A Note on Unawareness and Zero Probability

Jing Li

Department of Economics
University of Pennsylvania
3718 Locust Walk
Philadelphia, PA 19104
E-mail: jing.li@econ.upenn.edu

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Abstract

I study how choice behavior given unawareness of an event differs from choice behavior given subjective belief of zero probability on the event in an axiomatic framework. Depending on different types of unawareness the decision maker suffers, behavior is either incomparable with zero probability (in the case of pure unawareness), or allows for drastically different behavior than those under zero probability (in the case of partial unawareness). The key difference is that partial unawareness is consistent with dynamically inconsistent choice which is not possible under zero probability beliefs.

Keywords: unawareness, zero probability, dynamic consistency, information, conditional probability system

1   Introduction

It is a well-recognized fact that people may be unaware of some relevant uncertainties when making decisions. For example, most insurance companies in the 1970s were unaware of the harmful effects lead-based paint had on human body, which subsequently resulted in millions of dollars in compensations. The war against terrorism has been an extremely difficult endeavor precisely because we are unaware of many possible strategies the terrorists could employ. Understanding decision-making under unawareness thus is of great interest.

There are two interesting notions of unawareness. The first is unawareness of specific events, and the second is whether one is aware of the general issue that one may be unaware of *some* event. These are separate issues, while modeling the latter necessarily requires embedding the former in the model. In this note, I examine the simpler case where one is not only unaware of some specific events has specific unawareness, but also essentially unaware of such unawareness.[1] For simplicity, I drop the reference to the latter in the paper.

A frequently raised question is whether such unawareness is observationally equivalent to having full awareness with zero probability beliefs. The question arises from the observation that while one cannot take into account what happens in events of which one is unaware, neither does one care about what happens in events to which one assigns zero probability, as long as one is *rational*.[2] Indeed, this is the approach used in many papers to model agents with unawareness, for example, Modica, Rustichini and Tallon (1997), Ozbay (2006), among others. On the other hand, there is also a clear conceptual distinction between unawareness and zero probability that prompts many economists to explore models of unawareness explicitly. Nonetheless, all existing models of unawareness deliver results (in terms of behavior and outcomes) that can be obtained in models with zero-probability beliefs. Thus the goal of this note is twofold: I explore whether zero-probability belief systems are reasonable approximations of beliefs under unawareness; and what characterizes behavior under unawareness.

Epistemically, unawareness clearly differs from having zero probability. One is unaware of an event if one doesn't know it, and doesn't know he doesn't know it, and so on; while assigning zero probability to an event *requires* being aware of it. In a decision context, this distinction first translates to the availability of corresponding bets. While one cannot bet on an event of which one is unaware, one can certainly bet on an event to which one assigns zero probability. However, this observation has no behavioral content: a rational decision-maker (DM henceforth) never bets on an event to which he assigns zero probability anyways. The second observation is that unawareness has a natural symmetry property which cannot be satisfied in a probability system: one is unaware of an event if and only if one is unaware of the negation of it; while one cannot

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[1]The unawareness of unawareness is almost implied by the structure of the model – see the discussion in Section 4.

[2]In this context, rationality refers to expected utility maximization.

assign zero probability to both an event and its negation. It follows that while the DM can neither bet on nor against an event of which he is unaware, he would *always* want to bet against an event to which he assigns zero probability. However, this observation, too, has minimal behavioral content: testing it requires asking the DM to rank bets involving the event in question, which necessarily makes the DM aware of it. Moreover, in many situations, the DM could bet on an event that is equivalent to an event of which he is unaware. For example, imagine there are red, black and white balls in an urn, but Alex is unaware of the red balls. Although Alex cannot bet on either "the ball is red" or "the ball is not red", he could, and presumably would, bet on the event "the ball is either black or white", just as he would if he assigned zero probability to having red balls in the urn.[3] Therefore, I turn to examine the DM's choice behavior with respect to those bets of which he is aware. To have a meaningful comparison, I fix the choice behavior under full awareness as the benchmark.

First notice that under unawareness, the DM is necessarily also unaware of the choice set. A bet in its regular usage specifies an unambiguous outcome for each deterministic scenario. Call such bets the *objective bets*. In the previous example, an objective bet specifies what Alex receives for each color of the ball drawn from the urn. But then since Alex is unaware of the red balls, he is necessarily unaware of all such objective bets. On the other hand, arguably Alex can conceive bets on events such as "black ball" or "white ball." I call such "bets" perceived by the DM under unawareness the *subjective bets*. Note that a subjective bet is in fact not well-defined from a fully aware outside observer's perspective: every scenario the DM has in mind when he has unawareness cannot be deterministic from the observer's perspective (Li 2007a, Li 2007b). Thus the DM's ranking of the subjective bets reflect both how his likelihood assessments of those events of which he is aware, and his *perception* of the outcomes. To focus attention on how unawareness affects the DM's beliefs, I fix the DM's preferences for the outcomes.

