Waiver Exam - ECON 897 Final Exam

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Instructions

- KEEP THE EXAM CLOSED UNTIL TOLD TO BEGIN
- You have 180 minutes to complete the exam
- There are a total of 300 possible points on the exam
- READ THE QUESTIONS CAREFULLY
- If you use a theorem, make sure the assumptions are satisfied
- If a question states that you must prove any theorems used in your proof, be sure to actually prove them
- Be sure to answer the question asked, and not some construct of your imagination for which you will receive no credit
- Be sure to show all work
- Please write legibly
- GOOD LUCK!

Part I - Kurt

Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part. All questions for this part are on THIS PAGE.

1. (10 points) **Warm-up Questions**
   (a) Prove that every convergent sequence is Cauchy.
   (b) Let $f:(a,b)\to\mathbb{R}$, be differentiable. Prove that f is also continuous.

2. (50 points) **Metric Spaces, Continuity and Compactness** - Consider a metric space (X,d) and a continuous function $f: A\to B$, where $A\subseteq X$, $B\subseteq\mathbb{R}$, A, B non-empty.
   (a) *The Extreme Value Theorem*
      i. Prove that if A is compact, then $f(A)$ is compact. (*NOTE*: A is NOT a subset of $\mathbb{R}$.)
      ii. Prove that if A is compact, then $\arg\max_{a\in A} f\neq\emptyset$ (*Hint*: B IS a subset of $\mathbb{R}$.)
   (b) **Definition** Define the distance between a point x and a set A as:

   $$d(x,A):=\inf_{a\in A} d(x,a)$$

   Prove that for any non-empty set $A\subseteq X$, the function defined above, $d(\cdot,A): X\to\mathbb{R}_+$ is continuous. (*Hint*: Think of the function as $g(x):=d(x,A)$, and use triangle inequality and ε,δ definition of continuity).
   (c) Recall the definition of the diameter of a set:
   Definition The diameter, $\delta(\cdot)$, of a set, A, in a metric space, (X,d), is defined as:

   $$\delta(A)\equiv\sup\{d(x,y)|x,y\in A\}$$

   Remember, that a set $A\subseteq X$ is bounded if $\delta(A)<\infty$.
   Prove that any non-empty compact set A in the metric space (X,d) is bounded. (*Hint*: (a) & (b))

3. (30 points) **Derivatives**
   Let $f:[a,b]\to[a,b]$. Assume f has finite derivative on (a,b), and for all $x\in(a,b)$, $|f'(x)|<\alpha<1$, for some $\alpha\in\mathbb{R}_{++}$.
   (a) Prove that f is a contraction. (Recall, that a function is a contraction if $\exists\beta\in\mathbb{R}_+,\beta<1$ such that $d(f(x),f(y))<\beta d(x,y)$ for all x,y in the domain.)
   (b) Now assume that $f:\mathbb{R}\to\mathbb{R}$ and $|f'(x)|<1$ for all $x\in\mathbb{R}$.
      i. Does f have a fixed point? Prove your answer. (*Hint*: Drawing pictures may help)
      ii. If f has a fixed point is it unique? Prove your answer.

Part II - Fatih

Please use a NEW BLUE BOOK to answer the questions from this part. DO NOT include answers from Parts I or III in this blue book. Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part. The questions for this part are on THIS PAGE and THE FOLLOWING PAGE.

4. (30 points) **Definition** Let A be an n by n matrix. The *trace* of A is defined as the sum of its diagonal elements $tr(A) = \sum_{i=1}^{n} a_{ii}$.
   (a) Let A, B be n by n matrices. Is $tr(AB) = tr(BA)$? If so, then prove. Otherwise, provide a counterexample.
   (b) If A is real, symmetric, prove that the trace of A is the sum of its eigenvalues. [*Hint*: Spectral theorem may be helpful. If you decide to use it, please write down the statement of the theorem.] Note that this result holds for any square matrix.
   (c) Is it possible to have a real symmetric negative definite matrix with a negative trace? If so, provide an example. Otherwise, prove the resulting claim.
5. (20 points) Let P be a projection matrix. Show that $M = I - P$ is also a projection matrix. What space does M project on?
6. (20 points) Let A be an m by n matrix.
   (a) What should the null space of A be in order for the system $Ax = b$ to have *at most* one solution.
   (b) What should the dimension of the column space of A be in order for the system $Ax = b$ to have *at most* one solution.

