**University of Pennsylvania** **Economics 705, Spring 2018**

# Prelim Examination

## Friday June 8, 2018. Time limit: 150 minutes

**Instructions**:

(i) The number of points for each question is given below.

(ii) The exam is closed book and closed notes.

(iii) To receive full credit for your answers you have to explain your calculations. For example, when you use the weak law of large numbers (WLLN) or the central limit theorem (CLT), state it clearly.

(iv) You may state additional assumptions.

**Question 1:** Maximum Likelihood Estimation (40 Points)

Consider the following location-shift model:

$$Y_i = \theta + U_i, \quad U_i \sim N(0, 4). \tag{1}$$

(i) (4 Points) Derive the likelihood function $\mathcal{L}(\theta|Y)$ for an *iid* sample of $n$ observations.

(ii) (4 Points) Denote the log likelihood function as $l(\theta|Y)$. Derive the maximum likelihood estimator defined as $\hat{\theta} = \arg\max_{\theta \in \mathbb{R}} \; l(\theta|Y)$. Provide an explicit formula for $\hat{\theta}$.

(iii) (4 Points) Show that the score $s(\theta) = l'(\theta|Y)$ evaluated at $\theta = \theta_0$ has mean zero.

(iv) (4 Points) Compute the mean and the variance of $\hat{\theta}$.

(v) Propose a test for the null hypotheses $H_0 : \; \theta = 0$ versus the alternative that $\theta \neq 0$.

- (4 Points) the test statistic;
- (4 Points) the sampling distribution of the test statistic under $H_0$;
- (4 Points) the critical value that guarantees a type-I error $\alpha$;
- (4 Points) the power of the test against alternatives $\theta = \delta$.

(vi) (4 Points) Now suppose we impose the restriction $\theta \geq 0$. Provide the log likelihood function and an explicit formula for the maximum likelihood estimator $\widetilde{\theta} = \arg\max_{\theta \in \mathbb{R}^+} l(\theta|Y)$.

(vii) (4 Points) What is the sampling distribution of the maximum likelihood estimator $\widetilde{\theta}$?

**Question 2:** Central Limit Theorem (10 Points)

- Let $X_1, ..., X_k$ denote $k$ random variables. We do not know their distributions but we know their mean and variance, denoted by $\mu_1, ..., \mu_k$ and $\sigma_1^2, ..., \sigma_k^2$.
- Now we have $n$ *independent* data observations. Among them, $n_1$ observations are drawn from the distribution of $X_1$, $n_2$ observations are drawn from the distribution of $X_2, ..., n_k$ observations are drawn from the distribution of $X_k$. They satisfy $n = n_1 + ... + n_k$.
- Let $\overline{X}_n$ denote the sample mean of the $n$ observations.

Show the asymptotic distribution of $\overline{X}_n$ when $n$ is large. Clearly verify the conditions for each step of the proof.

**Question 3:** GMM (20 Points)

Take the model

$$\begin{aligned} Y_i &= X_i'\theta + U_i, \\ E(Z_i U_i) &= 0, \end{aligned}$$

where $Y_i \in R$, $X_i \in R^k$, and $Z_i \in R^l$, with $l \geq k$. Assume the observations are i.i.d. Consider the following statistic

$$\begin{aligned} J_n(\theta) &= n\overline{m}_n(\theta)' W_n \overline{m}_n(\theta) \\ \overline{m}_n(\theta) &= n^{-1} \sum_{i=1}^{n} Z_i (Y_i - X_i'\theta) \end{aligned}$$

for some weighting matrix $W_n \rightarrow_p W$.

(i) (4 Points) Consider the hypothesis

$$H_0 : \theta = \theta_0.$$

Derive the asymptotic distribution of $J_n(\theta_0)$ under $H_0$ as $n \to \infty$.

(ii) (4 Points) Write down an appropriate weighting matrix $W_n$ which takes advantage of $H_0$ and yields a known asymptotic distribution in part (i).

(iii) (4 Points) What is the asymptotic limit of $J_n(\theta_0)$ under the alternative $H_1 : \theta \neq \theta_0$?

(iv) (4 Points) Use your calculations from above to construct a confidence set for $\theta$ with asymptotic coverage probability equal to $1 - \alpha$.

(v) (4 Points) How do you define the exogeneity condition and the relevance condition of the instruments in this problem?

**Question 4:** Nonlinear Least Squares Estimation (20 points)

- Consider a linear model

$$Y_i = \beta X_i + U_i,\ E(U_i|X_i) = 0,\ \beta \in R,$$

  where the error term $U_i$ is conditionally heteroskedastic with the following parametric form

$$E(U_i^2|X_i) = m(X_i, \theta),\ \theta \in \Theta \subset R.$$

  The functional form $m(x, \theta)$ is known but the value of $\theta$ is unknown. We assume $m(x, \theta)$ is twice continuously differentiable in $\theta$, with the first and second order derivatives denoted by $m_\theta(x, \theta)$ and $m_{\theta\theta}(x, \theta)$, respectively.

- Let $\widehat{U}_i = Y_i - \widehat{\beta} X_i$, where $\widehat{\beta}$ is the OLS estimator. Let

$$\widehat{\theta} = \arg\min_{\theta \in \Theta} \sum_{i=1}^{n} \left( \widehat{U}_i - m(X_i, \theta) \right)^2,$$

  that is, $\widehat{\theta}$ is a nonlinear least squares estimator where the dependent variable is "estimated". Assume that we observe a random sample $(Y_i, X_i) : i = 1, ..., n$ generated by the model above.

- Suppose the consistency of $\widehat{\theta}$ has already been established.

  Derive the asymptotic distribution of $\widehat{\theta}$. You can assume additional regularity conditions (e.g., existence of moments) as necessary.

**Question 5:** Models with Limited Observations (10 Points)

Consider the following model

$$Y_i^* = X_i'\beta + u_i,\ U_i|X_i \sim iid\ N(0,\sigma^2),$$

where $X_i \in R^k$ with $k > 1$. Moreover, the $X_i$'s are also independent across $i$.

(i) (5 points) We do not observe $Y_i^*$, instead we observe

$$Y_i = \begin{cases} Y_i^* \text{ if } y_i^* \geq 0 \\ 0 \text{ if } y_i^* < 0, \end{cases}$$

Write down the log-likelihood function for the maximum likelihood estimator $\widehat{\beta}$.

(ii) (5 points) We do not observe $Y_i^*$, instead we observe

$$Y_i = \begin{cases} 1 \text{ if } y_i^* \geq 0 \\ 0 \text{ if } y_i^* < 0, \end{cases}$$

Write down the log-likelihood function for the maximum likelihood estimator $\widehat{\beta}$.