

Waiver Exam - ECON 897 Final Exam

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1 Instructions

- KEEP THE EXAM CLOSED UNTIL TOLD TO BEGIN
- You have 180 minutes to complete the exam
- There are a total of 300 possible points on the exam
- READ THE QUESTIONS CAREFULLY
- If you use a theorem, make sure the assumptions are satisfied
- If a question states that you must prove any theorems used in your proof, be sure to actually prove them
- Be sure to answer the question asked, and not some construct of your imagination for which you will receive no credit
- Be sure to show all work
- Please write legibly
- GOOD LUCK!

2 Part I - Kurt

Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part

1. (10 points) **Easy Questions**
 - (a) Let $\{x_n\}$ be a Cauchy sequence. Prove that the set of subsequential limits of $\{x_n\} = \emptyset$ or $\{x\}$ (a singleton).
 - (b) Let $f : (a, b) \rightarrow \mathbb{R}$, be differentiable. Prove that f is also continuous.
2. (20 points) **Metric Spaces** Consider the metric space (X, d) where $X = \mathbb{R}$ and $\forall x, y \in \mathbb{R}$, $d(x, y) = 1$ if $x \neq y$.
 - (a) Prove that the metric space (X, d) is complete.
 - (b) Prove that every subset of X is open.
3. (40 points) **Continuity and Compactness** Let $(X, d_X), (Y, d_Y)$ be metric spaces, $A \subseteq X$, $B \subseteq Y$, where Y is a totally ordered set. $f : A \rightarrow B$, is a continuous function.
 - (a) Let A be compact. Prove that $f(A)$ is compact. **NOTE:** The codomain of f is NOT a subset of \mathbb{R} . Use topological definition of compactness!
 - (b) Continue to assume that A is compact. Prove that $\arg \max_{a \in A} f \neq \emptyset$ and $\arg \min_{a \in A} f \neq \emptyset$, that is prove the EVT. (*Hint:* Prove by contradiction, and just prove for either max or min (since the proofs are wlog identical). Remember the topological definition of compactness and come up with an appropriate open covering.)
4. (30 points) **Derivatives** Let $f : [a, b] \rightarrow [a, b]$ is continuous, differentiable on (a, b) . Assume that $f'(x) < 1$ at every $x \in [a, b]$ such that $f(x) = x$. Prove that f has a fixed point and that it is unique. You must prove any theorems that you use in your proof. **NOTE:** This is not the same question as on Test 2, read it carefully. (*Hint:* Drawing pictures may be helpful)

3 Part II - Matt

Please use a NEW BLUE BOOK to answer the questions from this part. DO NOT include answers from Parts I or III in this blue book. Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part

5. True/False

Please tell me if the following statements are true or false. If true, provide a short proof. If false, provide a counter-example.

Note: For any given set $A \subseteq \mathbb{R}^n$, the set A' will be the derived set of A (or the set of limit points of A) and the set bdA will be the set of boundary points of A .

- (a) (10 points) For any set $X \subseteq \mathbb{R}^n$, $X' - X \subseteq bdX$.
- (b) (10 points) For any set $X \subseteq \mathbb{R}^n$, $X' \subseteq bdX$ iff $X' = \emptyset$.
- (c) (10 points) For any connected set $X \subseteq \mathbb{R}^n$, $X' \subseteq bdX$ iff $X' = \emptyset$.

6. (25 points) Connectedness and Continuity

Let $X \subseteq \mathbb{R}^n$ be a connected set. Let $f : X \rightarrow \mathbb{R}^m$ be a continuous function. Rigorously prove that $f(X)$ is a connected set.

7. Inverse Function Theorem

Throughout this problem, consider the multi-dimensional case with $n > 1$.

- (a) (5 points) Please state the Inverse Function Theorem (in \mathbb{R}^n).
- (b) (10 points) Let $X \subseteq \mathbb{R}^n$ and consider a C^1 function $f : X \rightarrow \mathbb{R}^n$ such that $f(X) \subseteq X$. Prove that the set $\{x \in X : \text{rank} Df(x) < n\}$ is closed in X .
- (c) (30 points) Define $X^* = \{x \in X : \text{rank} Df(x) = n\}$. Assume that X^* is nonempty. Assume further that $f(X^*) \subseteq X^*$. Rigorously prove that the set $\{x \in X^* : f(f(x)) = 0\}$ is discrete.

Note: A set $A \subseteq \mathbb{R}^n$ is discrete iff $A' = \emptyset$.

Hint: An element $x \in \mathbb{R}^n$ satisfies $f(f(x)) = 0$ iff $g(x, y) = 0$ for some appropriately defined mapping $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ s.t. $g(x, y) = (f(x) - y, f(y))$ where $y \in \mathbb{R}^n$.

4 Part III - Fatih

Please use a NEW BLUE BOOK to answer the questions from this part. DO NOT include answers from Parts I or II in this blue book. Answer all the questions on this page in the same blue book. Write your name, program and the part number on each blue book. READ the exam CAREFULLY and THINK before answering any question. This part is worth 1/3 of the points, so it is recommended that you allocate 60 minutes on this part

8. TRUE or FALSE: If you think the statement is true, provide a proof. Otherwise, provide a counterexample. Graphs might help.
- (15 points) The sum of two concave functions is concave.
 - (15 points) A quasi-concave function is continuous on an open set.
9. Consider the following problem:

$$\max_{c_1, c_2, x_1, x_2} u(c_1) + u(c_2)$$

$$\begin{aligned} \text{s. to } c_1 + x_1 &\leq x \\ c_2 + x_2 &\leq f(x_1) \\ c_t, x_t &\geq 0, \quad t = 1, 2 \end{aligned}$$

where $x \in \mathbb{R}_{++}$ is given and $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are nondecreasing continuous functions.

- (10 points) Describe what additional assumptions (if any) are necessary so that this problem has a solution.
 - (10 points) Describe what additional assumptions (if any) are necessary so that the solution, when it exists, is unique.
 - (10 points) Write down the Karush-Kuhn-Tucker conditions for this problem and state what *additional* assumptions (if any) are needed so that these conditions completely characterize the solution set.
 - (10 points) Assume that u is strictly increasing. Show that the optimal solution must satisfy $c_1 + x_1 = x$ and $c_2 + x_2 = f(x_1)$. This shows that we can express c_1 and c_2 in terms of x_1 and x_2 and that we can get rid of the first 2 constraints.
10. (30 points) Let X_1 and X_2 have the following density function:

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & \text{for } x_1, x_2 \geq 0 \\ 0 & \text{o/w} \end{cases}$$

Define Y_1 by $Y_1 = \frac{X_1}{X_1 + X_2}$. Obtain the probability density function of Y_1 .