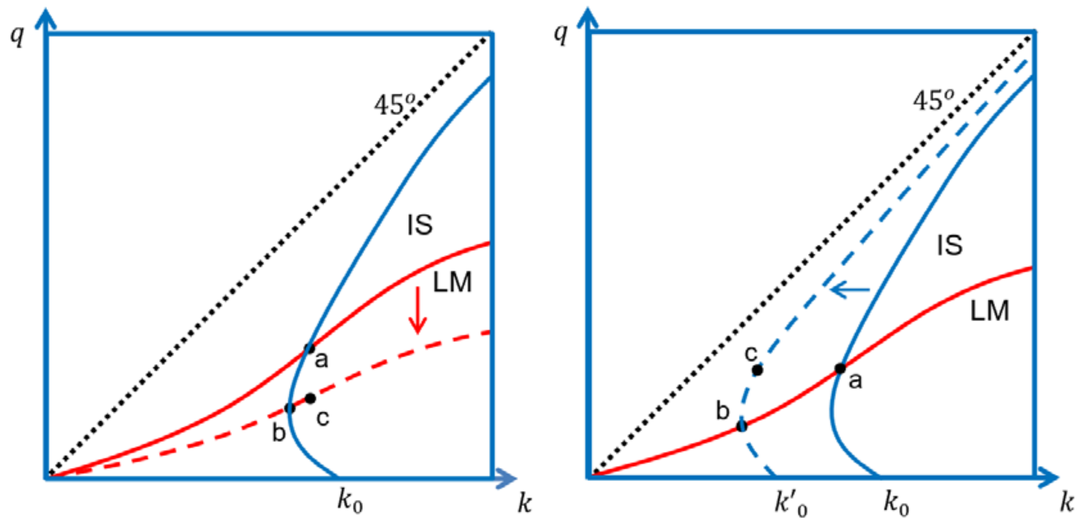
$$\frac{r+\delta}{1-\tau} = \psi f'(k) + \xi \varepsilon_H f'(k+q) + \xi \varepsilon_L f'(k-q). \tag{33}$$

This system is shown by the IS and LM curves in Figure 5. Then we have:

Proposition 8 *For the convenient parameterization with competitive search, monetary steady state exists iff* $\iota < \hat{\iota} \equiv \alpha_H \gamma_H (\varepsilon_H - \varepsilon_L) / \varepsilon_L$. *When it exists it is unique. If* $\tau = 0$ *perfect-credit equilibrium is efficient and monetary equilibrium is efficient iff* $\iota = 0$.

Similar to Proposition 5 existence requires ι below a threshold, but now the threshold is bigger; intuitively, this is because competitive search is a better trading arrangement. Also, although IS is nonmonotone, we get uniqueness even without the parameter conditions Proposition 5; heuristically, competitive search delivers uniqueness because posting internalizes complementarities in the choices of k and z (as discussed in Rocheteau and Wright 2005). The efficiency results follow immediately from comparing the planner problem to equilibrium with perfect credit and to equilibrium with money. In terms of policy, higher τ decreases both k and q; while higher ι decreases q but can in general increase or decrease k, depending on whether the LM curve intersects the IS curve in its increasing or decreasing region.[19]

[19] Again, directed search and posting is a relatively good way to organize markets, compared to random search and bargaining, but it does require communication and commitment. Actual capital markets are probably in between, with some trade better characterized by random search and bargaining, and other trade by directed search and posting. In principle, we could combine them in one model (see Lester 2011 or Bethune et al. 2018 for examples in goods markets), with shares disciplined by the data, but that is beyond the scope of this project.



Increase ι Increase τ

Figure 5: IS and LM with competitive search, monetary and fiscal policy

6.2 Integated Money Demand

So far, firms (or their owners) hold all the cash to finance secondary capital acquisition. In reality, households also use cash in goods markets, as modeled in many other papers. So consider an economy with two types, entrepreneurs and households, where the former are like the agents in the benchmark model, while the latter supply h and demand c in the CM, but also want a different good C traded among themselves in their own DM (we can instead have them trade with retailers and get similar results). Households sometimes need cash in their DM for the same reasons that firms need it in theirs. Hence, liquidity will generally be demanded by both firms and households, as in He and Zhang (2019).

Let households have the same preferences in the CM over c and h. Then as in much other monetary theory, in their DM they get utility $U(C)$ from a perishable good or service produced on the spot by others at unit cost in terms of disutility. Suppose they match bilaterally at random in their DM, and with little loss in generality suppose there is no direct barter because there are no double-coincidence meetings, only single-coincidence meetings, where one agent likes what his partner produces but not versa. In such meetings each household is a buyer or seller with equal probability. Moreover, we allow perfect credit in some of these meetings, say because the households know each other, while others require money, say because they do not know each other. Let the probabilities of credit and money meetings be σ_c and σ_m.

Households' CM problem is

$$W^h(a) = \max_{s,h,\hat{z}} \left\{ u(c) - Ah + \beta V^h(\hat{z}) \right\} \text{ st } c + (1+\pi)\hat{z} = wh + a$$

where $a = z - d - T$. In the DM,

$$\begin{aligned} V^h(z) &= W^h(a) + \sigma_m \left[U(C_m) - \frac{A}{w} p_m \right] + \sigma_m \left[\frac{A}{w} \tilde{p}_m - \tilde{C}_m \right] \\ &\quad + \sigma_c \left[U(C_c) - \frac{A}{w} p_c \right] + \sigma_c \left[\frac{A}{w} \tilde{p}_c - \tilde{C}_c \right], \end{aligned}$$

where the second two terms on the RHS are surpluses from buying and selling with cash, while the last two are surpluses from buying and selling on credit. In each case, the terms of trade are determined by Kalai bargaining with ς denoting buyers' share.

For credit meetings it is easy to verify $C_c = C^*$ and $p_c = p^*$, where $U'(C^*) = 1$ and $p^* A/w = (1-\varsigma) U(C^*) + \varsigma C^*$. For money meetings, $p_m = z$ and C_m solves $zA/w = (1-\varsigma) U(C_m) + \varsigma C_m$. It is standard to derive households' Euler equation $\iota = \sigma_m \lambda(c_m)$, where

$$\lambda(C_m) = \frac{\varsigma [U'(C_m) - 1]}{(1-\varsigma) U'(C_m) + \varsigma}$$

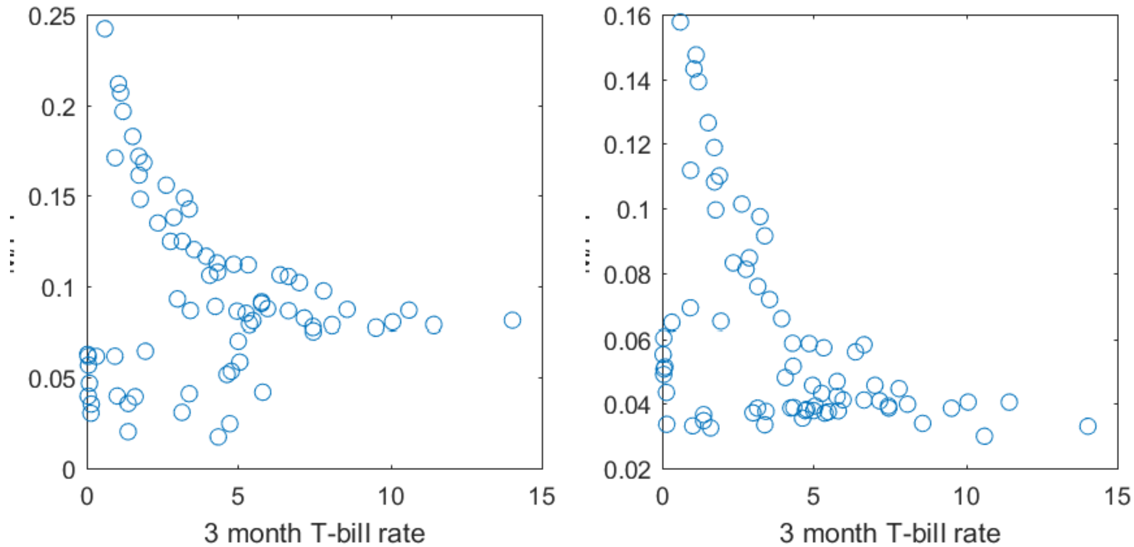
is their liquidity premium. Since the entrepreneur's problem is the same as before, steady state is determined by

