Microeconomic Theory I
Preliminary Examination
Suggested Solutions
University of Pennsylvania

August 8, 2016

Instructions

This exam has 4 questions and a total of 100 points.

Answer each question in a **SEPARATE** exam book.

If you need to make additional assumptions, state them clearly.

Be concise.

Write clearly if you want partial credit.

Good luck!

1. (25 pts) The inverse demand function for oil is given by a continuously differentiable function $P : \mathbb{R}_{++} \to \mathbb{R}_{++}$ satisfying $P' < 0$ and $P(x) \to \infty$ as $x \downarrow 0$. The price elasticity of the demand for oil is defined at any $x > 0$ as

$$e(x) := -\frac{P(x)}{P'(x)x}.$$

The total stock of oil below the ground is $0 < \bar{x} < \infty$. It is all owned by one oil company, which can extract it at zero cost. The firm's profit is zero if it sells no oil, and its profit is px if it sells an amount $x > 0$ at price p.

(a) (6 pts) Compare the competitive equilibrium (x^c, p^c) to the monopoly outcome (x^m, p^m) under (i) the assumption that $e(x) > 1$ for all $x \in (0, \bar{x}]$, and (ii) under the assumption that $e(\bar{x}) < 1$.

Soln: $(x^c, p^c) = (\bar{x}, P(\bar{x}))$. Proof: Given a price p, the consumer demands the x satisfying $P(x) = p$, and the firm supplies $\bar{x} = \arg\max_{0 \le x \le \bar{x}} px$. So supply = demand requires $x^c = \bar{x}$ and $p^c = P(\bar{x})$.

A monopoly chooses $x \in [0, \bar{x}]$ to maximize its revenue, which is given by $R(0) = 0$ and $R(x) = xP(x)$ for $x > 0$. Marginal revenue is

$$\begin{aligned} R'(x) &= P(x) + P'(x)x \\ &= \left(1 + \frac{P'(x)x}{P(x)}\right) P(x) = \left(1 - \frac{1}{e(x)}\right) P(x). \end{aligned}$$

Case (i) : $e(x) > 1$ for $x \in (0, \bar{x}]$. In this case $R'(x) > 0$ for all $x \in (0, \bar{x}]$. It follows that $R(\bar{x}) > R(x)$ for $x \in [0, \bar{x}]$. The monopoly outcome is thus $(x^m, p^m) = (\bar{x}, P(\bar{x}))$, the same as the competitive outcome.

Case (ii) : $e(\bar{x}) < 1$. In this case $R'(\bar{x}) < 0$, and so any maximizer of R on $[0, \bar{x}]$ is less than $\bar{x}$. Relative to the competitive firm, the monopoly produces less oil, $x^m < x^c$, at a higher price, $p^m = P(x^m) > p^c$. However, this statement is vacuous if the monopoly problem has no solution, which is possible. For example, R has no maximizer if $e(x) < 1$ for all $x \in (0, \bar{x}]$, as then R is a decreasing positive function on $(0, \bar{x}]$, with a discontinuity at 0 because $R(0) = 0$. ■

Now suppose there are two periods, $t = 1, 2$, and the firm discounts the second period at rate $r > 0$. The inverse demand function in period t is $P_t(x_t)$, which has the same properties as does the function P above. The firm's discounted payoff if it sells x_t in period t at price p_t is $p_1 x_1 + (1 + r)^{-1} p_2 x_2$, where (x_1, x_2) must satisfy $x_1 + x_2 \le \bar{x}$. Its profit in period t is 0 if $x_t = 0$.

(b) (6 pts) Suppose $(p_1^c, x_1^c, p_2^c, x_2^c)$ is a competitive equilibrium satisfying $x_1^c > 0$ and $x_2^c > 0$. Find a system of four equations this equilibrium must satisfy. Then compare x_1^c to x_2^c when $P_1(\cdot) = P_2(\cdot)$.

