Stability and Preference Alignment in Matching and Coalition Formation

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Abstract

We study frictionless matching and coalition formation environments. Agents have preferences over coalitions, and these preferences vary with an underlying, and commonly known, state of nature. Assuming that there is substantial variability of preferences, we show that there exists a core stable coalition structure in every state of nature if, and only if, agents' preferences are pairwise-aligned in every state of nature. In particular, we establish that there is a stable coalition structure if agents' preferences are generated by Nash bargaining over coalitional outputs. Looking at all stability-inducing rules for sharing outputs, we show that all of them may be represented by a profile of agents' bargaining functions, and that agents match assortatively with respect to these bargaining functions. We thus show that assortativeness is inherently related to stability, rather than being driven by particular modeling choices of agents' sharing of output. The framework allows us to show that the presence of complementarities and peer effects – two important phenomena whose analysis was missing from prior theoretical work on matching – overturns some of the well-known comparative statics of many-to-one matching.

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1 Introduction

Agents form coalitions in numerous environments: two-sided (many-to-one) matching between students and colleges, or between interns and hospitals, as well as one-sided formation of private clubs, partnerships, firms, and business alliances. In many of these environments agents have preferences over coalitions they can join; moreover they may care about the entire composition of their coalition, but not about other coalitions that form. The standard solution notion is stability: a partition of agents into coalitions is (core) stable if there does not exist a counter-factual coalition whose members prefer it to their coalitions in the partition.[1] Stability is an important property of coalition structures in part because of the empirical evidence that links lack of stability to market failures (Roth 1984).

We understand only partially when stable matchings or coalition structures exist. For instance, the literature on many-to-one matching generally assumes after Kelso and Crawford (1982) that (i) agents on one side of the market (say, firms) view the agents on the other side (workers) as gross substitutes, and (ii) there are no peer effects – that is, workers' preferences depend only on the firm they apply to and not on who their peers will be. An examination of matching with complementarities and peer effects has been missing.

The present paper introduces a unified framework to study coalition formation including many-to-one matching as well as one-sided coalition formation problems. We focus attention on matching settings in which each firm has a capacity to hire at least two workers, as well as on coalition formation problems in which every three agents can form a coalition.[2]

The paper aims to understand stability properties of environments, rather than of particular preference profiles. We thus look at a collection of states of nature, and ask whether stable coalition structures exist in all of them. In this respect, our approach resembles the social choice literature. We require that the environment is rich: there is

[1] Cf. Gale and Shapley (1962), Roth (1984), Roth and Peranson (1999), Buchanan (1965), Farrell and Scotchmer (1988), Hart and Moore (1990).

[2] For a firm to perceive two workers as complementary – or for the workers to experience peer effects – the firm must be able to employ at least two workers. Similarly, in one-sided coalition formation for an agent to see others as complementary, the coalition must be at least of size three. Hence our assumptions are without much loss of generality for the analysis of complementarities and peer effects. The assumptions also simplify the formulation of our other results, and the two main problems we are excluding – one-to-one matching and the roommate problem – are both fairly well understood.

enough variability among agents' possible preference profiles that – roughly speaking – any fixed agent can rank the coalitions he or she belongs to in all possible ways.

The main result says that there is a stable coalition structure in all states of nature if and only if agents' preferences are pairwise aligned in all states of nature. Agents' preferences are pairwise aligned if the restrictions of any two agents' preference relations to the set of proper coalitions to which they both belong coincide. For instance, a firm prefers a firm-and-one-worker coalition to a larger coalition if, and only if, the worker does. We further show that if pairwise alignment is satisfied and preferences are strict then the stable coalition structure is unique except for some cases of one-sided coalition formation in which there are two stable coalition structures.

As two applications of the main result, the paper analyzes sharing rules in matching and coalition formation (fully characterizing stability-inducing sharing rules and establishing an assortative structure that is common to all of them), and offers a tractable theory of matching with complementarities and peer effects.

In the first application, agents' payoffs in each coalition are determined by sharing the output produced by the coalition. A sharing rule is a collection of functions – one for each coalition and each of its members – that map output to agents' shares of the output. In this way, each profile of outputs determines a profile of agents' preferences over coalitions. The domain of preference profiles obtained as we vary the outputs is rich provided the sharing rule functions are continuous, monotonic, and each agent's share can be arbitrarily large when the coalition's output is sufficiently large. Hence, the main stability results imply that such a sharing rule leads to stable coalition formation problems if, and only if, it generates pairwise-aligned preference profiles.

Some sharing rules generate pairwise-aligned profiles and others do not. For instance, preferences are pairwise aligned if agents' shares in each coalition are determined through equal sharing, multi-agent Nash bargaining, Tullock's (1980) rent-seeking game, or egalitarian and Rawlsian sharing rules, but not if the shares are determined through the Kalai-Smorodinsky bargaining solution. Our stability results thus imply that if the shares are determined through, say, Nash bargaining then there always exists a stable coalition structure. They also imply that if the shares are determined through Kalai-Smorodinsky bargaining then there is no stable coalition structure for some output profiles.

To compare the above results to the literature, observe that Farrell and Scotchmer (1988) established the existence of stable coalition structures for equal sharing, and

Banerjee, Konishi, and Sönmez (2001) extended it to some other linear sharing rules.[3] However, the general connection between pairwise-aligned rules and stable outcomes has gone unnoticed. Surprisingly, even the fact that Nash bargaining implies stability was not observed.

The paper offers a useful characterization of the class of pairwise-aligned rules that share the output efficiently in each coalition. Each such sharing rule may be represented by a profile of agents' "bargaining functions." Similarly to Nash bargaining, the agents then share output in each proper coalition as if they were maximizing the product of the bargaining functions. The main thrust of the characterization remains true when the efficiency assumption is dropped. Furthermore, among efficient and monotonic sharing rules on proper coalitions, rules that generate pairwise-aligned preferences are precisely those which are consistent in the sense of Harsanyi (1959) and Thomson and Lensberg (1989). When translated in terms of consistency, the above characterization extends the main insight of Lensberg (1987) to many-to-one matching and other environments in our setting.[4]

The paper leverages the bargaining function representation of pairwise-aligned – and hence stability-inducing – sharing rules to show that the resulting stable coalition structures share some common characteristics. Most notably, the coalition structures generated by stability-inducing sharing rules are assortative. Agents sort themselves into coalitions according to the Aumann and Kurz (1977) fear-of-ruin coefficient of their bar-

[3]Farrell and Scotchmer (1988) study partnerships (coalitions) that share output equally among members. Since the per-agent payoff from a coalition is the same for all agents, Farrell and Scotchmer observe that there is a metaranking of all coalitions that agrees with agents' rankings. They then conclude that there exists a stable coalition structure because the coalition at the top of the metaranking cannot be blocked, and then the top coalition among the remaining agents cannot be blocked, etc. Furthermore, they show that the resulting stable coalition structure is assortative. When specialized to the sharing rules setting, our existence results go beyond Farrell and Scotchmer by replacing equal sharing with pairwise alignment, and by showing that in this way we obtain the entire class of stability-inducing sharing rules. A (relaxed) analogue of their metaranking structure is true in our setting; the substance of our uniqueness argument is the construction of such a relaxed metaranking. Below we discuss the present paper's results on assortativeness and compare them to Farrell and Scotchmer's. None of our other results (the necessity of pairwise alignment, and – discussed below – the characterization of pairwise-aligned rules in terms of bargaining functions, comparing stable structures across different sharing rules, and the applications to many-to-one matching) has a counterpart in Farrell and Scotchmer (1988) or in subsequent developments of their setting.

[4]This characterization and, more importantly, establishing the connection between consistency and stability (and consistency and assortativeness) are the paper's contributions to the literature on consistency. Prior work on consistency and the core is only superficially related: studies such as Peleg (1986) examine consistency properties of the core itself, assuming the core is non-empty (cf. Thomson's (2009) survey), while our results imply that the consistency of the underlying sharing rule is linked to the non-emptiness of the resultant core.

gaining functions. Agents with similar bargaining functions tend to belong to the same coalitions in the stable structure. Assortative matching has been extensively studied since Becker (1974), mostly in one-to-one problems, and Farrell and Scotchmer (1988) demonstrated it for coalition formation with equal sharing. The present paper's primary contribution to this literature is to show that assortativeness obtains not only for some special stability-inducing sharing rules, but instead is inherently linked to stability in sharing-rule environments.[5] Another notable characteristic of coalition structures – discussed in the concluding remarks – is that the exogeneity of the sharing rule endows the model with an element of hold-up: some beneficial coalitions may not form because agents with strong bargaining powers are not able to commit to adequately reward an agent with relatively weak bargaining power.

Finally, the paper compares stable coalitions across different stability-inducing and efficient sharing rules. Assuming that outputs are independently drawn from distributions with monotone hazard rates, the paper shows that the probability of a coalition being stable is larger when the bargaining functions of coalition members are more equal. We may conclude that, roughly speaking, more equal societies are more likely to have large coalitions.

The second application of the main results is to many-to-one matching. We depart from the standard treatment of many-to-one matching by allowing complementarities and peer effects, and we show that their presence changes the standard results. In our setting, firms see workers as complementary when the complementarity is embedded in the profile of outputs. Workers then care who their peers are. In contrast, as mentioned

[5]In particular, agents endowed with CRRA utilities and sharing output in Nash bargaining match in a positive assortative way on risk aversion. This remains true in one-to-one matching, and is of interest in the context of the large empirical literature on risk-sharing and its recent critique by Chiappori and Reny (2006). Observing that the literature relies on an implicit assumption that agents match in a positive assortative way on risk aversion, Chiappori and Reny show that this is impossible in a general model of one-to-one matching. Their result turns out to hinge on the assumption – shared by the empirical literature they criticize, but not by this paper – that agents can freely contract on the sharing of output. Our approach may also be seen as complementary to Legros and Newman's (2007) recent systematic study of assortativeness. In their model agents are exogenously ranked by their productivity type, and costly transfers at the matching stage are allowed. They establish a condition on the production function and costs of transfer that guarantee that agents match monotonically in their exogenous types. In contrast, the present paper starts with a sharing rule and shows that agents sort themselves according to the fear of ruin of their induced bargaining functions. The ordering is thus endogenous to the sharing rule, but nevertheless defined in a way that is consistent across all stability-inducing sharing rules. There are other differences between the models. Most importantly, Legros and Newman focus attention on one-to-one matching, while we look at many-to-one matching and one-sided coalition formation, they study a uniformly distributed continuum of agents on each side, while we look at any finite configuration of agents.

above, main expositions of matching – Gale and Shapley (1962), Kelso and Crawford (1982), Roth (1985), Roth and Sotomayor (1990), and Hatfield and Milgrom (2005) – impose variants of two assumptions: gross substitutes and lack of peer effects. Both of these assumptions are restrictive. The gross substitutes assumption says that if a firm wants to employ a worker w from a large pool of workers, then the firm wants to employ w from any smaller pool containing w. While in one-to-one matching this assumption is essentially equivalent to independence of irrelevant alternatives, in many-to-one matching it is surprisingly more restrictive. Imagine for instance that you want to hire a cook and a chauffeur. If both an expert cook and an expert chauffeur are available, you might want to hire the two of them over a generalist who can both cook and drive reasonably well. When, however, the chauffeur is not available, you might as well withhold the offer from the cook, and hire the generalist instead. Or, imagine you want to open a shop, and to pay the rent you need to hire enough personnel to keep it in operation throughout the week. If too few qualified candidates apply, you may choose not to hire anyone. The lack-of-peer-effects assumption says that workers (or students, or doctors) care only about the identity of the firm they match with, but not about the identity of their peers. By making this assumption the literature assumed away, for instance, workers' concerns about interactions in the workplace and ways in which one's peers influence one's workload. The assumption has also precluded the matching literature from contributing to the debate on whether peer effects affect schooling outcomes (Case and Katz 1991, Manski 1993, Sacerdote 2001, Angrist and Lang 2004, Duflo, Dupas, and Kremer 2009).

We show that allowing complementarities and peer effects changes the standard comparative statics of many-to-one matching. For instance, under substitutes retiring an agent from one side of the market benefits other agents on the same side, and hurts agents on the opposite side of the market (Crawford 1991). The paper shows that this result is no longer true in the presence of complementarities. The paper further shows that – irrespective of whether complementarities are present – as long as the pairwise alignment condition is satisfied, strategic agents reach a stable matching or coalition structure on the equilibrium path of a wide variety of algorithms and non-cooperative games such as the Gale and Shapley deferred acceptance algorithm. This positive finding also contrasts with the results on matching under the standard gross substitutes condition where strategic agents may end up in an unstable outcome (Roth 1985, Sönmez

1999).

The present paper is the first to propose the pairwise alignment assumption and analyze the impact of complementarities on comparative statics and strategic play. Prior results on complementarities in matching focused on the impossibility of obtaining existence when complementarities are allowed. For instance, in a very general model of matching with contracts, Hatfield and Milgrom (2005) and Hatfield and Kojima (2008) show that a variant of the gross substitutes condition is the most general condition that – when imposed on preferences of each agent separately – guarantees the existence of a stable matching.[6] Our results are consistent with their conclusion as the pairwise alignment assumption is a constraint on the relation between preferences of pairs of agents rather than on preferences of individual agents. For the study of complementarities, looking at relations between preferences, or how they are co-determined by the matching environment, is likely necessary.

Notable prior work that derives positive results that go beyond the standard assumptions of gross substitutes and lack of peer effects include Echenique and Oviedo (2006) and Echenique and Yenmez (2007). They are complementary to the present work: rather than analyzing environments that allow for complementarities, they construct algorithms that find stable matchings whenever they exist. Two other papers relax the assumption restricting peer effects. Maintaining substitutability, Dutta and Massó (1997) weakened the lack-of-peer-effects condition in two separate ways: (i) allowing exogenously "married" worker couples to prefer any coalition that includes their partner to any other coalition, and (ii) allowing peer effects to influence workers' preferences between two coalitions if the employer (firm) is the same but not otherwise (Revilla 2007 develops further this line of thought).

2 Example

Before turning to the more general model, let us examine the questions and results of the paper in the context of a simple matching environment with four agents and two states of nature. Looking at three illustrative sharing rules, we will get a preliminary sense of which rules give us stable coalition structures, and which do not. We will also see how the stable coalition structures depend on the sharing rule.

[6]Hatfield and Milgrom, and Kojima and Hatfield assume that there are no peer effects. Klaus and Klijn (2005), and Sönmez and Unver (2010) prove related impossibility results.