$$\begin{aligned} \iota &= \sigma_m \lambda(C_m), & (34) \\ \iota &= \alpha_H \gamma_H \Lambda(k, q), & (35) \\ \frac{r+\delta}{1-\tau} &= \gamma_H \varepsilon_H \left[\alpha_H f'(k+q) + (1-\alpha_H) f'(k) \right] & (36) \\ &\quad + \gamma_L \varepsilon_L \left[\alpha_L f'(k-q) + (1-\alpha_L) f'(k) \right]. \end{aligned}$$

Proposition 9 *For the convenient parameterization when households use money in goods markets, monetary steady state exists iff* $\iota < \max\{\sigma_m \lambda(0), \bar{\iota}\}$. *It is unique if either* θ *is not too small or* ι *is not too big. In terms of policy, Proposition 7 is valid as stated.*

Notice the system dichotomizes: C_m solves (34), then independently k and q are determined exactly as before. It is known how to break this kind of dichotomy in related models – e.g., make household utility nonseparable between C and c, or make k and input to production of C. That may well be interesting, but this simple version is certainly convenient because we can study firms' demand for money without reference to households' demand for money, hence rationalizing the way we ignored the latter in the benchmark model. In equilibrium, both households and firms use cash if $\iota < \min\{\sigma_m \lambda(0), \bar{\iota}\}$, only households use it if $\bar{\iota} < \iota < \sigma_m \lambda(0)$, and only firms use it if $\sigma_m \lambda(0) < \iota < \bar{\iota}$. In particular, if ι changes due to variation in monetary policy, it is possible to have firms use cash in low- but not high-inflation episodes, while household use it all the time. The money demand of firms and households predicted by our theory is consistent with data, as shown in Figure 6.[20]

[20] The vertical axis is money holdings by corresponding entities normalized by nominal GDP. All data are from FRED. For the money holding data of the households, we use the checkable deposits and currency data of the households. For that of the firms, we use the checkable deposits and currency data of non-financial business.



Household Money Demand Business Money Demand

Figure 6: Money Demand

6.3 A Somewhat Convenient Parameterization

In the interest of utmost tractability, in Section 5, $\varepsilon \in \{\varepsilon_L, \varepsilon_H\}$ and $F(k, h, \varepsilon)$ is linear in h. We now show the method works, if somewhat less easily, if we relax the second restriction by using $F(k, h, \varepsilon) = \varepsilon^{1-\eta_h} k^{\eta_k} h^{\eta_h}$, where $\eta_k > 0$, $\eta_h > 0$ and $\eta_k + \eta_h < 1$ (so we still have DRS). In this case CM profit is

$$\Pi(k, \varepsilon) = \max_h \varepsilon^{1-\eta_h} k^{\eta_k} h^{\eta_h} - wh.$$

Letting $B(w) \equiv \left(\frac{\eta_h}{w}\right)^{\frac{1}{1-\eta_h}}$, the solution is $h = \varepsilon B(w) k^{\eta}$, which means $\Pi(k, \varepsilon) = B(w) \varepsilon k^{\eta}$ takes over the role of $f(k)$ in Section 5.

Given w the previous approach yields two equations in (k, q)

$$\iota = \xi \Lambda(k, q), \tag{37}$$

$$\frac{r + \delta}{1 - \tau} = B(w) H(k, q), \tag{38}$$

where

$$\begin{aligned} H(k,q) &= \gamma_H \varepsilon_H \eta \left[\alpha\gamma_L (k+q)^{\eta-1} \Omega(k,q) + (1-\alpha\gamma_L) k^{\eta-1}\right] \\ &\quad + \gamma_L \varepsilon_L \eta \left[\alpha\gamma_H (k-q)^{\eta-1} \Gamma(k,q) + (1-\alpha\gamma_H) k^{\eta-1}\right], \end{aligned}$$

and Λ, Ω and Γ are similar to Section 5. To determine w, use the market clearing condition

$$\delta k + c + g = Y(w,k), \tag{39}$$

where $u'(c) = A/w$ and $Y(w,k)$ is output given w and k,

$$\begin{aligned} Y(w,k) &= \frac{B(w)}{1-\eta_h} \gamma_H \varepsilon_H \left[\alpha\gamma_L (k+q)^{\eta} + (1-\alpha\gamma_L) k^{\eta}\right] \\ &\quad + \frac{B(w)}{1-\eta_h} \gamma_L \varepsilon_L \left[\alpha\gamma_H (k-q)^{\eta} + (1-\alpha\gamma_H) k^{\eta}\right]. \end{aligned}$$

The system reduces to two equations in (k,q) as in Section 5, because we can solve w as a function of (k,q) using (38). Using this and $u'(c) = A/w$ to eliminate c and w from (39), we obtain

$$\begin{aligned} & Y\left(\eta_h \left[\frac{(1-\eta_h)(1-\tau) H(k,q)}{r+\delta}\right]^{\frac{1-\eta_h}{\eta_h}}, k\right) \\ &= \delta k + u'^{-1}\left(\frac{A}{\eta_h}\left[\frac{(1-\eta_h)(1-\tau) H(k,q)}{r+\delta}\right]^{-\frac{1-\eta_h}{\eta_h}}\right) + g. \end{aligned} \tag{40}$$

Now (k,q) is determined by (37) and (40), the LM and IS curves. At $q=0$, (40) reduces to

$$\frac{A}{\eta_h}\left[\frac{(1-\eta_h)(1-\tau) H(k,0)}{r+\delta}\right]^{-\frac{1-\eta_h}{\eta_h}} = u'\left\{\left[\frac{r+\delta}{(1-\tau)(1-\eta_h)\eta} - \delta\right] k - g\right\}.$$

The LHS is increasing in k and the RHS is decreasing in k. Also, if k small, the RHS large, so there exists a unique $k_0 > 0$ solving this condition. Similarly, one can show that there exists a unique $\bar{k}$ solving (40). This suggests IS starts at $(k,0)$ in (k,q) space and reaches the 45^o line at $(\bar{k},\bar{k})$. If ι is not too big, the LM curve starts at $(0,0)$ and goes to (∞,∞). By continuity, these two curves have to intersect at least once. While this version is complicated because c and w are endogenous, the economics is similar.

