Waiver Exam - ECON 897 Final Exam

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Instructions

- KEEP THE EXAM CLOSED UNTIL TOLD TO BEGIN
- You have 180 minutes to complete the exam
- There are a total of 300 possible points on the exam
- READ THE QUESTIONS CAREFULLY
- If you use a theorem, make sure the assumptions are satisfied
- If a question states that you must prove any theorems used in your proof, be sure to actually prove them
- Be sure to answer the question asked, and not some construct of your imagination for which you will receive no credit
- Be sure to show all work
- Please write legibly
- GOOD LUCK!

Part I - Mattis

> Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

Special Instructions for Mattis' part:

- Please do not refer to theorems that have not been proven in your bluebook UNLESS instructed to do so. This does not, of course, include definitions. You can use any equivalent, standard definition of a term, if not stated otherwise, but you have to stick in each exercise with the one you use and prove equivalent formulations if you also want to use them in the same exercise. You can use different (!equivalent!) definitions for different exercises.
- You can use all the results we have seen in class from Chapter 1-4 in the notes (logic, sets, real numbers and functions), all the properties of the absolute value, that any closed interval ($[a,b],(-\infty,a],[a,\infty)$) is closed and that any bounded, closed interval ($[a,b]$) is compact.

1. (15 pts) Let (X,d) be a metric space, $A \subseteq X$ a set with $X \setminus A$ open and $(x_n)_{n\in\mathbb{N}}$ a sequence in A that converges to a point $x \in X$. Show that x is contained in A.
2. (15 pts) Let (X,d) be a metric space, $K \subseteq X$ compact and $A \subseteq K$ closed. Show that A is compact.
3. Let (X,d) be a metric space.
   (a) (10 pts) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X with the property $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. Suppose it exists an $m \in \mathbb{N}$ such that A_m is compact. Prove that $\cap_{n\in\mathbb{N}} A_n$ is not empty.
   (b) (10 pts) Prove or give a counterexample: Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X with the property $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. Then $\cap_{n\in\mathbb{N}} A_n$ is not empty.
   (c) (10 pts) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of X with the property $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. Suppose it exists a decreasing sequence $(M_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $d(x,y) \le M_n$ for all $x,y \in A_n$ and $\lim_{n\to\infty} M_n = 0$. Show that $\cap_{n\in\mathbb{N}} A_n$ contains at most one element.
4. In this exercise we are going to prove that the real numbers are not countable in a different way from class.
   (a) (10 pts) Define what it means for a set to be (at most) countable, what it means to be finite and what it means for two sets to be equinumerous.
   (b) (10 pts) Argue that it is enough to show that $[0,1]$ is not countable to conclude that $\mathbb{R}$ is not countable. (You can use all results we have seen in class on cardinality beside the one that says that the real numbers are not countable. State the results you use clearly.)
   (c) (0 pts) In order to show that $[0,1]$ is not countable, assume the contrary. Then there is a bijection from the natural numbers into $[0,1]$ because $[0,1]$ is clearly not finite. Therefore we can write the elements of $[0,1]$ as a sequence $(a_n)_{n\in\mathbb{N}}$. (Nothing to prove here)
   (d) (20 pts) Use the previous exercise to prove that $[0,1]$ is not countable. (Hint: You could start by the observation that a_1 cannot be contained in all of the following three sets at the same time $[0,\frac{1}{3}],[\frac{1}{3},\frac{2}{3}],[\frac{2}{3},1]$).

Part II - Fatih

Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

1. (20 pts) Let $T : V \to V$ be a linear transformation. Suppose that there is a $v \in V$ such that $T^n(v) = 0$ but $T^{n-1}(v) \neq 0$ for some $n > 0$. Prove that $v, T(v), T^2(v), \cdots, T^{n-1}(v)$ are linearly independent.

2. (20 pts) Suppose there is a solution to $Ax = b$. Argue that the solution is unique if and only if the null space of A is $\{0\}$.

3. (20 pts) Define what it means for a matrix to be positive semi-definite and show that all of the eigenvalues of a positive semi-definite matrix are weakly positive.