In our example, there are two firms f and g, and two workers 1 and 2. They are forming coalitions in order to produce output. Each firm can employ either one or both workers. Agent a derives utility (or profit) $U_a(s)$ from obtaining share s of output. Let us assume that

$$U_f(s) = 5s^{\frac{1}{2}}, \quad U_g(s) = s, \quad U_1(s) = 16s^{\frac{1}{6}}, \quad U_2(s) = 4s^{\frac{1}{2}}.$$

There are two possible states of nature ω_1 and ω_2. Agents know the state when they form coalitions; in particular they know the outputs that each coalition can produce if formed. Denote by $\mathbf{y}(C,\omega)$ the output of coalition C produced in state ω. We will assume that coalition $\{g\}$ produces 0 in state ω_1 and 1 in state ω_2, and other single-agent coalitions produce output 0 regardless of state of nature. The outputs of two-agent coalitions do not depend on the state of nature: coalitions $\{g,1\}, \{g,2\}$ produce output 1, coalition $\{f,1\}$ produces 27, and coalition $\{f,2\}$ produces 25. The remaining two output levels depend on the state of nature in the following way:

$$\mathbf{y}(\{f,1,2\},\omega_1)) = 39, \qquad \mathbf{y}(\{g,1,2\},\omega_1)) = 36,$$

$$\mathbf{y}(\{f,1,2\},\omega_2)) = 27, \qquad \mathbf{y}(\{g,1,2\},\omega_2)) = 43.$$

The output in each coalition in the resulting many-to-one matching will be divided according to a sharing rule. One interpretation of the sharing rule setup is that the coalition formation and production take place on two different dates. On date 1, the agents learn the state of nature $\omega \in \{\omega_1, \omega_2\}$ and then form coalitions. On this date, they cannot make transfers conditional on joining a coalition, and cannot affect the payoffs they will obtain on date 2. In effect, on date 1, the agents' preferences over coalitions reflect their shares of the output produced on date 2.

We will look at the following three sharing rules.

Equal sharing. Agents share output equally. In state of nature ω, the share of agent i in coalition C is $\frac{\mathbf{y}(C,\omega)}{|C|}$.

Agents' shares of output in the four productive coalitions (rounded to the first decimal point) are listed in Table 1. Under equal sharing, there is a unique stable matching in state ω_1 and it takes the form $\{\{f,1\},\{g,2\}\}$. In state ω_2 the unique stable matching is $\{\{f\},\{g,1,2\}\}$. As first observed by Farrell and Scotchmer in their 1988 study of partnerships that share profits equally, it is not a coincidence that there is a stable

Table 1. Output shares and stable matchings for different sharing rules and states

	$\{f,1,2\}$	$\{f,1\}$	$\{f,2\}$	$\{g,1,2\}$	Stable matching
State ω_1					
Equal Sharing	13.0, 13.0, **13.0**	**13.5, 13.5**	12.5, 12.5	**12.0**, 12.0, 12.0	$\{f,1\},\{g,2\}$
Nash Bargaining	16.7, 5.6, **16.7**	**20.2, 6.7**	12.5, 12.5	**15.4**, 5.1, 15.4	$\{f,1\},\{g,2\}$
Kalai-Smorodinsky	**14.2,** 13.4, **11.4**	13.1, **13.9**	13.9, 11.1	**14.5**, 11.7, 9.7	$\{f,1,2\},\{g\}$
State ω_2					
Equal Sharing	9.0, 9.0, 9.0	**13.5,** 13.5	12.5, 12.5	**14.3**, **14.3, 14.3**	$\{f\},\{g,1,2\}$
Nash Bargaining	11.6, 3.9, 11.6	**20.2, 6.7**	12.5, 12.5	**18.4**, 6.1, **18.4**	$\{f,1\},\{g\},\{2\}$
Kalai-Smorodinsky	9.4, 10.0, 7.5	13.1, **13.9**	**13.9**, 11.1	**18.6**, 13.0, **11.4**	does not exist

Agents' shares in coalitions are listed in the same order as agents. The highest share of each agent is in bold.

matching in both states of nature when agents share output equally. Since each agent wants to be in a coalition with output per agent, $\frac{\mathbf{y}(C,\omega)}{|C|}$, as high as possible, no agent wants to change a coalition with the highest output per agent. We can hence set this coalition aside, and recursively construct a stable coalition structure.

Nash bargaining. Agents share output according to the Nash bargaining solution. In state of nature ω, members of C obtain shares s_a that maximize

$$\max_{s_a \geq 0} \prod_{a \in C} U_a(s_a)$$

subject to

$$\sum_{a \in C} s_a \leq \mathbf{y}(C,\omega).$$

We may thus interpret the function U_a as the composition of agent a's bargaining power and utility (or profit) function.

In both states of nature there is a unique stable matching: $\{\{f,1\},\{g,2\}\}$ in state ω_1 and $\{\{f,1\},\{g\},\{2\}\}$ in state ω_2. In particular, the stable matching is the same as under equal sharing in state ω_1, but different in state ω_2. It turns out that, again, it is not a coincidence that Nash bargaining leads to stable matchings in both states of nature. For any profile of outputs, we can construct a stable matching. The construction does not depend on the special power-function form of the utility functions U_a. Let us take any increasing, concave, and differentiable utility functions, normalize them so that $U_a(0) = 0$, and proceed in three steps. First, observe that the so-called fear-of-ruin coefficient (Aumann and Kurz 1977) $\chi_a(s_a) = \frac{U_a(s_a)}{U'_a(s_a)}$ is the same for every agent a in any given coalition C (because the first order condition in the Nash bargaining maximization equalizes $\frac{1}{\chi_a}$ and the Lagrange multiplier), and denote by χ_C this common fear of ruin. Second, observe that each agent's allocation s_i is increasing in the common fear of ruin χ_C of agents in coalition C. Third, conclude that no agent wants to change a coalition that maximizes χ_C and, therefore, we can set a coalition with maximal χ_C aside and look at coalition formation among the remaining agents. In this way, one can recursively construct a stable coalition structure.

Do all sharing rules lead to stable outcomes? The answer is no. Consider our last illustrative sharing rule.

Kalai-Smorodinsky bargaining. Agents share output according to the Kalai-Smorodinsky bargaining solution. In state of nature ω, member a of C obtains share

$$s_a = \frac{U_a\left(\mathbf{y}\left(C,\omega\right)\right)}{\sum_{b\in C} U_b\left(\mathbf{y}\left(C,\omega\right)\right)}\mathbf{y}\left(C,\omega\right).$$

Under Kalai-Smorodinsky bargaining, there is a unique stable matching $\{\{f,1,2\},\{g\}\}$ in state ω_1. In state ω_2, however, there is no stable matching. Indeed, any stable matching would need to include one of the coalitions $\{f,1\},\{f,2\},\{g,1,2\}$ because $\{f,1,2\}$ is dominated by both $\{f,1\}$ and $\{f,2\}$ and the remaining coalitions produce small or zero outputs. However, none of the coalitions $\{f,1\},\{f,2\},\{g,1,2\}$ can be part of a stable matching because

- coalition $\{g,1,2\}$ would be blocked by worker 1 and firm f,
- coalition $\{f,1\}$ would be blocked by firm f and worker 2, and finally,
- coalition $\{f,2\}$ would be blocked by worker 2 together with firm g and worker 1.

The above three sharing rules illustrate the results of the paper. First, some sharing rules consistently lead to stable outcomes, and others do not. The pairwise alignment of preferences turns out to be the key differentiating factor between equal sharing and Nash bargaining, which lead to stable outcomes in all states of nature, and the Kalai-Smorodinsky solution, which does not. Preferences are pairwise aligned for both of the stability-inducing sharing rules irrespective of the state of nature. The pairwise alignment fails, however, for the Kalai-Smorodinsky rule in state ω_1, as firm f prefers $\{f,1,2\}$ over $\{f,1\}$ while worker 1 has the opposite preference. This is not a coincidence: our first insight is that pairwise-aligned and stability-inducing rules coincide. Section 4 demonstrates this result for the more general setting of domains of ordinary preference profiles.

Second, we show that Nash bargaining is a typical example of stability-inducing sharing rules. Each of them may be described by endowing agents with a profile of increasing, differentiable, and log-concave bargaining functions, and letting them share the output so as to maximize the analog of the Nash product (see Subsection 5.1).

Third, notice that under Nash bargaining worker 2 is weaker than worker 1 and firm f is weaker than firm g. In both states, the weak worker matches with the weak firm, and the strong worker either matches with the strong firm, or remains unmatched. Subsection 5.2 shows that such an assortative structure is typical for stability-inducing sharing rules.

Fourth, the large coalition $\{g, 1, 2\}$ forms under equal sharing but no equally large coalition forms under the Nash bargaining in state ω_2. Again, this is illustrative of a typical situation. Results of Subsection 5.3 imply that the more equal the sharing among workers, the more likely such large coalitions are to be part of stable matching or coalition structure.

Finally, notice that workers 1 and 2 are complementary for firm g in state ω_2 under both equal sharing and Nash bargaining, and that each of the workers cares whether the other is also hired by the same firm. Hence, the setting encompasses environments in which there are complementarities and peer effects. We examine matching with complementarities and peer effects in Section 6.

3 Model

Let A be a finite set of agents and $\mathcal{C} \subseteq 2^A$ be a set of coalitions. A coalition C is proper if $C \neq A$. Each agent $a \in A$ has a preference relation $\precsim_a$ over coalitions C such that $a \in C$. The profile of preferences of agents in A is denoted $\precsim_A = (\precsim_a)_{a \in A}$. All references to a coalition in this paper will presume that the coalition belongs to $\mathcal{C}$, and all preference comparisons $C \precsim_a C'$ will presume that $a \in C \cap C'$.

Coalition structure μ is a partition of A into coalitions from $\mathcal{C}$. We will assume throughout that there are enough coalitions in $\mathcal{C}$ so that there exists at least one coalition structure. This assumption is satisfied if, for instance, every singleton set is a coalition.

We will use the following notion of (core) stability.

Definition 1. A coalition structure μ is blocked by a coalition C if each agent $a \in C$

strictly prefers C to the coalition in μ that contains a. A coalition structure is *stable* if no coalition blocks it.

Let $\mathbf{R}$ be a subset of the Cartesian product of sets of preference profiles of agents in A. We will call $\mathbf{R}$ a preference domain. We do not require that $\mathbf{R}$ is Cartesian.

Our main existence results formalize the comparison of Section 2 and taken together say that – with enough coalitions in $\mathcal{C}$ and preference profiles in $\mathbf{R}$ – all preferences in the domain admit stable coalition structures if, and only if, the preferences are pairwise aligned. Pairwise alignment is formally defined as follows.

Definition 2. Preferences are *pairwise aligned* if for all agents $a, b \in A$ and proper coalitions C, C' that contain a, b, we have

$$C \precsim_a C' \Longleftrightarrow C \precsim_b C'.$$

Preferences are *pairwise aligned over the grand coalition* if either $A \notin \mathcal{C}$, or $A \in \mathcal{C}$ and the above equivalence is true whenever C or C' equals A.

Notice that the pairwise alignment means that $C \sim_a C'$ iff $C \sim_b C'$, and $C \succ_a C'$ iff $C \succ_b C'$. The pairwise alignment of preferences in Nash bargaining was noted already by Harsanyi (1959), and it is also straightforward for the equal sharing rule. As noted above, the pairwise alignment fails in state ω_1 of Section 2 when agents share outputs via Kalai-Smorodinsky solution.

To complete the description of the environment, we will define the assumptions on $\mathcal{C}$ and $\mathbf{R}$ used in the stability-pairwise alignment equivalence. The assumptions imposed on the family of coalitions $\mathcal{C}$ are as follows

Definition 3. The family of coalitions $\mathcal{C}$ is *regular* if the set of agents A may be partitioned into two disjoint, possibly empty, subsets F (firms) and W (workers) so that $\mathcal{C}$ satisfies the following three assumptions:

C1. For any two different agents, there exists a coalition containing them if and only if at least one of the agents is a worker.

C2. For any workers a_1, a_2 and agent a_3, there exist proper coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ such that $C_{k,k+1} \ni a_k, a_{k+1}$ and $C_{1,2} \cap C_{2,3} \cap C_{3,1} \neq \emptyset$.

C3. (i) For any worker w and agent a, if $\{a, w\}$ is not a coalition then there are two different firms f_1, f_2 such that $\{f_1, a, w\}$ and $\{f_2, a, w\}$ are coalitions. (ii) No proper coalition contains W.

The partial overlap between C1 and C3(i) allows precise matching between assumptions and results: some of the results rely on only one or two of the assumptions. C2 is implied by the following easier-to-interpret requirement: for any agents $a_1, a_2 \in W$, $a_3 \in A$, there exist a proper coalition containing them.

Assumptions C1-C3 are satisfied in many coalition formation environments. In particular, the assumptions are motivated by the following two standard environments:

- The unconstrained one-sided coalition formation defined by $\mathcal{C} = 2^A - \{\emptyset\}$, and

- Many-to-one matching defined as follows: the set of agents is partitioned into two subsets F (interpreted as the set of firms, colleges, or hospitals) and W (interpreted as the set of workers, students, or doctors), each agent $f \in F$ is endowed with a capacity constraint M_f, and

$$\mathcal{C} = \{\{f\} \cup S : f \in F, S \subseteq W, |S| \leq M_f\} \cup \{\{w\} : w \in W\}.$$

The unconstrained coalition formation satisfies C1-C3 with $W = A$, $F = \emptyset$. We prove C2 by setting $C_{1,2} = C_{2,3} = C_{3,1} = \{a_1, a_2, a_3\}$. The remaining assumptions are straightforward. Many-to-one matching satisfies the assumptions as long as $M_f \in \{2, ..., |W| - 1\}$ and $|F| \geq 2$. Indeed, C1 is straightforward. C2 requires $M_f \geq 2$ for all $f \in F$, and is established by setting $C_{1,2} = C_{2,3} = C_{3,1} = \{a_1, a_2, a_3\}$ if $a_3 \in F$, and $C_{k,k+1} = \{a_k, a_{k+1}, f\}$ for some $f \in F$ if $a_3 \in W$. C3(i) requires $M_f \geq 2$ for at least two $f_1, f_2 \in F$, and is true as $\{a, w\}$ is a coalition if $a \in F$, and $\{f_1, a, w\}$ and $\{f_2, a, w\}$ are coalitions if $a \in W$. C3(ii) requires $M_f < |W|$ for all $f \in F$.[7]

Assumptions C1 and C2 are used to show that pairwise alignment is sufficient for stability, and assumptions C1 and C3 are used to show that pairwise alignment is necessary for stability. The equivalence between stability and pairwise alignment requires some assumptions on the family of coalitions, as illustrated by the following example of the roommate problem.

