Microeconomic Theory I
Preliminary Examination
University of Pennsylvania

SUGGESTED SOLUTIONS

June 2, 2014

Instructions

This exam has 4 questions and a total of 100 points.

Answer each question in a **SEPARATE** exam book.

If you need to make additional assumptions, state them clearly.

Be concise.

Write clearly if you want partial credit.

Good luck!

1. (25 pts) A strictly increasing utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ gives rise to a demand function $\mathbf{x}(\mathbf{p}, y) = (x_1(\mathbf{p}, y), x_2(\mathbf{p}, y))$. It is continuously differentiable in a neigborhood N of some $(\mathbf{p}^0, y^0) \gg \mathbf{0}$. Theory tells us much about the nature of such demand functions: use what it tells us to answer the following questions.

   (a) (3 pts) Write the definition of $\eta_i(\mathbf{p}, y)$, the income elasticity for good i.

   Soln: Dropping the understood arguments from η_i and x_i, we have

$$\eta_i := \frac{y}{x_i}\frac{\partial x_i}{\partial y}.$$

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   (b) (11 pts) Suppose the demand for good 1 takes the form

$$x_1(\mathbf{p}, y) = \alpha_1(\mathbf{p})g_1(y)$$

   for all $(\mathbf{p}, y) \in N$. What is the most that this implies about $\eta_i(\mathbf{p}, y)$ on N?

   Soln: The given functional form immediately implies that η_1 does not depend on prices:

$$\eta_1(\mathbf{p}, y) = \frac{y}{\alpha_1(\mathbf{p})g_1(y)}\left[\alpha_1(\mathbf{p})g_1'(y)\right] = \frac{yg_1'(y)}{g_1(y)}.$$

   But we can say more. Since $x_1(\mathbf{p}, y)$ is homogeneous of degree zero, Euler's law yields

$$p_1\frac{\partial x_1}{\partial p_1} + p_2\frac{\partial x_1}{\partial p_2} + y\frac{\partial x_1}{\partial y} = 0.$$

   Divide this by x_1 and rearrange to obtain

$$\eta_1(\mathbf{p}, y) = \frac{y}{x_1}\frac{\partial x_1}{\partial y} = -\left(\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} + \frac{p_2}{x_1}\frac{\partial x_1}{\partial p_2}\right).$$

   Substitute into this from the given form of $x_1(\mathbf{p}, y)$ to obtain

$$\eta_1(\mathbf{p}, y) = -\left(\frac{p_1}{\alpha_1(\mathbf{p})}\frac{\partial \alpha_1(\mathbf{p})}{\partial p_1} + \frac{p_2}{\alpha_1(\mathbf{p})}\frac{\partial \alpha_1(\mathbf{p})}{\partial p_2}\right).$$

   Hence, η_1 also does not depend on income. We conclude that a constant $e_1 \in \mathbb{R}$ exists such that for all $(\mathbf{p}, y) \in N$,

$$\eta_1(\mathbf{p}, y) = e_1.$$

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   (c) (11 pts) Now suppose in addition that in N, demand for good 2 takes the same form,

$$x_2(\mathbf{p}, y) = \alpha_2(\mathbf{p})g_2(y),$$

   and $\mathbf{x}(\mathbf{p}, y)$ satisfies the law of reciprocity:

$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}.$$

   What can you now say about the two income elasticity functions on N?

Soln: By the same logic as in (b), we know that $\eta_2(\mathbf{p}, y)$ is also a constant. But we can now show this, and more, in another way.

Intermediate Result. $\eta_2(\mathbf{p}, y) = \eta_1(\mathbf{p}, y)$ for $(\mathbf{p}, y) \in N$.

