Waiver Exam - ECON 897 Final Exam

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August 25, 2014

Instructions

- This is a closed-book test. No books or lecture notes are permitted.
- You have 180 minutes to complete the exam, the total score is 180.
- Read the questions carefully, and be sure to answer the questions asked.
- You can use all the results covered in all three parts.
- Please write legibly.
- Good luck!

1. This problem asks you to prove that the real line is connected. You CANNOT use the theorem that says: "$\mathbb{R}$ *is connected*".

   (a) **[5 points]** Let K be a non-empty closed subset of the real line. Show that, if it exists, $l.u.b.(K) \in K$. ($l.u.b.(K)$ stands for least upper bound of K.)

   (b) **[5 points]** Let U be a non-empty closed and open (*clopen*) subset of the real line. Use (a) to prove U cannot be bounded.

   (c) **[10 points]** Let U be a non-empty closed and open (*clopen*) subset of the real line. Prove that U must contain every real number, that is $U = \mathbb{R}$. Hence the real line is connected. [Hint: By contradiction. The sets $(-\infty, y)$ and $(-\infty, y]$ may be useful.]

2. Let the sets $B_n \subset \mathbb{R}^m$ for $n \in \mathbb{N}$ be defined as:

$$B_n = [-n, n] \times [-n, n] \times \cdots \times [-n, n].$$

   (a) **[5 points]** Prove that for every compact set $K \subset \mathbb{R}^m$, there exists some $n \in \mathbb{N}$ such that $K \subset B_n$.

   (b) **[10 points]** Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ be an open cover of $\mathbb{R}^m$. Show that for each B_n there exists an open cover $\mathcal{W} = \{W_\alpha\}_{\alpha \in A}$ of $\mathbb{R}^m$ indexed by the same set A as $\mathcal{V}$ such that the two following conditions hold simultaneously:

      i. For each $\alpha \in A$ the open scrap W_α is contained in the corresponding open scrap V_α of $\mathcal{V}$, i.e. $W_\alpha \subset V_\alpha$.

      ii. $B_n \cap W_\alpha \neq \emptyset$ only for a finite number of $\alpha_i \in A$.

   Hint: The set $B_n^c = \mathbb{R}^m \setminus B_n$ may be useful.

   (c) **[10 points]** Let $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$ be an open cover of $\mathbb{R}^m$. By induction show that there exists an *open* cover $\mathcal{W} = \{W_\alpha\}_{\alpha \in A}$ of $\mathbb{R}^m$ indexed by the same set A as $\mathcal{V}$ such that the two following conditions hold simultaneously:

      i. For each $\alpha \in A$ the scrap W_α is contained in the corresponding scrap V_α of $\mathcal{V}$.

      ii. For *each* compact set $K \subset \mathbb{R}^m$, $K \cap W_\alpha \neq \emptyset$ only for a finite number of $\alpha_i \in A$.

   Notice the same cover $\mathcal{W}$ must satisfy the conditions above *for all* compact sets $K \subset \mathbb{R}^m$.

3. **[20 points]** Let A be an $m \times n$ matrix. Assume $g : \mathbb{R}^m \to \mathbb{R}$ is twice differentiable. Define $f : \mathbb{R}^n \to \mathbb{R}$ as $f(x) = g(Ax)$. Calculate $D^2 f_x$.

4. **[20 points]** Let W be a nontrivial subspace of $\mathbb{R}^n$ and $W^\perp$ be its orthogonal complement. Suppose $P : \mathbb{R}^n \to W$ is the projection mapping from $\mathbb{R}^n$ onto W. Assume x is an arbitrary vector in $\mathbb{R}^n$. Prove

$$\sup_{\substack{z \in W^\perp \\ |z|=1}} x^T z = |x - Px|.$$

5. **[10 points]** Let X be a discrete random variable whose probability mass function f_X is symmetric with respect to 0, i.e. $f_X(x) = f_X(-x)$ for all $x \in \mathbb{R}$. Let

$$J = \begin{cases} 1 & \text{if } X > 0, \\ 0 & \text{if } X = 0, \\ -1 & \text{if } X < 0. \end{cases}$$

Show that $|X|$ and J are independent if $f_X(0) = 0$.

6. **[10 points]** Let $X_1, X_2, \cdots$ be iid with cdf

$$G(x) = 1 - e^{-x}, \; x \geq 0.$$

Show that $Y_n = \max\{X_1, \cdots, X_n\} - \ln(n)$ converges in distribution to a random variable with the following cdf:

$$F(x) = e^{-e^{-x}}.$$

7. Consider a consumer who lives for $T > 0$ periods. At the beginning of each period $1 \leq t \leq T$, this consumer must decide his consumption $c_t \geq 0$ for the current period and saving $s_t \geq 0$ for next period, given his saving s_{t-1} from previous period. Denote by $s_0 \geq 0$ the initial saving. The consumer maximizes his life time utility:

$$\begin{aligned} U(s_0) \equiv \max_{\{c_1, \cdots, c_T, s_1, \cdots, s_T\}} \quad & \sum_{t=1}^{T} u(c_t) \\ \text{subject to} \quad & c_t + s_t \leq f(s_{t-1}) \quad \forall 1 \leq t \leq T, \\ & c_t \geq 0, \; s_t \geq 0 \quad \forall 1 \leq t \leq T, \\ & s_0 \geq 0 \text{ is given,} \end{aligned}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is the momentary utility function and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is the production function. Assume both u and f are continuous.

(a) **[10 points]** Let

$$C(s_0) \equiv \left\{ (c_1, \cdots, c_T, s_1, \cdots, s_T) \in \mathbb{R}_+^T \times \mathbb{R}_+^T \middle| c_t + s_t \leq f(s_{t-1}), \; 1 \leq t \leq T \right\}$$

be the constraint set given initial saving s_0. Prove $C(s_0)$ is compact. Hence optimal solutions exist.

(b) **[10 points]** Prove the correspondence $C : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+^T \times \mathbb{R}_+^T$ is continuous.

(c) **[10 points]** Denote by $U : \mathbb{R}_+ \to \mathbb{R}$ the value function of the maximization problem. Prove U is continuous and increasing.

(d) **[15 points]** Assume u is differentiable on $(0, +\infty)$, $u'(c) > 0$ and $\lim_{c \to 0+} u'(c) = +\infty$. Assume f is strictly increasing and $f(0) = 0$. Prove any optimal solution must satisfy $c_t > 0$ for all $1 \leq t \leq T$ and $s_t > 0$ for all $1 \leq t \leq T - 1$.

(e) **[10 points]** In what follows, suppose both u and f are twice continuously differentiable on $(0, +\infty)$. Assume $u' > 0$, $u'' < 0$, $f' > 0$, $f'' < 0$, $f(0) = 0$ and $\lim_{c \to 0+} u'(c) = +\infty$. Prove the optimization problem has a unique solution.

(f) **[10 points]** Given any $s_0 > 0$, prove the optimal solution must satisfy

$$\frac{u'(c_t)}{u'(c_{t+1})} = f'(s_t) \quad \forall 1 \leq t \leq T - 1.$$

(g) **[10 points]** Consider $T = 2$. Prove both $c_1(s_0)$ and $c_2(s_0)$ strictly increase with s_0.