7. (30 points) Suppose an outstanding research economist (ORE) is interested in explaining business cycles. He comes up with a model to study business cycles. In the model economy, output is determined according to a production function. In order words, total output is denoted as Y and $Y = F(Ku, H) = (Ku)^{\alpha} H^{1-\alpha}$, where K is the level of capital in the economy, u is the capital utilization rate and H is the labor supply in the economy. Investment is risky, governed by a probabilistic shock ε. More precisely, the shock affects capital accumulation in the economy according to

$$K_{t+1} = K_t\,[1-\delta(u_t)] + i_t\,(1+\varepsilon_t),$$

where $\delta()$ is a strictly convex function capturing depreciation and i is the level of investment.

At this point, do not let the question scare you. All that the ORE is interested in is the response of labor supply h and capital utilization u to the investment shock ε_t, i.e. $\frac{dh}{d\varepsilon}$ and $\frac{du}{d\varepsilon}$.

The equations characterizing the behavior of these variables are as follows:

$$F_1(ku, h) = \frac{\delta'(u)}{1+\varepsilon} \tag{1}$$

$$F_2(ku, h) = G'(h) \tag{2}$$

The function G is also strictly convex. Note also that capital does not respond to ε. Using techniques learned in class, help the ORE by finding out the signs of $\frac{dh}{d\varepsilon}$ and $\frac{du}{d\varepsilon}$. [*Hint:* Regard the two equations above as a system and figure out the derivatives of interest. Try the Implicit Function Theorem.]

Part III - Mattis

Please use a NEW BLUE BOOK to answer the questions from this part. DO NOT include answers from Parts I or II in this blue book. Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part. The questions for this part are on THIS PAGE and THE FOLLOWING PAGE.

8. (70 points) **Optimization**

   In the following *subscripts* indicate *components* of vectors, *superscripts* different *households*. I will use the standard notation for the multiplication of two vectors, i.e. $px' = \sum_{i=1}^{d} p_i x_i$ for $p, x \in \mathbb{R}^d$.

   Suppose there is a world with two households and two goods. Denote the utility function of agent i by u^i. Assume that $u^i : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is continuously differentiable, concave and $(x_1, x_2) >> (y_1, y_2)$ implies $u^i(x_1, x_2) > u^i(y_1, y_2)$ for $i = 1, 2$.

   The total resources in the economy are $r \in \mathbb{R}^2_{++}$ (i.e. $r_1, r_2 > 0$) - both goods are available in strictly positive amounts.

   A planner wants to distribute the resources between the two households. In order to do this he is maximizing a weighted sum of the utilities of the two.

   (a) (15 points) Write out the Karush-Kuhn-Tucker Conditions for the following problem

   $$\max_{((x_1^1, x_2^1),(x_1^2, x_2^2)) \in \mathbb{R}^4_+} \lambda_1 u^1(x_1^1, x_2^1) + \lambda_2 u^2(x_1^2, x_2^2) \text{ s.t. } x_1^1 + x_1^2 \le r_1 \text{ and } x_2^1 + x_2^2 \le r_2$$

   and argue, that they are necessary and sufficient for a maximizer. (State separately which make them sufficient and which make them necessary.) Assume $\lambda_1, \lambda_2 \in \mathbb{R}_{++}$. You can assume, that u^i is defined on a open set containing $\mathbb{R}^2_+$.

