1. Endogenous Growth with Human Capital: Lucas-Uzawa

Consider the following model. There is a representative household with a utility function:

$$U(0) = \int_0^\infty e^{-\rho t} \log c(t)\, dt$$

There is no population growth.

The representative household has $k(t)$ units of physical capital and a level of human capital, $h(t)$, which evolves over time as:

$$\dot{h}(t) = A(1 - u(t))\, h(t) - \delta h(t)$$

where $0 \leq 1 - u(t) \leq 1$ is the fraction of human capital used by the household to further accumulate human capital. Note that $u(t)$ is a choice variable.

There is a firm that produces the final good renting $k(t)$, and a fraction $0 \leq u(t) \leq 1$ of $h(t)$ according to the technology:

$$y(t) = k(t)^\alpha (u(t)\, h(t))^{1-\alpha}$$

The final good can be used for consumption, $c(t)$, or for investment in physical capital:

$$\dot{k}(t) = k(t)^\alpha h(t)^{1-\alpha} - c(t) - \delta k(t)$$

1. Write down the budget constraint of the representative household.
2. Define an equilibrium for this economy.
3. Characterize the Balanced Growth Path (BGP) of this economy. Among other things, you need to:
    1. Find the optimality conditions of the household and the final good producer.
    2. Find the growth rate of the economy along the BGP.
    3. Show that u is constant along the BGP.
4. Solve the social planner's problem. Compare with the solution in step 2.