Thus the key notion is how subjective bets under unawareness are related to the objective ones. There are two cases to consider. For the first case, consider the following example. There are two relevant uncertainties, whether it rains and whether there is an earthquake, but Bob is unaware of the possibility of an earthquake. Bob's subjective bets only concern events such as "rain" or "no rain." Then the question is, when evaluating a subjective bet that says Bob is to receive $10 when it rains and Bob is to pay $10 when it does not rain, what does Bob have in mind about the payoff structure in terms of the objectively deterministic worlds? One plausible possibility is that Bob has in mind the bet where he is to receive $10 *whenever* it rains and pay $10 *whenever* it does not rain. This is the case where the DM simply neglects the details of which he is unaware in evaluating subjective bets. It follows that each subjective bet corresponds to one and only one objective bet, essentially identifying preferences

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[3]Of course, in the former case, Alex is unaware that his bet is equivalent to one wagering on "the ball is not red", while in the latter case he knows it.

under unawareness as a strict subset of preferences under full awareness. I refer to this case as "pure unawareness." It is not hard to see that the only implication of pure unawareness is the DM's inability to bet on or form beliefs about events of which he is unaware. Notice that since one cannot assign zero probability to both an event and its negation, pure unawareness of an event is incomparable with assigning zero probability to it.

For the second case, consider the urn example. Note there is a natural correlation between realizations of different uncertainties in this case. For example, a ball can *only* be either black, white, or red. If Alex is unaware of the red balls, he must implicitly assume no ball is red. One can view such unawareness as that the DM is aware of the relevant uncertainty, but is unaware of some possible resolutions. I refer to this case as "partial unawareness". Notice that under partial unawareness, there is a sense in saying the DM is unaware of some objective states. For example, let $\{b, w, r\}$ be the full state space for the urn example. Then Bob's subjective state space, from a decision perspective, can be viewed as the set $\{b, w\}$, consisting of those full states in which Bob's implicit assumption – no ball is red – is true, although he himself is unaware of this part of the description of the states. Bob's unawareness restricts him to reason about every event *relative to* this subjective state space. In this sense Bob's reasoning is confined in a subjective algebra of events that is a relativization of the objective algebra.[4] Intuitively, this is the case that begs a comparison with zero probability.

Unlike the previous two cases, absent of more detailed assumption regarding specific nature of the partial unawareness, one cannot pin down the connection between events of which the DM is aware given different awareness levels. For example, suppose Alex is unaware of the red balls because he suffers from color blindness that makes him unable to distinguish red and black. Then upon updating his state space, Alex would realize the event "the ball is black" that he perceives in the first period actually confounds two objective events: "the ball is black" and "the ball is red." Consequently, comparing to his valuation of the objective bets in the second period, his valuation of the subjective bets in the first period must have a systematic bias towards more weights on the consequences he receives in the state where the ball is black. For example, suppose Alex is indifferent between not betting and taking the bet "getting \$10 if the ball is black, paying \$10 if the ball is white" in the first period, then it seems plausible that he must be indifferent between not betting and taking the bet "getting \$10 if the ball is black *or red*, paying \$10 if the ball is white" in the second period. Alternatively, suppose Alex's color blindness makes him unable to distinguish between red and white instead. Then it seems in the second period Alex would be indifferent between not betting and taking the bet "getting \$10 if the ball is black, paying \$10 if the ball is white *or red*" instead. But absent of such additional assumptions regarding the nature

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[4]Technically, if one uses a subset to represent the DM's subjective state space, then each subjective state is a pair, consisting of the state itself and the subjective state space it belongs to. See Li (2007b) for details.

of unawareness, the only observation one can make is that Alex *somehow confounds* the red balls with either black balls or white balls or both, and in the second period, it must be the case the he weakly prefers on taking the bet "getting \$10 if the ball is black *or red*, paying \$10 if the ball is white" rather than not betting rather than taking the bet "getting \$10 if the ball is black, paying \$10 if the ball is white *or red*." In other words, the DM's valuation of a subjective bet under partial unawareness should fall between his valuation of a best-scenario completion of the subjective bet and a worst-scenario completion of it upon updating his state space.[5]

I formulate this discussion in a two-period Anscomb-Aumann model, where the DM ranks acts mapping states to lotteries in each period (Anscombe and Aumann 1963). The (finite) full state space, S_2, and hence the set of all objective bets, are only revealed to the DM in the second period. In the first period, the DM has a subjective state space S_1 and ranks all subjective bets. Under full awareness, the subjective state space and the full state space is connected as follows: $S_1 = S_2 \times U$ where U has a cardinality greater than 1; while under partial unawareness, one has $S_2 = S_1 \cup U_p$ where U_p has a cardinality greater than 1. It is worth emphasizing that there is no dynamic component in the situation: when ranking the subjective bets, the DM does not anticipate a second period where he updates his choice set.

