Microeconomic Theory II
Preliminary Examination Solutions

Exam date: June 5, 2017

1. **(40 points)** Consider a Cournot duopoly. The market price is given by $1 - q_1 - q_2$, where q_1 and q_2 are the quantities of output produced by the two firms. There are no costs.

   (a) Find the (Nash) equilibrium quantities of output. **[5 points]**

   Solution: Firm i's profits are $(1-q_1-q_2)q_i$, for first-order conditions of

$$\begin{aligned} 1-2q_1-q_2 &= 0 \\ 1-2q_2-q_1 &= 0, \end{aligned}$$

   which solve for

$$q_1 = q_2 = 1/3.$$

■

   (b) Suppose that firm 1's owner first hires a manager, after which the manager of firm 1 and owner of firm two simultaneously choose outputs q_1 and q_2. The manager of firm 1 is paid $\kappa\pi_1(q_1,q_2)+\lambda q_1 - B$, where q_1 is the quantity chosen by the manager, $\pi_1(q_1,q_2)$ is the profit earned in the duopoly game (given outputs q_1, and q_2), and κ, λ, and B are nonnegative constants chosen by the owner of firm 1. The outside option for the manager is 0. Assume that firm 2 observes the values of κ, λ, and B before the two firms simultaneously choose their outputs. What is the subgame perfect equilibrium of this game. Compare the result to the outcome of the Stackelberg model (without managers). **[15 points]**

   Solution: Note first that there is a distinct subgame corresponding to each choice of (κ,λ,B) by firm 1. In a subgame perfect equilibrium, we first solve for a Nash equilibrium (which turns out to be unique) of the subgame reached by (κ,λ,B). The manager of firm 1 chooses q_1 to maximize

$$\kappa(1-q_1-q_2)q_1+\lambda q_1 - B.$$

   First-order conditions are

$$\begin{aligned} \kappa(1-2q_1-q_2)+\lambda &= 0 \qquad \text{and} \\ 1-2q_2-q_1 &= 0, \end{aligned}$$

   which can be solved for best response functions

$$\begin{aligned} q_1 &= \frac{1+(\lambda/\kappa)-q_2}{2} \qquad \text{and} \\ q_2 &= \frac{1-q_1}{2}. \end{aligned}$$

The values of κ and λ thus shift firm 1 manager's best-response function. The solution is

$$\begin{aligned} q_1^*(\kappa,\lambda) &= \frac{2(\lambda/\kappa)+1}{3} \\ q_2^*(\kappa,\lambda) &= \frac{1-(\lambda/\kappa)}{3}. \end{aligned}$$

If $\lambda/\kappa < 1$, we have found the unique equilibrium of the subgame. If $\lambda/\kappa \geq 1$, prices are zero, and firm 2 optimally chooses to stay out of the market. It remains to determine the optimal values of (κ, λ, B) . Note that we can restrict attention to $\lambda/\kappa < 1$, since if $\lambda/\kappa \geq 1$, price is zero and firm 1 makes a loss.

Firm 1 will choose κ, λ, and B to maximize its profits of

$$(1-\kappa)\pi_1(q_1^*(\kappa,\lambda), q_2^*(\kappa,\lambda)) - \lambda q_1^*(\kappa,\lambda) + B.$$

Firm 1 wants to minimze B consistent with the manager accepting the contract and so the choice leaves the manager indifferent between accepting the contract and taking the outside option:

$$B = \kappa\pi_1(q_1^*(\kappa,\lambda), q_2^*(\kappa,\lambda)) + \lambda q_1^*(\kappa,\lambda),$$

and so the firm chooses κ and λ to maximize

$$\pi_1(q_1^*(\kappa,\lambda), q_2^*(\kappa,\lambda)).$$

In other words, firm 1 chooses κ and λ to achieve the Stackelberg outcome with firm 1 as leader, yielding $\lambda/\kappa = 1/4$, $q_1^* = 1/2$, $q_2^* = 1/4$. Note that only the ratio λ/κ is determined.