4. Job Search Model:

   Consider a worker who is looking for a job. Every period, he gets a job offer with probability $\alpha > 0$. The job offer w comes from a probability distribution with CDF $F(\cdot)$. Assume that the CDF is well behaved (e.g. it is differentiable). The worker decides whether to accept or to reject the offer. If he rejects the offer, he remains unemployed, and the government pays him b dollars in unemployment benefits. If the worker accepts the wage offer w, he gets paid this wage every period until he loses his job with probability δ. Every period, the worker eats his income, which is either his wage or the unemployment benefit. Worker's preferences are ordered according to utility function

   $$\sum_{t=0}^{\infty} \beta^t c_t,$$

   where c_t is his consumption in period t and $\beta < 1$ is the rate at which the worker discounts the future.

   (a) (10 pts) Show that the worker's job acceptance decision follows a cutoff rule; i.e. the worker accepts the offer if and only if the offer w is larger than equal to some reseration wage R.

   (b) (10 pts) Derive a function that implicitly defines R in terms of the model parameters (β, δ, b).

   (c) (10 pts) Show that you can cast the problem of finding R as a fixed point problem, i.e. R is a fixed point of some mapping. Show that this mapping is a contraction. Apply the contraction mapping theorem and state its conclusion. Make sure you verify all the assumptions of the theorem before you apply it.

   (d) (10 pts) Try to sign the following deriatives:

   $$\frac{dR}{db}, \frac{dR}{d\delta}, \frac{dR}{d\beta}$$

   If the sign of the derivative is ambiguous, state it. For fun (not for credit), try to interpret these results as an economist.

Part III - Kurt

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1. Define the bounded budget correspondence as:

$$\mathscr{B}_e(p) = \{x \in \mathbf{R}^n_+ : p \cdot x \le p \cdot e, \;\; x \le \bar{x}\}$$

   for some $\bar{x} \in \mathbf{R}^n_{++}$.

   (a) (15 pts) Prove that if $e >> 0$, then $\mathscr{B}_e(p)$ is compact valued, convex valued, uhc and lhc

   (b) (5 pts) If $e \ge 0$ which of the above conditions no longer hold? Prove your answer.

2. Let $f : \mathbb{R}^n \Rightarrow \mathbb{R}$ be a concave function.

   (a) (10 pts) Prove that the hypograph of f is closed and convex.

   (b) (5 pts) State the Supporting Hyperplane Theorem

   (c) **Definition** A vector $g \in \mathbb{R}^n$ is a **subgradient** of f at x_0 if

$$g^T(x - x_0) \ge f(x) - f(x_0) \quad \forall x \in \mathbb{R}^n$$

   (5 pts) Prove that there exists a subgradient of f at every point $x \in \mathbb{R}$. Note that in general the subgradient may be different at every point.

   Hint: Use (a) & (b)

   (d) (15 pts) If in addition f is differentiable at x_0 prove that there is a unique subgradient at x_0 and that it is equal to the gradient $Df(x_0)$. Note: This is actually an if and only if relationship, but you only need to prove the if part as stated (and in general only holds on the interior of the domain, but here the domain is $\mathbb{R}^n$).

   (e) (10 pts) Prove the following theorem:

   Theorem (Benveniste and Scheinkman) Let $V : \mathbb{R}^n \to \mathbb{R}$ be concave, let $x_0 \in \mathbb{R}$, and let B_ϵ be the epsilon neighborhood of x_0. If there is a concave, differentiable function $W : B_\epsilon \to \mathbb{R}$, with $W(x_0) = V(x_0)$ and with $W(x) \le V(x)$ for all $x \in B_\epsilon$, then V is differentiable at x_0 and $DV(x_0) = DW(x_0)$.

   Hint: Use (c) and (d)

   (f) (10 pts) Let X be a continuous random variable and assume that $\mathbb{E}[X]$ and $\mathbb{E}[f(X)]$ exist and are finite. Prove that $f(\mathbb{E}[X]) \ge \mathbb{E}[f(X)]$ (this is called Jensen's Inequality).

   Hint: Use what you've already shown and remember for any two functions g, h, if $g(X) \ge h(X)$ $\forall X$ then $\mathbb{E}[g(X)] \ge \mathbb{E}[h(X)]$.

3. (20 pts) Let X be a convex and compact set, $A = (0,1) \subseteq \mathbb{R}$, $f : X \times A \to \mathbb{R}$, f concave, $f \in \mathscr{C}^1$. Let

$$V(\alpha) := \max_{x \in X} f(x,\alpha) \qquad X^*(\alpha) := \arg\max_{x \in X} f(x,\alpha)$$

   Prove that V is well defined, concave and differentiable on A. *Hint: 2 (e)*