7 Quantitative Analysis

We now analyze the quantitative implications of the model using the Cobb-Douglas technology in Section 6.3 and a CM utility function $u(c) = \log(c)$. Also ε has a two-point distribution. Moreover, we assume here that half of the firms can access the used capital market in any given period. We also introduce a labor tax τ_h and denote the capital tax by τ_k. We calibrate an quarterly model to the US economy from 1984 onwards.[21]

Table 1 shows the targets and calibrated parameter values. Many of these are standard, so we focus on others. Low-productivity firms are assumed 60% as productive as high-productivity firms, and with mean normalized to 1 we end up with $\varepsilon_H = 1.25$ and $\varepsilon_L = 0.75$. The credit limit D is set to match a 20% down payment ratio and the meeting probability α is set to match the spending on used capital as a fraction of total investment spending. There is little data on the bargaining power θ, so as a benchmark we set it to 0.6, capturing that buyers have a slightly higher bargaining power, but then we consider robustness on this dimension.

[21] We choose this period because the ratio of expenditure on used capital to total investment has a trend prior to 1984 (Cui 2017). The other series are chosen to be consistent with this. The annual real interest rate is obtained by substracting inflation from the Aaa bond yield, both from the FRED database. The benchmark calibration does not exclude the period of the financial crisis, but results are robust to that detail. We also tried a monthly calibration and found similar results.

Parameter	Targets	Value
β	Annual Real Rate 4.03%	0.9902
μ	Annual Inflation Rate 2.69%	0.0067
A	Hours Worked 1/3	31.41
δ	Standard	0.1
τ_h	Average Labor Income Tax	0.22
τ_k	Average Capital Income Tax	0.3
γ_H	Symmetry	0.5
ε_H	Normalization	1.25
ε_L	Productivity Ratio 0.6	0.75
τ_g	Government Spending	0.17
η_h	Labor Share 0.6	0.6168
η_k	Annual Capital Output Ratio 1.45	0.3202
D	Down Payment 20%	0.8244
θ	Benchmark	0.6
α	LE Ratio 30%	.9102

Table 1: Parameters for Quantitative Analysis

7.1 Monetary Policy

Figure 6 shows how ι changes the economy across steady states at the calibrated parameters, where ι is the annualized nominal interest rate. Except for the L/E ratio, all variables are normalized such that they take the value 1 at $\iota = 0$, the Friedman rule. The total capital stock, output and total employment are non-monotone in ι, decreasing for small ι and increasing for large ι. At the current nominal interest $\iota = 6.72\%$, higher ι decreases the capital stock, employment and output, consistent with Keynesian thinking, although for different reasons, as we said above. The real price of used capital is increasing in ι for most values of ι, a consequence of two forces: higher ι reduces q and k for the most ι; and both effects raises the value of an additional unit of capital to firms. Reallocation shuts down if ι is around 14%, which corresponds to an inflation rate around 10%.[22]

[22] If we embed the housholds into the model as described in Section 6.2, households may still hold money for DM transactions even if ι is above 14%.

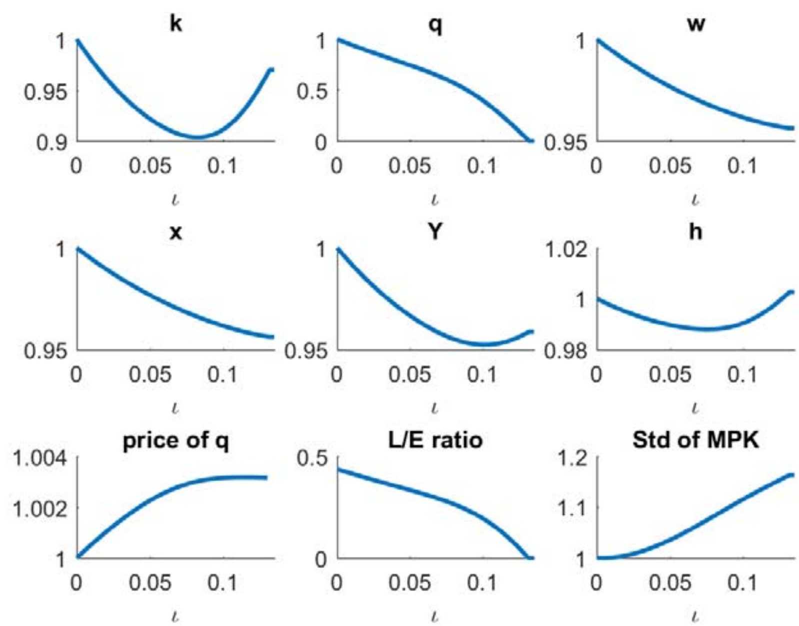


Figure 6: Effects of Monetary Policy on Real Variables

We also plot the standard deviation of the cross-sectional of marginal product of capital (MPK). It is sometimes considered as a measure of misallocation. Although it is difficult to discern, the standard deviation is non-monotone in ι, first decreasing and then increasing with ι. Note that ι can be interpreted as a financial friction, because higher ι reduces liquidity, which has similar effects as a lower borrowing limit. Therefore, our result suggests that a lower standard deviation of MPK does not necessarily imply less frictions. Moreover, it does not necessarily imply a higher output because here, if ι is around 10%, increasing it raises both output and the standard deviation of MPK. This is due to general equilibrium effects of ι on k. If ι increases, it is harder to trade in the used capital market, so investment in the primary market goes up, increasing the capital stock and output.

Figure 7 shows impact on welfare of monetary policy measured in the usual way – the percentage of consumption that a household is willing to give up to go to $\iota = 0$ from some $\iota > 0$. Notice the welfare cost is increasing in ι. Hence, the standard deviation of MPK is not monotone in the welfare. Moreover, the welfare cost is very large. A 10% annual inflation can reduce welfare by around 5.5%. This is similar to the numbers in Lagos and Wright (2005) in a setting where inflation taxes household consumption, although more recently, using other models and methods, those numbers have been revised downward. Here ι affects the accumulation of capital through the secondary market, and that can have a big welfare effect.

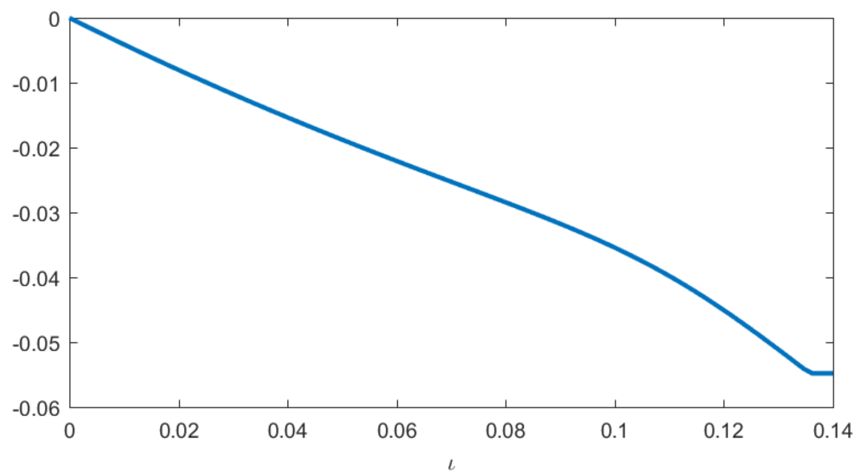


Figure 7: Effects of Monetary Policy on Welfare

7.2 Stylized Facts

Now we show how the model can match some stylized facts in the literature: capital reallocation is procyclical; capital mismatch is countercyclical; the price of capital in the secondary market is procyclical; and spending on capital in this market, as a fraction of total investment, is procyclical. We consider two exogenous changes that could drive the cycle: increasing productivity and expanding credit conditions. It turns out we need both.