By encouraging its manager to choose an output larger than would maximize profits (since $\lambda > 0$), firm 1 is encouraging firm 2 to be less aggressive (which is what a Stackelberg leader does by committing to a quantity larger than the best reply to the follower's chosen quantity). ∎

(c) Now suppose that both owners hire managers, simultaneously making public the terms ($(\kappa_1, \lambda_1, B_1)$ and $(\kappa_2, \lambda_2, B_2)$) of the managers' contracts, after which the managers simultaneously choose outputs. Solve for the equilibrium. **[10 points]**

Solution: Now there is a distinct subgame corresponding to each choice of $(\kappa_1, \lambda_1, B_1)$ by firm 1 and $(\kappa_2, \lambda_2, B_2)$ by firm 2. Set $\widehat{\lambda}_i := \lambda_i/\kappa_i$. The best-response functions for the two firms are now

$$\begin{aligned} q_1 &= \frac{1+\widehat{\lambda}_1 - q_2}{2} \\ q_2 &= \frac{1+\widehat{\lambda}_2 - q_1}{2}, \end{aligned}$$

which can be solved for

$$q_1^* = \frac{1+2\widehat{\lambda}_1-\widehat{\lambda}_2}{3}$$
$$q_2^* = \frac{1+2\widehat{\lambda}_2-\widehat{\lambda}_1}{3}$$

As before B_i is chosen to leave the managers indifferent between accepting the contracts, and the firm i chooses $\widehat{\lambda}_i$ to maximize its profits. Firm i's profit is

$$\left(1-\frac{2+\widehat{\lambda}_i+\widehat{\lambda}_j}{3}\right)\left(\frac{1+2\widehat{\lambda}_i-\widehat{\lambda}_j}{3}\right),$$

giving a first-order condition that reduces to

$$1+4\widehat{\lambda}_i+\widehat{\lambda}_j=0,$$

giving $\widehat{\lambda}_1=\widehat{\lambda}_2=1/5$. ■

(d) How do firms' outputs and profits in the previous part compare to the those of the Nash equilibrium without managers? Explain your answer. Notice that your answer to the previous two parts should allow you to make these comparisons without calculating equilibrium outputs and profits. **[5 points]**

Solution: In equilibrium, both firms subsidize output. Equilibrium output must then be higher, and hence profits lower, than in a Nash equilibrium without subsidies. The firms are effectively playing a Prisoners' Dilemma. ■

(e) Suppose that a law is proposed making it illegal to disclose the compensation contracts of managers of firms. Given that the owners always have the option of not disclosing such contracts, why would such a law have any effect? Would you expect the owners of the two firms to support this law? **[5 points]**

Solution: Either firm would prefer to be the only firm offering an observable contract, but both fare worse when both do so. Nonetheless, in the absence of the law, it is not an equilibrium to have only one firm disclosing the contract. The firms would support such a law. ■

2. This question concerns an asynchronous version of an infinitely repeated game, with stage game given by

	A	B
A	$3,3$	$0,0$
B	$0,0$	$1,1$

.

The discount factor is given by $\delta \in (0,1)$, and time is indexed by $t = 0,1,2,\ldots$. For $t > 0$, player 1 (the row player) chooses in odd periods, and player 2 (the column player) chooses in even periods. So, for $t > 0$, each player's action is fixed for *two* periods. Period 0 is different in that both players choose simultaneously in that period, and player 1 gets to choose again in the next period, period 1.