Soln: x_t^c must be what the consumers demand in period t at price p_t^c, and so

$$p_1^c = P_1(x_1^c), \quad p_2^c = P_2(x_2^c). \tag{1}$$

As these prices are positive, the price-taking firm must find it optimal to sell all its oil:

$$x_1^c + x_2^c = \bar{x}.$$

Thus, $x_1 = x_1^c$ maximizes $p_1^c x_1 + (1+r)^{-1} p_2^c(\bar{x} - x_1)$ on $[0, \bar{x}]$. Since $0 < x_1^c < \bar{x}$, the FOC is $p_1^c - (1+r)^{-1} p_2^c = 0$, which rearranges to the Hotelling no-arbitrage condition that the equilibrium price of oil increases over time at the rate of interest:

$$p_2^c = (1+r) p_1^c.$$

The equilibrium $(p_1^c, x_1^c, p_2^c, x_2^c)$ satisfies these four displayed equations, and they can in principle be solved to find the equilibrium.

If P_1 and P_2 are the same function, then because it is a decreasing function and $p_2^c > p_1^c$, (1) implies $x_2^c < x_1^c$. Thus, more oil is consumed in the present than in the future, due to discounting and the no-arbitrage condition. ■

(c) (6 pts) Again allowing P_1 and P_2 to be different functions, assume now that for some $\underline{e} > 1$, the elasticities satisfy $e_t(x_t) > \underline{e}$ for all $x_t \in (0, \bar{x}]$ and $t = 1, 2$. Suppose $(p_1^m, x_1^m, p_2^m, x_2^m)$ is a monopoly outcome satisfying $x_1^m > 0$ and $x_2^m > 0$. Find a system of four equations this outcome must satisfy.

Soln: The firm's revenue function in period t is given by $R_t(x_t) = x_t P_t(x_t)$ for $x_t \in (0, \bar{x}]$, and $R_t(0) = 0$. The monopoly outputs maximize

$$R_1(x_1) + (1+r)^{-1} R_2(x_2)$$

subject to $x_1 + x_2 \le \bar{x}$. Because both elasticities exceed 1, each R_t is an increasing function (see the solution to (a)). The constraint thus binds, and so $x_1 = x_1^m$ maximizes

$$R_1(x_1) + (1+r)^{-1} R_2(\bar{x} - x_1)$$

subject to $0 \le x_1 \le \bar{x}$. Since $x_1^m \in (0, \bar{x})$, the FOC is $R_1'(x_1^m) - (1+r)^{-1} R_2'(\bar{x} - x_1^m) = 0$. Substitute x_2^m for $\bar{x} - x_1^m$ and rearrange to obtain the monopoly no-arbitrage condition:

$$R_2'(x_2^m) = (1+r) R_1'(x_1^m). \tag{2}$$

The monopoly outcome $(p_1^m, x_1^m, p_2^m, x_2^m)$ satisfies this equation and $x_1^m + x_2^m = \bar{x}$, together with $p_1^m, = P_1(x_1^m)$ and $p_2^m = P_2(x_2^m)$. ■

(d) (7 pts) Under the additional assumption that both P_1 and P_2 have constant elasticities, e_1 and e_2, satisfying $e_1 \ge e_2 > 1$, how does $(p_1^m, x_1^m, p_2^m, x_2^m)$ compare to $(p_1^c, x_1^c, p_2^c, x_2^c)$?

Soln: Using the expression for R' shown in the solution to (a), we can now write the monopoly no-arbitrage condition (2) as

$$\left(1 - \frac{1}{e_2}\right) P_2(x_2^m) = (1+r) \left(1 - \frac{1}{e_1}\right) P_1(x_1^m),$$

and so

$$\frac{P_2(\bar{x} - x_1^m)}{(1+r) P_1(x_1^m)} = \frac{1 - \frac{1}{e_1}}{1 - \frac{1}{e_2}} \ge 1,$$

where the inequality follows from $e_1 \ge e_2 > 1$. The competitive no-arbitrage condition is