First, consider only a productivity shock. To generate a boom, we increase ε_L but keep ε_H fixed, appealing to the findings in Kehrig (2015) that dispersion in productivity is countercyclical, with firms at the lower end more effected by business cycles.[23] The results are shown in Figure 8. The model is consistent with many stylized facts except that spending on the used capital is counterfactual, i.e. the L/E ratio decreases as output increases. Intuitively, higher ε_L leads to less capital reallocation because the low productivity firms are less willing to sell used capital. This is true as long as ε_H does not increase as much as ε_H. Now suppose only credit conditions improve, as shown in Figure 9. This makes the price of used capital counterfactual, i.e. it decreases with output. Now buyers can buy more in a boom because credit conditions are better, and due to DRS buyers are willing to pay less for additional capital, so the average unit price drops.

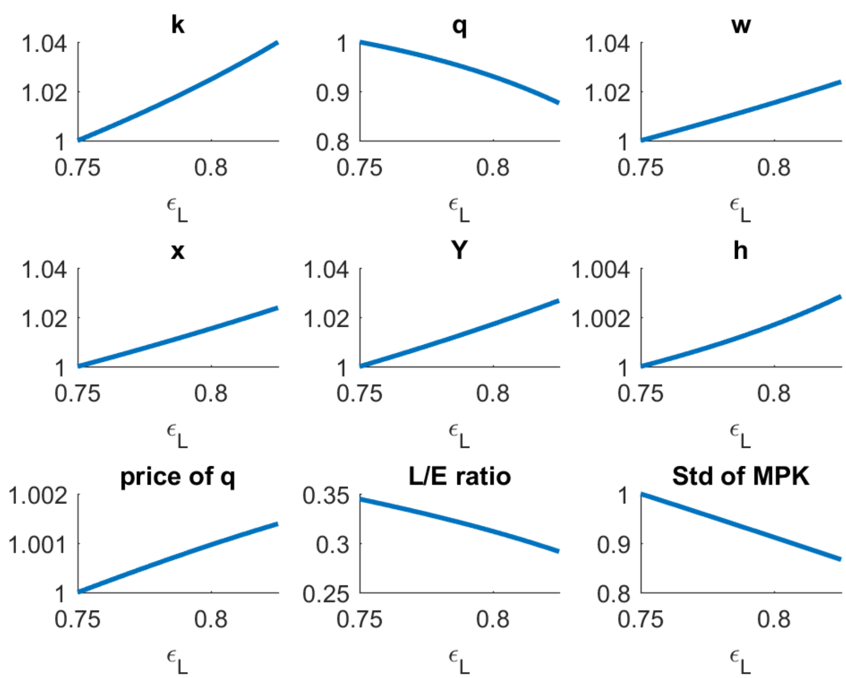


Figure 8: Effects of ε_L on Real Economy

[23] As Kehrig (2015) says: "First, crosssectional productivity dispersion is countercyclical; the distribution of total factor productivity levels across establishments is about 12% more spread-out in a recession than in a boom. Second, the bottom quantiles of the productivity distribution are more cyclical than the top quantiles. In other words, the countercyclicality of productivity dispersion is mostly due to a higher share of relatively unproductive establishments during downturns." While we do not calibrate to match these numbers, we use the general idea.

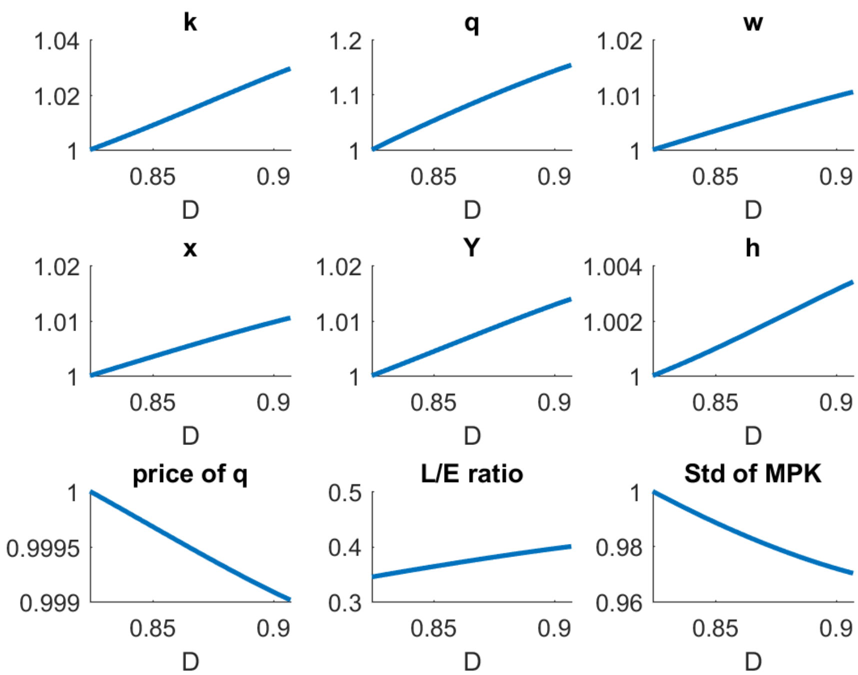


Figure 9: Effects of Credit Conditions on the Economy

Figure 10 shows what happens if both productivity and credit conditions improve. This can generate all the stylized facts. Improved credit conditions increase in reallocation in a boom, but the negative effect on the price of used capital is off-set by increased ε_L. These two effects together generate all the qualitative stylized facts. This is consistent with Eisfeldt and Shi (2018), who also show that a productivity shock is not able to generate increasing reallocation but expansion in credit conditions can.[24] We also mention that only expansion in credit conditions is not likely to increases price of used capital if only the traded capital can be used as collateral. However, credit conditions would increase the

[24] They wrote "...aggregate productivity shocks alone are unlikely to generate a realistic business cycle correlation for capital reallocation; higher aggregate productivity alone does not lead to greater capital reallocation in either a frictionless model, or a model with financial or real trading frictions. In contrast, relaxing financial constraints increases reallaocation."

price of old capital if the bargaining power to the seller is close to 0.

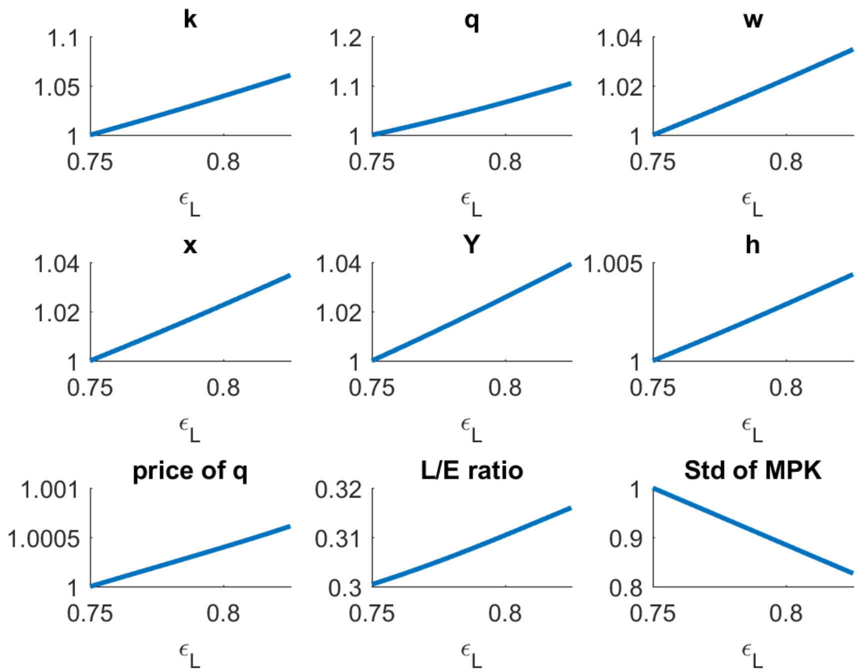


Figure 10: Effects of both Credit Conditions and ε_L on the Economy

8 Conclusion

This paper explored the determination of capital investment and reallocation in dynamic general equilibrium. The theory included frictional secondary markets with credit or monetary exchange, and different microstructures including random search and bargaining plus directed search and posting. For each specification we provided relatively strong results on existence, uniqueness, efficiency and policy. The framework is tractable: it can be reduced to two equations for capital and money – or, if one prefers, for investment and reallocation. Depending on parameters, decreasing the nominal interest rate can stimulate real investment and output, consistent with Keynesian macroeconomics, even if our approach to microfoundations is very different. In some versions of the model, inflation above the Friedman rule is optimal because, while it hinders the secondary market, it encourages investment in the primary market.