[7]The marriage problem is a well-known special case of many-to-one matching defined by $M_f = 1$ for all $f \in F$. While the marriage problem does not satisfy conditions C2 and C3, it is primarily for the sake of simplicity of exposition: the equivalence between stability and the pairwise alignment obtains for the marriage problem. The pairwise alignment of preferences is satisfied for the marriage problem in a trivial way, and Gale and Shapley (1962) showed that the marriage problem always admits a stable matching (coalition structure). On the other end of the matching literature, many-to-many matching is not a coalition formation problem, and our results do not apply.

Example 1. The roommate problem is the coalition formation problem in which $\mathcal{C} = \{C \subseteq A,\ |C| \leq 2\}$. Any preference profile in the roommate problem is pairwise aligned, but the existence of a stable coalition structure is not assured. For instance, there is no stable coalition structure if $A = \{a_1, a_2, a_3\}$, all agents prefer any two-agent coalition to being alone, and their preferences among two-agent coalitions are such that

$$\{a_1, a_2\} \succ_{a_2} \{a_2, a_3\} \succ_{a_3} \{a_3, a_1\} \succ_{a_1} \{a_1, a_2\}.$$

The assumptions we will impose on the domain of preferences $\mathbf{R}$ are as follows.

Definition 4. A domain of preference profiles $\mathbf{R}$ is *rich* if it satisfies the following assumptions:

R1. For any profile $\precsim_A \in \mathbf{R}$, any agent a, and any three different coalitions C_0, C, C_1, if $C_0 \precsim_a C_1$ and $a \in C$, then there is a profile $\precsim'_A \in \mathbf{R}$ such that $C_0 \precsim'_a C \precsim'_a C_1$, and all agents' $\precsim'_A$-preferences between pairs of coalitions not including C are the same as their $\precsim_A$-preferences.

R2(i). For any $\precsim_A \in \mathbf{R}$ and two different coalitions C, C_1, there is a profile $\precsim'_A \in \mathbf{R}$ such that $C \prec'_a C_1$ for all $a \in C \cap C_1$ and all agents' $\precsim'_A$-preferences between pairs of coalitions not including C are the same as their $\precsim_A$-preferences.

R2(ii). For any $\precsim_A \in \mathbf{R}$, any agents a, b, and any three different coalitions C_0, C, C_1, if $C_0 \prec_a C \sim_b C_1$, then there is a profile $\precsim'_A \in \mathbf{R}$ such that $C_0 \prec'_a C \prec'_b C_1$, and all agents' $\precsim'_A$-preferences between pairs of coalitions not including C are the same as their $\precsim_A$-preferences.

The assumptions R1 and R2 formalize the requirement that there is substantial variability, or richness, among preference profiles. Assumption R1 requires that for any preference profile in $\mathbf{R}$ and any coalition C, there is a (shocked) preference profile in $\mathbf{R}$ in which C is ranked just below another coalition C_1 by the relevant agent. Assumption R2(i) postulates the existence of a shocked profile in which a coalition C is ranked below another coalition C_1 by all relevant agents. Assumption R2(ii) postulates that there is a shocked profile in which an indifference is broken; this last assumption is trivially satisfied if agents' preferences are always strict.

Examples of rich preference domains include the domain of preference profiles generated in equal sharing and Nash bargaining when we vary output functions $\mathbf{y} : \mathcal{C} \to (0, \infty)$. The domain of profiles generated in Kalai-Smorodinsky bargaining is rich provided each agent's utility is unbounded above (or provided each agent's utility is bounded above); the Kalai-Smorodinsky domain might fail condition R1 if some agents' utilities are unbounded above, while others are bounded above. More generally, consider a setting in which agents' payoffs are determined by a state of nature $\omega \in \times_{C \in \mathcal{C}} \Omega_C$, and the payoffs in coalition C depend only on the C-coordinate of the state of nature. For each coalition C and agent $a \in C$, consider a payoff mapping $D_{a,C}^{\text{payoff}}$ from Ω_C to payoffs of agent a. Then, R1 is satisfied as long as the set of outcomes $D_{a,C}^{\text{payoff}}(\Omega_C)$ does not depend on coalition C but only on agent a. Assumption R2 is satisfied if we additionally require that for each $C \in \mathcal{C}$, the set Ω_C is an open interval in R, and the payoff mapping is continuous and strictly monotonic. Other examples satisfying both R1 and R2 include the domain of all strict preference profiles and the domain of all preference profiles.

Assumption R1 is used to show that pairwise alignment is sufficient for stability, and assumption R2 is added to show that pairwise alignment is necessary for stability. To see that the relation between stability and pairwise alignment needs some assumptions on the domain of preferences, recall the Kalai-Smorodinsky sharing rule from Section 2. In state ω_2, there is no stable matching even though, in this state, the pairwise alignment holds. At the same time, in state ω_1 the pairwise alignment fails even though, in this state, there is a stable matching. The next section and Appendix B provide more details on the role of assumptions C1-C3 and R1-R2.

4 Main Results: Stability in Preference Domains

Our main existence results are as follows:

Theorem 1. *Suppose that the family of coalitions $\mathcal{C}$ satisfies C1 and C2, and that the preference domain $\mathbf{R}$ satisfies R1. If all preference profiles in $\mathbf{R}$ are pairwise aligned then (i) all $\precsim_A \in \mathbf{R}$ admit a stable coalition structure, and (ii) the stable coalition structure is unique for any profile of strict preferences $\precsim_A \in \mathbf{R}$ that is pairwise aligned over the grand coalition.*

Theorem 2. *Suppose that the family of coalitions $\mathcal{C}$ satisfies C1 and C3, and that the preference domain $\mathbf{R}$ satisfies R1 and R2. If all profiles from $\mathbf{R}$ admit stable coalition structures, then all profiles from $\mathbf{R}$ are pairwise aligned.*

Theorem 1 relies on part R1 of the richness assumption. To develop an understanding of the role of this assumption let us look again at the failure of stability under the Kalai-Smorodinsky sharing rule from Section 2. Theorem 1 implies that the preference profile $\precsim_{\{f,g,1,2\}}$ of agents using the Kalai-Smorodinsky rule to share outputs in state ω_2 cannot be embedded in an R1-rich domain of pairwise-aligned preference profiles. To check this corollary notice that if this profile belonged to an R1-rich domain of pairwise-aligned profiles, then there would exist a pairwise-aligned profile $\precsim'_{\{f,g,1,2\}}$ such that

$$\{f,1\} \precsim'_f \{f,1,2\} \precsim'_f \{f,2\},$$

and all agents' $\precsim'_{\{f,g,1,2\}}$ -preferences between pairs of coalitions not including $C = \{f,1,2\}$ would be the same as their $\precsim_{\{f,g,1,2\}}$ -preferences. Then, the pairwise alignment of $\precsim'_{\{f,2\}}$ would imply that $\{f,1,2\} \precsim'_2 \{f,2\} \prec'_2 \{g,1,2\}$, and the transitivity of $\precsim'_2$ and the pairwise alignment of $\precsim'_{\{1,2\}}$ would give

$$\{f,1,2\} \prec'_1 \{g,1,2\} \prec'_1 \{f,1\}.$$

However, then the $\precsim'_{\{f,1\}}$-preferences of agents 1 and f between coalitions$\{f,1,2\}$ and $\{f,1\}$ would violate pairwise alignment. This contradiction shows that $\precsim_{\{f,g,1,2\}}$ cannot be embedded in any R1-rich domain of pairwise-aligned preferences.

The first step of the proof of Theorem 1 generalizes the above argument to show that the pairwise alignment, R1, C1 and C2 imply that there are no 3-cycles for any $\precsim_A \in \mathbf{R}$. A 3-cycle, or generally, an n-cycle is any configuration of proper coalitions $C_1, ..., C_n$ and agents $a_1, ..., a_n$ such that (subscripts modulo n)

$$C_i \precsim_{a_i} C_{i+1} \text{ for } i = 1, ..., n, \text{ with at least one preference strict.} \tag{1}$$

For instance, Section 2 discussion of the Kalai-Smorodinsky rule shows that in state ω_2 coalitions $\{g,1,2\}, \{f,1\}, \{f,2\}$ form a 3-cycle.

The main step of the proof uses the lack of 3-cycles, R1, and C1 to show that lack of n-cycles implies lack of $(n+1)$-cycles, and hence that there are no n-cycles for $n = 2, 3, ...$ The final step of the proof is to observe that the lack of n-cycles implies both the existence and – with the added assumptions of strict preferences and pairwise-alignment over the grand coalition – the uniqueness of stable coalition structure. The uniqueness relies on the added assumptions. For instance, if $|A| \geq 3$ and $\mathcal{C} = \{C \subseteq A, C \neq \emptyset\}$, then there is a domain of pairwise-aligned profiles that contains a strict preference profile that (i) is not pairwise aligned over the grand coalition, and (ii) allows both the grand coalition A and a coalition structure of proper coalitions to be stable.

The proof of the necessity of pairwise alignment (Theorem 2) roughly reverses the steps of the proof of its sufficiency. First, assuming R2 we show that stability implies lack of 3-cycles C_1, C_2, C_3 such that $C_j \cap C_i$ are singletons for $j \neq i$. Then, assuming C1, C3, and R1, we show that the lack of such 3-cycles implies pairwise alignment. To get a sense of this proof, consider the preference profile that obtains in state ω_1 when agents share outputs in Kalai-Smorodinsky bargaining, and assume that the environment contains a third worker, 3. Since agents f and 1 differ in their preference ranking of coalitions $\{f,1\}$ and $\{f,1,2\}$, Theorem 2 implies that this preference profile cannot be embedded in a rich domain of stability-inducing profiles. We will check this claim directly under an additional assumption that all preference profiles are strict. Any domain satisfying R1, R2(ii), and containing the ω_1-profile, contains a preference profile $\precsim_A$ such that $\{f,1\} \prec_f \{f,3\} \prec_f \{f,1,2\}$ and $\{f,1,2\} \prec_1 \{g,1,3\} \prec_1 \{f,1\}$. Consider the case $\{f,3\} \prec_3 \{g,1,3\}$; the other case is symmetric. In this case, $\{f,3\} \prec_3 \{g,1,3\} \prec_1 \{f,1\} \prec_f \{f,3\}$ is a 3-cycle. Because of R2(i), we may assume that members of coalitions $\{f,3\}, \{g,1,3\}, \{f,1\}$ strictly prefer them to any coalition other than these three. Any stable coalition structure would then need to contain one of these three coalitions, but each one of them is blocked by one of the other two. Hence, there is no stable coalition structure.

Appendix A gives the proofs and Appendix B discusses the trade-offs involved in relaxation of the assumptions. For instance, regularity assumption C2 may be dropped in Theorem 1 at the cost of replacing pairwise alignment with lack of 3-cycles. In some problems, such as many-to-one matching, the richness assumptions may be relaxed to take account of the additional structure of such problems. The results hold true for other stability concepts such as pairwise stability and group stability in many-to-one matching. Finally, the above map of the proofs implies that no preference profile in a rich domain of pairwise aligned profiles admits an n-cycle. Appendix B also demonstrates that every profile that does not admit n-cycles may be embedded in a rich domain of pairwise-aligned profiles.

5 Applications: Sharing Rules

This section applies the existence results of the previous section to an analysis of stability-inducing sharing rules. As in Section 2, we will look at instances of our general setup, in which each coalition produces output $\mathbf{y}(C) \in R_+ = [0, +\infty)$. The mapping from coalitional outputs to agents' preferences is determined by a sharing rule. A *sharing rule* is a collection of functions $D_{a,C} : R_+ \to R_+$, one for each coalition C and each of its members $a \in C$, that map the output of C into the share of output obtained by agent a. We will be assuming that the shares are feasible, $D_{a,C}(y) \leq y$. A sharing rule is pairwise aligned if the preference profiles that it generates are pairwise aligned for every profile of outputs. In Section 2 we have seen two examples of pairwise-aligned sharing rules (equal sharing and Nash bargaining) and one instance of a sharing rule that violates pairwise alignment (Kalai-Smorodinsky).

Theorems of Section 4 imply the following

Corollary 1. *Suppose that the family of coalitions $\mathcal{C}$ satisfies C1-C3 and the sharing rule D is such that for every $a \in C \in \mathcal{C}$ the function $D_{a,C}$ is strictly increasing, continuous, and $\lim_{y\to+\infty} D_{a,C}(y) = +\infty$. Then, there is a stable coalition structure for each profile of outputs if, and only if, the sharing rule is pairwise-aligned.*

For instance, in the environment of Section 2, the assumptions of the corollary are satisfied by all three sharing rules. To prove the corollary, first notice that the pairwise alignment yields stability because the domain of preference profiles generated by D satisfies R1, and hence Theorem 1 is applicable. Second, to prove the converse implication, notice that the restriction of the sharing rule D to profiles of strictly positive outputs satisfies R1 and R2, and hence Theorem 2 implies that agents' preferences over coalitions with strictly positive outputs are pairwise aligned. This implies that D is pairwise aligned because all agents strictly prefer any relevant coalition with strictly positive output to any coalition with zero output, and are indifferent between any two relevant coalitions with zero output (because $D_{a,C}(y) = 0$ iff $y = 0$ for every $a \in C \in \mathcal{C}$).

5.1 Pareto-efficient sharing rules: a characterization

A sharing rule is *Pareto-efficient* if $\sum_{a\in C} D_{a,C}(y) = y$ for any $C \in \mathcal{C}$ and $v \geq 0$. The equal sharing and Nash bargaining are efficient. Efficient pairwise-aligned sharing rules may be characterized as follows

Proposition 1. *Suppose that the family of coalitions $\mathcal{C}$ satisfies C1 and C2, and the sharing rule D is such that for every $a \in C \in \mathcal{C}$ the function $D_{a,C}$ is strictly increasing, continuous, and $\lim_{y\to+\infty} D_{a,C}(y) = +\infty$. The rule D is pairwise-aligned and efficient if, and only if, for all $a \in A$ there exist increasing, differentiable, and strictly log-concave functions $U_a : R_+ \to R_+$ such that $\frac{U_a}{U_a'}(0) = 0$, and*

$$(D_{a,C}(y))_{a\in C} = \arg \max_{\sum_{a\in C} s_a \le y} \prod_{a\in C} U_a(s_a)$$

for all $y \in R_+$ and proper coalitions C.