   (b) Now, assume that the planner has solved the problem for some λ_1, λ_2 and allocates the resources between the two agents according to that solutions. We know that every solution $((r_1^1, r_2^1), (r_1^2, r_2^2))$ to the problem in a) has the following property:

   $$(\star) \quad \text{There is no } \left\{(y_1^1, y_2^1), (y_1^2, y_2^2)\right\} \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \text{ such that:}$$
   $$\begin{aligned} & y_1^1 + y_1^2 = r_1 \\ & y_1^2 + y_2^2 = r_2 \\ & u^i(y_1^i, y_2^i) \ge u^i(r_1^i, r_2^i) \quad \text{for } i = 1, 2 \\ & u^i(y_1^i, y_2^i) > u^i(r_1^i, r_2^i) \quad \text{for either } i = 1 \text{ or } i = 2 \end{aligned}$$

   (You do not have to prove this. If you are familiar with Pareto efficiency, it is exactly, what this property says.).

The planner now wants to set prices in such a way that both households like the amounts he gave them at least as much as any other they could obtain by selling and buying goods given that chosen price. Let $(r_1^i, r_2^i) \in \mathbb{R}^2_{++}$ be the quantities of the two goods agent i got from the planner. Notice, that both are assumed to be strictly positive. Suppose, that these quantities fulfill the condition $(\star)$.

i. (10 points) Prove that

$$K^i := \left\{ z \in \mathbb{R}^2 : r^i + z \geq 0 \text{ and } u^i(r^i + z) > u^i(r^i) \right\}$$

is convex (yes, I want the second inequality to be strict).

ii. (5 points) Prove, that

$$K := K^1 + K^2 := \left\{ (z_1, z_2) \in \mathbb{R}^2 : \exists (z_1^1, z_2^1) \in K_1, (z_1^2, z_2^2) \in K_2 \text{ s.t. } (z_1, z_2) = (z_1^1, z_2^1) + (z_1^2, z_2^2) \right\}$$

is convex.

iii. (10 points) Prove, that $(0,0) \notin K$. (Hint: here you need to use the condition $(\star)$)

Lemma 1 *There exists a $p \in \mathbb{R}^2 \setminus \{0\}$, so that $pz' \geq 0$ for all $z \in K$.*
(Nothing for you to prove, but you should recognize this is true from the supporting hyperplane theorem)

iv. (10 points) Show, that $p > 0$, i.e. $p_1, p_2 \geq 0$ and either $p_1 > 0$ or $p_2 > 0$ (or both).

v. (10 points) Show, that if for $(y_1, y_2) \in \mathbb{R}^2_+$ we have $u^i(y_1, y_2) > u^i(r^i)$, then $p(y_1, y_2)' \geq p(r_1^i, r_2^i)'$.

vi. (10 points) Show, that if for $(y_1, y_2) \in \mathbb{R}^2_+$ we have $u^1(y_1, y_2) > u^i(r^i)$, then $p(y_1, y_2)' > p(r_1^i, r_2^i)'$ by using the continuity of u^i.

Congratulations, you proved the second welfare theorem!

9. (30 points) **Measure Theory**

(a) (15 points) Let $(\Omega, \mathscr{F}, \mu)$ be a measure space with μ a finite measure. Show, that for every decreasing (i.e. $A_n \supset A_{n+1}$) sequence of sets $(A_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$ we have $\lim_{n\to\infty} \mu(A_n) = \mu(\cap_{n\in\mathbb{N}} A_n)$. Hint: Remember, that for every measure μ and every increasing sequence of sets in its domain $(A_n)_{n\in\mathbb{N}}$ we have $\lim_{n\to\infty} \mu(A_n) = \mu(\cup_{n\in\mathbb{N}} A_n)$.

(b) (5 points) Take $\cap_{n\in\mathbb{N}}(-\infty, x + a_n]$ and $\cup_{n\in\mathbb{N}}(-\infty, x - a_n]$ for $x \in \mathbb{R}$ and $(a_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}_+$ with limit 0. Rewrite the two sets as intervals. Explain.

(c) (10 points) Take a finite measure μ on $(\mathbb{R}, \mathscr{B})$ (where $\mathscr{B}$ is the Borel-σ-algebra) and define a function $F(x) := \mu((-\infty, x])$ for all $x \in \mathbb{R}$. Use a) and b) to show, that for any $x \in \mathbb{R}$ and any decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with limit x we have $\lim_{n\to\infty} F(x_n) = F(x)$ and that $\lim_{x\to-\infty} F(x) = 0$.