Pure unawareness thus can be formulated by identifying each subjective bet f with the objective bet f' that yields the same lottery in every "completion" of a subjective state. Adding this axiom to the standard Anscomb-Aumann axioms applied to the preferences in each period amounts to identifying the first-period choice set as a subset of the second-period choice set, and hence there is no beliefs distortion under unawareness with respect to the benchmark case of full awareness. Let μ_i denote the DM's subjective beliefs over S_i, then for all $E \subseteq S_1$, $\mu_1(E) = \mu_2(E \times U)$.

I formulate partial unawareness by comparing the certainty equivalent of a subjective bet with different objective bets that extends the subjective bet to the full state space. The resulting representations are characterized by the following connection in beliefs: for all $E \subseteq S_1$, $\mu_1(E) \geq \mu_2(E)$. In particular, this connection allows one to talk about a measure of degree of incompleteness of each subjective state: there exists $\alpha : S_1 \to [0, 1]$ such that $\mu_2(\{s\}) = \alpha(s)\mu_1(s)$ for all $s \in S_1$, where $\alpha(s)$ is unique if and only if $\mu_1(s) \neq 0$.[6]

Such beliefs encompasses a naive case of zero probability: $\mu_2(U_p) = 0$ if and only if $\alpha(s) = 1$ for all $s \in S_1$. A more sophisticated zero-probability interpretation

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[5]It is worth noting that the dichotomy of pure unawareness and partial unawareness is only meaningful in the context of decision-making. Epistemically, partial unawareness is not fundamentally different from pure unawareness. However, given correlations between realizations of uncertainties, unawareness of an uncertainty could entail additional complication of unawareness of logical deductions. For example, not only Alex is unaware of both "the ball is red" and "the ball is not red", but also he is unaware that "the ball is black or white" (of which he is aware) is equivalent to "the ball is not red." See Galanis (2006) for a thorough discussion of unawareness of logical deductions.

[6]For notational ease, I omit the brackets when having a singleton set as the argument of μ_i.

takes into account the informational content of S_2: the DM is aware of the full state space S_2 in both periods, and expresses preferences (extended to the correct choice set) *conditional on different information* $S_i, i = 1, 2$ à la Myerson (1986b).[7] But if S_2 is only information regarding which state obtains, then one should expect the preferences to satisfy an "event consistency" axiom: if the DM prefers the subjective bet f to g, and the objective bets f' and g' in the second period coincide with f and g respectively on S_1, and yield the same payoffs on U_p, then the DM should prefer f' to g'. But then the Myerson result applies: for any $E \subseteq S_1$, $\mu_1(E) = \mu_2(E|S_1)$. Beliefs under partial unawareness encompasses this case as well: set $\alpha(s) = \mu_2(S_1)$ yields the conditional probabilities.

Thus, partial unawareness alone imposes very little restriction on the DM's subjective beliefs. In particular, the key difference between the partial unawareness axiom and event-consistency axiom is that the latter identifies *dynamically consistent* behavior. In the unawareness environment, as long as α is not constant, that is, there is some correlation between the unforeseen scenarios and the foreseen scenarios, then there always exist bets that differ only on the foreseen scenarios such that the DM's preferences for them would not be the same under different levels of awareness. Moreover, such dynamic inconsistency is very different from those in the standard non-expected utilities models in the sense that ex ante, the DM would like to have tomorrow's preferences instead of today.

Finally, while unawareness of an event is not behaviorally equivalent to assigning zero probability to it, it is always possible to *model* behavior under partial unawareness by *constructing a model involving updating on zero probability*. More specifically, one can take the state space to be the disjoint union of S_1 and S_2, let the DM's preferences conditional on the information S_1 and S_2 (appropriately extended to the expanded domain) match the two period preferences in the unawareness case respectively, and the preferences conditional on the entire state space reflects a subjective belief of zero probability on S_2. Under this construction, the DM is fully aware of all states, including both the subjective states and the objective states, but first assigns probability one to S_1 in the first period, and then upon receiving the information that true state lies in the prior zero probability event S_2, the DM updates to assign probability one to S_2. The caveat of this approach is that beliefs are disjoint by construction and hence are entirely unrestrictive.

There has been little attempt in exploring beliefs under unawareness from a decision-theoretic perspective. Ahn and Ergin (2007) explore a model in which the DM's subjective beliefs for different sub-algebras of events are solicited, and the connection between these beliefs are interpreted as reflecting the effects of framing, and possibly unawareness. In the context of the above example, Ahn and Ergin compares Bob's subjective beliefs for "rain" and "no rain" with his beliefs for "rain, earthquake",

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[7]Given the time structure, one interpretation is that the signal S_1 is false information and later is corrected in the second period.

"rain, no earthquake" and "no rain", and interpret the discrepancy between the probability for "rain" in the first beliefs system and the sum of the probabilities for "rain, earthquake" and "rain, no earthquake" in the second beliefs system as an evidence that Bob was unaware of earthquake when he ranks only bets measurable with respect to only the events "rain" and "no rain." In contrast, in this paper, I compare the subjective probability Bob assigns to "rain" under pure unawareness of earthquake with that he assigns to "rain" when he is aware of the earthquake.

