Microeconomic Theory II
Preliminary Examination Solutions

Exam date: June 1, 2015

1. **[25 points]** This question considers a sequence of games of increasing complexity based on the following simultaneous move game to be played by ROWAN and COLIN:

	L	C	R
U	$4,3$	$0,1$	$3,2$
D	$7,0$	$1,1$	$4,-1$

(a) What are the Nash equilibria? **[5 points]**

Solution: D is strictly dominant for ROW, and C is COL's unique best reply to D. Hence, DC is the unique Nash equilibrium. ■

(b) In the first modification, ROW makes his choice from $\{U, D\}$ publicly before COL makes his choice from $\{L, C, R\}$. Are any of the Nash equilibrium outcomes you found in part 1(a) Nash equilibrium outcomes of this game of perfect information? What are the subgame perfect equilibria? **[5 points]**

Solution: The Nash equilibrium outcome from part 1(a), DC, is a Nash equilibrium outcome of this game of perfect information: The strategy profile in which COL plays C after any choice by ROW, and ROW plays C is a Nash equilibrium.

L is COL's only best reply to U, and since ROW prefers the outcome UL to DC, the only subgame perfect outcome is UL, and the unique subgame perfect equilibrium is (U, LC), where L is played after U and C is played after D. ■

(c) In the second modification, keep the sequencing from part 1(b), but allow ROW to change his mind, but at a cost: that is, after COL has chosen his action, ROW can change his action at a cost of 2. This cost (if incurred) is subtracted from his payoffs. What are the subgame perfect equilibria? **[5 points]**

Solution: The unique subgame perfect equilibrium is that Row chooses U, COL plays R after U and C after D, and ROW does not change his action from U if Col had chosen C or R, but does change his action if Col had chosen L. If ROW had chosen D initially, he does not change subsequently. The outcome is UR. ■

(d) In the final modification, suppose that ROW's decision on whether to change his action and COL's choice are simultaneous, but otherwise the sequencing of the game in part 1(c) is unchanged. What are the subgame perfect equilibria? Compare and contrast your answer here with your answer from part 1(c). **[10 points]**

Solution: If Row first plays D, then Col plays C and Row will not change his action (since U is now even more strictly dominated), yielding a ROW payoff of 1.

If Row first plays U, then in the second stage, the following simultaneous move game is played, where U^U means no change and D^U means change from U to D:

	L	C	R
U^U	$4,3$	$0,1$	$3,2$
D^U	$5,0$	$-1,1$	$2,-1$

In this game R is strictly dominated for Col. Moreover, there is no pure strategy equilibrium. The mixed strategy equilibrium is $(\frac{1}{3}\circ U^U+\frac{2}{3}\circ D^U, \frac{1}{2}\circ L+\frac{1}{2}\circ C)$, with an expected payoff for Row of 2. This exceeds the payoff from D of 1, so the unique subgame perfect equilibrium, is

$$\left((U, \frac{1}{3}\circ U^U+\frac{2}{3}\circ D^U, D^D), (\frac{1}{2}\circ L+\frac{1}{2}\circ C, C)\right),$$

i.e., Row plays U, followed by the mixed strategy equilibrium just described, and if Row had played D, Row does not change and Col plays C.

This is a subgame perfect equilibrium in behavior strategies. There is no pure strategy subgame perfect equilibrium in this game. ■

2. **[35 points]**In a procurement auction, firms compete for the right to provide a good to the government. Each firm has a private cost of provision, with firms' costs independently distributed. The firms participating in the auction simultaneously submit bids, with the government awarding the contract to the lowest bid (as long as the lowest bid is no greater than some bound $\bar{B}>0$), and paying that bid to the winning firm. Suppose there are two possible firms. Each possible firm participates in the auction with probability $p\in(0,1)$, with each firm's participation determined independently. Firm i's cost c_i is randomly drawn from the interval $[\underline{c}_i, \bar{c}_i]$ according to the distribution function F_i, with density f_i. When a firm submits his or her bid, he or she does not know if the other possible firm is participating.

