University of Pennsylvania **Economics 705, Spring 2014**

Prelim Examination

Friday June 6, 2014 Time limit: 150 minutes

Instructions:

(i) The exam consists of two parts. The total number of points for each part is 50. The number of points for each question is given below.

(ii) The exam is closed book and closed notes.

(iii) To receive full credit for your answers you have to explain your calculations. **You may state additional assumptions.**

Part I

Question 1: TRUE or FALSE Questions (9 Points)

(i) (3 points) TRUE or FALSE? $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ implies that the random variables X and Y are independent. Explain your answer.

(ii) (3 Points) Consider the regression $y_i = \theta x_i + u_i$, where x_i is scalar. Suppose that $u_i|x_i \sim N(0, \sigma^2)$.
TRUE or FALSE? The larger σ^2, the more precise is the OLS estimator. Explain your answer.

(iii) (3 Points) Suppose that X is a random variable with cdf $F_X(x)$.
TRUE or FALSE? Then the transformed random variable $Y = F_X(X) \sim U[0,1]$, where $U[0,1]$ is the uniform distribution on the unit interval. Explain your answer by deriving the distribution of Y.

Question 2: Point Estimation (21 Points)

Consider the model given by the conditional distribution $Y|\theta \sim N(\theta, 1)$ and the marginal distribution $\theta \sim N(0, 1/\lambda)$. Note that the sample size is $n = 1$.

This question involves three different estimators of θ, denoted by $\hat{\theta}_B$, $\hat{\theta}_*$, and $\hat{\theta}_{mle}$.

Throughout this question, we will consider the quadratic loss function

$$L(\theta, \delta) = (\theta - \delta)^2. \tag{1}$$

(i) (4 Points) Derive the posterior distribution of $\theta|Y$.

(ii) (3 Points) Derive the Bayes estimator $\hat{\theta}_B$ under the loss function in (1).

(iii) (2 Points) Is the Bayes estimator $\hat{\theta}_B$ an unbiased estimator of θ?

(iv) (4 Points) Now consider the alternative estimator

$$\hat{\theta}_* = \begin{cases} 0 & \text{if } |Y| < 1 \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

Derive the distribution $\hat{\theta}_*|\theta$.

(v) (4 Points) TRUE or FALSE? $\hat{\theta}_*$ attains a lower integrated risk than $\hat{\theta}_B$. Provide a detailed explanation for your answer.

(vi) (4 Points) Is there an unambiguous ranking of the maximum likelihood (MLE) estimator $\hat{\theta}_{mle} = Y$ and the estimator $\hat{\theta}_*$ (defined in (2)) based on their frequentist risk properties. Provide a detailed explanation for your answer.

Question 3: Testing and Asymptotics (20 Points)

Consider a regression model with a single regressor:

$$y_i = x_i\theta + u_i, \quad u_i|x_i \sim N(0,1), \quad i = 1, \ldots, n \tag{3}$$

where (x_i, u_i) are *iid*. You may assume the existence of higher-order moments of (x_i, u_i).

(i) (4 Points) Derive the likelihood function for θ as well as the maximum likelihood estimator (MLE) $\hat{\theta}_{mle}$.

(ii) (5 Points) Show that the MLE is consistent and derive its limit distribution.

(iii) (6 Points) Show that the likelihood ratio test for the hypothesis $H_0: \theta = \theta_0$ against the alternative $H_1: \theta \neq \theta_0$ is equivalent to the t-test for $H_0: \theta = \theta_0$. Here "equivalent" means that the acceptance and rejection regions of the two tests are identical.

(iv) (5 Points) Show that the power of the t-test (and hence the likelihood ratio test) against any fixed alternative $\theta_1 \neq \theta_0$ converges to one as $n \longrightarrow \infty$.

Part II

Question 4: Linear Instrumental Variable Regression (10 Points)

Consider the instrumental variable problem:

$$Y_i = \alpha_0 + X_i\beta_0 + X_i^2\gamma_0 + e_i,$$

where Y_i and X_i are both scalar, $\mathbb{E}(e_i) = 0$. Suppose $\mathbb{E}(X_i e_i) \neq 0$ and $\mathbb{E}(X_i^2 e_i) \neq 0$, but there exists a scalar valued random variable Z_i such that $\mathbb{E}(e_i|Z_i) = 0$.