Proof. There are two ways to show this. The first is to use $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}$, the Slutsky equations,

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial y} \text{ for } i \neq j,$$

and the fact that Hicksian demand always satisfies the law of reciprocity, $\frac{\partial h_2}{\partial p_1} = \frac{\partial h_1}{\partial p_2}$, to obtain

$$x_1 \frac{\partial x_2}{\partial y} = x_2 \frac{\partial x_1}{\partial y}.$$

Dividing this by $x_1 x_2$ and multiplying by y yields

$$\eta_2(\mathbf{p}, y) = \frac{y}{x_2} \frac{\partial x_2}{\partial y} = \frac{y}{x_1} \frac{\partial x_1}{\partial y} = \eta_1(\mathbf{p}, y).$$

The second way to prove $\eta_2 = \eta_1$ is to use the given functional forms for x_1 and x_2. Using those forms, the equality $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}$ becomes

$$\frac{\partial \alpha_1}{\partial p_2} g_1 = \frac{\partial \alpha_2}{\partial p_1} g_2.$$

Differentiating this equality with respect to y yields

$$\frac{\partial \alpha_1}{\partial p_2} g_1' = \frac{\partial \alpha_2}{\partial p_1} g_2'.$$

Dividing this equality by the preceding one yields $g_1'/g_1 = g_2'/g_2$, and hence

$$\eta_1(\mathbf{p}, y) = \frac{y g_1'}{g_1} = \frac{y g_2'}{g_2} = \eta_2(\mathbf{p}, y).$$

Final Result. $\eta_2(\mathbf{p}, y) = \eta_1(\mathbf{p}, y) = 1$ for $(\mathbf{p}, y) \in N$.

Proof. We know the demand function satisfies Walras' Law: $p_1 x_1 + p_2 x_2 = y$. Differentiate this with respect to y to obtain

$$\begin{aligned} p_1 \frac{\partial x_1}{\partial y} + p_2 \frac{\partial x_2}{\partial y} = 1 \;&\Rightarrow\; \frac{p_1 x_1}{y} \frac{y}{x_1} \frac{\partial x_1}{\partial y} + \frac{p_2 x_2}{y} \frac{y}{x_2} \frac{\partial x_2}{\partial y} = 1 \\ &\Rightarrow\; s_1 \eta_1 + s_2 \eta_2 = 1, \end{aligned}$$

where $s_i = p_i x_i / y$ is the share of income spent on good i. This last expression, since $\eta_1 = \eta_2$ (by the Intermediate Result) and $s_1 + s_2 = 1$ (by Walras' Law), implies that $\eta_1 = \eta_2$ in the neighborhood N. ■

2. (25 pts) A competitive firm uses hops to make beer via a production function $f : \mathbb{R} \to \mathbb{R}$. The price of beer is $p > 0$ and the cost of new hops is $w > 0$. The firm has $x_0 > 0$ hops left over from last year, but an unknown amount of these old hops will either grow or spoil before production begins this year; the amount of them that will be usable is $x_0 + \theta\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is a nondegenerate random variable with mean zero, $\theta > 0$ is a parameter to allow easy comparative statics, and $\theta\tilde{\varepsilon} \in (-x_0, x_0)$. The firm will purchase more hops, $x \geq 0$, to maximize its expected profit,

$$\mathbb{E}\left\{pf(x_0 + \theta\tilde{\varepsilon} + x) - wx\right\}.$$

Assume f is smooth with derivatives $f' > 0$ and $f'' < 0$, and that the solution, $x^*(x_0, w, p, \theta)$, is positive. Make, if necessary, additional reasonable assumptions under which the signs of the partial derivatives,

$$x^*_{x_0},\ x^*_w,\ x^*_p,\ x^*_\theta,$$

can be determined, and find their signs under those assumptions.

