University of Pennsylvania **Economics 705, Spring 2012**

Prelim Examination

Friday June 8, 2012, Time limit: 150 minutes

Instructions:

(i) The exam consists of two parts. The total number of points for each part is 50. The number of points for each question is given below.

(ii) The exam is closed book and closed notes.

(iii) To receive full credit for your answers you have to explain your calculations. **You may state additional assumptions.**

Part I

Question 1: Probability Density Functions (7 Points)

Two random variables, X, and Y, have a joint distribution characterized by the following probability density function (pdf):

$$p(x,y) = cxy^2, \quad 0 < x < 1,\ 0 < y < 1.$$

(i) (3 Points) Find the value of the normalization constant c that ensures that $p(x,y)$ is a proper (pdf).

(ii) (2 Points) Derive the density associated with the conditional distribution of $X|Y$, denoted by $p(x|y)$.

(iii) (2 Points) Are the random variable X and Y independent? Explain.

Question 2: Frequentist Inference with Two Observations (15 Points)

Consider the following experiment:

$$X_1, X_2 \sim iidN(\theta, 1)$$

(i) (3 Points) Derive the likelihood function and the maximum likelihood estimator for this experiment.

(ii) (2 Points) What is the sampling distribution of the maximum likelihood estimator?

(iii) (2 Points) Provide a definition of a confidence interval and derive a 95% confidence interval for the above experiment.

(iv) (8 Points) Derive the following statistics for the hypothesis $\theta = \theta_*$ and their sampling distribution under the null hypothesis: Wald (W) test statistic, the likelihood ratio (LR) test statistic, and the lagrange multiplier (LM) test statistic.

Question 3: Inference with One Observation (14 Points)

Consider the following experiment:

$$X|\theta \sim N(\theta, 1), \quad \theta \sim N(0, 1/\lambda).$$

(i) (4 Points) Derive the posterior distribution $\theta|X$.

(ii) (2 Points) Provide a definition of a credible interval and derive a 95% credible interval for the above experiment.

(iii) (2 Points) Derive the Bayes estimator $\hat{\theta}_B(X)$ under the loss function

$$L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2.$$

(iv) (1 Point) What is the relationship between the Bayes estimator and the maximum likelihood estimator?

(v) (3 Points) Compute the frequentist risk associated with the Bayes estimator: $R(\theta, \hat{\theta}_B)$.

(vi) (2 Points) Recall that the Bayes estimator depends on the prior precision λ. Based on the frequentist risk, does there exist a clear ranking of the Bayes estimators that obtain for different values of λ? Explain.

Question 4: Linear Regression Model (14 Points)

Consider the linear regression model

$$y_i = x_i\theta + u_i, \quad u_i|x_i \sim iid(0,1), \quad x_i \geq \epsilon > 0, \quad \mathbb{E}[x_i^2] = Q.$$

Moreover, the x_i's are also independent across i. Notice that we assumed that the conditional variance of u given x is known to be one. Moreover, $k = 1$ and both x_i and θ_i are scalar.

(i) (3 Points) Show that the OLS estimator $\hat{\beta}$ is consistent.

(ii) (2 Points) Does the probability limit of $\hat{\beta}$ change if one changes the assumption $\mathbb{E}[u_i|x_i] = 0$ to $\mathbb{E}[u_i|x_i] \neq 0$?

(iii) (2 Points) Derive the limit distribution of the OLS estimator $\hat{\beta}$.

(iv) (5 Points) Now consider the alternative estimator

$$\tilde{\beta} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i}$$

Is this estimator consistent? Derive its limit distribution.

(v) (2 Points) Which estimator, $\hat{\beta}$ or $\tilde{\beta}$, is preferable? Explain.

Part II

Question 5: Linear Model with Endogeneity (16 points)

Consider estimation of the model

$$y_i = x_i'\beta + u_i$$

using instruments z_i in the just-identified case. Assume $x_i \in R^k$ with $k > 1$ and $\{(x_i, z_i, u_i) : i = 1, ..., n\}$ are *iid*.

(i) (2 points) Define the Two Stage Least Squares (TSLS) estimator $\hat{\beta}$.

(ii) (3 points) What essential assumptions must the instruments satisfy for the TSLS estimator to be consistent?

(iii) (6 points) Derive the limit distribution of the TSLS estimator. Be clear when applying the weak law of large numbers (WLLN) and the central limit theorem (CLT).

(iv) (5 points) Suppose $k = 4$ and denote $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)'$. Provide a procedure to test $H_0 : \beta_1 = \beta_3^2$ and $\beta_2 = -\beta_4$ vs H_1 : at least one equality does not hold.

Question 6: Binary Choice Model with Endogeneity (17 Points)

Consider the binary choice model

$$y_i = 1\{x_i'\beta > u_i\},$$

where $x_i \in R^k$ and $u_i \in R$. Because some of the regressors are endogenous, $\mathbb{E}[y_i|x_i] \neq \Phi(x_i'\beta)$. Suppose there exist some variables $z_i \in R^l$, $l > k$, such that

$$\mathbb{E}[y_i - \Phi(x_i'\beta)|z_i] = 0.$$

Assume $\{(x_i, z_i, u_i) : i = 1, ..., n\}$ are *iid* and $\mathbb{E}[y_i - \Phi(x_i'\beta^*)|z_i] \neq 0 \; \forall \beta^* \neq \beta$ on a compact parameter space.

(i) (3 Point) Provide a GMM sample criterion function for a consistent estimator of β. This estimator does not have to be efficient.

(ii) (6 Points) For your estimator above, give the limit distribution of the GMM estimator $\widehat{\beta}$.

(iii) (4 Points) Give the optimal weight matrix and a way to calculate it. Explain why this weight matrix is optimal (no formal proof is needed).

(iv) (4 Points) How do you test whether the population GMM criterion function corresponding to that in (i) is 0 at the true value β.

Question 7: Censored Observations (17 Points)

Consider the following model

$$y_i^* = x_i'\beta + u_i, \;\; u_i|x_i \sim iid\ N(0, \sigma^2),$$

where $x_i \in R^k$ with $k > 1$. Moreover, the x_i's are also independent across i. We do not observe y_i^*, instead we observe

$$y_i = \begin{cases} y_i^* \text{ if } y_i^* \geq \lambda \\ \lambda \text{ if } y_i^* < \lambda, \end{cases}$$

where λ is a known constant.

(i) (5 Point) Write down the log-likelihood function for the maximum likelihood estimator $\widehat{\beta}$.

(ii) (6 Points) Give the limit distribution of the maximum likelihood estimator $\widehat{\beta}$.

(iii) (3 Points) Estimate the standard error of the maximum likelihood estimator.

(iv) (3 Points) Write down the LR test statistic and its limit distribution for $H_0 : \beta = 0$ vs $H_0 : \beta \neq 0$.

END OF EXAM