The note is organized as follows. Section 2 investigates how unawareness affect the DM's beliefs regarding events of which he is aware. I discuss both the benchmark case of pure unawareness and the main interest, the case of partial unawareness. Section 3 contrasts the case of unawareness with zero probability models. Section 4 discusses caveats of the decision-theoretic approach and potential interesting issues in this environment. I conclude in Section 5.

2 Beliefs under Unawareness

Let Z denote an arbitrary set of prizes and $\Delta(Z)$ the set of simple lotteries over Z.[8] Let S_i denote the state space for period $i, i = 1, 2$. Let S_2 be finite. Let $\succcurlyeq^i$ denote the DM's preference orderings over the period i choice set $C_i = (\Delta(Z))^{S_i}$. Let $\succ^i$ and $\sim^i$ denote the asymmetric and symmetric parts of $\succcurlyeq^i$, respectively. Let l denote a generic element in $\Delta(Z)$. Slightly abusing notation, I also use l to denote the constant act that yields l in every state. As usual, convex combination of acts is defined by taking convex combination state-wise: for all $\alpha \in [0, 1]$ and any $f, g \in C_i, [\alpha f + (1-\alpha)g](s) = \alpha f(s) + (1-\alpha)g(s)$ for all $s \in S_i$.

Fixing $f, g \in C_i$ and $E \subseteq S_i$, I say the DM *prefers* f to g *conditional on* E, denoted by $f \succcurlyeq^i_E g$, if $f' \succcurlyeq^i g'$ where $f'(s) = g'(s)$ for all $s \notin E$ and $f'(s) = f(s)$ and $g'(s) = g(s)$ for all $s \in E$. An event $E \subseteq S_i$ is *Savage-null* under $\succcurlyeq^i$ if $f \sim^i_E g$ for all $f, g \in C_i$. A state s is said to be *non-null* if $\{s\}$ is not Savage-null.

First I postulate $\succcurlyeq^i$ satisfies the standard Anscomb-Aumann axioms.

AA.1 (weak order): $\succcurlyeq^i$ is transitive and complete;

AA.2 (continuity): for all $g \in C_i$, the sets $\{f \in C_i : g \succcurlyeq^i f\}$ and $\{f \in C_i : f \succcurlyeq^i g\}$ are closed;

AA.3 (independence): for all $f, g, h \in C_i$, $f \succ^i g, \alpha \in (0, 1)$ implies $\alpha f + (1-\alpha)h \succ^i \alpha g + (1-\alpha)h$;

AA.4 (non-triviality): there exist $f, g \in C_i$ such that $f \succ^i g$;

AA.5 (state-independence): for all non-null $s, t \in S_i$, $l_1 \succcurlyeq^i_{\{s\}} l_2$ if and only if $l_1 \succcurlyeq^i_{\{t\}} l_2$ for all constant acts $l_1, l_2 \in \Delta(Z)$.

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[8]A simple lottery is a lottery that has finite support.

Proposition 1 (Anscombe and Aumann (1963)): *The axioms AA.1-5 are necessary and sufficient for $\succcurlyeq^i, i = 1, 2$ to have the following representation: for all $f \in C_i$,*

$$V_i(f) = \sum_{s \in S_i} \mu_i(s) u_i(f(s)), \tag{2.1}$$

where $u_i : \Delta(Z) \to \mathbb{R}$ is linear in probabilities and unique up to affine transformation, and $\mu_i : 2^{S_i} \to [0, 1]$ is the unique subjective probability on S_i.

I refer to μ_2 as the DM's *latent beliefs* under full awareness, and use it as the reference point in discussing the DM's subjective beliefs under unawareness, μ_1. Since the utilities for lotteries and the beliefs are jointly determined, in order to focus on the effects of unawareness on beliefs alone, I fix the DM's preferences for prizes across the two periods by requiring the DM to rank the constant acts in both periods in exactly the same way.

U.1 (foreseen consequences): for all $l_1, l_2 \in \Delta(Z), l_1 \succcurlyeq^1 l_2 \iff l_1 \succcurlyeq^2 l_2$.

Intuitively, this axiom says the DM has no unawareness of possible consequences. Upon updating his subjective state space to S_2, his valuation of all lotteries remain the same. Adding axiom U.1 to AA.1-5 results in setting $u_1 = u_2 = u$ in the above representations.

The key axiom concerns how the DM's subjective bets relate to the objective ones, or, equivalently, how the DM's subjective state space relates to the full state space. First, I consider the case of pure unawareness as a benchmark.

2.1 Pure unawareness.

Let $S_2 = S_1 \times U$, where $\#(U) > 1$. The interpretation is, the DM is unaware of those uncertainties whose resolutions are described by U. If a subjective bet specifies a lottery l as the consequence in a subjective state s, the DM perceives it to mean that he receives l *whenever E is true*. Under this perception, each subjective bet is identified by one and only one objective bet. Let $G : C_1 \to C_2$ be defined as follows: for any $f \in C_1$, $G(f)((s, u)) \equiv f(s)$ for all $s \in S_1$ and $u \in U$. Thus, pure awareness amounts to the following axiom:

U.2 (pure unawareness): for any $f, g \in C_1$, $f \succcurlyeq^1 g \Leftrightarrow G(f) \succcurlyeq^2 G(g)$.