Soln: The FOC satisfied by $x^*(x_0, w, p, \theta)$ is

$$p\mathbb{E}f'(x_0 + \theta\tilde{\varepsilon} + x^*(x_0, w, p, \theta)) - w = 0.$$

Differentiating it "totally" with respect to each of the four parameters gives us our comparative statics results. Letting $\tilde{z} = x_0 + \theta\tilde{\varepsilon} + x^*$, we obtain the following:

(a) For x_0 :

$$p\mathbb{E}f''(\tilde{z})(1 + x^*_{x_0}) = 0 \quad \Rightarrow \quad x^*_{x_0} = -1 < 0.$$

(b) For w :

$$p\mathbb{E}f''(\tilde{z})x^*_w - 1 = 0 \quad \Rightarrow \quad x^*_w = \frac{1}{p\mathbb{E}f''(\tilde{z})} < 0.$$

(c) For p :

$$\mathbb{E}f'(\tilde{z}) + p\mathbb{E}f''(\tilde{z})x^*_p = 0 \quad \Rightarrow \quad x^*_p = \frac{-\mathbb{E}f'(\tilde{z})}{p\mathbb{E}f''(\tilde{z})} > 0.$$

(d) For θ :

$$p\mathbb{E}f''(\tilde{z})\left(\tilde{\varepsilon} + x^*_\theta\right) = 0 \quad \Rightarrow \quad x^*_\theta = \frac{\mathbb{E}f''(\tilde{z})\tilde{\varepsilon}}{-\mathbb{E}f''(\tilde{z})}.$$

So the sign of x^*_θ is the sign of $\mathbb{E}f''(\tilde{z})\tilde{\varepsilon}$. To sign it, we use the logic of the precautionary savings problem (Q4 of PS6 in 701A, 2013). Since $\mathbb{E}\tilde{\varepsilon} = 0$, we have

$$\begin{aligned}\mathbb{E}f''(\tilde{z})\tilde{\varepsilon} &= \mathbb{E}f''(x_0 + \theta\tilde{\varepsilon} + x^*)\tilde{\varepsilon} \\ &= \mathbb{E}\left[f''(x_0 + \theta\tilde{\varepsilon} + x^*) - f''(x_0 + x^*)\right]\tilde{\varepsilon}.\end{aligned}$$

As $\theta > 0$, this expression is positive (negative) if f'' is an increasing (decreasing) function. We conclude that we can sign x^*_θ under either assumption:

$$x^*_\theta \begin{cases} > 0 & \text{if } f''' > 0 \\ < 0 & \text{if } f''' < 0 \end{cases}.$$

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3. (25 pts) Consider an exchange economy with two goods and two households with endowments

$$e^1 = (1,0) \text{ and } e^2 = (0,1).$$

In each of the following specifications of utility functions, state whether a competitive equilibrium exists. If any do exist, compute them all. If none exist, explain why not and which part of the standard existence proof fails (be specific about which assumption fails).

(a) $u^1(x_1, x_2) = \min(x_1, 2x_2),\ u^2(x_1, x_2) = \min(2x_1, x_2)$

Soln: There are three equilibrium price vectors: $p = (1,1),\ p = (1,0)$, and $p = (0,1)$. At $p = (1,1)$ the equilibrium allocation is $\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right)$. At $p = (1,0)$ the equilibrium allocations are

$$\left\{(1,x),(0,1-x) : \tfrac{1}{2} \leq x \leq 1\right\}.$$

At $p = (0,1)$ the equilibrium allocations are

$$\left\{(x,1),(1-x,0) : 0 \leq x \leq \tfrac{1}{2}\right\}.$$

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(b) $u^1(x_1, x_2) = \max(x_1, x_2),\ u^2(x_1, x_2) = \max(x_1, x_2)$

Soln: The agents' preferences are not convex, so our general existence does not apply. Nonetheless, two competitive equilibria exist. One is $p = (1,1)$ and $x = ((1,0),(0,1))$ $(= e)$, and the other is $p = (1,1)$ and $x = ((0,1),(1,0))$. ■

(c) $u^1(x_1, x_2) = (x_1)^2 + (x_2)^2,\ u^2(x_1, x_2) = x_1 x_2$

Soln: In this case an equilibrium does not exist because agent 1's preferences are not convex. At any strictly positive price vector, agent 1's optimal bundle is a boundary point while agent 2's is interior. There cannot be an equilibrium with a price of 0 for either good since both agents have strictly monotonic preferences for both goods. ■

