Waiver Makeup Exam - ECON 897 Final Exam

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Instructions

- KEEP THE EXAM CLOSED UNTIL TOLD TO BEGIN
- You have 180 minutes to complete the exam
- There are three parts of the exam. Please answer them in separate blue books.
- READ THE QUESTIONS CAREFULLY
- If you use a theorem, make sure the assumptions are satisfied
- If a question states that you must prove any theorems used in your proof, be sure to actually prove them
- Be sure to answer the question asked, and not some construct of your imagination for which you will receive no credit
- Be sure to show all work
- Please write legibly
- GOOD LUCK!

Part I - Selman

Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the exam, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

1. Define the distance of two sets A,B in $\mathbb{R}^k$ as follows:

$$d(A,B) = \inf\{\|a-b\| \ : \ a \in A, b \in B\}.$$

Here $\|.\|$ denotes the euclidean metric.

Prove or disprove the following:

(a) (15 points) Given two closed sets A,B, there exists two elements $a \in A$ and $b \in B$ such that $d(A,B) = \|a-b\|$.

(b) (15 points) Given two compact sets A,B, there exists two elements $a \in A$ and $b \in B$ such that $d(A,B) = \|a-b\|$.

2. Given two sets A,B in $\mathbb{R}^k$,

(a) (15 points) Prove that $\partial(A \cup B) \subset \partial A \cup \partial B$.

(b) (15 points) Prove that $\partial(A \cap B) \subset \partial A \cup \partial B$.

(c) (15 points) Prove or disprove that $\partial(A \cup B) \cup \partial(A \cap B) = \partial A \cup \partial B$.

3. (25 points) Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $\mathbb{R}$ such that $\lim a_n = a$, $\lim b_n = b$. Let $c_n = \max\{a_n, b_n\}$. Prove that $\lim c_n = \max\{a,b\}$.

Part II - Ju

Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the exam, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

1. **(15 points)** Let A be an $m \times n$ matrix. Assume there exists an $n \times m$ matrix B such that $AB = I_m$. Show that for any $b \in \mathbb{R}^m$, the system of linear equations $Ax = b$ always has a solution.

2. Let A be an $n \times n$ symmetric, positive definite matrix. We know A is invertible.

   (a) **(15 points)** Show λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

   (b) **(10 points)** Show A^{-1} is symmetric and positive definite.

3. **(25 points)** Let $f : [a,b] \to \mathbb{R}$ be twice differentiable. Assume $\sup_{x\in[a,b]} |f''(x)| \le M$ for some constant M. Assume also f achieves its global maximum at some point x^* in (a,b). Prove

$$\left|f'(a)\right| + \left|f'(b)\right| \le M(b-a).$$

   (Hint: Use the fact $f'(x^*) = 0$.)

4. Consider a utility maximization problem. There is only one consumption good and the consumer gains utility $u(c)$ if he consumes $c \ge 0$. But to purchase this good, the consumer has to work to earn money. If he works $h \ge 0$ hours, he gets wh unit of money where $w > 0$ is the wage rate and incurs $d(h)$ disutility. Assume the price of the consumption good is $p > 0$. Given (p,w), the consumer maximizes his utility subject to the budget constraint:

$$\max_{c\ge 0,\, h\ge 0} \quad u(c) - d(h)$$
$$s.t. \quad pc \le wh$$

   Assume the two functions $u,h : \mathbb{R}_+ \to \mathbb{R}_{++}$ are twice continuously differentiable and $u'(c) > 0$, $d'(h) > 0$ for all $c,h \ge 0$. Suppose for all $(p,w) \in \mathbb{R}^2_{++}$, a unique interior solution to the maximization problem exists. We denote it by $(c^*(p,w), h^*(p,w)) \in \mathbb{R}^2_{++}$.

   (a) **(10 points)** Prove $pc^*(p,w) = wh^*(p,w)$ for all $(p,w) \in \mathbb{R}^2_{++}$, i.e. at optimal solution the budget constraint is binding.

   (b) **(10 points)** Write down Kuhn-Tucker conditions for the optimization problem and use the assumption that the solution is interior to show

$$\frac{u'(c^*(p,w))}{d'(h^*(p,w))} = \frac{p}{w}.$$

   (c) **(15 points)** Calculate $\frac{\partial c^*}{\partial p}$ and $\frac{\partial h^*}{\partial w}$. You can assume the conditions for implicit function theorem hold. (Hint: (a) and (b) give you two equations.)

Part III - Mattis

Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

1. For each of the following statements state if it is true or false. If it is true prove it, if it is false provide a counterexample.

   (a) (20 points) Let $f : X \to \mathbb{R}$ be strictly concave and X be not empty, convex and open. Let $A \subseteq X$ be a compact, non empty set. Then f has a unique maximizer in X.

   (b) (20 points) Every single-valued, lower hemicontinuous correspondence is upper hemicontinuous.

   (c) (20 points) Let A be non empty, convex sets in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$ is not lie in the interior of A. Then there is a $p \in \mathbb{R}^n \setminus \{0\}$ such that $px > py$ for all y in the interior of A.

2. Let (X_1, X_2) be two random variables with joint pdf

$$f(x_1,x_2) = \frac{1}{2\pi\sqrt{(det(A))}} exp(-0.5(x-\mu)A^{-1}(x-\mu)')$$

   for all $x = (x_1, x_2) \in \mathbb{R}^2$, where A is a positive definite, 2×2 matrix and $\mu \in \mathbb{R}^2$. $det(A)$ denotes the determinante of A and A^{-1} the inverse of A.

   (a) (20 points) Let B be an invertible matrix. Show that the joint pdf of $(Y_1, Y_2) = B(X_1, X_2)'$ is of the same form as the one of (X_1, X_2).

   (b) (20 points) Compute mean and variance of Y_1.