Proposition 2 *Axioms AA.1-5 and U.1-2 are necessary and sufficient for $\succcurlyeq^i, i = 1, 2$, to be represented as in (2.1), and in addition, (1) $u_1 = u_2$; (2) for all $E \subseteq S_1$, $\mu_1(E) = \mu_2(E \times U)$.*

The proof is straightforward and hence omitted.

Proposition 2 says, pure unawareness is equivalent to a measurability constraint: the DM's preferences over the subjective bets are identified by a subset of his preferences over the objective bets. The DM's subjective beliefs under pure unawareness is the restriction of his latent beliefs on events of which he is aware. In terms of the DM's choice behavior, the only implication of pure unawareness is the incompleteness of his choice set. Pure unawareness is simply not comparable to assigning zero probability.

2.2 Partial unawareness.

Alternatively, in the first period, the DM may have some implicit assumptions in mind, such that each subjective state corresponds to one full state. Let $S_2 = S_1 \cup U_p$, where $U_p \neq \emptyset$ contains scenarios none of the DM's subjective state corresponds to.[9] In this case, not only is the DM unaware of some uncertainty, but also he implicitly assumes it to be a particular certainty. Upon revelation of the full state space in the second period, the DM becomes aware of his own implicit assumption and reevaluates every event with respect to the expanded universal event.

Let $l_1[E]l_2[S_i \setminus E]$ denote the (subjective or objective) bet that yields the lottery l_1 on the event E and l_2 on its complement $S_i \setminus E$. Given any $f \in C_i$, let $\bar{f}$ denote its *certainty equivalent* under $\succcurlyeq^i$ (which exists by the Anscomb-Aumann axioms), i.e., $\bar{f} \in \Delta(Z)$ is the constant act such that $f \sim^i \bar{f}$.

Consider a subjective bet $l_1[E]l_2[S_1 \setminus E]$. Suppose $l_1 \succ^1 l_2$. From an outside observer's perspective, this subjective bet leaves states in $S_2 \setminus S_1$ unspecified; while from the DM's perspective in the first period, it specifies the payoff for every possible scenario. Intuitively, the DM implicitly "confounds" scenarios described in U_p when reasoning about his subjective states in S_1. Since the DM is fully aware of all consequences, this means the DM must have in mind implicitly an objective bet where the consequence specified in any scenario of which the DM is unaware must coincide with that in *some* subjective state. Thus, fixing a subjective bet $f \in C_1$, I say an objective bet $g \in C_2$ is an *extension* of f, or g *extends* f, if $g(s) = f(s)$ for all $s \in S_1$ and $g(s) \in \{f(s) : s \in S_1\}$ for all $s \in U_p$. Without additional assumptions about the nature of the DM's partial unawareness, that the DM perceives each subjective bet to be *some* objective extension of it seems to be the only thing that one can say.

U.3 (partial unawareness): for all $E \subseteq S_1$, $l_1 \succ^1 l_2$,

$$l_1[E \cup U_p]l_2[S_1 \setminus E] \succcurlyeq^2 \overline{l_1[E]l_2[S_1 \setminus E]} \succcurlyeq^2 l_1[E]l_2[S_2 \setminus E]. \quad (\dagger)$$

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[9]Strictly speaking, the set S_1 is not the DM's subjective state space; rather, each state $s \in S_1$ is a full state *corresponding to a subjective state given his implicit assumption regarding the uncertainties of which he is unaware.* For simplicity, I do not make this distinction in the text, and simply refer to S_1 as the subjective state space.

This axiom essentially says, the DM's valuation of a subjective bet falls between its best scenario extension and worst scenario extension. To see this, notice $\overline{l_1[E]l_2[S_1 \setminus E]}$ is the certainty equivalent of the subjective bet $l_1[E]l_2[S_1 \setminus E]$, and serves as a "price tag" for the latter that can be taken to the second period for comparison. The objective bet $l_1[E \cup U_p]l_2[S_1 \setminus E]$ is the best-scenario extension of $l_1[E]l_2[S_1 \setminus E]$, stipulating that the DM is to receive the better lottery l_1 in all states in U_p; while $l_1[E]l_2[S_2 \setminus E]$ is the worst-scenario extension, where the DM receives the worse lottery l_2 on U_p.

Proposition 3 *Axioms AA.1-5, U.1 and U.3 are necessary and sufficient for $\succcurlyeq^i, i = 1, 2$, to be represented as in (2.1), and in addition, (1) $u_1 = u_2$; (2) for all $s \in S_1$, $\mu_2(s) = \alpha(s)\mu_1(s)$, where $\alpha : S_1 \to [0, 1]$ is unique if and only if $\mu_1(s) = 0$. Moreover, $\alpha(s) = 1$ for all non-null $s \in S_1$ if and only if U_p is Savage-null under $\succcurlyeq^2$.*

Proof: Only need to prove sufficiency for 2. Let $V_i(f) = \sum_{s \in S_i} \mu_i(s)u(f(s))$ represent $\succcurlyeq^i$.