4. (25 pts) Consider an economy with a linear production set. Suppose there are 2 agents and 4 commodities. Commodities 1 and 2 are consumption goods while commodities 3 and 4 are skilled and unskilled labor. Suppose there are 4 possible activities transforming skilled and unskilled labor into commodities 1 and 2:

$$a_1 = (1, 0, -3, 0) \quad a_2 = (0, 1, -1, 0)$$
$$a_3 = (1, 0, 0, -4) \quad a_4 = (0, 1, 0, -2)$$

Suppose there is a single firm whose production set Y is the closed cone generated by these activity vectors:

$$Y = \{y \in R^L : y = \Sigma_{m=1}^{4} \alpha_m a_m \text{ for some } \alpha \in R_+^4\}$$

The agents' utility functions are $u^1(c_1, c_2, c_3, c_4)$ and $u^2(c_1, c_2, c_3, c_4)$, and their endowments are $e^1 = (0, 0, 1, 0)$ and $e^2 = (0, 0, 0, 2)$.

(a) Define a competitive equilibrium.

Soln: A competitive equilibrium is a specification of prices p, consumption bundles $\bar{c}^1$ and $\bar{c}^2$, and production plan $\bar{y}$ such that consumers are maximizing utility subject to budget constraints, firms are maximizing profit subject to feasibility, and markets clear:

$$\bar{c}^i \in \arg\max_{c} u^i(c) \text{ such that } p \cdot c \leq p \cdot e^i, i = 1, 2$$

$$p \cdot \bar{y} \in \arg\max_{y \in Y} p \cdot y$$

$$\bar{c}^1 + \bar{c}^2 = e^1 + e^2 + \bar{y}$$

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(b) Show that if activities a_2, a_3, and a_4 are used in equilibrium, a_1 cannot be used.

Soln: Since the activities are linear, profit for any equilibrium, $p \cdot a_m = 0$ for any activity used. Normalize prices with $p_4 = 1$; then since a_3 and a_4 are used, it must be that $p_1 = 4$ and $p_2 = 2$; this and the fact that a_2 is used means that $p_3 = 2$. The profit from a_1 is then $p_1 \cdot 1 - p_3 \cdot 3 = -2$, hence it cannot be used in equilibrium. ■

(c) Show that it cannot be the case that a_1, a_3, and a_4 are used in equilibrium.

Soln: As in the answer to (b), if activities a_3 and a_4 are used, $p_1 = 4$ and $p_2 = 2$; if a_1 is used, $p_3 = 4/3$. Then the profit from $a_2 = p_2 \cdot 1 - p_3 \cdot 3 = 2 - 4/3 > 0$. Hence a_1 is profitable, which is impossible for equilibrium. ■

(d) Suppose now that agents' utility functions are

$$u^1(c_1, c_2, c_3, c_4) = \log(c_1) + \log(c_2)$$
$$u^2(c_1, c_2, c_3, c_4) = \log(c_1) + \log(c_2)$$

Compute a competitive equilibrium in which activities a_2, a_3 and a_4 have 0 profit, and hence might be used in equilibrium.

Soln: We saw from the answer to (b) that the prices in this case must be $p_1 = 4$, $p_2 = 2$, $p_3 = 2$ and $p_4 = 1$. Since agents do not get utility from their endowments they must sell them in equilibrium. Given the prices, each agent will have income 2; given her symmetric Cobb-Douglas utility function, she will spend equal amounts on goods 1 and 2. Hence, each agent will consume $(c_1, c_2) = (1/4, 1/2)$. and the aggregate quantities of these goods must be 1/2 and 1. Since activity 1 is not used, agent 1's input must be used on activity 2 only, yielding 1 unit of good 2. As this is the entire aggregate amount of good 2 that is being produced, activity a_4 is not used in equilibrium (even though it has profit 0). All of agent 2's labor must be allocated to activity a_3, yielding the 1/2 unit of total good 1. ■