Economics 897
Waiver Exam

Part I: Matthew Hoelle   Part II: KyungMin (Teddy) Kim
Part III: Fatih Karahan

University of Pennsylvania
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Steinberg Hall- Dietrich Hall 1206
9 AM - 12 Noon

Instructions

- You will have 180 minutes to complete this examination.
- You have 4 blue books. It is imperative that you use one for Part I, one for Part II, one for Part III, and one for scratch. Indicate on the front of each blue book what is contained inside (e.g. Part I). Additional scratch books may be requested.
- Write your name on all blue books.

These directions pertain to Part I only. Part I is found entirely on this single page. Part I is worth 100 points. As this is the first of three parts, it is recommended that you spend 60 minutes on Part I. Number clearly each question you are answering. Answer all questions thoroughly.

1. **Boundary and Interior**

   (a) (8 points)
   For any set V, show that $V \subseteq bdV \cup intV$.

   (b) (8 points)
   For any set V, show that $bdV \cap intV = \emptyset$.

2. **Implications of Continuity?**

   By definition, a function $f : A \to B$ with $A \subseteq \mathbb{R}^1$ and $B \subseteq \mathbb{R}^1$ is differentiably strictly increasing if $f'(x) > 0 \ \forall x \in A$, is differentiably strictly decreasing if $f'(x) < 0 \ \forall x \in A$, and is differentiably strictly monotonic if it is either differentiably strictly increasing or differentiably strictly decreasing.

   (a) (12 points)
   Consider a function $f : C \to D$ with $C \subseteq \mathbb{R}^1$ and $D \subseteq \mathbb{R}^1$. If f is 1-1 and continuous and C is connected, show that f is differentiably strictly monotonic.

   (b) (16 points)
   Prove or give a counter-example to the following statement:
   Consider a function $f : A \to B$ with $A \subseteq \mathbb{R}^1$ and $B \subseteq \mathbb{R}^1$. If f is onto and continuous and $C \subseteq A$ is open, then $f(C)$ is open in B.

   (c) (16 points)
   Prove or give a counter-example to the following statement:
   Consider a function $f : A \to B$ with $A \subseteq \mathbb{R}^1$ and $B \subseteq \mathbb{R}^1$. If f is bijective and continuous and $C \subseteq A$ is connected and open, then $f(C)$ is open in B.

3. **Inverse Function Theorem (40 points)**

   State and prove the Inverse Function Theorem in $\mathbb{R}^1$. As a hint, you may use the fact that any open set in $\mathbb{R}^1$ can be written as the arbitary union of open, connected sets. In math, if $A \subseteq \mathbb{R}^1$ is open, then $A = \cup_i A_i$ for some sets $A_i \subseteq \mathbb{R}^1$ that are open and connected.

1. (15) X : a compact metric space, Y : a closed subset of X. Prove that Y is compact.

2. (15) Suppose C_1 and C_2 are connected and $C_1 \cap C_2 \neq \emptyset$. Prove that $C_1 \cup C_2$ is connected.

3. (15) X is a random variable defined on a probability space $(\Omega, \mathcal{F}_0, P)$ such that $E(|X|) < \infty$, and $\mathcal{F}$ is a σ-field such that $\mathcal{F} \subseteq \mathcal{F}_0$. Let $var(X|\mathcal{F}) = E\left(X^2|\mathcal{F}\right) - E(X|\mathcal{F})^2$. Show that
$$var(X) = E(var(X|\mathcal{F})) + var(E(X|\mathcal{F})).$$

4. (15) Suppose $X = (X_1, X_2, X_3, X_4)$ has a joint probability density function
$$f_X(x_1, x_2, x_3, x_4) = 24 \exp(-(x_1 + x_2 + x_3 + x_4)), 0 < x_1 < x_2 < x_3 < x_4 < \infty.$$
Consider the transformation $U_1 = X_1, U_2 = X_2 - X_1, U_3 = X_3 - X_2, U_4 = X_4 - X_3$. Derive a joint probability density function of $U = (U_1, U_2, U_3, U_4)$.

5. Consider two functions $W_1, W_2 : [0, k] \to R_+$ $(k < 1)$ that satisfy the following properties.
(i) $W_1(0) = W_2(0)$ and $W_1(k) = W_2(k) = 0$.
(ii) W_1 is strictly concave.
(iii) W_2 is strictly decreasing and strictly convex.

Define $\gamma : [0, k] \to [0, k]$ so that $\gamma(x) > x$ and $W_2(x) = W_1(\gamma(x))$. Let $\gamma^0(x) = x$, and recursively define $\gamma^k(x) = \gamma\left(\gamma^{k-1}(x)\right)$ for $k \geq 2$.

   (a) (20) Let N be the value such that $\sum_{k=0}^{N} \gamma^k(0) < 1$ and $\sum_{k=0}^{N+1} \gamma^k(0) \geq 1$. In addition, let n be the smallest integer that is greater than $1/k$. Prove that for any $n \leq m \leq N$, there exists x^m such that $\sum_{k=0}^{m-1} \gamma^k(x^m) = 1$.
   (b) (20) Suppose $f : [0, k] \to R_+$ is C^1 and a strictly concave function, and $n \leq m < N$. Show that
$$\sum_{k=0}^{m-1} f\left(\gamma^k(x^m)\right) \leq \sum_{k=0}^{m} f\left(\gamma^k\left(x^{m+1}\right)\right).$$

1. (30 pts) Linear Algebra

   (a) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Are the columns of A independent? Is $b \in C(A)$?

   (b) A square matrix A is said to be nilpotent if $\exists k > 1$ such that $A^k = 0$ but $A^{k-1} \neq 0$. Prove that if A is a square nilpotent matrix, then 0 is the only eigenvalue of A.

   (c) Show that if A is a symmetric and positive definite matrix, then so is A^{-1}.

2. (20 pts) Prove the following separating hyperplane theorem: Let X and Y be two closed, convex, nonempty, bounded subsets of $\mathbb{R}^n$ such that $X \cap Y = \varnothing$. Then there exists a hyperplane $H(p, a)$ that strictly separates X and Y, i.e. $px > a \ \forall x \in X$ and $py < a \ \forall y \in Y$. (Hint: Try to transform this into a point-set separation problem and apply the theorem proved in class.)

3. (20 pts) Consider the following maximization problem:

$$\begin{aligned} \max \ & u(c_1, c_2) \\ \text{s.to} \ & c_1 + c_2 \leq I \end{aligned}$$

   where $u : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}$ with $u(c_1, c_2) = c_1 - \frac{1}{c_2}$

   (a) Does there necessarily exist a solution to this problem?

   (b) Fully characterize the set of optimal solutions. State any theorems you use, and make sure that the hypotheses of the relevant theorems hold.

4. (15 pts) Let $\Gamma : X \rightrightarrows Y$ be a single-valued correspondence, i.e. $\Gamma(x) = \{y\}$. Show that Γ is lhc if and only if Γ is continuous when viewed as a function.

5. (15 pts) State the Theorem of the Maximum and show that if any of the hypotheses of the theorem fails to hold, then the conclusion may fail to hold.