In the proposition, we allow $U_a'(0) = +\infty$. We will refer to functions U_a as *bargaining functions*. In Nash bargaining they are simply agents' utility (or profit) functions. As illustrated at the end of the next subsection, the main thrust of the result remains true when we drop the efficiency assumption. Also, an inspection of the proof shows that the representation remains true if C equals the grand coalition A provided that $A \in \mathcal{C}$ and agents' preferences are additionally pairwise aligned over the grand coalition. In what follows, we will use the term *regular sharing rule* to refer to sharing rules which are aligned over the grand coalition, and such that all functions $D_{a,C}$ are strictly increasing, continuous, and $\lim_{y\to+\infty} D_{a,C}(y) = +\infty$.

Proposition 1 extends the main insight of Lensberg (1987) onto our setting, and onto many-to-one matching in particular. Lensberg constructed a representation resembling that of Proposition 1 for consistent and efficient sharing rules (consistency and pairwise alignment of sharing rules are closely related as discussed in the introduction). Proposition 1 does not follow from the earlier result because Lensberg effectively required agents' preferences to be pairwise aligned across a substantially larger space of output-sharing problems than are available in our context.[8]

The above results imply the following

Corollary 2. *Suppose that the family of coalitions $\mathcal{C}$ satisfies C1-C3 and the sharing*

[8]The results are logically independent. On one hand, Lensberg imposes neither monotonicity nor our limit assumption; thus his result is not implied by Theorem 1. On the other hand, Lensberg's result cannot be applied in the present context because his sharing rules are defined for (i) any finite subset of an infinite set of agents (excluding for instance many-to-one matching in which two firms cannot form a coalition) and (ii) for all convex choice sets as opposed to choice sets below a hyperplane (in other words he looks at sharing rules that are well-behaved across all choice problems with non-transferable utility, while the present paper assumes only that the sharing rules are well-behaved across choice problems with transferable utility). Both of these differences mean that Lensberg requires the sharing rules to be well-behaved across a substantially larger space of problems than are available in our context.

rule D is regular. There is a stable coalition structure for each preference profile induced by the sharing rule if, and only if, for all $a \in A$ there exist increasing, differentiable, and strictly log-concave functions $U_a : R_+ \to R_+$ such that $\frac{U_a}{U'_a}(0) = 0$, and

$$\left(D_{a,C}(y)\right)_{a \in C} = \arg \max_{\sum_{a \in C} s_a \leq y} \prod_{a \in C} U_a(s_a)$$

for all $y \in R_+$ and coalitions C.

As an illustration of the above characterization results, consider a linear sharing rule

$$D_{a,C}(y) = d_{a,C}\, y$$

where shares $d_{a,C}$ are positive constants such that $\sum_{a \in C} d_{a,C} = 1$. Corollary 1 implies that the linear sharing rule admits stable coalition structures in all states of nature if, and only if, the shares $d_{a,C}$ satisfy the proportionality condition

$$\frac{d_{a,C}}{d_{b,C}} = \frac{d_{a,C'}}{d_{b,C'}}$$

for all $C, C' \neq A$ and $a, b \in C \cap C'$.[9] This observation may be rephrased in terms of Nash bargaining. The Nash bargaining example of Section 2 leads to linear sharing of value if agents' utilities are $U_a(s) = s^{\lambda_a}$ for some agent-specific constants (bargaining powers) λ_a. The resulting shares satisfy the above proportionality condition, and any profile of shares $\{d_{i,C}\}_{C \in \mathcal{C} - \{A\}}$ that satisfies the proportionality condition may be interpreted as generated by Nash bargaining. Thus, a profile of shares $\{d_{i,C}\}_{C \in \mathcal{C} - \{A\}}$ guarantees the existence of stable matching for all $\mathbf{y} : \mathcal{C} \to R_+$ if, and only if, the shares may be represented as an outcome of Nash bargaining.

5.2 Assortative matching and coalition formation

When shares are divided by a stability-inducing sharing rule, agents sort themselves into coalitions in a predictably assortative way. Let us start by looking at Nash bargaining in which each agent a is endowed with an increasing, concave, and differentiable utility function U_a normalized so that $U_a(0) = 0$. In this setting, agents sort themselves into

[9] In the context of one-sided coalition formation, Banerjee, Konishi, and Sönmez (2001) proved a slightly weaker variant of one of the above implications: assuming the proportionality condition for all coalitions C, C' they show that a stable coalition structure exists. The converse implication is new.

coalitions according to their fear of ruin and their productivity. Recall that Aumann and Kurz's (1977) fear-of-ruin coefficient is defined as $\frac{U_a(s)}{U_a'(s)}$. We will say that agent a has higher fear of ruin than agent b if $\frac{U_a(s)}{U_a'(s)} \geq \frac{U_b(s)}{U_b'(s)}$ for all $s > 0$, and agent a has strictly higher fear of ruin if strict inequality holds for all $s > 0$.

We will assume that each agent $a \in A$ is endowed with productivity type θ_a from a space Θ of types, and the output $\mathbf{y}(C)$ is fully determined by the size of C and productivity types of members of C. In particular, if C_a and $C_b = C_a \cup \{b\} - \{a\}$ are coalitions and $\theta_a = \theta_b$, then $\mathbf{y}(C_a) = \mathbf{y}(C_b)$. We will furthermore assume that the space Θ is endowed with a partial ordering on types, and that the output is strictly increasing in the partial ordering on $\theta_a \in \Theta$ (keeping productivity types of agents in $C - \{a\}$ constant). Finally, we will assume that the family of coalitions is symmetric, that is for any agents a and b who are on the same side of the market, if a coalition C contains a but not b, then $(C \cup \{b\}) - \{a\}$ is a coalition. We say that two agents are on the same side of the market if both are workers or if both are firms.

In this environment, the resulting stable coalition structure is assortative: agents with high productivity and high fear of ruin form coalitions together. Formally,

Proposition 2. *Assume that the family of coalitions is symmetric and that the outputs are increasing in agents' productivity. Let C_1 and C_2 belong to the same stable coalition structure. For any agents $a_1, b_1 \in C_1$ and $a_2, b_2 \in C_2$ such that a_1 and a_2 are on the same side of the market and b_1 and b_2 are on the same side of the market, if a_1 is more productive and has higher fear of ruin than a_2, with at least one of the relations being strict, then it is not possible that b_2 is more productive and has higher fear of ruin than b_1, with at least one of the relations being strict.*

The proof of Proposition 2 is by a straightforward indirect argument. Assume that there are coalitions C_1, C_2 and agents a_1, a_2, b_1, b_2 that falsify the proposition. Denote by s_a the share of output of agent a. Recall from the analysis of Nash bargaining in Section 2 that $\frac{U_a(s)}{U_a'(s)}$ takes a common value χ_{C_1} for all agents in C_1, and a common value χ_{C_2} for all agents in C_2, and that agents always prefer coalitions with higher χ. Because of the symmetry between assumptions on C_1 and C_2, we may assume that $\chi_{C_1} \geq \chi_{C_2}$. Moreover, by symmetry of the family of coalitions, $C = (C_1 - \{b_1\}) \cup \{b_2\}$ is a coalition. Since b_2 is more productive and more risk averse than b_1, with at least one of the relations being strict, we must have $\chi_C > \chi_{C_1}$. Because agents' preferences over coalitions are aligned with χ, all agents in $C \cap C_1$ prefer it to C_1, and agent b_1 prefers C to C_2. Thus, C would be a blocking coalition, a contradiction. QED

As an example of an application of Proposition 2, consider the case of agents endowed with identical utility functions who differ in their productivity types $\theta_a \in \Theta$. Such agents sort themselves in terms of productivity. This case of Proposition 2 generalizes the main result of Farrell and Scotchmer (1988).[10]

As another example, consider the case when $\mathbf{y}(C)$ depends on C only through its cardinality $|C|$ and agents have power utilities, $U_a(s) = s^{\lambda_a}$. Then the coefficients λ_a may be interpreted as agents' bargaining powers, and the result says that agents sort themselves on bargaining powers: for any two coalitions in the stable coalition structure, the largest bargaining power of a worker in one of them is weakly lower than the smallest bargaining power of a worker in the other coalition; and similarly for firms. As an aside, notice that the assortative structure implies that the differences among bargaining powers of agents within coalitions of a stable coalition structure are suppressed relative to the differences of bargaining powers among all agents.

Provided the coalition structure satisfies C1-C3, the assortative structure of Proposition 2 remains true for all stability-inducing and efficient sharing rules characterized in Corollary 1 and Proposition 1. Proposition 1 allows us to define analogues of U_a for all such sharing rules, and the above proof of Proposition 2 extends to the more general case without any changes. Furthermore, the assortative structure remains true for all stability-inducing sharing rules if – in addition to agents' impact on productivity – we explicitly account for agents' impact on the inefficiency. This is so because the inefficient sharing rules may be viewed as dividing the effective output (defined as the sum of agents shares) in an efficient way. Proposition 1 allows us thus to find the bargaining functions for the induced efficient sharing rule, and – productivity and inefficiency factors held constant – agents sort themselves according to the fear-of-ruin coefficient of their bargaining functions. Finally, if the fear-of-ruin coefficients of two bargaining functions, U_a, U_b, are not comparable because the relation between $\frac{U_a}{U_a'}(s)$ and $\frac{U_b}{U_b'}(s)$ depends on the stake s, then agents still sort themselves according to their fear of ruin calculated at relevant stakes.

[10]Farrell and Scotchmer analyzed the equal sharing rule (which corresponds to assuming all agents are endowed with identical utility functions) and imposed a linear relationship between outputs and one-dimensional productivity types.

5.3 Comparative statics: how inequality among members of a coalition decreases the chances the coalition is stable

We will study how the sharing rule impacts the formation of any given coalition L. We will assume that outputs $\mathbf{y}(C)$ are independently drawn from absolutely continuous log-concave distributions on R_+. Log-concavity means that the logarithm of the c.d.f. is concave, and is equivalent to a monotone hazard rate condition. Many of the distributions studied in economics, including the uniform distribution on $[0,1]$ and the exponential distribution, satisfy this property. We will allow the distribution of $\mathbf{y}(C)$ to depend on C through its cardinality $|C|$ but not otherwise. Finally, we will impose the following symmetry assumption on the family of coalitions: for any workers a and b, if a coalition C contains a but not b, then $(C \cup \{b\}) - \{a\}$ is a coalition.

As we will see, under these assumptions the equal sharing rule maximizes the probability of coalition L being stable in the case $L = A$.[11] An analogue of this claim holds true for any other coalition $L \neq A$, but its formulation requires some care. There are two forces that increase the probability that $L \neq A$ belongs to a stable coalition structure: equality among workers in L and the relative bargaining strength of members of L when compared to other agents. The probability of L being stable is maximized as we approach the limit in which workers in L share the output equally, but would get nothing in any coalition containing an agent from $A - L$. Since no regular sharing rule accomplishes this limit, we will prove the maximal property of equal sharing while controlling for the relative strengths of members of L vis-à-vis other agents.

Proposition 3. *Assume that the family of coalitions $\mathcal{C}$ is symmetric and satisfies C1-C3, that L is a coalition, and that outputs $\mathbf{y}(C)$ are independently drawn from size-dependent log-concave distributions on R_+. There is a partition of the class of stability-inducing, efficient, and regular sharing rules such that each element of the partition contains a unique sharing rule D that equalizes shares of workers in L for all output levels, and the probability of L being stable under D is weakly higher than under any other sharing rule from the element of the partition (and strictly higher if the distributions are strictly log-concave). In particular, if $L = A$, then the equal sharing rule maximizes the probability of A being stable among all stability-inducing, efficient, and regular sharing rules.*

[11] We will be studying regular sharing rules. For such rules, the probability of a particular coalition structure is well defined when $\mathbf{y}(C)$ are drawn from continuous distributions. Theorem 1 implies that the stable coalition structure is generically unique when the sharing rule is regular, and it is easy to extend the argument to conclude that in the current setting it is unique with probability 1.

The proof of Proposition 3 allows us to construct a partial ordering on all stability-inducing efficient sharing rules, such that the probability of L being stable is increasing in the partial ordering. In lieu of a heuristic of the proof, let us look at such an ordering restricting attention to the following class of linear sharing rules: each agent a is endowed with bargaining power $\lambda_a > 0$, and coalition C divides output y so that the share of agent a is

$$D_{a,C}(y) = \frac{\lambda_a}{\sum_{b \in C} \lambda_b} y.$$

The probability of coalition L forming is then decreasing in inequality among bargaining powers of workers in L. Denote by $\lambda_{(i)}$ the i-th highest value of λ_w, among workers $w \in L$. We will keep the bargaining powers of firms and of agents not in L proportional to $\sum_{w \in L \cap W} \lambda_w$, and assess the inequality among workers in L with the following partial order

$$(\lambda_w)_{w \in L \cap W} \geq (\lambda'_w)_{w \in L \cap W} \iff \frac{\lambda_{(i)}}{\lambda_{(i+1)}} \geq \frac{\lambda'_{(i)}}{\lambda'_{(i+1)}}, \quad i = 1, ..., |L \cap W|;$$

the ordering is strict if at least one of the above inequalities is strict.

Proposition 4. *Assume that the family of coalitions is symmetric, and that outputs* $\mathbf{y}(C)$ *are independently drawn from size-dependent log-concave distributions on* R_+. *Then, the probability that the coalition* L *is stable is decreasing in the above-defined partial order on the profiles of bargaining powers. The probability is strictly decreasing if output distributions are strictly log-concave on* R_+.

Let us sketch the proof for the case of $L = A$. The appendix gives omitted parts of the argument. First note that rescaling all bargaining powers by a constant changes neither the ordering nor agents' payoffs, and hence we may assume that $\sum_{a \in A} \lambda_a = \sum_{a \in A} \lambda'_a$. Assume that there are two workers, say a and b, whose bargaining powers differ, $\lambda_a < \lambda_b$. Take any coalition C. If $a, b \in C$ or $a, b \notin C$ then a small increase in λ_a and an offsetting decrease in λ_b that keeps the sum of the two powers constant does not change the probability that A is blocked by C. If $a \in C$ but $b \notin C$ then $C \cup \{b\} - \{a\}$ is a coalition by the symmetry of the family of coalitions, and we use the log-concavity of the distributions to show that the above adjustment of the two bargaining powers decreases the joint probability that A is blocked. Hence, the above adjustment of bargaining powers increases the product of probabilities that A is not blocked by any coalition, and hence the probability that A is stable. To conclude the proof we then show that if one profile of bargaining powers is dominated by another in the above partial ordering, then there is a finite sequence of adjustments of bargaining powers that connects the two.

6 Applications: Matching

6.1 Complementarities in matching

The results of the preceding sections are applicable to many-to-one matching situations with complementarities and peer effects. In the example of Section 2, we have seen that firm g but not firm f treats the two workers as complementary in state ω_2 both under equal sharing and Nash bargaining, and that each worker cares whether the other one works for the same firm. In general, under equal sharing or Nash bargaining, the firm's preferences may treat two or more workers as complementary as no assumption is needed on the profile of outputs $\mathbf{y}$. The peer effects are inherent to both equal sharing and Nash bargaining: workers care about which other workers belong to their coalition.