Fix $s \in S_1$ and let $l_1 \succ^1 l_2$. Suppose $\{s\}$ is Savage-null under $\succcurlyeq^1$ and hence $\mu_1(s) = 0$. Then $l_1[\{s\}]l_2[S_1 \setminus \{s\}] \sim^1 l_2$, or $\overline{l_1[\{s\}]l_2} = l_2$. By U.3, $l_2 \succcurlyeq^2 l_1[\{s\}]l_2[S_2 \setminus \{s\}]$. By U.1, $l_1 \succ^2 l_2$, but then we must have $\mu_2(s) = 0$.

Suppose $\{s\}$ is not Savage-null. Then $\mu_1(s) \neq 0$. By U.3, we have

$$\mu_1(s)u(l_1) + (1 - \mu_1(s))u(l_2) \geq \mu_2(s)u(l_1) + (1 - \mu_2(s)u(l_2).$$

Set $\alpha(s) = \frac{\mu_2(s)}{\mu_1(s)} \in [0, 1]$. Uniqueness of $\alpha(s)$ follows from the uniqueness of μ_1 and μ_2.

Suppose U_p is Savage-null under $\succcurlyeq^2$. Then for all $E \subseteq S_1$, $l_1[E \cup U_p]l_2[S_1 \setminus E] \sim^2 l_1[E]l_2[S_2 \setminus E]$. By axiom U.3, we have $\overline{l_1[E]l_2[S_1 \setminus E]} \sim^2 l_1[E]l_2[S_2 \setminus E]$, which then indicates $\mu_1(E)u(l_1) + (1 - \mu_1(E))u(l_2) = \mu_2(E)u(l_1) + (1 - \mu_2(E))u(l_2)$. It follows $\mu_1(E) = \mu_2(E)$. But then for all $s \in S_1$ such that $\{s\}$ is not Savage-null, we must have $\alpha(s) = 1$.

For the converse, observe that $\alpha(s) = 1$ implies $\mu_2(s) = \mu_1(s)$. But then $\mu_2(S_1) = \mu_1(S_1) = 1$, and hence $S \setminus S_1$ is Savage-null under $\succcurlyeq^2$. $\square$

The number $\alpha(s)$ can be interpreted as the DM's *degree of partial unawareness* of s. The smaller $\alpha(s)$ is, the more significant the role of those unaware scenarios the DM confounds into the subjective state s. It is worth emphasizing that, absent of additional assumptions regarding the nature of unawareness, beliefs under partial unawareness are rather unrestrictive. For example, suppose Bob's latent beliefs are such that each color has probability $\frac{1}{3}$, then being partially unaware of red balls, Bob's subjective beliefs can be anything that assigns a probability between $\frac{1}{3}$ and $\frac{2}{3}$ to the ball being black and to the ball being white, respectively.

3 Partial Unawareness and Zero Probability Beliefs

Partial unawareness begs the comparison with zero probability belief under full awareness, i.e., the DM is always aware of S_2, but simply assigns zero probability to U_p in the first period. Then the key question is how one interprets the informational content of the signal S_2. I consider three cases.

First consider the standard interpretation that S_2 is the signal that all states in S_2 are possible. Given the DM's full awareness of S_2, such information is trivial. To recast the previous model in this story, one can view the first-period preferences $\succcurlyeq^1$ over C_1 as resulting from omitting U_p, which is null; while in the second-period, the DM is asked to explicitly rank all available bets in S_2. For any $f \in C_2$, let $f|_{S_1}$ denote the restriction of f on S_1. Preferences in the two periods have the following natural connection:

Z.0 (naive zero-probability): for all $f, g \in C_2$, $f|_{S_1} \succcurlyeq^1 g|_{S_1} \Rightarrow f \succcurlyeq^2 g$.

It is easy to see partial unawareness encompasses this case: adding Z.0 to AA.1-5 is equivalent to adding B1, B3 to AA.1-5, plus an additional requirement that U_p be Savage-null under $\succcurlyeq^2$.

However, one can argue the above is not the relevant comparison. In the case of unawareness, S_2 is an informative signal, while in the above story, S_2 is not. Thus a more desirable comparison case is to reinterpret S_2 as new information that somehow induce the DM to update his beliefs and assign a positive probability to U_p. Note in this case, S_2 is not information in the standard usage of this term: it *expands* the set of non-null states instead of narrowing it. Consider two bets f, g in the second period that coincide on the event U_p. No matter what interpretation one gives to the signal S_2, as long as the DM is fully aware of S_2 in the first period, this signal does not add anything to the DM's understanding of the environment, and hence his ranking of f and g should coincide with that of their counterparts in the first period. Formally, I postulate the following axiom.