We also argued that common measures of mismatch related to productivity dispersion do not necessarily capture frictions. Further, we showed how to account for some stylized facts. All of these results help us better understand issues related to investment and reallocation, and to the effects of monetary and fiscal policy. In terms of future research, one could further pursue quantitative analysis. For this one should perhaps relax a few special assumptions – like i.i.d. shocks, or having only two realizations – that were made here to build simple examples illustrating the ideas. One can also add aggregate shocks. It might be interesting to additionally examine endogenous growth in this framework, perhaps allowing liquid assets other than currency to facilitate trade, and perhaps allowing financial intermediation. Additionally, it might be interesting to combine models with frictional capital and frictional labor markets. All of this is left for future work.

Appendix

Proof of Proposition 4: We want to show IS and CR cross uniquely in (k, q) space at $k > 0$ and $q > 0$. First, let $\Delta = (\partial q/\partial K)_{|IS} - (\partial q/\partial K)_{|CR}$ and derive

$$\Delta = \frac{\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q) + \psi f''(k)/\xi}{\Phi(k,q)} - \frac{\Phi(k,q)}{\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)}.$$

Consider case (i): $\Phi(k,q) < 0$ when the curves cross. Letting $\approx$ indicate both sides take the same sign, we have

$$\begin{aligned} \Delta &\approx [\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]^2 \\ &\quad + [\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]\psi f''(k)/\xi - \Phi(k,q)^2 \\ &> [\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]^2 - \Phi(k,q)^2. \end{aligned}$$

As the RHS can be shown to be positive, IS is steeper than CR when they cross. Consider next case (ii): $\Phi(k,q) > 0$ when the curves cross. Then

$$\begin{aligned} \Delta &\approx -[\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]^2 \\ &\quad - [\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]\psi f''(k)/\xi + \Phi(k,q)^2 \\ &< -[\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)]^2 + \Phi(k,q)^2. \end{aligned}$$

As the RHS can be shown to be negative, IS is again steeper than CR when they cross.

Hence, in either case, given IS and CR are single-valued functions, they cannot cross more than once. To show they cross, notice CR satisfies $Q(0) = 0$ and $0 < Q(k) \leq k \; \forall k > 0$. Also, as $k \to \infty$, $k - Q(k) \to \infty$, because $\varepsilon_H f'[k + Q(k)] > \varepsilon_H f'(k) \to 0$, implying $\varepsilon_L f'[k - Q(k)] \to 0$. Similarly, the IS curve satisfies $K(0) > 0$ and $K(q) - q \to c < \infty$ as $q \to \infty$. Now having the curves cross is equivalent to finding a solution to

$$Q \circ K(q) - q = 0, \tag{41}$$

where $\circ$ denotes the composite of functions. Notice $Q \circ K(0) > 0$, and as $q \to \infty$

$$Q \circ K(q) - q = Q \circ K(q) - K(q) + K(q) - q \to c - \infty = -\infty.$$

Hence there exists $q^* > 0$ solving (41), and the curves cross at q^* and $K(q^*)$. ■

Proof of Proposition 5: As long as ι is not too big, (28) defines q as a function of k, say $q = Q(k)$ where

$$\begin{aligned} Q(k) &= \frac{f'(k+q)\,f''(k-q) - f'(k-q)\,f''(k+q)}{f'(k+q)\,f''(k-q) + f'(k-q)\,f''(k+q)} \\ &\simeq \frac{\partial}{\partial q}\left[f'(k+q)\,f'(k-q)\right]. \end{aligned}$$

Notice $Q(0) = 0$, $Q(k) > 0$ if $k > 0$, and $Q'(k) < 1$. Similarly, (29) defines k as a function of q, say $k = K(q)$. Let k_0 satisfy $r + \delta = (1-\tau)(\gamma_H \varepsilon_H + \gamma_L \varepsilon_L) f'(k_0)$. Then $K(0) = k_0$. In addition, let $\bar{k}$ satisfy

$$\begin{aligned} \frac{r+\delta}{1-\tau} &= \gamma_H \alpha_H \theta \varepsilon_H f'\left(2\bar{k}\right) + \gamma_H \varepsilon_H (1-\alpha_H) f'\left(\bar{k}\right) \\ &\quad + \frac{\gamma_L \alpha_L (1-\theta)}{\theta} \varepsilon_H f'\left(2\bar{k}\right) + \gamma_L \varepsilon_L f'\left(\bar{k}\right). \end{aligned}$$

Then $\bar{k} = K\left(\bar{k}\right)$. Any q solves $Q \circ K(q) = q$ is an equilibrium. Notice $Q \circ K(0) = Q(k_0) > 0$ and $Q \circ K\left(\bar{k}\right) = Q\left(\bar{k}\right) < \bar{k}$. Then, by continuity, there exists at least one equilibrium.

If $\theta = 1$, one can check the FOC's are sufficient. Therefore, (k, q) is a steady state equilibrium iff it satisfies

$$\iota = \alpha_H \gamma_H \left[\frac{\varepsilon_H f'(k+q)}{\varepsilon_L f'(k-q)} - 1\right] \tag{42}$$

$$r + \delta = \gamma_H \alpha_H \varepsilon_H f'(k+q) + (1-\alpha_H)\gamma_H \varepsilon_H f'(k) + \gamma_L \varepsilon_L f'(k). \tag{43}$$

Uniqueness follows if $K'(q)\,Q' \circ K(q) < 0$ whenever $Q \circ K(q) = 0$. To check this, notice

$$\begin{aligned} K_2'(q)\,Q_1' \circ K_2(q) \simeq & -f'(k+q)\,f''(k-q)\,\gamma_H \alpha_H \varepsilon_H f''(k+q) \\ & - \gamma_H \alpha_H \varepsilon_H f''(k+q)\,f'(k-q)\,f''(k+q) \\ & - \left[f'(k+q)\,f''(k-q) + f'(k-q)\,f''(k+q)\right] \\ & \times \left[(1-\alpha_H)\gamma_H \varepsilon_H f''(k) + \gamma_L \varepsilon_L f'(k)\right] < 0. \end{aligned}$$

This proves uniqueness.