(i) (3 points) Propose instruments for estimating β_0 and γ_0 and show they satisfy the necessary exogeneity restrictions.

(ii) (3 points) For the instruments proposed above, state the relevant rank condition.

(iii) (4 points) State how to construct an efficient GMM estimator with the optimal weight matrix.

Question 5: Linear Model with Binary Endogenous Variable (20 Points)

Consider the following model

$$Y_i = X_i\beta + e_i,$$

where $X_i \in R$ is an endogenous dummy variable. We assume

$$X_i = 1\{Z_i\delta_0 + v_i > 0\},$$

where $Z_i \in R$ is independent of e_i and v_i, $\mathbb{E}(Z_iX_i) \neq 0$, $\mathbb{E}(e_i) = \mathbb{E}(v_i) = 0$, e_i and v_i are possibly correlated. We have i.i.d. observations $\{X_i, Z_i, Y_i\}_{i=1}^n$.

(i) (4 Points) Consider the IV estimator of β_0 :

$$\widehat{\beta}_{IV} = \frac{\sum_{i=1}^n Z_iY_i}{\sum_{i=1}^n Z_iX_i}.$$

Is this estimator consistent without any distribution assumption on v? Explain your reasoning in detail.

(ii) (4 points) What is the asymptotic distribution of $\widehat{\beta}_{IV}$?

(iii) (6 points) Now assume v has a standard normal distribution. We estimate $W_i(\delta_0) = \Phi(Z_i\delta_0) = \mathbb{E}(X_i|Z_i)$ by probit. Let $\widehat{\delta}$ be the maximum likelihood estimator of δ_0 in the probit model. Consider the two stage least squares estimator

$$\widehat{\beta}_{TSLS} = \frac{\sum_{i=1}^n W_i(\widehat{\delta})Y_i}{\sum_{i=1}^n W_i(\widehat{\delta})^2}.$$

Is this estimator consistent? Explain your reasoning in detail.

(iv) (3 points) What will happen to $\widehat{\beta}_{TSLS}$ if the distribution of v is not standard normal but a probit model is used? Explain.

(v) (3 points) Maintain the assumption that v has a standard normal distribution and consider the infeasible estimator $\widehat{\beta}_{TSLS}(\delta_0)$. Are $\widehat{\beta}_{TSLS}(\delta_0)$ and $\widehat{\beta}_{TSLS}$ asymptotically equivalent? Explain.

Question 6: Censored Regression Model (20 points)

Consider a censored regression model

$$Y_i^* = \alpha_0 + X_i'\beta_0 + e_i,$$

where e_i is independent of X_i and $e_i \sim N(0,1)$. The dependent variable Y_i^* is not always observable. Instead, we observe the censored random variable Y_i, where

$$Y_i = \begin{cases} Y_i^* \text{ if } e_i > c, \\ 0 \text{ if } e_i \leq c, \end{cases}$$

for some known scalar constant c.

Recall that if $Z \sim N(0,1)$, $\mathbb{E}(Z|Z > c) = \lambda(-c)$, where $\lambda(c) = \phi(c)/\Phi(c)$ and Φ, ϕ are the cdf and pdf of a standard normal distribution, respectively.

(i) (2 points) Compute $P(Y_i = 0|X_i)$ under the assumption of the model.

(ii) (4 points) Compute $\mathbb{E}(Y_i|X_i, Y_i \neq 0)$ and $\mathbb{E}(Y_i|X_i)$ in the model.

(iii) (4 points) Suppose you run OLS using only the observations $\{Y_i, X_i\}_{i=1}^n$ for which Y_i^* is uncensored. The regression equation is

$$Y_i = \alpha + X_i'\beta + v_i.$$

Find the probability limit of your OLS estimator.

(iv) (5 points) Instead of the censoring form above, suppose it takes the form

$$Y_i = \begin{cases} Y_i^* \text{ if } Y_i^* > 0, \\ 0 \text{ if } Y_i^* \leq 0. \end{cases}$$

Is the OLS estimator on the uncensored sample deliver a consistent estimator for β_0? Explain.

(v) (5 points) For the new censoring form in part (iv), construct the maximum likelihood estimator.

END OF EXAM