Waiver Exam - ECON 897 Final Exam

Selam Erol Ju Hu Mattis Gornemann

August 31, 2012

Instructions

- KEEP THE EXAM CLOSED UNTIL TOLD TO BEGIN
- You have 180 minutes to complete the exam
- There are three parts of the exam. Please answer them in separate blue books.
- READ THE QUESTIONS CAREFULLY
- If you use a theorem, make sure the assumptions are satisfied
- If a question states that you must prove any theorems used in your proof, be sure to actually prove them
- Be sure to answer the question asked, and not some construct of your imagination for which you will receive no credit
- Be sure to show all work
- Please write legibly
- GOOD LUCK!

Part I - Selman

Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the exam, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

1. (40 points) Define the distance of two sets A, B in $\mathbb{R}^k$ as follows:

$$d(A,B) = \inf\{\|a-b\| \ : \ a \in A, b \in B\}.$$

   Here $\|.\|$ denotes the euclidean metric.

   Prove or disprove the following:

   Given two closed sets A, B, there exists two elements $a \in A$ and $b \in B$ such that $d(A,B) = \|a-b\|$.

2. (30 points) Given two sets A, B in $\mathbb{R}^k$, prove that $\partial(A \cup B) \subset \partial A \cup \partial B$.

3. (30 points) Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $\mathbb{R}$ such that $\lim a_n = a$, $\lim b_n = b$. Let $c_n = \max\{a_n, b_n\}$. Prove that $\lim c_n = \max\{a, b\}$.

Part II - Ju

> Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the exam, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

1. Let A be an $n \times n$ square matrix. Assume:

$$x^T A x = 0, \qquad \forall x \in \mathbb{R}^n. \tag{1}$$

   (a) **(5 points)** Prove all diagonal components of A are $0 \in \mathbb{R}$.

   (b) **(5 points)** Show by example that condition (**??**) does not imply $A = 0$ (zero matrix).

   (c) **(10 points)** Prove if in addition A is symmetric, condition (**??**) implies $A = 0$ (zero matrix).

2. **(20 points)** Let A and B be two $n \times n$ square matrices. Prove if λ is an eigenvalue of AB, it is an eigenvalue of BA. (Hint: You may want to consider the matrix BAB.)

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real valued function. We say f has increasing differences if for all $y_2 > y_1$, the difference

$$f(x, y_2) - f(x, y_1)$$

   is increasing in x. Assume f is twice continuously differentiable.

   (a) **(10 points)** Express $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ in terms of limit and f (In your final answer, do not use first order derivatives).

   (b) **(10 points)** Use (a) to prove if f has increasing differences, then

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} \geq 0 \tag{2}$$

   for all $(x, y) \in \mathbb{R}^2$.

   (c) **(10 points)** Prove condition (**??**) is also sufficient for increasing differences, i.e. if condition (**??**) holds for all $(x, y) \in \mathbb{R}^2$, then f has increasing differences.

4. **(30 points)** Assume $f : [a, b] \times \mathbb{R}$ is twice continuously differentiable and has increasing differences defined in Question 3. Assume also $\frac{\partial^2 f(x,y)}{\partial x^2} < 0$ for all $x \in [a, b]$ and $y \in \mathbb{R}$. Given these assumptions, we know for any $y \in \mathbb{R}$, the function $f(\cdot, y) : [a, b] \to \mathbb{R}$ has a unique global maximum in $[a, b]$. We denote it by $x^*(y)$. Assume further $x^*(y) \in (a, b)$ for all $y \in \mathbb{R}$. Prove $x^*(y)$ is increasing in y.

Part III - Mattis

Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

1. (15 points) For the following statements state if it is true or false. If it is true prove it, if it is false provide a counterexample. If two correspondences $\Gamma_1 : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, \Gamma_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are both upper and lower hemi continuous, then $\Gamma_1 + \Gamma_2 : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $(\Gamma_1+\Gamma_2)(x) = \{y \in \mathbb{R}^n : \exists y_1 \in \Gamma_1(x), y_2 \in \Gamma_2(x) \text{ s.t. } y = y_1+y_2\}$ is upper and lower hemi continuous.

2. Suppose there is a firm who has to borrow in order to produce. Given an interest rate $R \in (0,\infty)$ it solves the problem

$$\max_{c,b\geq 0} \text{u(c) s.t. } f(b) \geq c + Rb$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ strictly increasing and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly concave, continuous and strictly increasing. Assume $u(0) = f(0) = 0$. Suppose there is a $b^* > 0$ such that $f(b^*) = Rb^*$.

   (a) (10 points) Does this problem have a solution (prove or give a counterexample)?;

   (b) (10 points) If a solution exists, is it unique (prove or give a counterexample)?;

   (c) (20 points) Write down the Karush-Kuhn-Tucker conditions for this problem and state what additional assumptions (if any) are needed so that these conditions completely characterize the solution set (explain);

   (d) (10 points) Prove that the solution can found by first maximizing $f(b) - Rb$ over $b \geq 0$ and setting $c = f(b) - Rb$.

3. Let X_1, X_2 be two random variables with joint probability density function

$$f(x_1,x_2) = \begin{cases} exp(-\frac{x_1 x_2}{1-x_1})\frac{x_2}{(1-x_1)^2} & (x_1,x_2) \in (0,1)^2 \\ 0 & \text{otherwise} \end{cases}$$

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   (a) (15 points) Determine the joint probability density funtion of the random variable $Y_1 = \frac{X_1 X_2}{1-X_1}$ and $Y_2 = X_2$;

   (b) (10 points) Determine the marginal distribution of Y_1 and Y_2;

   (c) (10 points) Find the conditional expectation of Y_1 given Y_2.