Now we have

$$\begin{aligned}
\iota &= \alpha_H \gamma_H \theta \left[\varepsilon_H f'(k+q) - \varepsilon_L f'(k-q)\right]/D && (44)\\
\frac{r+\delta}{1-\tau} &= \gamma_H \varepsilon_H \left[\alpha \gamma_L f'(k+q)\,\Omega(k,q) + (1-\alpha\gamma_L) f'(k)\right] && (45)\\
&\quad + \gamma_L \varepsilon_L \left[\alpha \gamma_H f'(k-q)\,\Gamma(k,q) + (1-\alpha\gamma_H) f'(k)\right],
\end{aligned}$$

$$\begin{aligned}
\Omega(k,q) &= \left[(1-\theta)\,\varepsilon_H f'(k) + \theta \varepsilon_L f'(k-q)\right]/D\\
\Gamma(k,q) &= \left[(1-\theta)\,\varepsilon_H f'(k+q) + \theta \varepsilon_L f'(k)\right]/D
\end{aligned}$$

If $\theta > 0$ and $\iota = 0$, (28) and (29) reduces to

$$\begin{aligned}
0 &= \varepsilon_H f'(k+q) - \varepsilon_L f'(k-q)\\
\frac{r+\delta}{1-\tau} &= \gamma_H \left\{\alpha\gamma_L \left[(1-\theta)\,\varepsilon_H f'(k) + \theta \varepsilon_L f'(k-q)\right] + (1-\alpha\gamma_L)\,\varepsilon_H f'(k)\right\}\\
&\quad + \gamma_L \left\{\alpha\gamma_H \left[(1-\theta)\,\varepsilon_H f'(k+q) + \theta \varepsilon_L f'(k)\right] + (1-\alpha\gamma_H)\,\varepsilon_L f'(k)\right\}.
\end{aligned}$$

Partially differentiate the RHS of these equations wrt q and k to obtain

$$\begin{aligned}
\hat{\mathbf{J}}_{11} &= \varepsilon_H f'(k+q) + \varepsilon_L f''(k-q)\\
\hat{\mathbf{J}}_{12} &= \varepsilon_H f''(k+q) - \varepsilon_L f''(k-q)\\
\hat{\mathbf{J}}_{21} &= \alpha\gamma_L\gamma_H \left\{(1-\theta)\left[\varepsilon_H f''(k) + f''(k+q)\right] - \theta\varepsilon_L \left[f''(k-q) - f''(k)\right]\right\}\\
\hat{\mathbf{J}}_{22} &= \alpha\gamma_L\gamma_H \left\{(1-\theta)\left[\varepsilon_H f''(k) + f''(k+q)\right] + \theta\varepsilon_L \left[f''(k-q) + f''(k)\right]\right\}\\
&\quad + \varepsilon_H \gamma_H (1-\alpha\gamma_L) f''(k) + \varepsilon_L \gamma_L (1-\alpha\gamma_H) f''(k).
\end{aligned}$$

Obviously, $\hat{\mathbf{J}}_{11} < 0$ and $\hat{\mathbf{J}}_{22} < 0$. One can show that, at the equilibrium k and q,

$$K_2'(q)\,Q_1' \circ K_2(q) - 1 = \frac{\hat{\mathbf{J}}_{12}}{\hat{\mathbf{J}}_{11}}\frac{\hat{\mathbf{J}}_{21}}{\hat{\mathbf{J}}_{22}} - 1 \simeq \hat{\mathbf{J}}_{12}\hat{\mathbf{J}}_{21} - \hat{\mathbf{J}}_{11}\hat{\mathbf{J}}_{22} < 0$$

which implies uniqueness for $\iota = 0$. By continuity this also holds for ι not too big. ■

Proof of Proposition 7: One can show that welfare is determined by

$$\begin{aligned}
\mathbf{W} &= -(r+\delta)\,k + \gamma_H \varepsilon_H \alpha_H f(k+q) + \gamma_H \varepsilon_H \left[1-\alpha_H\right] f(k)\\
&\quad + \gamma_L \varepsilon_L \alpha_L f(k-q) + \gamma_L \varepsilon_L \left[1-\alpha_L\right] f(k),
\end{aligned}$$

which is the total benefit of capital minus the cost. To show the optimal ι^* is positive for some θ, we only need check $\left(\partial \boldsymbol{W}/\partial\iota\right)|_{\iota=0} > 0$. Notice that

$$\frac{\partial \boldsymbol{W}}{\partial \iota}|_{\iota=0} = N_k \frac{\partial k}{\partial \iota}|_{\iota=0}$$

where N_k is the net marginal benefit of capital:

$$\begin{aligned} N_k &= -(r+\delta) + \gamma_H \varepsilon_H \alpha_H f'(k+q) + \gamma_H \varepsilon_H (1-\alpha_H) f'(k) \\ &\quad + \gamma_L \varepsilon_L \alpha_L f'(k-q) + \gamma_L \varepsilon_L (1-\alpha_L) f'(k). \end{aligned}$$

Notice $NB_k = 0$ if $\theta = \theta^*$. Now $\theta > \theta^*$ implies $k < k^*$ and $NB_k > 0$ and $\theta < \theta^*$ implies $k > k^*$. Therefore, $\theta < \theta^*$ implies $\left(\partial \boldsymbol{W}/\partial\iota\right)|_{\iota=0} > 0$ iff $\left(\partial k/\partial\iota\right)|_{\iota=0} < 0$, and $\theta > \theta^*$ implies $\left(\partial \boldsymbol{W}/\partial\iota\right)|_{\iota=0} > 0$ iff $\left(\partial k/\partial\iota\right)|_{\iota=0} > 0$.

Next, totally differentiate (28)-(29) and evaluate the result at the equilibrium (k, q) with $\iota = 0$. It turns out to be easier to rewrite (29) using $\iota = \gamma_H \alpha_H \Lambda(k, q)$ and obtain

$$\begin{aligned} r+\delta &= \alpha\gamma_H\gamma_L\theta\varepsilon_H f'(k+q) - \iota(1-\theta)\varepsilon_H\left[f'(k+q) - f'(k)\right] \\ &\quad + \alpha\gamma_L\gamma_H(1-\theta)\varepsilon_L f'(k-q) - \iota(1-\theta)\varepsilon_L\left[f'(k) - f'(k-q)\right] \\ &\quad + \psi f'(k). \end{aligned} \tag{46}$$

Then use (28) and (46) and the fact that $\varepsilon_H f'(k+q) = \varepsilon_L f'(k-q)$ to obtain, at $\iota = 0$,

$$\begin{bmatrix} \Upsilon_0 & \Upsilon_0 \\ \Upsilon_1 & \Upsilon_2 \end{bmatrix} \begin{bmatrix} \partial q \\ \partial k \end{bmatrix} = \begin{bmatrix} 1 \\ -(1-\theta)(\varepsilon_H - \varepsilon_L) f'(k) \end{bmatrix} \partial\iota$$

where

$$\begin{aligned} \Upsilon_0 &= \alpha\gamma_H\gamma_L\theta\left[\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)\right]/D \\ \Upsilon_1 &= \alpha\gamma_H\gamma_L\left[\theta\varepsilon_H f''(k+q) - (1-\theta)\varepsilon_L f''(k-q)\right], \\ \Upsilon_2 &= \alpha\gamma_H\gamma_L\theta\varepsilon_H f''(k+q) + \alpha\gamma_L\gamma_H(1-\theta)\varepsilon_L f''(k-q) \\ &\quad + \psi f''(k). \end{aligned}$$

The determinant of the Jacobian matrix is equal in sign to

$$\begin{aligned} &\alpha\gamma_H\gamma_L\left[\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)\right]\left[\theta\varepsilon_H f''(k+q) + (1-\theta)\varepsilon_L f''(k-q)\right] \\ &-\alpha\gamma_H\gamma_L\Phi(k,q)\left[\theta\varepsilon_H f''(k+q) - (1-\theta)\varepsilon_L f''(k-q)\right] \\ &+\alpha\gamma_H\gamma_L\psi\left[\varepsilon_H f''(k+q) + \varepsilon_L f''(k-q)\right] f''(k). \end{aligned}$$