The presence of complementarities means that some of the standard comparative statics derived in the theory of many-to-one matching under the standard assumptions of gross substitutes and lack of peer effects no longer hold true.[12] A major standard comparative static result says that removing an agent from one side of the market weakly increases the payoffs of the other agents on the same side of the market, and weakly decreases payoffs of agents on the other side (Crawford 1991). In contrast, even if we assume that agents' payoffs are determined in equal sharing or in Nash bargaining and that the stable matching is unique, removing a worker may lead to a change of the stable matching that results in some firms obtaining higher payoffs, and some workers obtaining lower payoffs. For instance, consider equal sharing in state ω_2 of the example of Section 2. The unique stable matching is $\{\{f\},\{g,1,2\}\}$ when all agents are available, and $\{\{f,2\},\{g\}\}$ when worker 1 is not available. Thus, firm f benefits and worker 2 loses when worker 1 is removed. Similarly, removing a firm may increase the payoffs of some workers, and decrease the payoff of some firms as illustrated below.

[12] We consider only many-to-one matchings that satisfy C2: any firm and any two workers can form a coalition together. For the purposes of our discussion, this is not a strong restriction as complementarities and peer effects can be present only in many-to-one matching situations in which some firms can be matched with two (or more) workers. By Theorem 1 the stable matching is unique in our setting if we adapt the standard assumption that agents' preferences are strict. The uniqueness implies that the lattice structure of the set of stable matchings and the so-called rural hospital theorem (Roth 1984, Roth and Sotomayor 1990) are true in our setting. As usual the properties are not true when indifference is allowed.

Example 2. Consider three firms f_1, f_2, f_3 and two workers w_1, w_2. Assume that agents share outputs equally, and that the outputs are such that

$$\mathbf{y}(\{f_1, w_1\}) = 2, \quad \mathbf{y}(\{f_2, w_2\}) = 1, \quad \mathbf{y}(\{f_3, w_1, w_2\}) = 2,$$

and all other outputs equal zero. The unique stable matching is $\{\{f_1, w_1\}, \{f_2, w_2\}, \{f_3\}\}$ when all agents are available, and $\{\{f_2\}, \{f_3, w_1, w_2\}\}$ when firm f_1 is not available. Thus, removing firm f_1 decreases the payoff of firm f_2 and increases the payoff of worker w_2.

Weak versions of the standard comparative statics remain true: it is straightforward to check that adding a worker weakly improves the payoff for at least one firm, and adding a firm weakly improves the payoff for at least one worker.

6.2 Strategic play

Under the substitutes condition, the Gale and Shapley (1962) deferred acceptance algorithm produces a stable coalition structure in a many-to-one matching provided agents act truthfully. However, firms have incentives to be strategic, and in general the outcome of the coalition formation process depends on the details of the non-cooperative game agents play, and does not need to be stable (see, for instance, Roth 1985, Roth and Sotomayor 1990, and Sönmez 1999). The following result shows that under pairwise alignment the details of the process of coalition formation are less important.

Proposition 5. *Consider a non-cooperative game (extensive or normal form) among agents from A and a mapping $\hat{\mu}$ from agents' strategies Σ to coalition structures such that for each coalition $C \in \mathcal{C}$ there is a profile of strategies σ_C of agents in C such that $C \in \hat{\mu}(\sigma)$ for all strategy profiles $\sigma \in \Sigma$ that agree with σ_C on C. If the family of coalitions satisfies C1 and C2, and agents' preferences come from a rich domain of pairwise-aligned preference profiles, then for every stable coalition structure μ, there is a Strong Nash Equilibrium σ such that $\mu = \hat{\mu}(\sigma)$.*[13]

[13] We are implicitly assuming that each agent's payoff is uniquely determined by the coalition in $\hat{\mu}(\sigma)$ the agent belongs to. A profile of players' strategies σ is a Strong Nash Equilibrium if there does not exist a subset of players that can improve the payoffs of all its members by a coordinated deviation, while players not in the subset continue to play strategies from σ (Aumann 1959). Similar results are true for the Strong Perfect Equilibrium of Rubinstein (1980), and the Coalition-Proof Nash Equilibrium of Bernheim, Peleg, and Whinston (1987).

Proposition 5 relies on the alignment of agents' preferences but not on the many-to-one structure. The converse claim that any Strong Nash Equilibrium gives a stable coalition structure is straightforward and does not require any assumptions on preferences.

The coalition formation game based on the Gale and Shapley deferred acceptance algorithm satisfies the assumption of Proposition 5 as does the single-round-of-application game in which each worker applies for one or no jobs and then each firm selects its workforce from among its applicants. The assumption is also satisfied by the algorithm of Echenique and Yenmez (2007) that takes as input agents' preferences and generates stable matching whenever it exists. In particular, in our setting truthful reporting is an equilibrium of the game induced by Echenique and Yenmez' algorithm.

The proposition shows that the assumption of Cartesian preference profile domain is crucial for the main result of Sönmez (1999). Assuming that the domain of preference profiles is the Cartesian product of the domains of agents' preferences, Sönmez shows that truthful reporting in equilibrium of a large class of games encompassing coalition formation is possible only when the core is essentially single-valued, that is only when the agents are indifferent among all coalitions in the core. As explained after Theorem 1, in general, even if preferences are strict, the core is not essentially single-valued in Proposition 5 with $\mathcal{C} = 2^A - \{\emptyset\}$. Similarly as in our comparison to Hatfield and Milgrom (2005), it is the lack of Cartesian structure that explains why Sönmez' insight does not extend to our setting.

7 Concluding Remarks

We have seen which sharing rules and, more generally, which preference domains guarantee existence of stable coalition structures, and analyzed properties of such sharing rules and preference domains. Let us conclude with a brief overview of three possible research areas to apply the results of the paper.

7.1 Hold-up

Hold-up problems resulting from contractual incompleteness have been studied in some detail following Grossman and Hart (1986) and Hart and Moore (1990). The hold-up in this literature results from inadequate specific investments made by coalition members (after the coalitions form). Within the constraint imposed by non-contractability of specific investments, the further assumptions made by the literature force the coalition structure to be (constrained) efficient.[14]

Our framework – and non-transferable utility coalition formation in general – gives rise to a different type of hold-up problem: hold-up caused by inflexible sharing of output. Consider Nash bargaining with constant bargaining powers. An agent may be better off with a lower rather than higher bargaining power (other things held equal), as a highly productive coalition may not form because the high bargaining power makes the agent unable to compensate other agents on par with their outside options. Relatedly, the stable coalition structure does not necessarily maximize the sum of agents' payoffs. For instance, a highly productive coalition may not form because the lack of side payments makes it privately suboptimal for an agent with a relatively low bargaining power to join it. Both of these problems illustrate the hold-up inherent to the model in which at the matching stage the anticipated sharing of output is fixed, and agents cannot make or contract on side payments when forming coalitions. In particular, this hold-up problem may lead to inefficiency in the coalition structure itself.

The hold-up caused by inflexible sharing of output is far from being a solely theoretical possibility. For instance, Baker, Gibbons, and Murphy (2008) report that a series of interviews with practitioners involved in formation of alliances (coalitions) among firms led them to conclude that lack of flexibility in dividing payoffs that accrue directly to firms in an alliance ("spillover payoffs") – rather than the alliance itself – is a major factor determining the form and performance of alliances, more so than the standard hold-up problems of inadequate specific investments the literature focused on.

The present paper contributes to this research area by showing which hold-up problems resulting from inflexible sharing of output may be solved for stable outcomes.

7.2 Synergies among start-ups in venture capital portfolios

Sørensen (2007) argues that many-to-one matching between start-ups and (lead) venture capital firms is an example of a problem that exhibits severe contractual incompleteness. To model such matching, Sørensen relies on the standard Gale and Shapley (1964) model of many-to-one matching. This model forces him to assume that there are no synergies

[14]Reference to coalition structures is in keeping with Hart and Moore's (1990) general setup; the subsequent literature focused on their example in which coalitions are effectively formed by two parties only.

among start-ups in a venture capital firm's portfolio; no synergies are allowed in both the venture capital payoff (complementarities) nor in the start-ups' payoffs (peer effects). Synergies among start-ups, if present, would bias upward his estimates of the synergy between venture capital and start-ups. The model proposed here may lend itself to a reassessment of synergies among start-ups and venture capital.

7.3 Sharing rules as an instrument of market design

The literature on the design of the matching markets that follows Gale and Shapley (1962) and Roth (1984) has been primarily concerned with algorithms used in the matching process. The algorithms were designed with a goal to make the resulting matching stable. The algorithm used in the centralized matching is the primary tool to achieve stability, but it is not the only one.

For instance, consider the often-studied environment in which stability matters: the matching between residents and US hospitals described by Roth (1984) and Roth and Peranson (1999). The matching is organized by the Association of American Medical Colleges, the Council on Medical Education of the American Medical Association, and the American Hospital Association, all playing the role of a social planner. The medical associations want the resulting matching to be stable because, historically, the lack of stability led to the unraveling of the resident-hospital matching process. The main instrument used by the associations to achieve stability is the matching algorithm. However, the associations also regulate the residency system in other ways that influence residents' and hospitals' payoffs and thus affect the stability of the matching (even if achieving stability is not the immediate reason the regulations are agreed on). A recent example is the 2003 regulation by the Accreditation Council for Graduate Medical Education that limits residents' working hours. It affected the agents' payoffs – the majority of residents surveyed by Niederee, Knudtson, Byrnes, Helmer, and Smith (2003), and Brunworth and Sindwani (2006) supported the restriction, while the majority of teaching hospitals' faculty opposed it.[15]

The regulations are like sharing rules, or preference domains, in that they influence payoffs but do not depend on the payoff-relevant factors – states of nature – that are ex

[15] A well publicized intervention in the residency process was the Bell Commission's restriction on the number of consecutive hours residents may work. The restriction was imposed after Libby Zion died in the New York Hospital while cared for by two sleep-deprived medical residents (cf. Robins (1995)). I am grateful to Alvin Roth for directing me to these examples.

ante unknown to the medical associations. While our model lacks the institutional detail to apply it to the medical matching market, its results may be viewed as a step towards understanding what tools – other than the matching algorithm – may be employed to achieve stability in market design.

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Appendix A. Proofs

Lemma 1. *Let $\succsim_A$ be a preference profile such that the coalition structure $\mu \neq \{A\}$ is stable and let $\succsim'_A$ be a preference profile such that*

$$C \succsim'_a C' \iff C \succsim_a C'$$

for $a \in C, C' \in \mathcal{C} - \{A\}$. Then, either $\{A\}$ or μ is stable with respect to $\succsim'_A$. Moreover, if μ is the unique $\succsim_A$-stable coalition structure and $C \succsim_a A$ for any $a \in C \in \mathcal{C}$, then there are no $\succsim'_A$-stable coalition structures other than μ and $\{A\}$.

Proof. Take any $\succsim_A$-stable coalition structure μ. Because $\succsim'_A$ is equivalent to $\succsim_A$ on $\mathcal{C} - \{A\}$, hence no coalition other than A can $\succsim'_A$-block μ. Thus, either μ is $\succsim'_A$-stable or is $\succsim'_A$-blocked by A. In the latter case,

$$A \succ'_a \mu(a) \text{ for } a \in A$$

where $\mu(a)$ denotes the coalition of agent a in coalition structure μ. Now, if $\{A\}$ were not $\succsim'_A$-stable then there would be a coalition $C \neq A$ such that

$$C \succ'_a A \text{ for } a \in C.$$

The two displayed preferences would then imply that $C \succ'_a \mu(a)$ and hence $C \succ_a \mu(a)$ for $a \in C$, contrary to stability of μ. Thus, if μ is not $\succsim'_A$-stable then A is. The uniqueness claim is straightforward. QED

Lemma 2. *If there are no n-cycles for any $n = 3, 4, \ldots$ then there is a stable coalition structure.*[16]

[16]The proof shows something more: if there are no n-cycles, $n = 3, 4, \ldots$, in which all agents in $C_k \cap C_{k+1}$

Proof. By Lemma 1, to show that lack of n-cycles for any $n = 3, 4, \ldots.$ implies that there is a stable coalition structure, it is enough to prove this claim under the additional assumption that either $A \notin \mathcal{C}$ or A is $\precsim_A$-worst choice for each agent. We will proceed by induction with respect to $|A|$. For $|A| = 1$ the claim is true. For the inductive step, assume that the claim is true whenever the number of agents is less than $|A|$. By way of contradiction, let us also assume that there is no stable coalition structure on A. Then, for any coalition $C \in \mathcal{C}$, there must exist a proper coalition C' that blocks C, that is $C \cap C' \neq \emptyset$ and all agents $a \in C \cap C'$ strictly prefer C' to C. Indeed, if there was a coalition C that is not blocked, then the following coalition structure would be stable: C and coalitions that form a stable structure on $A - C$. Hence every coalition can be blocked. Let us thus start with coalition C_1 and find a proper coalition C_2 that blocks C_1, and then a proper coalition C_3 that blocks C_2, etc. Since there is a finite number of coalitions in $\mathcal{C}$, and all are proper, there is an n-cycle. Moreover, n must be larger than two, as C_2 cannot be blocked by C_1. This contradiction completes the proof. QED

Lemma 3. *Let $\mathcal{C}$ satisfy C1 and C2, and $\mathbf{R}$ satisfy R1. If all profiles in $\mathbf{R}$ are pairwise-aligned then no profile in $\mathbf{R}$ admits a 3-cycle.*

Proof. By way of contradiction, assume that there are proper coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ and agents a_1, a_2, a_3 such that

$$C_{3,1} \prec_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C_{3,1}.$$

Condition C1 implies that at least two of the agents a_1, a_2, a_3 are workers, and then C2 implies the existence of an agent a_0 and proper coalitions $C'_{1,2}, C'_{2,3}, C'_{3,1}$ such that $C'_{k,k+1} \ni a_k, a_{k+1}$ and $a_0 \in C'_{1,2} \cap C'_{2,3} \cap C'_{3,1}$.