(a) Prove that the strategy profile in which each player always plays B is not a subgame perfect equilibrium for any δ. **[10 points]**

Solution: Let (s_1, s_2) be the strategy profile in which each player always plays B. In particular, this means each player plays B after *every* history. Consider player 1 in period 1 after player 2 had chosen A in period 0. By following the strategy, player 1 receives a payoff of δ, while by deviating to A this period (and so in period 2), and then playing B thereafter, player 1 has payoff (since player 2, by assumption, plays B thereafter),

$$3(1-\delta)+\delta^2,$$

and

$$\begin{aligned} 3(1-\delta)+\delta^2 &> \delta \\ \iff 3-4\delta+\delta^2 &> 0 \\ \iff (3-\delta)(1-\delta) &> 0, \end{aligned}$$

which is true for all $\delta \in (0,1)$. ∎

(b) Prove that there exists a δ^* such that the infinite repetition of BB is a subgame perfect equilibrium outcome if $\delta < \delta^*$. What is the value of δ^*? **[10 points]**

Solution: Consider the profile that specifies that both players choose B in period 0; if B was chosen the previous period by player i, then player j chooses B this period; and if A was chosen the previous period by player i, then player j chooses A this period.

It is clearly optimal (for any δ) for player j to play A if player i had played A in the previous period.

Suppose player i had played B in the previous. By following the strategy, player i receives a payoff of 1, while by deviating to A this period (and so in the next period), and then following the strategy, player i has payoff (since player j, by assumption, is also following the strategy) of 3δ, and

$$1 \geq 3\delta \iff \delta < \frac{1}{3} =: \delta^*.$$

A similar argument shows that player 2 does not want to deviate in period 0 for $\delta < \frac{1}{3}$. Finally, player 1 will not deviate in period 0, since B is a myopic best reply to B, and 1's choice in period 0 has no future payoff implications. ∎

Questions 3 and 4 concern the same setting, but are otherwise independent.

In the common setting, two players jointly own an asset (with equal shares) and are bargaining to dissolve the joint ownership. Suppose v_i is the private value that player i assigns to the good, and suppose v_1 and v_2 are independently drawn from the interval $[0,1]$, according to the distributions F_i, with densities f_i, for $i = 1,2$. Efficiency requires that player i receive the asset if $v_i > v_j$. If the partnership is not dissolved, player i receives a payoff of $\frac{1}{2}v_i$.

3. The players bargain as follows: the two players simultaneously submit bids, with the higher bidder winning the object (with ties resolved by a fair coin flip), and the winner pays the loser the winning bid.

   (a) What are the interim payoffs for each bidder? **[5 points]**

   Solution: Since there is always a winner in this bargaining game, the partnership is always dissolved. Suppose bidder j is playing σ_j. Then i's interim payoff from bidding b_i with value v_i can be written as either

$$U_i(v_i, b_i; \sigma_j) = \int_{\sigma_j(v_j)<b_i} (v_i - b_i) d F_j(v_j) + \int_{\sigma_j(v_j)>b_i} \sigma_j(v_j)\, d F_j(v_j) + \frac{v_i}{2} \Pr\{\sigma_j(v_j) = b_i\}$$

   or as

$$\begin{aligned} U_i(v_i, b_i; \sigma_j) = E[v_i - b_i \mid \sigma_j(v_j) < b_i] \Pr\{\sigma_j(v_j) < b_i\} \\ + E\left[\, \sigma_j(v_j) \middle| \sigma_j(v_j) > b_i \right] \Pr\{\sigma_j(v_j) > b_i\} + \frac{v_i}{2} \Pr\{\sigma_j(v_j) = b_i\}. \end{aligned}$$

   ■

   (b) Suppose (σ_1, σ_2) is a Bayes-Nash equilibrium of the auction, and assume σ_i is strictly increasing and differentiable function for $i = 1,2$. Describe the pair of differential equations the strategies must satisfy. **[10 points]**

   Solution: Since σ_j is strictly increasing, and there are no atoms, $\Pr\{\sigma_j(v_j) = b_i\} = 0$. Interim payoffs can then be written as

$$U_i(v_i, b_i; \sigma_j) = (v_i - b_i) F_j(\sigma_j^{-1}(b_i)) + \int_{v_j > \sigma_j^{-1}(b_i)} \sigma_j(v_j)\, f_j(v_j) d v_j.$$