   (a) Conditional on participating, what are the interim payoffs of firm i? What are firm i's interim payoffs (*not* conditional on participating)? **[5 points]**

   Solution: The analysis of procurement auctions is **very** similar to that of standard first price sealed bid auctions, except the *lowest* bid wins, and the winning bidder's ex post payoff is the bid less the private cost (value).

   Suppose firm j is playing σ_j. Conditional on participating, i's interim payoff from bidding b_i with cost c_i is

$$\begin{aligned} U_i(c_i, b_i; \sigma_j) = &(1-p)(b_i-c_i) \\ &+p\left\{(b_i-c_i)\Pr\{\sigma_j(c_j)>b_i\}+\frac{1}{2}(b_i-c_i)\Pr\{\sigma_j(c_j)=b_i\}\right\}. \end{aligned}$$

[-2 points if ties ignored]

Interim payoffs are

$$pU_i(v_i, b_i; \sigma_j).$$

■

(b) Suppose (σ_1, σ_2) is a Nash equilibrium of the auction, and assume σ_i is a strictly increasing and differentiable function, for $i = 1, 2$. Describe the pair of differential equations the strategies must satisfy. **[5 points]**

Solution: Since σ_j is strictly increasing, and there are no atoms, $\Pr\{\sigma_j(c_j) = b_i\} = 0$. Since i's participation is unaffected by his/her own bidding behavior, maximizing the unconditional interim payoffs is equivalent to maximizing the interim payoffs *conditional* on participation. For the remainder of the question, interim payoffs should be interpreted as conditional on participation. Interim payoffs can then be written as

$$U_i(c_i, b_i; \sigma_j) = (1-p)(b_i - c_i) + p(b_i - c_i)(1 - F_j(\sigma_j^{-1}(b_i))).$$

[1 points]

The first order condition is

$$0 = (1-p) + p(1 - F_j(\sigma_j^{-1}(b_i))) - p(b_i - c_i) f_j(\sigma_j^{-1}(b_i))(\sigma_j'(\sigma_j^{-1}(b_i)))^{-1}.$$

[2 points]

At the equilibrium, $b_i = \sigma_i(c_i)$, and so

$$\sigma_j'(\sigma_j^{-1}(\sigma_i(c_i)))[1 - p + p(1 - F_j(\sigma_j^{-1}(\sigma_i(v_i))))] = p(\sigma_i(c_i) - c_i) f_j(\sigma_j^{-1}(\sigma_i(v_i))).$$

[2 points]

■

(c) Suppose c_1 and c_2 are uniformly and independently distributed on $[0, 1]$. Describe the differential equation a symmetric, strictly increasing, and differentiable equilibrium bidding strategy must satisfy. **[5 points]**

Solution: Substituting $F_i(c_i) = c_i$, $f_i = 1$ and $\sigma_1 = \sigma_2 = \tilde{\sigma}$ yields, for all $c \in [0, 1]$,

$$\tilde{\sigma}'(c)(1 - pc) = p(\tilde{\sigma}(c) - c).$$

■

(d) Solve for the equilibrium strategy satisfying the differential equation found in part 2(c). **[10 points]**

Solution: The differential equation can be rewritten as

$$\tilde{\sigma}'(c)(1 - pc) - p\tilde{\sigma}(c) = -pc,$$

or

$$\frac{d}{dc}\tilde{\sigma}(c)(1-pc)=-pc,$$

so integrating yields

$$\tilde{\sigma}(c)(1-pc)=-\frac{pc^2}{2}+k,$$

where k is the constant of integration.

