ECON 897 - Waiver Exam
August 22, 2016

Important: This is a closed-book test. No books or lecture notes are permitted. You have **180** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

1. Let (x_n) a sequence in $\mathbb{R}$.

   (a) Prove that (x_n) has a monotone (either increasing or decreasing) subsequence. [**6 points**]

   (b) Prove that every bounded and monotone sequence in $\mathbb{R}$ converges. [**5 points**]

   (c) Conclude that any bounded sequence in $\mathbb{R}^m$ has a convergent subsequence. [**5 points**]

   (d) Can you conclude from this argument that any closed and bounded subset of $\mathbb{R}^m$ is compact? [**4 points**]

2. Let (M, d_M) and (N, d_N) two metric spaces and $f : M \to N$ a function.

   (a) Prove that the following four conditions are equivalent for continuity of f:

      i. $f^{pre}(K)$ is a closed set of M for any closed subset K of N.

      ii. For all subsets $B \subseteq N$, one gets $\overline{f^{pre}(B)} \subseteq f^{pre}(\overline{B})$.

      iii. For all subsets $A \subseteq M$, one gets $f(\overline{A}) \subseteq \overline{f(A)}$.

      Hint: Prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. Argue $A \subseteq f^{pre}(f(A))$. [**15 points**]

   (b) Conclude that if $f : M \to N$ is a homeomorphism, then for any $A \in M$, $f(\overline{A}) = \overline{f(A)}$. [**5 points**]

3. Let W be a finite-dimensional subspace of a vector space V and let $T : V \longrightarrow W$ be the projection of V on W. Prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$. **[7 points]**

4. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Prove that:

$$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$$

[7 points]

5. Let $u : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ be continuous, quasi-concave and strictly increasing[1].

   (a) Let $x^* \in \mathbb{R}^n_{++}$.

      i. Prove that there exists a $p \in \mathbb{R}^n$ and $M \in \mathbb{R}$ such that $p \cdot x^* \leq M$ and $p \cdot x \geq M$ for all x such that $u(x) \geq u(x^*)$. **[5 points]**

      ii. Prove that $p \cdot x > M$ if x is such that $u(x) > u(x^*)$. **[5 points]**

      iii. Prove that, in fact, $p \in \mathbb{R}^n_+$. **[5 points]**

   (b) Now, given $p \in \mathbb{R}^n_+$ denote by $x(p, w)$ the correspondence that solves the problem:

$$\max_{x \geq 0} u(x), \quad s.t. \quad p \cdot x \leq w$$

      i. Prove that the demand correspondence $x(p, w)$ is convex-valued. If, in addition, u is strictly quasi-concave, prove that $x(p, w)$ is a single element.
      [6 points]

      Now, define the indirect utility function as $v(p, w) = u(x(p, w))$.

      ii. Show that $v(p, w)$ is quasiconvex. **[5 points]**

[1] Definition: A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is strictly increasing if $f(x) > f(y)$ whenever $x >> y$, where $x >> y$ means that $x_i > y_i$ for all $i \in \{1, \ldots, n\}$.

6. Consider the optimization problem, with continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}$ and $U \subset \mathbb{R}^n$ open,

$$\max f(x), \quad s.t.\ x \in D = U \cap \{x : Ax - b \geq 0\}. \tag{P}$$

where A is a $m \times n$ matrix, and b is a $m \times 1$ vector. We prove that x^*, the solution to (P), satisfies constraint qualification condition in the following way:

(a) Show that for $x, x' \in D$, $x \neq x'$, there exists number $\bar{t}$, $0 < \bar{t} < 1$ such that for all $t \in (0, \bar{t})$, $tx + (1-t)x' \in D$. [**5 points**]

(b) Suppose that there exists $x^+ \in D$ with $Df(x^*) \cdot x^+ > Df(x^*) \cdot x^*$. Using the above result, show that there exists $\tilde{x} \in D$ with $f(\tilde{x}) > f(x^*)$. [**5 points**]

(c) Therefore, x^* is a solution to

$$\max Df(x^*) \cdot x, \quad s.t.\ x \in D.$$

Assume $Df(x^*) \neq 0$. Show that it cannot be $Ax^* - b \gg 0$. ($y \gg 0$ for $y \in \mathbb{R}^n$ means that $y_i > 0$ for all i.) [**5 points**]

(d) From (c), we have

$$Ax^* - b = \begin{bmatrix} A' \\ A'' \end{bmatrix} x^* - \begin{bmatrix} b' \\ b'' \end{bmatrix}$$

with $A'x^* - b' = 0$ and $A''x^* - b'' > 0$. (It may be $A' = A$ and $b' = b$.) Show that x^* solves

$$\max_x Df(x^*)x \quad s.t.\ x \in D' = U \cap \{x : A'(x - x^*) \geq 0\}.$$

Then we have proven the statement. [**5 points**]

(e) Write down the Kuhn-Tucker first-order conditions for this problem. Is the Kuhn-Tucker condition necessary for the solution? Sufficient? [**5 points**]

For the next three questions, let $X_1, X_2, \ldots, X_n$, $n > 2$ be independent sample from distribution $F_X(x) = x$, $0 < x < 1$.

7. Verify that $P(X_1 < X_j, \text{ for all } j \neq 1) = \frac{1}{n}$ using law of iterative expectations. [**5 points**]

8. Show that $X_{(n-1)}$ defined by the second largest element in the sample, has pdf

$$f_{X_{(n-1)}}(x) = n(n-1)x^{n-2}(1-x)$$

[**5 points**]

9. Let $Z_n = nX_{(2)}$, where $X_{(2)}$ is the second smallest element in the sample. Does Z_n converge in distribution? What is the limit distribution? Use

$$\lim_{n\to\infty} a_n = a, \quad \Longrightarrow \quad \lim_{n\to\infty} \left(1 + \frac{a_n}{n}\right)^n \to e^a.$$

[**5 points**]