If $C'_{1,2} = C'_{2,3}$ then we can assume that $C'_{1,2} = C'_{2,3} = C'_{3,1} = C'$. We obtain a contradiction in the same way as in the analysis of the Kalai-Smorodinsky example

strictly prefer C_{k+1} to C_k then there is a stable coalition structure. Also, since lack of n-cycles implies lack of k-cycles for all $k \leq n$, Lemma 2 shows that if there are no n-cycles for odd integers $n \geq 3$ then there is a stable coalition structure. For the roommate problem, Lemma 2 follows from Chung's (2000) "no-odd-rings" condition; however his proof relies on the structure of the roommate problem. Lemma 2 also strengthens a result from Farrell and Scotchmer (1988): they assume an existence of a complete ordering on all coalitions that satisfy an equivalence counterpart of implication 3 from the proof of Lemma 5 below, and show that there exists a stable coalition structure. Banerjee, Konishi, and Sönmez (2001) relaxed the Farrell and Scotchmer's ordering condition to a requirement that among the coalitions formed by any subset of agents, there is a "top" coalition that is preferred by its members to all alternatives. A relaxed version of their top coalition property is true in our setting as shown in the proof of Proposition 5. Their result is logically independent of Lemma 2.

presented after the statement of Theorem 1.

If $C'_{1,2}, C'_{2,3}, C'_{3,1}$ are all different, then R1 implies that there is a pairwise-aligned profile $\precsim'_A$ such that

$$\begin{aligned} C_{3,1} &\precsim'_{a_1} C'_{1,2} \precsim'_{a_1} C_{1,2}, \\ C_{1,2} &\precsim'_{a_2} C'_{2,3} \precsim'_{a_2} C_{2,3}, \\ C_{2,3} &\precsim'_{a_3} C'_{3,1} \precsim'_{a_3} C_{3,1}, \end{aligned} \tag{2}$$

and all agents' $\precsim'_A$-preferences between pairs of coalitions not including $C'_{1,2}, C'_{2,3}, C'_{3,1}$ are the same as their $\precsim_A$-preferences. Pairwise alignment of $\precsim'_A$ gives $C'_{3,1} \precsim'_{a_1} C_{3,1}$, and thus

$$C'_{3,1} \precsim'_{a_1} C'_{1,2}.$$

Similarly

$$C'_{1,2} \precsim'_{a_2} C'_{2,3}, \quad \text{and} \quad C'_{2,3} \precsim'_{a_3} C'_{3,1}.$$

Because agent a_1 strictly prefers $C_{1,2}$ over $C_{3,1}$, at least one of the preference relations in (2) must be strict, and thus at least one preference relation above is strict. Hence, the pairwise alignment implies that

$$C'_{3,1} \precsim'_{a_0} C'_{1,2} \precsim'_{a_0} C_{2,3} \precsim'_{a_0} C'_{3,1},$$

with at least one preference relation strict, which is a contradiction. QED

Lemma 4. *Let $\mathcal{C}$ satisfy C1, and* $\mathbf{R}$ *satisfy R1, and $n \in \{3, 4, ...\}$. If no profile in* $\mathbf{R}$ *admits a* 3*-cycle, then no profile in* $\mathbf{R}$ *admits an n-cycle.*

Proof. For an inductive argument, fix $m \geq 4$ and assume that preferences in $\mathbf{R}$ do not admit n-cycles for $n = 3, ..., m-1$. We will argue that preferences in $\mathbf{R}$ do not admit m-cycles. By way of contradiction, assume that there is $\precsim_A \in \mathbf{R}$ that admits an m-cycle $C_{m,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} ... \precsim_{a_m} C_{m,1}$. Then a_1 or a_2 is a worker as otherwise C1 implies that $a_1 = a_2$ and

$$C_{m,1} \precsim_{a_1} C_{2,3} \precsim_{a_3} ... \precsim_{a_m} C_{m,1}$$

is an $(m-1)$-cycle contrary to the inductive assumption. By symmetry, we may assume that $a_1 \in W$. Assumption C1 then implies that there is C such that $a_1, a_3 \in C$. We will consider two cases.

Case $C = C_{i,i+1}$, for some $i = 1, ..., m$. Look at $C_{1,2}, C_{2,3}, C$ and conclude from the lack of 3-cycles that one of the following three subcases would obtain.

- $C_{1,2} \prec_{a_1} C = C_{i,i+1}$. Then $C_{i,i+1} \precsim_{a_{i+1}} C_{i+1,i+2} \precsim_{a_{i+2}} ... \precsim_{a_m} C_{m,1} \precsim_{a_1} C_{i,i+1}$, and the last preference is strict because $C_{m,1} \precsim_{a_1} C_{1,2} \prec_{a_1} C = C_{i,i+1}$. For the same reason, $i \neq 1, m$. Thus, there would be an $(m - i + 1)$-cycle, contrary to the inductive assumption.

- $C_{2,3} \succ_{a_3} C = C_{i,i+1}$. Then $C_{i,i+1} \precsim_{a_3} C_{3,4} \precsim_{a_4} ... \precsim_{a_i} C_{i,i+1}$ and the first preference is strict because $C \prec_{a_3} C_{2,3} \precsim_{a_3} C_{3,4}$. For the same reason, $i \neq 2, 3$. Thus, there would be an n-cycle with $n = i - 2 \bmod m$, contrary to the inductive assumption.

- $C \sim_{a_1} C_{1,2} \sim_{a_2} C_{2,3} \sim_{a_3} C$. Then $C \precsim_{a_3} C_{3,4} \precsim_{a_4} ... \precsim_{a_m} C_{m,1} \precsim_{a_1} C$ with at least one strict preference inherited from the m-cycle $C_{m,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} ... \precsim_{a_m} C_{m,1}$. Thus, there would be an $(m - 1)$-cycle, contrary to the inductive assumption.

Case $C \neq C_{i,i+1}$ for all $i = 1, ..., m$. By R1 there is a pairwise-aligned preference profile $\precsim'_A \in \mathbf{R}$ such that all preferences along the m-cycle are preserved and

$$C_{m,1} \precsim'_{a_1} C \precsim'_{a_1} C_{1,2}.$$

We cannot have $C \prec'_{a_3} C_{2,3}$ for then $C \prec'_{a_3} C_{2,3} \precsim'_{a_3} C_{3,4}$ and hence $C \prec'_{a_3} C_{3,4} \precsim'_{a_4} ... \precsim'_{a_m} C_{m,1} \precsim'_{a_1} C$ would be an $(m-1)$-cycle. Thus, $C \succsim'_{a_3} C_{2,3}$, and

$$C \precsim'_{a_1} C_{1,2} \precsim'_{a_2} C_{2,3} \precsim'_{a_3} C.$$

The lack of 3-cycles implies that all agents above are indifferent. But then $C \precsim'_{a_3} C_{3,4} \precsim'_{a_4} ... \precsim'_{a_m} C_{m,1} \precsim'_{a_1} C$ would be an $(m-1)$-cycle with at least one strict preference inherited from the m-cycle $C_{m,1} \precsim_{a_1} C_{1,2} \precsim_{a_2} ... \precsim_{a_m} C_{m,1}$. This contradiction completes the proof. QED

Lemma 5. *If preference profile $\precsim_A$ has no n-cycles for $n = 2, 3, ...,$ and agents' preferences are strict, then there is at most one stable coalition structure different than $\{A\}$.*

Proof. Let us define a partial ordering $\trianglelefteq$ on proper coalitions as follows: $C \trianglelefteq C'$ iff there exists a sequence of proper coalitions $C_{i,i+1} \in \mathcal{C}$ such that $C = C_{1,2}$, $C' = C_{m,m+1}$, and for each $i = 2, ..., m$ there is an agent $a_i \in C_{i-1,i} \cap C_{i,i+1}$ that weakly prefers $C_{i,i+1}$ to $C_{i-1,i}$. The relation is strict if for at least one $i = 2, ..., m$ all agents $a \in C_{i-1,i} \cap C_{i,+1}$

strictly prefer $C_{i,i+1}$ to $C_{i-1,i}$. Notice that all proper coalitions C, C' with a non-empty intersection are comparable, and

$$\text{if } C \trianglelefteq C' \text{ then } C \precsim_a C' \text{ for all } a \in C \cap C'. \tag{3}$$

The relation is transitive by construction. It is also acyclic in the sense of preference theory: given transitivity, a relation is acyclic if there are no coalitions C and C' such that $C \trianglelefteq C'$ and $C \triangleright C'$.

To prove the uniqueness claim, first consider the case $A \notin \mathcal{C}$. Let $C_1, C_2, ..., C_k$ be maximal coalitions in ordering $\trianglelefteq$. By lack of 2-cycles and preference strictness, the coalitions $C_1, ..., C_k$ are disjoint. By (3) and strictness of preferences, the maximal coalitions must be a part of every stable coalition structure, and by induction there is a unique stable coalition structure. Lemma 1 completes the argument for the case $A \in \mathcal{C}$. QED

Proof of Theorem *1*. The existence claim follows from Lemmas 3, 4, and 2. The uniqueness claim follows from Lemma 5. QED

Lemma 6. *Let* $\mathbf{R}$ *satisfy R2. If all profiles* $\precsim_A \in \mathbf{R}$ *admit a stable coalition structure, then there is no* $\precsim_A \in \mathbf{R}$ *and* 3*-cycle* $C_{3,1} \prec_{a_1} C_{1,2} \precsim_{a_2} C_{2,3} \precsim_{a_3} C_{3,1}$ *such that* $C_{i-1,i} \cap C_{i,i+1} = \{a_i\}$ *for* $i = 1, 2, 3$.

Proof. By way of contradiction, assume that there exist $\precsim_A \in \mathbf{R}$ and 3-cycle of coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ prohibited by the lemma. Notice that coalitions $C_{i,i+1}$ are all different because if $C_{i-1,i} = C_{i,i+1}$ then $C_{i-1,i} = C_{i,i+1} = \{a_i\}$, and hence $a_1 = a_2 = a_3$, and this agent's preferences would be cyclic. We will modify the preference profile and construct a profile in $\mathbf{R}$ that does not admit a stable coalition structure. At each step of the procedure let us continue to denote the current profile by $\precsim_A$.

First, use R2(ii) with $C = C_{1,2}$ to find a preference profile $\precsim_A \in \mathbf{R}$ such that $C_{3,1} \prec_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \precsim_{a_3} C_{3,1}$. Then, use R2(ii) with $C = C_{2,3}$ to find $\precsim_A$ such that $C_{3,1} \prec_{a_1} C_{1,2}$, $C_{1,2} \prec_{a_2} C_{2,3}$, and $C_{2,3} \prec_{a_3} C_{3,1}$.

Last, one-by-one, for all coalitions C that (i) contain an agent from $C_{1,2} \cup C_{2,3} \cup C_{3,1}$, and (ii) are different than $C_{1,2}, C_{2,3}, C_{3,1}$, use R2(i) to find $\precsim_A \in \mathbf{R}$ such that $C \prec_a C_{k,k+1}$ for $a \in C \cap C_{k,k+1}$, $k = 1, ..., 3$.

The resulting profile of preferences belongs to $\mathbf{R}$ and does not admit a stable coalition structure. This completes the proof. QED

Lemma 7. *Assume that* $\mathbf{R}$ *satisfies R1 and no profile in* $\mathbf{R}$ *admits a 3-cycle prohibited by Lemma 6. If agents* a, b, c *and coalitions* C, C', C_a, C_b *are such that*

$$C_a \cap C = C_a \cap C' = \{a\}\,,$$
$$C_b \cap C = C_b \cap C' = \{b\}\,,$$
$$C_a \cap C_b = \{c\}$$

then $C \precsim_a C' \Longrightarrow C \precsim_b C'$ *for all* $\precsim_A \in \mathbf{R}$.

Proof. By way of contradiction, assume that

$$C \precsim_a C' \text{ and } C' \prec_b C.$$

Because $C_a \neq C_b$, the condition R1 implies that there is $\precsim_A \in \mathbf{R}$ (we will continue using the symbol $\precsim_A$ for the new profile) such that

$$C \precsim_a C_a \precsim_a C' \text{ and } C' \precsim_b C_b \precsim_b C,$$

and preferences between coalitions other than C_a, C_b are unchanged. Notice that at least one above preference of b is strict. Assume that

$$C' \prec_b C_b;$$

the argument in the other case is symmetric. Since C, C_b, C_a cannot form a prohibited 3-cycle we have

$$C_b \precsim_c C_a.$$

Then, however, the coalitions C', C_b, C_a form a prohibited 3-cycle; a contradiction that proves the claim. QED

Lemma 8. *Let* $\mathcal{C}$ *satisfy C1 and C3, and* $\mathbf{R}$ *satisfy R1. If no profile in* $\mathbf{R}$ *admits a 3-cycle prohibited by Lemma 6, then all profiles in* $\mathbf{R}$ *are pairwise aligned.*

Proof. Take any proper coalitions C, C' such that $C \precsim_a C'$. We are to show the following claim: $C \precsim_b C'$ for any $b \in C \cap C'$. In particular, we may assume that $b \neq a$ and $C' \precsim_b C$.

Step 1. We will prove the claim assuming that $a, b \in W$ and $|F| \leq 1$. Consider two cases:

(i) Case $W \nsubseteq C \cup C'$. Take $c \in W - (C \cup C')$. By C1 and C3(i), $C_a = \{a, c\}$ and $C_b = \{b, c\}$ are coalitions. The assumptions of Lemma 7 are satisfied, and our claim follows.

(ii) Case $W \subseteq C \cup C'$. By C1 and C3(i), $\{a, b\}$ is a coalition. Because of R1, we can assume that

$$C \precsim_a \{a, b\} \precsim_a C'.$$

By C3(ii), $W \nsubseteq C \cup \{a, b\}$, and thus the first preference above and case (i) imply that $C \precsim_b \{a, b\}$. Similarly, $\{a, b\} \precsim_b C'$. By transitivity $C \precsim_b C'$.

Step 2. We will prove the claim assuming that a or b is in F. The argument resembles Step 1.

(i) Case $W \nsubseteq C \cup C'$. Assumption C1 implies that one of the agents a, b is a worker. Assume that $a \in F, b \in W$; the argument in the other case is similar. Take $c \in W - (C \cup C')$. By C1 and C3(i), $C_a = \{a, c\}$ is a coalition and either $C_b = \{b, c\}$ is a coalition or there is $f \in F - \{a\}$ such that $C_b = \{b, c, f\}$ is a coalition. In both cases the assumptions of Lemma 7 are satisfied, and our claim follows.

(ii) Case $W \subseteq C \cup C'$. The argument follows the argument of Step 1(ii) word-by-word.

Step 3. We will prove the claim assuming that $a, b \in W$ and $|F| \geq 2$. Consider two cases.

(i) Case $F \nsubseteq C \cup C'$. Take $c \in F - (C \cup C')$. By C1 and C3(i), $C_a = \{a, c\}$ and $C_b = \{b, c\}$ are coalitions, and our claim follows from Lemma 7.