   The first order condition (with respect to b_i) is

$$\begin{aligned} 0 = -F_j(\sigma_j^{-1}(b_i)) + (v_i - b_i) f_j(\sigma_j^{-1}(b_i))(\sigma_j'(\sigma_j^{-1}(b_i)))^{-1} \\ - \sigma_j(\sigma_j^{-1}(b_i)) f_j(\sigma_j^{-1}(b_i))(\sigma_j'(\sigma_j^{-1}(b_i)))^{-1}. \end{aligned}$$

   Simplifying,

$$0 = -F_j(\sigma_j^{-1}(b_i)) + \left(v_i - b_i - \sigma_j(\sigma_j^{-1}(b_i))\right) f_j(\sigma_j^{-1}(b_i))(\sigma_j'(\sigma_j^{-1}(b_i)))^{-1}.$$

At the equilibrium, $b_i = \sigma_i(v_i)$, and so

$$\sigma_j'(\sigma_j^{-1}(\sigma_i(v_i)))F_j(\sigma_j^{-1}(\sigma_i(v_i))) = \Big(v_i - \sigma_i(v_i) - \sigma_j(\sigma_j^{-1}(\sigma_i(v_i)))\Big) f_j(\sigma_j^{-1}(\sigma_i(v_i))).$$

■

(c) Suppose v_1 and v_2 are uniformly and independently distributed on $[0,1]$. Describe the differential equation a symmetric increasing and differentiable equilibrium bidding strategy must satisfy, and solve it. [**Hint:** conjecture a functional form.] [**10 points**]

Solution: Substituting $F_i(v_i) = v_i$, $f_i = 1$ and $\sigma_1 = \sigma_2 = \tilde{\sigma}$ yields, for all $v \in [0,1]$,

$$\tilde{\sigma}'(v)v = (v - 2\tilde{\sigma}(v).$$

Suppose $\tilde{\sigma}(v) = kv$. Then,

$$kv = (v - 2kv),$$

so that $3k = 1$, that is

$$\tilde{\sigma}(v) = \frac{v}{3}$$

solves the differential equation. ■

(d) Prove the strategy identified in part 3(c) is a symmetric equilibrium strategy. [**5 points**]

Solution: If player j is following the strategy

$$\tilde{\sigma}(v) = v/3,$$

then for any $b_i \in [0,1/3]$, $F(\tilde{\sigma}^{-1}(b_i)) = 3b_i$, and so interim payoff for i is

$$\begin{aligned} U_i(v_i, b_i; \tilde{\sigma}) &= (v_i - b_i)3b_i + \frac{1}{3}\int_{3b_i}^{1} v_j \, dv_j \\ &= 3(v_i - b_i)b_i + \frac{1 - 9b_i^2}{6}, \end{aligned}$$

which is a quadratic in b_i, with b_i^2 having a negative coefficient. That is, U_i is strictly concave in $b_i \in [0,1/3]$. Since bidder j never bids more than $1/3$, it is clear that a bid above $1/3$ is not a profitable deviation. Hence, $\tilde{\sigma}$ is a symmetric equilibrium strategy. ■

4. (a) Describe the class of Groves mechanisms to efficiently dissolve the partnership. What attractive property do Groves mechanisms have in terms of strategic behavior?

[**5 points**]

Solution: Given reports of valuations (v_1, v_2), a direct mechanism specifies ownership shares, with x_i denoting i's share, and transfers t_i to i. A mechanism (x,t) is *efficient* if

$$x_i(v_i, v_j) = \begin{cases} 1, & v_i > v_j, \\ 0, & v_j > v_i. \end{cases}$$

The payoff to player i under the mechanism (x, t) from the report $(\hat{v}_1, \hat{v}_2)$ is

$$x_i(\hat{v}_i, \hat{v}_j)v_i + t_i(\hat{v}_i, \hat{v}_j).$$

Groves mechanisms are the class of mechanisms (x, t) with efficient dissolution x and transfers

$$t_i(v_i, v_j) = x_j(v_i, v_j)v_j + k_i(v_j).$$

The attractive property is that truth telling is a dominant strategy equilibrium of the direct mechanism. ■