$$\frac{P_2(\bar{x} - x_1^c)}{(1+r) P_1(x_1^c)} = 1.$$

Hence,

$$\frac{P_2(\bar{x} - x_1^m)}{P_1(x_1^m)} \ge \frac{P_2(\bar{x} - x_1^c)}{P_1(x_1^c)},$$

with equality iff $e_1 = e_2$. This and $P_t' < 0$ imply $x_1^m \geq x_1^c$, and hence $x_2^m \leq x_2^c$. Therefore, if $e_1 > e_2$ then

$$p_1^m < p_1^c, \quad x_1^m > x_1^c, \quad p_2^m > p_2^c, \quad x_2^m < x_2^c,$$

and these are all equalities if $e_1 = e_2$. ■

2. (25 pts) Consider a Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ that has derivatives $u' > 0$ and $u'' < 0$, and exhibits DARA (decreasing absolute risk aversion). Prove each of the following.

   (a) (10 pts) (Lemma) For any $k \in \mathbb{R}$ and any random gamble $\tilde{y}$,

   $$\mathbb{E}u(\tilde{y}) = u(k) \;\Rightarrow\; \mathbb{E}u(\tilde{y} + a) > u(k + a) \;\forall a > 0.$$

   Soln: Proof 1. Because $\mathbb{E}u(\tilde{y}) = u(k)$, "agent" u is indifferent between the random gamble $\tilde{y}$ and the deterministic gamble δ_k. Let u_a be the utility function defined by $u_a(z) = u(z+a)$. Then for $a > 0$, DARA implies that u_a is strictly more risk averse than u (essentially by Pratt's theorem). Hence, by definition of "strictly more risk averse," the indifference of u between $\tilde{y}$ and δ_k implies that u_a strictly prefers the former. Hence,

   $$\mathbb{E}u(\tilde{y} + a) = \mathbb{E}u_a(\tilde{y}) > u_a(k) = u(k + a).$$

   Proof 2. Let u_a be the utility function defined by $u_a(z) = u(z + a)$. Let $c(a)$ be the certainty equivalent $C(\tilde{y}, u_a)$, so that $\mathbb{E}u_a(\tilde{y}) = u_a(c(a))$. Note that $c(0) = k$. Since u_a exhibits DARA because u does, Pratt's Theorem implies $c(a)$ is an increasing function. Hence, for any $a > 0$ we have $c(a) > k$. Thus,

   $$\mathbb{E}u(\tilde{y} + a) = \mathbb{E}u_a(\tilde{y}) = u_a(c(a)) > u_a(k) = u(k + a).$$

   ■

Even if you were unable to prove the "Lemma" in (a), feel free to use it to prove (b)-(d) below.

   (b) (5 pts) Let $\tilde{x}$ be a random gamble, and let $b(w)$ be the maximum price the agent is willing to pay for $\tilde{x}$ when her wealth is w. Then $b(w)$ increases in w.
   Soln: Let $\hat{w} > w$. The buy price $b(w)$ is defined by

   $$\mathbb{E}u(w + \tilde{x} - b(w)) = u(w).$$

   By (a), adding $\hat{w} - w$ to the arguments on both sides of this equality yields

   $$\mathbb{E}u(\hat{w} + \tilde{x} - b(w)) > u(\hat{w}) = \mathbb{E}u(\hat{w} + \tilde{x} - b(\hat{w})),$$

   where the equality comes from the definition of $b(\hat{w})$. Hence, $b(w) < b(\hat{w})$. ■

   (c) (5 pts) Let $\tilde{x}$ be a random gamble, and let $s(w)$ be the minimum price at which the agent is willing to sell $\tilde{x}$ when her wealth is w. Then $s(w)$ increases in w.
   Soln: Let $\hat{w} > w$. The sell price $s(w)$ is defined by

   $$\mathbb{E}u(w + \tilde{x}) = u(w + s(w)).$$

   By (a), adding $\hat{w} - w$ to the arguments on both sides of this equality yields

   $$\mathbb{E}u(\hat{w} + \tilde{x}) > u(\hat{w} + s(w)).$$

Thus, since $u(\hat{w}+s(\hat{w})) = \mathbb{E}u(\hat{w}+\tilde{x})$, we have

$$u(\hat{w}+s(\hat{w})) > u(\hat{w}+s(w)),$$

which implies $s(\hat{w}) > s(w)$. ■

(d) (5 pts) Now let $\tilde{x}$ be a random gamble that is valuable at wealth w, in the sense that $\mathbb{E}u(w+\tilde{x}) > u(w)$. Then $s(w) > b(w)$, where $b(w)$ and $s(w)$ are defined in (b) and (c) from this $\tilde{x}$ and w.