(ii) Case $F \subseteq C \cup C'$. By C1, there is $c \in F - C'$, and then C1 and C3(i) imply that $\{a, c\}$ and $\{b, c\}$ are coalitions. If $c \notin C$ then Lemma 7 concludes the argument. Consider $c \in C$. By R1 we can assume that

$$C \precsim_a \{a, c\} \precsim_a C' \text{ and } C' \precsim_b \{b, c\} \precsim_b C.$$

Step 2 applied to $\{b, c\} \precsim_b C$ gives $\{b, c\} \precsim_c C$, and similarly we derive $C \precsim_c \{a, c\}$. By transitivity,

$$\{b, c\} \precsim_c \{a, c\}.$$

Since we also know that $\{a,c\} \precsim_a C' \precsim_b \{b,c\}$, the lack of prohibited 3-cycles gives

$$C' \sim_a \{a,c\} \sim_c \{b,c\} \sim_b C'.$$

Putting together what we have shown above about the preferences of c we see that $C \precsim_c \{a,c\} \sim_c \{b,c\} \precsim_c C$, and thus, $\{b,c\} \sim_c C$. Step 2 then implies that $\{b,c\} \sim_b C$. This indifference, and the above-displayed indifference of b imply that $C \sim_b C'$. This ends the proof of the lemma because Steps 1-3 cover all possible situations. QED

Proof of Theorem 2. The theorem follows from Lemmas 6 and 8. QED

Proof of Proposition 1. The assumptions on U_a guarantee that the maximization problem $\max_{(s_a)_{a\in C}} \log\left(\prod_{a\in C} U_a(s_a)\right)$ is concave and has an interior solution. The resulting sharing rule is pairwise aligned and efficient. The remaining implication is the non-trivial part of the proposition. Let us thus assume that a sharing rule D is pairwise aligned and efficient, and construct the Nash-like representation. We will also assume that $W \neq \emptyset$ as otherwise C1 would imply that only singleton-sets are coalitions, and the efficiency of D would imply that any profile of U_a represents D.

For each proper coalition C and agents $a, b \in C$ let us define the function $t_{b,a} : R_+ \to R_+$ by

$$t_{b,a}(D_{a,C}(y)) = D_{b,C}(y), \ y \geq 0.$$

The function maps the share agent a obtains when C produces y into the share agent b obtains at the same output level. The function is well-defined because $D_{a,C}$ is onto R_+ and the strict monotonicity of $D_{a,C}$ guarantees that $D_{a,C}(y) = D_{a,C}(y')$ implies $y = y'$, and hence $D_{b,C}(y) = D_{b,C}(y')$. Moreover, function $t_{b,a}$ is strictly increasing and continuous (by monotonicity and continuity of $D_{a,C}$ and $D_{b,C}$). Finally, function $t_{b,a}$ does not depend on C because $D_{a,C}(y) = D_{a,C'}(y')$ and the pairwise alignment of D imply that $D_{b,C}(y) = D_{b,C'}(y')$.

We will now construct $W_a = \log U_a$, $a \in A$, in three steps. Let w^* be an arbitrary reference worker. Notice that the function $t_{w^*,b}$ is defined for all agents b, that $t_{w^*,b}$ is invertible, and the function $f : (0,\infty) \to (0,\infty)$ given by

$$f(t) = \min_{b\in A} \left[(t_{w^*,b})^{-1}(t)\right]^{-\frac{1}{2}}, \ t > 0,$$

is continuous and strictly decreasing (because $t_{w^*,b}$ are continuous and strictly increasing), and $f(s) \to +\infty$ as $s \to 0+$ (because the inverse function $t^{-1}_{w^*,b}(s) \to 0$ as $s \to 0$). Thus, the auxiliary functions functions $\psi_a : (0,\infty) \to (0,\infty)$ given by

$$\psi_a = f \circ t_{w^*,a}.$$

are positive, continuous, strictly decreasing, and $\psi_a(s) \to +\infty$ as $s \to 0+$. Functions ψ_a are integrable at 0 because they are positive and bounded above by $(0,\infty) \ni s \to s^{-\frac{1}{2}}$. Define

$$W_a(s) = \int_0^s \psi_a(\tau)\, d\tau,$$

and observe that W_a are strictly increasing, strictly concave, differentiable, and $W'_a(0) = \lim_{s\to 0} \psi_a(s) = +\infty$. Thus, $U_a = \exp \circ W_a$ are strictly increasing, strictly log-concave, differentiable, and $\frac{U_a}{U'_a}(0) = \frac{1}{W'_a}(0) = 0$.

It remains to take an arbitrary coalition C and show that $(D_{a,C}(y))_{a\in C}$ is equal to the solution of the following maximization problem,

$$\arg \max_{\sum_{a\in C} s_a \leq y} \sum_{a\in C} W_a(s_a).$$

The maximization problem is concave and hence has a solution. Furthermore, $W'_a(0) = +\infty$ guarantees that the solution is internal and satisfies the first order Lagrange condition $\psi_a(\tilde{s}_a) = \lambda$. The first order condition can be rewritten as $t_{w^*,a}(\tilde{s}_a) = f^{-1}(\lambda)$, or

$$\tilde{s}_a = t_{a,w^*}\left(f^{-1}(\lambda)\right).$$

Since t_{a,w^*} is strictly monotonic, $\tilde{s}_a$ is uniquely determined by this equation and the tight feasibility constraint

$$\sum_{a\in C} \tilde{s}_a = y$$

(the constraint is tight because W_a are strictly increasing).

If C does not contain any workers, then C1 implies that C is a singleton coalition, and the Pareto efficiency of D is enough to yield the claim. Otherwise, fix a worker $w \in C$ and notice that for agents $a \in C$, we have $D_{a,C} = t_{a,w} \circ D_{w,C}$. Since C1-C2 and R1 are satisfied, Lemma 3 implies that $t_{a,w^*} \circ t_{w^*,w} = t_{a,w}$, and hence,

$$D_{a,C}(y) = t_{a,w^*}\left(t_{w^*,w} \circ D_{w,C}(y)\right) = t_{a,w^*}(x)$$

for $x = t_{w^*,w} \circ D_{w,C}(y)$. Notice that x does not depend on a, and can be written as $f^{-1}(\lambda)$ for some λ, and thus the system of equations for $D_{a,C}(y)$ is identical to the analogous equations for $\tilde{s}_a$ above, and that the Pareto efficiency of $D_{a,C}(y)|_{a\in C}$ is identical to the tight feasibility constraint on $\tilde{s}_a$. The uniqueness of solutions to these equations means that $\tilde{s}_a = D_{a,C}(y)$, what was to be proved. QED

Proof of Proposition 3. The representation of Corollary 2 is applicable because $\mathcal{C}$ satisfies C1-C3. Hence all regular, efficient, and stability-inducing sharing rules may be represented by a profile of agents' bargaining functions, U_a. Let d_a be the inverse function of $\frac{U_a'}{U_a}$. Let us refer to functions d_a as *demand* functions because $d_a(p)$ may be interpreted as the demand of an agent with utility $\log U_a$ who faces price p per unit of output. Notice that agent a's share s_a of output y in coalition $C \ni a$ satisfies $s_a = d_a(p)$ where p is determined by the market-clearing condition $\sum_{a\in C} d_a(p) = y$. Agents prefer coalitions with lower market-clearing price p, and the sharing rule is fully characterized by the profile of demands $(d_a)_{a\in A}$. For instance, if $U_a(s) = s^{\lambda_a}$ then $d_a(p) = \frac{\lambda_a}{p}$ and the market clearing price in coalition of such agents is $p = \frac{1}{y}\sum_{a\in C}\lambda_a$.

Notice that every demand function $d_a : R_+ \to R_+$ is a decreasing bijection, and any decreasing bijection d_a corresponds to a bargaining function $U_a(s) = \exp \circ \int_1^s d_a^{-1}(t)\,dt$ that is strictly increasing, differentiable, log-concave, and such that $\frac{U_a}{U_a'}(0) = 0$. Consequently, two demand profiles $(d_a)_{a\in A}$ and $\left(\tilde{d}_a\right)_{a\in A}$ represent the same sharing rule if, and only if, there is an increasing bijection $T : R_+ \to R_+$ such that $\tilde{d}_a = d_a \circ T$ for every $a \in A$. We can thus normalize the demand function by assuming that $\sum_{a\in A} d_a(p) = \frac{1}{p}$. In this way we obtain a one-to-one correspondence between sharing rules and demand profiles. Let us partition the sharing rules into subclasses such that normalized demands d_b, $b \in F \cup (W - L)$, and the sum $\sum_{a\in L} d_a$ are the same for each rule in the subclass. It remains to prove that among rules in an element of the partition, the unique rule in which $d_w = d_{w'}$ for workers $w, w' \in L$ maximizes the probability that L belongs to a stable coalition structure.

We may restrict attention to output profiles that result in strict preferences of agents among subcoalitions of $A - L$, as this event happens with probability 1. Furthermore, we may prove the claim conditional on a fixed profile of outputs for L and coalitions of agents in $A - L$. Because of strict preferences and the regularity of the sharing rule, Theorem 1 implies that there is a unique stable coalition structure on $A - L$. Let $\left\{C^1, ..., C^k\right\}$ be this stable coalition structure. Hence, whenever L belongs to a stable coalition structure, the stable structure is $\{L, C^1, ..., C^k\}$. We will refer to this event as L being stable.

We will start with any sharing rule $(d_a)_{a\in A}$, select a pair of workers $w, w' \in L \cap W$, and adjust the sharing rule so that both workers w, w' are endowed with demand functions $\frac{d_w + d_{w'}}{2}$. The resulting demand-function representation is normalized and the resultant sharing rule belongs to the same element of the partition as $(d_a)_{a\in A}$. Let us check that the probability of L being stable is larger for the adjusted sharing rule (and strictly larger if the distributions are strictly log-concave). The independence of output distributions implies that the probability that L is stable equals to the product of probabilities that L is not blocked by coalitions $C \neq L$ such that $C \cap L \neq \emptyset$. The adjustment does not change the probability that L is blocked by C if C contains both or neither of workers w, w' because the market-clearing prices of L, coalitions disjoint with L, and C, stay the same in every profile of outputs. We will show that the adjustment increases the joint probability that L is not blocked by coalitions C_w or $C_{w'}$ such that $w \in C_w$ and $w' \notin C_w$ and $C_{w'} = C_w \cup \{w'\} - \{w\}$; by symmetry either both sets $C_w, C_{w'}$ are coalitions or none is. Let us denote the set $C_w - \{w\} = C_{w'} - \{w'\}$ by S (this set is not necessarily a coalition), and let p^* be the minimum among market-clearing prices in coalition L and these among coalitions C^i, $i = 1, ..., k$, that have a non-empty intersection with set S. Coalitions C_a, C_b do not block $\{L, C^1, ..., C^k\}$ precisely when their market clearing prices are above p^*. Denote by F_m the c.d.f. of the probability distribution from which outputs of a coalition of size m are drawn. The probability that before adjustment L is not blocked by C_w and $C_{w'}$ equals

$$F_{|C_w|}\left(y \leq d_w(p^*) + \sum_{a\in S} d_a(p^*)\right) F_{|C_{w'}|}\left(y \leq d_{w'}(p^*) + \sum_{a\in S} d_a(p^*)\right),$$

and by concavity of $\log \circ F_{|C_w|} = \log \circ F_{|C_{w'}|}$ the probability after adjustment

$$F_{|C_w|}\left(y \leq \frac{d_w + d_{w'}}{2}(p^*) + \sum_{a\in S} d_a(p^*)\right) F_{|C_{w'}|}\left(y \leq \frac{d_w + d_{w'}}{2}(p^*) + \sum_{a\in S} d_a(p^*)\right)$$

is higher.

As we repeat the above procedure in such a way that every possible pair of workers a, b is selected infinitely many times, the demand function of each worker from L converges point-wise to $\frac{1}{|L\cap W|}\sum_{w\in L\cap W} d_w$ (the demand functions of other agents do not change). The probability that L is stable is higher for the sharing rule that is the point-wise limit of the procedure than for the original sharing rule $(d_a)_{a\in A}$. This is so because the distributions of outputs are absolutely continuous, demands are decreasing, and hence the probability of L being stable is continuous in the point-wise metric on the demand profiles $(d_a)_{a\in A}$. QED

Proof of Proposition 4. First consider the case $L = A$. There are two places in the sketch of the proof in the main text that require a supporting argument. First consider coalition C that contains a but not b. We need to show that the probability that A is not blocked by one (or both) of the coalitions $C_a = C$ and $C_b = C \cup \{b\} - \{a\}$ is decreasing when we replace λ_a with $\lambda_a + \epsilon$ and λ_b with $\lambda_b - \epsilon$ so that

$$\lambda_a < \lambda_a + \epsilon \leq \lambda_b - \epsilon < \lambda_b.$$

We will refer to such changes of bargaining powers as an $(\lambda_a, \lambda_b, \epsilon)$-adjustment. We will show that $(\lambda_a, \lambda_b, \epsilon)$-adjustment increases the probability that A is not blocked conditional on the output of A being equal to an arbitrary $y \geq 0$. The argument follows the same logic as an analogous argument in Proposition 3. Denote by F_k the c.d.f. of the probability distribution from which outputs of a coalition of size k are drawn. Conditional on the output of A being equal to y, and keeping the original bargaining powers, the grand coalition is not blocked by either C_a and C_b if

$$\frac{y}{\sum_{i\in A}\lambda_i} \geq \frac{\mathbf{y}\left(C_a\right)}{\sum_{i\in C_a}\lambda_i} \text{ and } \frac{y}{\sum_{i\in A}\lambda_i} \geq \frac{\mathbf{y}\left(C_b\right)}{\sum_{i\in C_b}\lambda_i}.$$

By independence, the conditional probability of these two inequalities is equal to the product

$$F_{|C_a|}\left(\frac{\sum_{i\in C_a}\lambda_i}{\sum_{i\in A}\lambda_i}y\right) F_{|C_b|}\left(\frac{\sum_{i\in C_b}\lambda_i}{\sum_{i\in A}\lambda_i}y\right).$$

Similarly, the conditional probability that A is not blocked after we adjustλ_a and λ_b equals

$$F_{|C_a|}\left(\frac{\epsilon + \sum_{i\in C_a}\lambda_i}{\sum_{i\in A}\lambda_i}v\right) F_{|C_b|}\left(\frac{-\epsilon + \sum_{i\in C_b}\lambda_i}{\sum_{i\in A}\lambda_i}v\right),$$

and is higher than the probability before adjustment because of the assumption that $\log \circ F_{|C_a|} = \log \circ F_{|C_b|}$ is concave. The probability increase is strictly positive if $\log \circ F_{|C_a|} = \log \circ F_{|C_b|}$ is strictly concave.