(b) Prove that there is a Groves mechanism satisfying ex ante budget balance and interim individual rationality. **[15 points]**

Solution: The mechanism satisfies *ex ante budget balance* if

$$E\Big\{t_1(v_1, v_2) + t_2(v_1, v_2)\Big\} = 0.$$

The payoff to player i under the mechanism (x, t) from the report $(\hat{v}_1, \hat{v}_2)$ is

$$x_i(\hat{v}_i, \hat{v}_j)v_i + t_i(\hat{v}_i, \hat{v}_j).$$

The mechanism satisfies *individual rationality* if, for all $v_i \in [\underline{v}, \bar{v}]$,

$$E_{v_j}\Big\{x_i(v_i, v_j)v_i + t_i(v_i, v_j)\Big\} \geq \tfrac{1}{2}v_i.$$

A Groves mechanism satisfies individual rationality if

$$E_{v_j}\Big\{x_i(v_i, v_j)v_i + x_j(v_i, v_j)v_j + k_i(v_j)\Big\} \geq \tfrac{1}{2}v_i,$$

$$\text{i.e.,} \quad E_{v_j}\Big\{\Big(x_i(v_i, v_j) - \tfrac{1}{2}\Big)v_i + x_j(v_i, v_j)v_j + k_i(v_j)\Big\} \geq 0,$$

so individual rationality is satisfied if

$$\min_{v_i} E_{v_j}\Big\{\Big(x_i(v_i, v_j) - \tfrac{1}{2}\Big)v_i + x_j(v_i, v_j)v_j\Big\} \geq -E_{v_j}k_i(v_j).$$

We now evaluate the left side: From part 4(a), and ignoring ties (since these have zero probability)

$$\begin{aligned}
&E_{v_j}\Big\{\Big(x_i(v_i, v_j) - \tfrac{1}{2}\Big)v_i + x_j(v_i, v_j)v_j\Big\} \\
&\qquad = \frac{v_i}{2}\Pr(v_i > v_j) + E_{v_j}\left\{v_j - \frac{v_i}{2}\,\middle|\, v_j > v_i\right\}\Pr(v_j > v_i) \\
&\qquad = \frac{v_i}{2}F_j(v_i) - \frac{v_i}{2}(1 - F_j(v_i)) + E_{v_j}\left\{v_j\,\middle|\, v_j > v_i\right\}\Pr(v_j > v_i) \\
&\qquad = v_i(F_j(v_i) - \tfrac{1}{2}) + \int_{v_i}^{1} v_j f_j(v_j)\, dv_j. \qquad (1)
\end{aligned}$$

Differentiating with respect to v_i gives

$$F_j(v_i) - \tfrac{1}{2} + v_i f_j(v_i) - v_i f_j(v_i) = F_j(v_i) - \tfrac{1}{2},$$

and the function is minimized at v_i^* solving

$$F_j(v_i^*) = \tfrac{1}{2}$$

(we know the first order condition is sufficient for minimization because the second derivative is nonengative).

Moreover, the value of (1) at v_i^* is

$$\int_{v_i^*}^{1} v_j f_j(v_j)\, d v_j =: \kappa_i^* > 0,$$

so individual rationality is satisfied by any $E_{v_j} k_i(v_j) \geq -\kappa_i^*$.

We can choose k_i and k_j to satisfy individual rationality and ex ante budget balance if

$$E\Big\{x_1(v_1, v_2)v_1 + x_2(v_1, v_2)v_2 - \kappa_1^* - \kappa_2^*\Big\} \leq 0.$$

This inequality is equivalent to

$$E\Big\{x_1(v_1, v_2)v_1 + x_2(v_1, v_2)v_2\Big\} \leq \kappa_1^* + \kappa_2^*, \tag{2}$$

Assume $F_1 = F_2$, so that $v_1^* = v_2^* = v^*$. (It is true in the asymmetric case as well, as long as the asymmetries are not severe, but the calculations are more involved.) See Figure 1.