Soln: Again, the buy price $b(w)$ is defined by

$$\mathbb{E}u(w+\tilde{x}-b(w)) = u(w).$$

Because $\mathbb{E}u(w+\tilde{x}) > u(w)$, we have $b(w) > 0$. Therefore, by (a), adding $b(w)$ to the arguments on both sides of this equality yields

$$\mathbb{E}u(w+\tilde{x}) > u(w+b(w)).$$

Thus, since $u(w+s(w)) = \mathbb{E}u(w+\tilde{x})$, we have

$$u(w+s(w)) > u(w+b(w)),$$

which implies $s(w) > b(w)$. ■

3. (25 pts) Consider an exchange economy with two consumers and two goods. Good x is a perfectly divisible numeraire. Good y, in contrast, is *indivisible,* that is, consumers can only consume it in nonnegative integer amounts. The utility of consumer $i = 1, 2$ from consuming a bundle (x^i, y^i) of the two goods is given by $u^i(x^i, y^i) = x^i + v^i(y^i)$, where $v^i(\cdot)$ is a function on nonnegative integers. Assume that

$$v^i(2) > v^i(1) = v^i(0) = 0, \text{ and } v^i(y) = v^i(2) \text{ for } y > 2.$$

(Think of good y as chopsticks where the value of only one is 0.) Assume also that

$$v^2(2) \leq v^1(2) \leq 10.$$

The initial endowment of consumer $i = 1, 2$ is (e^i_x, e^i_y). Assume the total endowment of good y is $e^1_y + e^2_y = 2$, and that $e^1_x = e^2_x = 20$.

(a) (4 pts) Describe the Pareto efficient allocations in this economy.

Soln: If $v^1(2) = v^2(2)$, an allocation is efficient if and only if it is non-wasteful and one agent gets both units of y. If instead $v^1(2) > v^2(2)$, then an allocation is efficient if and only if it is non-wasteful, and either agent 1 gets both units of y, or agent 2 gets both units and $x^1 < v^2(2)$. ■

(b) (4 pts) Write conditions for a Walrasian equilibrium for this economy.

Soln: Let $p \geq 0$ be an equilibrium price. If $p = 0$, each agent would demand at least 2 units of y, and so market demand would exceed supply. So $p > 0$. This implies that each agent's demand for y is 0, 1, or 2. If $2p < v^2(2)$, then each agent would optimally demand $y = 2$ (which he could afford because his income $20 + pe^i_y$ exceeds the cost $2p$ of purchasing $y = 2 : 20 + pe^i_y > 10 \geq v^i(2) > 2p$). Again, they cannot both demand 2 units of y in equilibrium, and so $2p \geq v^2(2)$. If $2p > v^1(2)$, then both agents would optimally demand 0 units of y, and so market demand would be less than the market supply of 2. Hence, a necessary condition for p to be an equilibrum price is

$$v^2(2) \leq 2p \leq v^1(2).$$

It is also necessary for one agent to get both units of y. If $v^2(2) = v^1(2)$, this can be either agent. If instead $v^2(2) < v^1(2)$, agent 1 must get both units of y, for if agent 2 were to demand a positive amount of y, then $2p = v^2(2) < v^1(2)$, and so agent 1 would demand 2 units and demand would exceed supply. ■

(c) (4 pts) Does a Walrasian equilibrium always exist for such an economy? Either prove that it does or give a counterexample.