The remaining supporting argument is given by the following

Claim. If $\sum_{a\in A}\lambda_a = \sum_{a\in A}\lambda'_a$ and $(\lambda_a)_{a\in A} > (\lambda'_a)_{a\in A}$ then there exists a finite sequence of $(\lambda_a, \lambda_b, \epsilon)$-adjustments that transforms $(\lambda'_a)_{a\in A}$ into $(\lambda_a)_{a\in A}$ (that is there are bargaining power profiles $\lambda^k \in R^A_+$, $k = 1, ..., n \geq 2$ such that $\lambda^1 = \lambda'$, $\lambda^n = \lambda$, and λ^{k+1} equals λ^k except for two coordinates $a, b \in C$ such that $\lambda^k_a < \lambda^{k+1}_a \leq \lambda^{k+1}_b < \lambda^k_b$, and $\lambda^{k+1}_a - \lambda^k_a = \lambda^k_b - \lambda^{k+1}_b$).

Proof. We may assume that $\lambda'_1 < ... < \lambda'_n$ because the ordering between λ' and λ is independent of permutation (or, renaming) of agents in A. Then also $\lambda_1 < ... < \lambda_n$ because $\lambda > \lambda'$. Notice that $\lambda_1 > \lambda'_1$ as otherwise $\lambda_i \leq \lambda'_i$ for all $i = 1, ..., n$ with some inequalities strict, contrary to λ and λ' having the same sum of coordinates. Similarly, $\lambda_n < \lambda'_n$.

Let $\lambda^1 = \lambda'$ and define a to be the maximal subscript such that $\lambda^1_a = \lambda^1_1$ and b be the minimal subscript such that $\lambda^1_b = \lambda^1_n$. Let $\epsilon = \min\left(\lambda^1_b - \lambda^1_{b-1}, \lambda^1_b - \lambda_1, \lambda^1_{a+1} - \lambda^1_a, \lambda_1 - \lambda^1_a\right)$. Let λ^2 be given by the $(\lambda^1_a, \lambda^1_b, \epsilon)$-adjustment of λ^1 (in words, the adjustment is raising the a-coordinate and lowering the b-coordinate as long as λ^2_a is weakly lower than $\lambda^1_{a+1}, \lambda_n$, and λ^2_b weakly higher than $\lambda^1_{a+1}$and λ_1). Notice that

$$\lambda^1 < \lambda^2 \leq \lambda,$$

and

$$\sum_{i=1,\dots,n} \left|\lambda_i - \lambda^1_i\right| > \sum_{i=1,\dots,n} \left|\lambda_i - \lambda^2_i\right|.$$

If $\lambda^2 \neq \lambda$ then we construct λ^3 via the same procedure with λ^2 substituted for λ^1. The analogues of the above-displayed relations continue to hold. Since all λ^k_a, λ^k_b come from a finite grid, the iterations terminate with some $\lambda^k = \lambda$. This ends the proof of the claim.

Finally, consider the case $L \neq A$. Again, we may assume that $\sum_{a\in L}\lambda_a = \sum_{a\in L}\lambda'_a$. The argument that ϵ-adjustments increase the probability of L being stable follows the same lines as an analogous argument in the proof of Proposition 3. The analogue of the above claim on sequences of ϵ-adjustments remains true, and concludes the proof.[17] QED

[17] We can define a similar partial ordering on all regular, efficient, and stability inducing sharing rules. Take, for instance, $L = A$ and represent the sharing rule in the demand form of the proof of Proposition 3. The ordering is then such that $(d_a)_{a\in A}$ dominates $(d'_a)_{a\in A}$ if $\frac{d_{(i)}(p)}{d_{(i+1)}(p)} \geq \frac{d'_{(i)}(p)}{d'_{(i+1)}(p)}, i = 1, ..., |A|$; where the order statistics $d_{(i)}, d'_{(i)}$, are taken point-wise (independently for every price p). The comparison of probabilities of A being stable under the two sharing rules would follow the construction from Proposition 4 with the proviso that (d_a, d_b, ϵ)-adjustments would be defined for functions $\epsilon : R_+ \to R_+$ that preserve monotonicity of $d_a + \epsilon$ and $d_b - \epsilon$.

Proof of Proposition 5. Fix agents' preference profile and a stable coalition structure $\{C_1, ..., C_k\}$. To show that the coalition structure is implementable as a Strong Nash Equilibrium consider first the case $C_1 = A$. By (ii) there is a profile of strategies of agents σ such that $A \in \mu(\sigma)$; this profile must be a Strong Nash Equilibrium. If $C_1 \neq A$ then all coalitions C_i are proper. By Lemmas 3 and 4, there are no n-cycles for $n = 2, 3, ...$ Notice that this implies that at least one of the coalitions $C_1, ..., C_k$ is weakly preferred by its members to all other proper coalitions. Indeed, if not then let C_1' be a proper coalition that is strictly preferred to C_1 by an agent a_1 from $C_1 \cap C_1'$. Since $\{C_1, ..., C_k\}$ is stable, there must be a coalition C_i and agent a_1' that weakly prefers C_i to C_1'. We could repeat the procedure, and define C_2' to be a proper coalition that is strictly preferred to C_i by at least one agent $a_2 \in C_i \cap C_2'$. Since there are a finite number of coalitions, in this way we would eventually construct an n-cycle. The contradiction proves that there is a coalition C_{i_1} that is weakly preferred by all its members to any other proper coalition. We can recursively re-index coalitions $C_{i_1}, C_{i_2}, ..., C_{i_k}$ so that C_{i_j} is weakly preferred by all its members to any proper coalition of agents in $A - \left(C_{i_1} \cup ... \cup C_{i_{j-1}}\right)$. By (ii), there is a profile σ_{C_i} of strategies of agents from C_i that leads to formation of C_i. Profiles σ_{C_i} put together form a Strong Nash Equilibrium. QED

Appendix B. Material for Online Appendix

Properties of profiles in rich pairwise-aligned domains

Proposition 6. *If the family of coalitions satisfies C1 and C2, then a preference profile $\precsim_A$ belongs to a rich domain of pairwise-aligned profiles iff $\precsim_A$ admits no n-cycles, $n = 2, 3, ...$*

Proof. One implication follows from Lemmas 3, 4 of the proof of Theorem 1. To prove the remaining implication, we will show that the domain of all preference profiles that do not admit n-cycles, $n = 2, 3, ...$ satisfies R1 and R2. Fix a profile $\precsim_A$ that does not admit n-cycles, $n = 2, 3, ...$

To prove that R1 is satisfied, we will use an extension of the partial ordering $\trianglelefteq$ constructed in the proof of Lemma 5 (the construction did not rely on the strictness of preferences assumed in Lemma 5). We will recursively construct the extension so that all proper coalitions are comparable while maintaining transitivity and acyclicity. Let relation $\trianglelefteq^k \subset \mathcal{C} \times \mathcal{C}$ be a transitive and acyclic extension of $\trianglelefteq$. Take any C and C' that are not comparable under $\trianglelefteq^k$ (if there are no such coalitions then the extension is complete). Let $\trianglelefteq^{k+1} \subset \mathcal{C} \times \mathcal{C}$ be the smallest transitive extension of $\trianglelefteq^k \cup \{(C, C')\}$. Relation $\trianglelefteq^{k+1}$ is transitive by definition and it is acyclic as otherwise either $\trianglelefteq^k$ would violate acyclicity or C and C' would be comparable under $\trianglelefteq^k$. Since there is a finite number of coalitions this process will terminate producing the postulated extension of $\trianglelefteq$. The extension satisfies relation (3) from the proof of Lemma 5 because all pairs of proper coalitions C and C' with non-empty intersection are comparable under $\trianglelefteq$.[18] In the remainder of the proof let us refer to the extension as $\trianglelefteq$.

To prove R1, we will take three different coalitions C_0, C, C_1 and agent a such that $C_0 \precsim_a C_1$, and construct a preference profile $\precsim'_A$ whose existence is postulated in R1. At least one of the coalitions C_0, C_1 is proper, and because of symmetry we may assume that $C_1 \neq A$. Define the preference profile $\precsim'_A$ so that it coincides with $\precsim_A$ for pairs of coalitions $C', C'' \in \mathcal{C} - \{C\}$, and for coalitions $C' \in \mathcal{C} - \{C\}$ and $a \in C' \cap C$ let $C' \prec_a C$ if $C' = A$, and otherwise let

$$
\begin{aligned}
C \prec'_a C' \quad & \text{if} \quad C_1 \lhd C', \\
C \succ'_a C' \quad & \text{if} \quad C_1 \rhd C', \\
C \sim'_a C' \quad & \quad\quad \text{otherwise.}
\end{aligned}
$$

The profile $\precsim'_A$ proves R1 because $C_0 \precsim'_a C \sim'_a C_1$ and $\precsim'_A$ does not admit n-cycles. The latter claim is true because if there was an n-cycle $C_{m,1} \prec'_{a_1} \dots \precsim'_{a_{m-1}} C_{m-1,m} \precsim'_{a_m} C_{m,1}$ then $C'_{m,1} \lhd \dots \trianglelefteq C'_{m-1,m} \trianglelefteq C'_{m,1}$ for coalitions $C'_{i,i+1} = C_{i,i+1}$ when $C_{i,i+1} \neq C$ and $C'_{i,i+1} = C_1$ when $C_{i,i+1} = C$. This is however impossible as $\trianglelefteq$ is acyclic.

To prove R2(i), take two different coalitions C, C_1 and define the preference profile $\precsim'_A$ so that it coincides with $\precsim_A$ for pairs of coalitions $C', C'' \in \mathcal{C} - \{C\}$, and $C \prec'_a C'$ whenever $C' \in \mathcal{C} - \{C\}$ and $a \in C' \cap C$. If $\precsim'_A$ admitted an n-cycle $C_{m,1} \prec'_{a_1} \dots \precsim'_{a_{m-1}} C_{m-1,m} \precsim'_{a_m} C_{m,1}$ then one of the coalitions $C_{k,k+1| \bmod m}$, $k = 1, \dots, m$ would need to be C (since $\precsim_A$ does not admit n-cycles), but C is not weakly preferred to any coalition.

To prove R2(ii), take three different coalitions C_0, C, C_1 and agents a, b such that $C_0 \prec_a C \sim_b C_1$, and define the preference profile $\precsim'_A$ so that it coincides with $\precsim_A$ for pairs of coalitions $C', C'' \in \mathcal{C} - \{C\}$, and for $C' \in \mathcal{C} - \{C\}$ and $a \in C' \cap C$ let $C \prec'_a C'$ if $C \precsim_a C'$ and $C \succ'_a C'$ otherwise. If $\precsim'_A$ admitted an n-cycle $C_{m,1} \prec'_{a_1} \dots \precsim'_{a_{m-1}} C_{m-1,m} \precsim'_{a_m} C_{m,1}$

[18] The resultant ordering may be seen as a weak counterpart of the ordering constructed by Farrell and Scotchmer (1988) for equal sharing.

then we would need to have $C = C_{m,1}$ (as otherwise the coalitions would form an n-cycle under $\precsim_A$), and hence $C_{m,1} \prec'_{a_1} ... \precsim'_{a_{m-1}} C_{m-1,m} \prec'_{a_m} C_{m,1}$ (as no agent is $\precsim'_A$ -indifferent between C and another coalition). Then, however, $C_{m,1} \precsim_{a_1} ... \precsim_{a_{m-1}} C_{m-1,m} \prec_{a_m} C_{m,1}$ would be an n-cycle, a contradiction. QED

The above argument shows, in particular, that a preference profile $\precsim_A$ belongs to an R1-rich domain of pairwise-aligned profiles iff $\precsim_A$ admits no n-cycles, $n = 2, 3, ...$ The argument may be adapted to show that the domain of all strict preference profiles that do not admit n-cycles is also rich.

Relaxing assumptions on the family of coalitions $\mathcal{C}$ and the roommate problem

We may relax C2 and C3 at the cost of replacing pairwise alignment by less transparent conditions. For instance, for families of coalitions satisfying C1 and preference profile domains satisfying R1, if no profile admits a 3-cycle, then all profiles admit a stable coalition structure (by Lemmas 2 and 4). For the roommate problem, assumption R2 and Lemma 6 provide the reverse implication, and we obtain:

Corollary 3. *Suppose that $\mathcal{C} = \{C \subseteq A, |C| \leq 2\}$ and that domain of preferences* $\mathbf{R}$ *satisfies R1 and R2. Then, the following three statements are equivalent:*
* *All profiles in* $\mathbf{R}$ *admit a stable matching.*
* *No profile in* $\mathbf{R}$ *admits a 3-cycle.*
* *No profile in* $\mathbf{R}$ *admits an n-cycle, $n = 3, 4, ...$*

Cardinal vs. ordinal formulation

The richness conditions R1 and R2 imposed on the domain of preferences capture the properties of monotone and surjective sharing rules (see Corollary 1) that are needed for the equivalence between pairwise alignment and stability. Richness is weaker than monotonicity and surjectivity. In fact, some R1 and R2 rich domains of ordinary profiles cannot be reinterpreted in terms of cardinal utilities. An example is given in the working paper draft, Pycia (2005).

Relaxing richness in matching

In many-to-one matching, one might expect that the division of output in coalitions of a firm and a worker is different than the division of output in larger coalitions if workers compete against each other in the internal bargaining. We can take this into account, by imposing an R1-like condition only on coalitions with two, or more, workers.

Proposition 7. *Assume that* $\mathbf{R}$ *is a domain of preferences in a many-to-one matching problem satisfying C2, and that for any* $\precsim_A \in \mathbf{R}$*, any agent* $a \in A$*, and coalitions* C, C' *such that* $a \in C, C'$ *and* $|C|, |C'| \geq 3$*, there exists a profile* $\precsim'_A \in \mathbf{R}$ *such that* $C \sim'_a C'$ *and all agents'* $\precsim'_A$*-preferences between pairs of coalitions not including* C *are the same as their* $\precsim_A$*-preferences. If all preference profiles in* $\mathbf{R}$ *are pairwise-aligned, then they admit stable coalition structures.*

The working paper draft, Pycia (2007), gives the proof. In particular, the proof shows that there is a stable matching if in any subset of agents either there is a coalition that is weakly preferred to by all its members to all other coalitions of agents in the subset, or there is a group G of one- and two-member coalitions that are weakly preferred by all their members to any coalition of agents in the subset that is not in G.

Other stability concepts

The existence results of this paper are not specific to core stability of Definition 1. The results hold true for other stability concepts such as pairwise stability and group stability in many-to-one matching. The working paper draft, Pycia (2007), gives details.