First note that the left side of (2) is simply $E\max\{v_1, v_2\}$, and is the expectation is the integral over the square in Figure 1.

The right side of (2) is

$$\begin{aligned}
\kappa_1^* + \kappa_2^* &= \int_{v_2^*}^{1} v_1 \; dF_1(v_1) + \int_{v_1^*}^{1} v_2 \; dF_2(v_2) \\
&= \iint_{C\cup D\cup E} v_1 \; dF_2(v_2)dF_1(v_1) + \iint_{B\cup C\cup D} v_2 \; dF_1(v_1)dF_2(v_2) \\
&= \iint_{B\cup C\cup D\cup E} \max\{v_1, v_2\} \; dF_2(v_2)dF_1(v_1) + \iint_{C\cup D} \min\{v_1, v_2\} \; dF_1(v_1)dF_2(v_2) \\
&\geq \iint_{B\cup C\cup D\cup E} \max\{v_1, v_2\} \; dF_2(v_2)dF_1(v_1) + \iint_{C\cup D} v^* \; dF_1(v_1)dF_2(v_2).
\end{aligned}$$

But since $\Pr(A) = \Pr(C \cup D)$, we have

$$\iint_{C\cup D} v^* \; dF_1(v_1)dF_2(v_2) \geq \iint_{A} \max\{v_1, v_2\} \; dF_1(v_1)dF_2(v_2),$$

verifying (2). ■

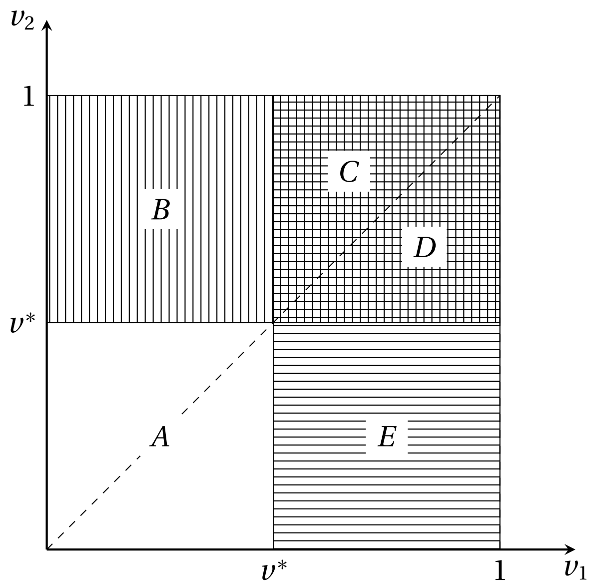

Figure 1: Figure for Question 4(b). The vertically shaded region ($B \cup C \cup D$) is the region of integration for κ_1^*, while the horizontally shaded region ($C \cup D \cup E$) is the region of integration for κ_2^*.

(c) Is there a Groves mechanism satisfying ex post budget balance? Provide a proof. **[10 points]**

Solution: There is no Groves mechanism satisfying individual rationality and ex post budget balance: Ex post budget balance requires for all v_1 and v_2,

$$x_1(v_1, v_2)v_1 + x_2(v_1, v_2)v_2 + k_1(v_2) + k_2(v_1) = 0.$$

(Note that there is no seller to "break the budget.") Choose any $v_1 < v_2 < v_1'$. Then ex post budget balance implies

$$v_1' + k_1(v_2) + k_2(v_1') = 0$$

and

$$v_2 + k_1(v_2) + k_2(v_1) = 0,$$

so that

$$v_2 = v_1' + k_2(v_1') - k_2(v_1),$$

a contradiction (simply choose $v_2 \neq v_1' + k_2(v_1') - k_2(v_1)$). ■