Soln: An equilibrium always exists. Any p satisfying $v^2(2) \leq 2p \leq v^1(2)$ is an equilibrium price. For any such p, a corresponding equilibrium allocation is

$$y^1 = 2, \quad x^1 = 20 + pe_x^i - 2p, \quad y^2 = 0, \quad x^2 = 20 + pe_x^2.$$

■

(d) (4 pts) If a Walrasian equilibrium exists for such an economy, is it Pareto efficient? Either explain why it is or provide a counterexample.

Soln: Yes, any Walrasian equilibrium allocation is efficient. From part (a) we saw that an allocation is efficient if it is non-wasteful and gives both units of y to an agent who has the highest $v^i(2)$ From part (b) we see that any equilibrium allocation has these properties. ■

(e) (9 pts) Suppose we replace the assumption $v^i(1) = 0$ with $v^i(1) > 0$, keeping all the other assumptions. Will a Walrasian equilibrium now always exist? Either explain why or give a counterexample.

Soln: Now an equilibrium need not exist. Consider the case in which

$$v^1(0) = 0, \; v^1(1) = 1, \; v^1(2) = 12, \text{ and}$$
$$v^2(0) = 0, \; v^2(1) = 8, \; v^2(2) = 10.$$

Suppose an equilibrium exists with price p and an allocation (y^1, y^2) of good y. Then $y^1 + y^2 = 2$, and $y^1 \in \{0, 1, 2\}$. Consumer maximization gives us the following:

$$y^1 = 2 \;\Rightarrow\; 2p \leq v^1(2) \;\Rightarrow\; p \leq 6 \;\Rightarrow\; y^2 \geq 1$$
$$\Rightarrow\; y^1 + y^2 \geq 3, \text{ contradiction.}$$

$$y^1 = 0 \;\Rightarrow\; 2p \geq v^1(2) \;\Rightarrow\; p \geq 6 \;\Rightarrow\; y^2 \in \{0, 1\}$$
$$\Rightarrow\; y^1 + y^2 \leq 1, \text{ contradiction.}$$

$$y^1 = 1 \;\Rightarrow\; v^1(1) - p \geq \max\{0, v^1(2) - 2p\}$$
$$\Rightarrow\; p \leq v^1(1) = 1 \text{ and } p \geq v^1(2) - v^1(1) = 11, \text{ contradiction.}$$

An equilibrium therefore does not exist. ■

4. (25 pts) Consider a two-period period GEI model of an exchange economy with a single commodity per state. There are 3 states and 2 assets. The assets pay

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{pmatrix}.$$

(a) (7 pts) If the price of asset 1 is $q_1 = 1$, what prices for asset 2 are consistent with no arbitrage?

Soln: To preclude arbitrage there must be nonnegative state prices α for which asset prices q satisfy $q = \alpha^T A$. Thus for nonnegative state prices α we must have

$$\begin{aligned} \alpha_1 + 3\alpha_3 &= 1 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= q_2 \end{aligned}$$

No arbitrage implies $q_2 \geq 1/3$. There is no upper bound since there is no upper bound on α_2. ∎

(b) (8 pts) Suppose now that the price of each asset is 1. What state prices for state 3 are consistent with these asset prices?

Soln: No arbitrage guarantees nonnegative state prices that satisfy

$$\begin{aligned} \alpha_1 + 3\alpha_3 &= 1 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= 1 \\ or \ \ \alpha_1 &= 1 - 3\alpha_3 \\ \alpha_2 &= -1 + 5a_3 \end{aligned}$$

From nonnegativity of state prices we then have $\alpha_3 \leq 1/3$ and $\alpha_3 \geq 1/5$. ∎

(c) (10 pts) Suppose again that the asset prices are both 1. Suppose also that there is a call option that allows an agent to purchase one unit of asset 1 for 2. What prices for this call option are consistent with these asset prices?

Soln: At strike price 2 the agent will purchase the asset only in state 3, when it is worth 3. His net gain in that state is 1. The state price of state 3 was computed above to be between 1/5 and 1/3, which is then the range of prices for the call option that are consistent with no arbitrage. ∎