One can check that the above expression is positive, so by Cramer's rule,

$$\begin{aligned}\frac{\partial k}{\partial \iota}|_{\iota=0} &\simeq -\theta(1-\theta)\left[\varepsilon_H f''(k+q)+\varepsilon_L f''(k-q)\right](\varepsilon_H-\varepsilon_L)f'(k)\\ &\quad -\left[\theta\varepsilon_H f''(k+q)-(1-\theta)\varepsilon_L f''(k-q)\right]\varepsilon_H f'(k+q).\end{aligned}$$

The first term is positive and the second can be positive or negative depending on θ. If $\theta=1$, the first term is 0 while the last term is positive. By continuity, there exists $\theta_H>\theta^*$ such that $(\partial k/\partial\iota)|_{\iota=0}>0\ \forall\theta>\theta_H$ and hence $(\partial\mathbf{W}/\partial\iota)|_{\iota=0}>0$. Similarly, if $\theta=0$ then $(\partial k/\partial\iota)|_{\iota=0}<0$, and by continuity, $\theta<\theta_L\leq\theta^*$ implies $(\partial k/\partial\iota)|_{\iota=0}<0$ and $(\partial\mathbf{W}/\partial\iota)|_{\iota=0}>0$. In either case, $\iota>0$ yields higher welfare than $\iota=0$, but does not achieve the first best because q is not efficient.

Now suppose that $f(k)=k^\eta$. At $\iota=0$, one can show that $q=ck$ where

$$c=\frac{(\varepsilon_H/\varepsilon_L)^{\frac{1}{1-\eta}}-1}{(\varepsilon_H/\varepsilon_L)^{\frac{1}{1-\eta}}+1}=\frac{\varepsilon_H^{\frac{1}{1-\eta}}-\varepsilon_L^{\frac{1}{1-\eta}}}{\varepsilon_H^{\frac{1}{1-\eta}}+\varepsilon_L^{\frac{1}{1-\eta}}}.$$

From this we obtain

$$\begin{aligned}\frac{\partial k}{\partial \iota}|_{\iota=0} &\simeq \theta(1-\theta)\left[\varepsilon_H(1+c)^{\eta-2}+\varepsilon_L(1-c)^{\eta-2}\right](\varepsilon_H-\varepsilon_L)\\ &\quad +\left[\theta\varepsilon_H(1+c)^{\eta-2}-(1-\theta)\varepsilon_L(1-c)^{\eta-2}\right]\varepsilon_H(1+c)^{\eta-1}.\end{aligned}$$

The RHS is a quadratic function of θ with roots

$$\begin{aligned}\theta_1 &= \frac{1+\frac{\varepsilon_H(1+c)^{\eta-1}}{\varepsilon_H-\varepsilon_L}+\sqrt{\left[1+\frac{\varepsilon_H(1+c)^{\eta-1}}{\varepsilon_H-\varepsilon_L}\right]^2-4\frac{\varepsilon_H\varepsilon_L(1-c)^{\eta-2}(1+c)^{\eta-1}}{\left[\varepsilon_H(1+c)^{\eta-2}+\varepsilon_L(1-c)^{\eta-2}\right](\varepsilon_H-\varepsilon_L)}}}{2},\\ \theta_2 &= \frac{1+\frac{\varepsilon_H(1+c)^{\eta-1}}{\varepsilon_H-\varepsilon_L}-\sqrt{\left[1+\frac{\varepsilon_H(1+c)^{\eta-1}}{\varepsilon_H-\varepsilon_L}\right]^2-4\frac{\varepsilon_H\varepsilon_L(1-c)^{\eta-2}(1+c)^{\eta-1}}{\left[\varepsilon_H(1+c)^{\eta-2}+\varepsilon_L(1-c)^{\eta-2}\right](\varepsilon_H-\varepsilon_L)}}}{2}.\end{aligned}$$

Notice that $\theta_1>1$ and $\theta_2\in(0,1)$ if

$$\frac{\varepsilon_H\varepsilon_L(1-c)^{\eta-2}(1+c)^{\eta-1}}{\left[\varepsilon_H(1+c)^{\eta-2}+\varepsilon_L(1-c)^{\eta-2}\right](\varepsilon_H-\varepsilon_L)}<\frac{\varepsilon_H(1+c)^{\eta-1}}{\varepsilon_H-\varepsilon_L}.$$

This condition is equivalent to

$$\frac{\varepsilon_L(1-c)^{\eta-2}}{\varepsilon_H(1+c)^{\eta-2}+\varepsilon_L(1-c)^{\eta-2}}<1,$$

which always holds. Therefore, $(\partial k/\partial\iota)\,|_{\iota=0} > 0$ if $\theta \in (\theta_2, 1]$ and $(\partial k/\partial\iota)\,|_{\iota=0} < 0$ if $\theta \in [0, \theta_2)$.

We now show that $\theta_2 > \theta^*$. Notice

$$\begin{aligned}
\theta^* &= \frac{\varepsilon_H - B^{\,1-\eta}}{\varepsilon_H - \varepsilon_L}, \\
\theta_2 &= \frac{\varepsilon_H - \varepsilon_L + B^{\,1-\eta} - \sqrt{\left[\varepsilon_H - \varepsilon_L + B^{\,1-\eta}\right]^2 - 2\frac{B^{\,1-\eta}\varepsilon_H^{\frac{1}{1-\eta}}(\varepsilon_H - \varepsilon_L)}{B}}}{2\left(\varepsilon_H - \varepsilon_L\right)}.
\end{aligned}$$

where B is defined in the statement of this Proposition. Therefore,

$$\theta_2 - \theta^* \simeq 3B^{1-\eta} - (\varepsilon_H + \varepsilon_L) - \sqrt{B^{\,2(1-\eta)} + (\varepsilon_H - \varepsilon_L)\left[2B^{1-\eta} + \varepsilon_H - \varepsilon_L - 2\frac{B^{\,1-\eta}\varepsilon_H^{\frac{1}{1-\eta}}}{B}\right]}.$$

This is positive if

$$3B^{1-\eta} - (\varepsilon_H + \varepsilon_L) > \sqrt{B^{\,2(1-\eta)} + (\varepsilon_H - \varepsilon_L)\left[2B^{1-\eta} + \varepsilon_H - \varepsilon_L - 2\frac{B^{\,1-\eta}\varepsilon_H^{\frac{1}{1-\eta}}}{B}\right]}.$$

After some algebra, one can show this is equivalent to

$$\left[2B^{1-\eta} - (\varepsilon_H + \varepsilon_L)\right]\left[4B^{1-\eta} - (\varepsilon_H + \varepsilon_L)\right] > -B^{\,1-\eta}\frac{\varepsilon_H^{\frac{1}{1-\eta}} - \varepsilon_L^{\frac{1}{1-\eta}}}{B}\left(\varepsilon_H - \varepsilon_L\right) + \left(\varepsilon_H - \varepsilon_L\right)^2. \tag{47}$$