It remains to determine k : The highest cost firm only wins the auction if his competitor is not present, and so he must bid the highest allowable bid $\bar{B}$, that is,

$$\tilde{\sigma}(1)=\bar{B},$$

implying

$$k=\bar{B}(1-p)+\frac{p}{2}.$$

This largest bid needs to be large enough to cover the firm's costs, so this requires

$$\bar{B}\geq 1.$$

So,

$$\tilde{\sigma}(c)=\frac{\bar{B}(1-p)}{(1-pc)}+\frac{p(1-c^2)}{2(1-pc)}.$$

■

(e) Prove the strategy found in part 2(d) is a symmetric equilibrium strategy. **[10 points]**

Solution: Suppose firm j is following the strategy

$$\tilde{\sigma}(c)=\frac{\bar{B}(1-p)}{(1-pc)}+\frac{p(1-c^2)}{2(1-pc)}.$$

Note first that firm i does not find it optimal to bid less than $\tilde{\sigma}(0)$, since a bid of $\tilde{\sigma}(0)$ ensures that the firm wins the auction with probability 1. Moreover, since $\tilde{\sigma}(1)=\bar{B}$, firm i cannot bid above the maximum of j's bid. [2 points]

It remains to show that i with cost c_i does not prefer making some other bid in the interval $[\tilde{\sigma}(0),\bar{B}]$. Since for any $b_i\in[\tilde{\sigma}(0),\bar{B}]$, there is a $\hat{c}\in[0,1]$ such that $b_i=\tilde{\sigma}(\hat{c})$, this is equivalent to showing that $\hat{c}=c_i$ maximizes

$$\begin{aligned}(b_i-c_i)(1-p)+p(b_i-c_i)(1-\tilde{\sigma}^{-1}(b_i)) &= (\tilde{\sigma}(\hat{c})-c_i)(1-p)+p(\tilde{\sigma}(\hat{c})-c_i)(1-\hat{c})\\ &= (\tilde{\sigma}(\hat{c})-c_i)(1-p\hat{c})\\ &= \bar{B}(1-p)+\frac{p}{2}(1-\hat{c}^2)-c_i(1-p\hat{c}).\end{aligned}$$

Since this expression is strictly concave in $\hat{c}$ and (trivially) maximized at $\hat{c}=c_i$, $\tilde{\sigma}$ is a symmetric equilibrium strategy. [8 points]

■

3. **[33 points]** A monopoly airline is faced with a traveler who may be a business traveler (B) or a leisure traveler (L). The probability that she is of type B is λ. There are two periods. At the begining of period one, the traveler learns her type (B or L). This type determines the probability distribution F_B or F_L that will determine her valuation for the ticket (for example, $F_B(v)$ is the probability that the B type will draw a value of v or less).

   The seller and the traveler contract at the end of period 1.

   At the beginning of period two, the traveler privately learns her value for the ticket and then decides whether to travel. The cost of flying a passenger is c. Both parties are risk neutral and there is no discounting.

   A partially refundable ticket consists of a pair (p, r) where p is a payment from the buyer to the airline for the ticket at the end of period one, and r is a refund that can be claimed by the buyer from the seller at the end of period two if the buyer chooses not to travel. The traveler's ex-post payoff from purchasing this contract is therefore $v - p$ if she chooses to travel, and $r - p$ if not. Note that p may be larger than r.

   Suppose $\lambda = \frac{2}{3}$, $c = 1$, F_B is the uniform distribution on $[0,1] \cup [2,3]$ and F_L is the uniform distribution on $[1,2]$.

   (a) Suppose the seller offers a single fully refundable ticket, i.e. $p = r$. What is the optimal ticket price to maximize the seller's expected profit, and how much profit does the seller make? **[8 points]**

   Solution: Note that a fully refundable ticket is equivalent to selling a ticket at a posted price after the buyer has learned her value. Given the assumptions, the buyer's value is distributed as $U[0,3]$.

   In this case, the posted price solves:

$$\max_{0 \le p \le 3} (p-1)\frac{3-p}{3}.$$

   Solving, we get $p = 2$ and an optimal profit $\frac{1}{3}$. ∎

   (b) Suppose instead the seller offers two kinds of tickets (p_B, r_B) and (p_L, r_L). The buyer may choose either contract, or the outside option of not flying (which gives her 0 utility). Describe carefully the conditions so that the B type buyer weakly prefers the (p_B, r_B) contract, while the L type buyer weakly prefers the (p_L, r_L) contract. **[10 points]**

   Solution: Note that in period 2, having signed a contract (p, r) the buyer will choose to travel whenever

$$v - p \ge r - p,$$

   i.e. whenever $v \ge r$, that is, his value exceeds the refund offered.