Z.1 (event consistency): let $f, g \in C_2$ be such that $f(s) = g(s)$ for all $s \in U_p$, then
$f|_{S_1} \succcurlyeq^1 g|_{S_1} \Rightarrow f \succcurlyeq^2 g$.

Proposition 4 *(**Myerson (1986b)**): Axioms AA.1-5 and Z.1 are necessary and sufficient for $\succcurlyeq^i, i = 1, 2$, to be represented as in (2.1), and in addition, $u_1 = u_2$; and $\mu_1(s) = \mu_2(s|S_1)$.*

That is, the first-period beliefs are conditional probabilities. Note Z.1 implies U.3, thus Proposition 3 applies: this is the special case where α is a constant function, given by $\alpha(s) = \mu_2(S_1)$ for all $s \in S_1$. In terms of behavior, Z.1 characterizes dynamic consistency: conditional on events in S_1, the DM's preferences remain the same. In contrast, behavior under partial unawareness tends to be dynamically inconsistent.

One can always find a pair of bets for which the DM has different preference orderings in the two periods *conditional on events of which the DM is aware in both periods*, as long as α is not constant.

How to understand this? Intuitively, there is a fundamental conceptual difference between updating on new *factual* information, i.e., information on which events have occurred, and updating on new *awareness* information, i.e., information on which events should be included in the probability space. Given state-independent preferences for lotteries, beliefs are naturally additively separable in events under different factual information, due to the fact that states are mutually exclusive. However, when there is partial unawareness, the problem is precisely the DM's inability to separate states, and hence events. New awareness information is not about revealing relevant facts, but rather, is about revealing the hidden correlation between those states of which the DM is aware. Consequently, while dynamic consistency characterizes the choice behavior under different factual information, dynamic inconsistency tends to arise in environment involving partial unawareness.

Finally, I note that while unawareness of an event differs from assigning zero probability to an event per se, one can always write down *some* model of zero probability that *matches* the behavior under unawareness.[10] The idea is to construct a state space encompassing both S_1 and S_2, and reinterpreting the DM's preferences in the two periods as conditioning on disjoint events. More specifically, given $\succcurlyeq^i$ over $C_i, i = 1, 2$, I can define another model *à la* Myerson (1986b) as follows. Let S be the disjoint union of S_1 and S_2, and $C = (\Delta(Z))^S$ be the choice set for both periods. For all $f, g \in C$, define $f \succcurlyeq_S g$ if and only if $f|_{S_1} \succcurlyeq^i g|_{S_1}$, and $f \succcurlyeq_{S_2} g$ if and only if $f|_{S_2} \succcurlyeq^i g|_{S_2}$. Then one can view the two-period beliefs system as a *conditional probability system*(CPS) $\mu : 2^S \times \{S, S_2\} \to [0, 1]$, such that $\mu(S_2|S) = 0$.[11] Intuitively, this model describes the following: On the expanded state space S, the DM has a prior belief of zero probability on S_2. In the second period, the DM receives the zero-probability signal S_2, updates his beliefs to assign probability 1 to S_2 instead.

The caveat of this approach is that, given disjoint beliefs in the two periods, behavior can be made to match anything one wishes. Therefore to give this model any content, one needs to impose additional structures that capture the essence of unaware, for example, the counterparts of U.2-3. But doing this necessarily requires one to examine a model of unawareness, which defeats the very purpose of writing a zero probability model for unawareness.

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[10]Note this construction surpasses both pure unawareness and partial unawareness.

[11]Fix a finite state space Ω. A *conditional probability system*(CPS) is any function $\mu : 2^\Omega \times 2^\Omega \backslash \{\emptyset\} \to [0, 1]$ such that, for all $X, Y, Z \subseteq \Omega$, $Z \neq \emptyset$, $\mu(\cdot|Z)$ is a probability measure over Ω such that (1) $\mu(\Omega|Z) = \mu(Z|Z) = 1$; (2) $X \subseteq Y \subseteq Z$, $Y \neq \emptyset \Rightarrow \mu(X|Z) = \mu(X|Y)\mu(Y|Z)$.

4 Discussion

In this section, I discuss the limitations of the above results, in particular those imposed by the standard decision-theoretic framework itself, as well as some conceptual issues of unawareness.

4.1 Unawareness and (bounded) rationality.

In this paper, unawareness is treated as an informational constraint. There is a sense in saying the DM in this model is perfectly rational with respect to his own awareness: He forms probabilistic beliefs about relevant uncertain events and maximizes subjective expected utilities under such beliefs. He is also Bayesian within the boundary of his awareness. The dynamic inconsistency in his behavior under partial unawareness arguably reflects a notion of rationality in this environment: Whenever there is correlation between resolutions of aware uncertainties and unaware uncertainties, states in the subjective state space cannot be mutually exclusive from an outsider's perspective, which the DM, if rational, should recognize in his retrospection in the second period.

4.2 Foreseen consequences and unforeseen contingencies.

A key element in the model of unawareness is foreseen consequences: both the set of prizes and the DM's preferences over the prizes are fixed across periods. Without this assumption, beliefs under unawareness are not identified in the model.