By Hölder's inequality, $B^{\,1-\eta} \geq (\varepsilon_H + \varepsilon_L)/2$. This means that the LHS is positive. For the RHS, notice

$$\begin{aligned}
& -B^{\,1-\eta}\frac{\varepsilon_H^{\frac{1}{1-\eta}} - \varepsilon_L^{\frac{1}{1-\eta}}}{B}\left(\varepsilon_H - \varepsilon_L\right) + \left(\varepsilon_H - \varepsilon_L\right)^2 \\
\simeq\; & -B^{\,1-\eta}\left(\varepsilon_H^{\frac{1}{1-\eta}} - \varepsilon_L^{\frac{1}{1-\eta}}\right) + \left(\varepsilon_H - \varepsilon_L\right)B \\
\leq\; & -\frac{(\varepsilon_H + \varepsilon_L)}{2}\left(\varepsilon_H^{\frac{1}{1-\eta}} - \varepsilon_L^{\frac{1}{1-\eta}}\right) + \left(\varepsilon_H - \varepsilon_L\right)\left(\frac{\varepsilon_H^{\frac{1}{1-\eta}}}{2} + \frac{\varepsilon_L^{\frac{1}{1-\eta}}}{2}\right) \\
\simeq\; & -\left(\varepsilon_H + \varepsilon_L\right)\left(\varepsilon_H^{\frac{1}{1-\eta}} - \varepsilon_L^{\frac{1}{1-\eta}}\right) + \left(\varepsilon_H - \varepsilon_L\right)\left(\varepsilon_H^{\frac{1}{1-\eta}} + \varepsilon_L^{\frac{1}{1-\eta}}\right) \\
=\; & \frac{2}{\varepsilon_H^{\frac{1}{\eta-1}}\varepsilon_L^{\frac{1}{\eta-1}}}\left(\varepsilon_H^{-\frac{\eta}{1-\eta}} - \varepsilon_L^{-\frac{\eta}{1-\eta}}\right) < 0.
\end{aligned}$$

Hence (47) holds. Then we have established that $\theta_2 > \theta^*$. ■

Equilibrium with Competitive Search: Then the Lagrangian for (31) is

$$\begin{aligned}\mathcal{L} &= \frac{\alpha(n)}{n}A\{(1-\tau)\varepsilon_H[f(k_b+q)-f(k_b)]-z\} \qquad (48)\\ &+\omega\{\alpha(n)Az-\alpha(n)A\varepsilon_L(1-\tau)[f(k_s)-f(k_s-q)]-v_s\}.\end{aligned}$$

The FOC's are

$$0 = \frac{\alpha(n)}{n}\varepsilon_H f'(k_b+q)-\lambda\alpha(n)\varepsilon_L f'(k_b-q) \qquad (49)$$

$$0 = \frac{\alpha(n)-n\alpha'(n)}{n^2}\left[\varepsilon_H f(k_b+q)-\varepsilon_H f(k_b)-\frac{z}{1-\tau}\right] \qquad (50)$$

$$-\lambda\alpha'(n)\left[\frac{z}{1-\tau}-\varepsilon_L f(k_s)+\varepsilon_L f(k_s-q)\right]$$

$$v_s = \alpha(n)\left[\frac{z}{1-\tau}-\varepsilon_L f(k_s)+\varepsilon_L f(k_s-q)\right]. \qquad (51)$$

Using (49) to eliminate ω from (50), we obtain

$$\begin{aligned}\frac{z}{1-\tau} &= \frac{e(n)\varepsilon_H f'(k_b+q)\varepsilon_L[f(k_s)-f(k_s-q)]}{e(n)\varepsilon_H f'(k_b+q)+[1-e(n)]\varepsilon_L f'(k_s-q)} \qquad (52)\\ &+\frac{[1-e(n)]\varepsilon_L f'(k_s-q)\varepsilon_H[f(k_b+q)-f(k_b)]}{e(n)\varepsilon_H f'(k_b+q)+[1-e(n)]\varepsilon_L f'(k_s-q)}\end{aligned}$$

where $e(n)$ is the elasticity of the matching function. It is interesting (although common in related models) to note that (52) says the payment $p=z$ is the same as as the outcome of Nash bargaining when buyer's bargaining power is $\theta = e(n)$. Given this, n and q solve (51) and (52).

DM value function is Because $V(z,k,\varepsilon_H) = v_b + W(z,k,\varepsilon_H)$, by the envelope theorem

$$\begin{aligned}V_1(z,k,\varepsilon_H) &= \frac{\alpha(n)}{n}A\frac{\varepsilon_H f'(k+q)-\varepsilon_L f'(k-q)}{\varepsilon_L f'(k-q)}+A\\ V_2(z,k,\varepsilon_H) &= \frac{\alpha(n)}{n}A(1-\tau)[\varepsilon_H f'(k+q)-\varepsilon_H f'(k)]\\ &+A(1-\tau)[\varepsilon_H f'(k)+1-\delta].\end{aligned}$$

From the expression for $V(z,k,\varepsilon_L)$ we obtain

$$\begin{aligned}V_1(z,k,\varepsilon_L) &= A\\ V_2(z,k,\varepsilon_L) &= \alpha(n)\varepsilon_L A(1-\tau)[f'(k-q)-f'(k)]\\ &+A(1-\tau)[\varepsilon_L f'(k)+1-\delta].\end{aligned}$$

Then combine with the first-order conditions in the CM, we obtain (32) and (33).

Proof of Proposition 8: First, q is a continuous function of k, $q = Q(k)$ where

$$Q'(k) = -\frac{f''(k+q)\, f'(k-q) - f'(k+q)\, f''(k-q)}{f''(k+q)\, f'(k-q) + f'(k+q)\, f''(k-q)}$$

can be positive or negative depends on the sign of $f''(k+q)\, f'(k-q) - f'(k+q)\, f''(k-q)$. In addition, if $k \to 0$, $Q(k) \to 0$ and if $k \to \infty$, $Q(k) \to \infty$ and $k - Q(k) \to \infty$. Equation (33) defines k as a continuous function of q: $k = K(q)$. Notice that $K(0) = k_0$ where k_0 solves

$$\frac{r+\delta}{1-\tau} = (\gamma_H \varepsilon_H + \gamma_L \varepsilon_L)\, f'(k_0).$$

If $q \to \infty$, $K(q) \to \infty$ and $K_2(q) - q \to c < \infty$ where c solves

$$\frac{r+\delta}{1-\tau} = \alpha(n)\, \gamma_L \varepsilon_L f'(c).$$

Any k that satisfies $K \circ Q(k) - k = 0$ is an equilibrium. Notice $K \circ Q(0) = k_0 > 0$. In addition, if $k \to \infty$

$$K \circ Q(k) - k = K \circ Q(k) - Q(k) + Q(k) - k \to c - \infty.$$

This means that $K \circ Q(k) - k < 0$ for k sufficiently large. By the intermediate value theorem, an equilibrium exists. At the equilibrium k and $q = Q(k)$,

$$\begin{aligned}
& \frac{\partial}{\partial k}\left[K \circ Q(k) - k\right] \\
= \; & K' \circ Q(k)\, Q'(k) - 1 \\
\simeq \; & -\left\{\gamma_H \left[1 - \frac{\alpha(n)}{n}\right] \varepsilon_H + \gamma_L \left[1 - \alpha(n)\right] \varepsilon_L\right\} f''(k) \frac{\partial}{\partial k} f'(k+q)\, f'(k-q) \\
& -2\frac{\alpha(n)}{n} \gamma_H \varepsilon_H f''(k+q) \frac{\partial}{\partial k} f'(k+q)\, f'(k-q).
\end{aligned}$$

Notice that all terms are negative. Therefore, $K' \circ Q(k)\, Q'(k) - 1 < 0$. In this case, $K \circ Q(k) - k = 0$ for at most one k and uniqueness follows. ■

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