   Given the offered contracts, we have first the usual IR constraint that the agent's intended contract must offer a higher surplus to her than the outside option, i.e.,

$$\mathbb{E}_x[v - p_x | v \ge r_x](1 - F_x(r_x)) + (r_x - p_x)(F_x(r_x)) \ge 0,$$

where $\mathbb{E}_x$ is the expectation operator with respect to distribution F_x.

Further, we need that each of the business and leisure types prefer the contract offered to them than the other contract, i.e.,

$$\begin{aligned} & \mathbb{E}_B[v - p_B | v \geq r_B](1 - F_B(r_B)) + (r_B - p_B)(F_B(r_B)) \\ \geq\ & \mathbb{E}_B[v - p_L | v > r_B](1 - F_B(r_L)) + (r_L - p_L)(F_B(r_L)), \end{aligned}$$

for the business traveler not to want to purchase the leisure traveler's contract. The converse IC constraint is analogous and omitted. ■

(c) Suppose the seller offers a contract $(1.5, 0)$, i.e. an advance payment of 1.5 and no refund, which is intended for the L type. Calculate the optimal contract to offer the B type to maximize overall expected profit. Of course, the offered contracts must satisfy the constraints you derived above. What is the expected profit of the seller from offering these contracts? *(Hint: Guess as always that only the IC constraint corresponding to the B type purchasing L's constract binds, and optimize. Verify the other constraints later.)*

[15 points]

Solution: Note that this contract is IR for the L type. Further, a buyer purchasing this contract always travels, resulting in a net profit of 0.5 for a seller whenever this contract is purchased.

The remaining problem for the seller is therefore

$$\begin{aligned} & \max_{p_B, r_B} (p_B - 1)(1 - F_B(r_B)) + (p_B - r_B) F_B(r_B) \\ = & \max_{p_B, r_B} p_B - (1 - F_B(r_B)) - r_B F_B(r_B) \end{aligned}$$

subject to the constraints above.

Note that if B buyer purchases the L contract, it nets him a surplus of 0.

Guessing as suggested in the hint:

$$\mathbb{E}_B[v - p_B | v \geq r_B](1 - F_B(r_B)) + (r_B - p_B)(F_B(r_B)) = 0.$$

Solving, we have:

$$\mathbb{E}_B[v; v \geq r_B] + r_B(F_B(r_B)) = p_B$$

Substituting p_B into the objective function, we have

$$\max_{r_B} \mathbb{E}_B[v; v \geq r_B] - (1 - F_B(r_B)).$$

Note that we can evaluate the objective function for each of r_B in $[0,1]$, $[1,2]$ and $[2,3]$. Note that it is increasing in the first interval, constant in the second, and decreasing in the third. Therefore it is maximized at $r_B = 1$.

Substituting back in, we have $p_B = 1.75$.

Finally, note that both types make no surplus! Further, the profit of the seller is $(1.75 - 1)\frac{2}{3} + 0.5\frac{1}{3} = \frac{2}{3}$, i.e. twice the full refund contract! The verification that other constraints are satisfied is mechanical, and omitted. ■

4. [**27 points**] An insurance agent, Heidi Ductible is considering whether to insure a truck driver Etienne Wheeler, and faces a simple moral hazard problem. Mr. Wheeler can act in two ways, either "recklessly"or "cautiously." If he is cautious, then the probability of damage is $\frac{1}{3}$, while if he is reckless, the probability of damage is $\frac{2}{3}$. Being cautious involves an effort cost of $e_c = \frac{1}{3}$ utils to the driver, while being reckless involves no cost. Mr Wheeler's initial wealth is $w_0 = 100$ and values wealth according to $\sqrt{w}$. His overall utility therefore is $\sqrt{w} - e$ where w is his eventual wealth level, and e is the effort cost. If he gets into an accident, it costs 51.

   (a) If Mr. Wheeler could not buy insurance, will he choose to be cautious or reckless and what is his utility. **[5 points]**

   Solution: His final wealth if an accident occurs is 49, whereas if it doesn't, it remains at a 100.