Are foresee consequences a reasonable assumption in this environment? In general, the choice objects are actions not acts, and the DM evaluates consequences induced by the actions given the relevant contingencies instead of monetary payoffs, which requires a more direct formulation of Z. But then given the presence of unforeseen contingencies, unforeseen consequences seem unavoidable. For example, if the DM is unaware of the possibility of an earthquake, he is certainly also unaware of the catastrophic destruction an earthquake brings about. From this perspective, assuming foreseen consequences at the presence of unforeseen contingencies seems not without loss of generality.

On the other hand, notice one can always interpret Z as money, then it seems compelling that Z should be foreseen. This is precisely the device of this model: The DM is asked to bet on all conceivable events, which removes unforeseen consequences, and makes it possible to solicit beliefs under unawareness. This assumption does impose extra burden on another axiom, namely the state-independence axiom, by postulating that the DM's preferences for money do not depend on the states, even unforeseen ones.

4.3   Unforseen Options

The most salient feature in the unawareness environment is actually the unavailability of some bets, namely those explicitly involving events of which the DM is unaware.[12] In other words, the DM has an incomplete choice set under unawareness. In contrast, in any standard model, the choice set is complete, and hence all possible rankings are conceivable. This distinction has important implications in behavior.

First, in models with full unawareness, there is no real dynamic inconsistency. The DM can perceive all possible information he may receive and hence makes a decision for every possible scenario. If there is any dynamic inconsistency in his behavior, he *anticipates precisely this particular dynamic inconsistency to occur.* For example, in a model with hyperbolic discounting, the DM plans to start saving tomorrow, yet when tomorrow arrives, he postpones it and plans to do it the next day, and so on. In a sense, such behavior is not dynamically inconsistent: if one makes the DM to write out a plan conditional on the arrival of every day for the next year *on day zero*, "start saving tomorrow" will appear on every day's to-do list and "start saving today" will never appear. This is not the case in an environment with unawareness. The DM never anticipates any specific change of choice in the first period. Rather, his dynamic inconsistency comes as a surprise to even himself. The key aspect is, awareness information, by definition, is beyond one's imagination. In this sense there is true dynamics in this environment. It is worth emphasizing that, the difference of the two cases is not that whether the DM can anticipate himself to be dynamically inconsistent, – that can happen in either environment, in fact, one may argue that is the interesting case of unawareness – instead, the difference is that whether the DM can anticipate the *specific* incidences of dynamic inconsistency.

Second, note the type of dynamic inconsistency in an unawareness environment and that in a full awareness environment are very different. In a full awareness model, dynamic inconsistency tends to be an undesirable consequence of lack of self-control power that the DM would like to avoid; while in the case of partial unawareness, if anything, the DM would like to make room for such dynamic inconsistency to occur because these are the choices made with better information. Using the language of the multi-selves story, in the first case, today's DM prefers to make decisions for his future selves, while in the second case, today's DM prefers to be his future self, and would like to leave room for his future self to override his own decision today. The first case leads to a preference for a commitment device, while the second case leads to a preference for flexibility.

Third, in addition to the dynamic inconsistency issue, the presence of unforeseen options gives rise to the possibility that all foreseen options may be strictly dominated by some unforeseen option. This feature brings in very different dynamic consider-

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[12]One immediate implication is that, even if the DM is irrational, whatever it means, he will never bet on an event of which he is unaware, but he could certainly bet on it if he simply assigns zero probability to it.

ations from the ones in standard models. Intuitively, a DM who understands such incompleteness of his choice set would want to postpone decisions as much as possible, in case some foreseen options become available. In contrast, any model with full awareness that matches the behavior of the DM under unawareness has to have the feature that the counterparts of those unforeseen options are necessarily dominated by the foreseen ones. In fact, since the choice set is the primitives of any standard decision theory model, modeling a DM who contemplates the incompleteness of his choice set may require a drastic departure from the standard approach.

5   Conclusion

While unawareness is a realistic cognitive restriction that seems to have broad implications in many important economic situations, it is commonly regarded to be behaviorally equivalent to having zero probability beliefs. In this paper, I carefully compare, in an axiomatic decision theory framework, the case of unawareness and models of zero probability beliefs. I show that on the one hand, unawareness does have implications for one's beliefs with respect to events of which one is aware, on the other hand, such restrictions are rather weak, and resulting beliefs have much less structures than zero probability beliefs. The axiomatic framework also enables me to organize novel issues brought up by unawareness, as well as point out which implicit assumptions of the standard decision theory models will have to be relaxed in order to address them. Dekel, Lipman and Rustichini (1998b) point out the need of "(finding) an 'Ellsbergian paradox' that indicates behaviors that are due to unforeseen contingencies and that conflict with standard subjective (non)expected utility models, thus helping focus the search for alternatives." While this note does not provide such a "Ellsbergian paradox", it points out structural assumptions in the standard models that preclude producing such a paradox, and possible directions one may find one.

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