   If he drives cautiously, his utility is therefore $\frac{7}{3} + \frac{20}{3} - \frac{1}{3} = \frac{26}{3}$. If he drives recklessly, his net utility is $\frac{14}{3} + \frac{10}{3} = \frac{24}{3} < \frac{26}{3}$.

   Therefore he will drive cautiously. His net utility will then be $\frac{26}{3}$. ■

   (b) Ms. Ductible can only offer full insurance, i.e. if Mr. Wheeler gets into an accident, the insurance must pay out the full cost of the accident if Mr. Wheeler is driving cautiously. However, she can put a monitoring tool in Mr. Wheeler's truck that checks his driving, and does not pay him anything if he is driving recklessly. She is risk neutral and maximizes the price minus any expected damages. What is the maximum she can charge for insurance? **[5 points]**

   Solution: If Mr. Wheeler does not buy insurance, he will drive cautiously, as we argued above. This results in a net utility of $\frac{26}{3}$.

   If Mr. Wheeler does buy insurance, he might as well drive cautiously (he will be out the insurance premium plus any damage otherwise). Therefore Ms. Ductible can charge a price as large as:

$$\sqrt{100-p} - \frac{1}{3} = \frac{26}{3}$$
$$\text{i.e. } \sqrt{100-p} = 9$$
$$\text{i.e. } p = 19.$$

   Note that the price exceeds the expected loss ($\frac{51}{3} = 17$). ■

   (c) Now suppose this monitoring tool does not exist, so that the insurance must always pay out the full cost of an accident if it occurs. Argue that there is no possibility of a full insurance contract that has non-negative expected profits for Ms Ductible, which Mr Wheeler would accept, in the absence of the monitoring tool. **[7 points]**

   Solution: Note that now, if the driver purchases insurance, he might as well drive recklessly, since the insurance will make him whole and saves on cautious driving effort costs.

Therefore, the expected cost to Ms. Ductible of providing full insurance is $\frac{2}{3} \times 51 = 34$. Even if she offers the insurance at cost, Mr. Wheeler's net utility is $\sqrt{66} < \frac{26}{3}$. Therefore he will not accept any contract that is (weakly) profitable for her. ■

(d) Write down the incentive and participation constraints a contract must satisfy if it is to induce cautious driving, and Ms. Ductible's optimization problem for the profit maximizing contract that induces cautious driving. Find the profit maximizing contract that has an up-front payment $p = 116/9$. Show that this contract achieves strictly positive expected profits. **[10 points]**

Solution: To ensure Mr. Wheeler drives cautiously after purchasing the insurance, it must be the case that:

$$\frac{1}{3}\sqrt{49-p+d}+\frac{2}{3}\sqrt{100-p}-\frac{1}{3} \geq \frac{2}{3}\sqrt{49-p+d}+\frac{1}{3}\sqrt{100-p},$$
$$\text{i.e.}\quad \sqrt{100-p}-\sqrt{49-p+d} \geq 1.$$

To ensure Mr. Wheeler wishes to purchase the insurance, it must be the case that:

$$\frac{1}{3}\sqrt{49-p+d}+\frac{2}{3}\sqrt{100-p}-\frac{1}{3} \geq \frac{26}{3},$$
$$\text{i.e.}\quad \sqrt{49-p+d}+2\sqrt{100-p} \geq 27.$$

Therefore Ms. Ductible's problem is:

$$\begin{aligned} &\max_{p,d}\, p-\frac{1}{3}d \\ \text{s.t.}\quad &\sqrt{100-p}-\sqrt{49-p+d} \geq 1, \\ &\sqrt{49-p+d}+2\sqrt{100-p} \geq 27. \end{aligned}$$

Taking $p = 116/9$, the problem becomes:

$$\begin{aligned} &\min_{d}\, d \\ \text{s.t.}\quad &\sqrt{49-p+d} \leq \frac{25}{3} \\ &\sqrt{49-p+d} \geq \frac{25}{3} \end{aligned}$$

Therefore, $d = \frac{625+116-441}{9} = \frac{100}{3}$. This contract achieves profits of $\frac{16}{9}$. ■