The War of Information[†]

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Abstract

Two advocates with opposing interests provide costly information to a voter who must choose between two policies. Players are symmetrically informed and information flows continuously as long as either advocate is willing to uncur its cost. In the unique subgame perfect equilibrium, an advocate's probability of winning is decreasing in his cost. When costs are different, increasing the low-cost advocate's cost benefits the voter. We analyze court proceedings with our model and show that the optimal burden of proof favors the high-cost advocate. If one advocate is informed, equilibrium yields a *signaling barrier*, a threshold that bounds the voter's beliefs no matter how much information is revealed.

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1. Introduction

A political party proposes a new policy, for example, a new health care plan. Opposing interest groups provide information to convince voters of their respective positions. This process continues until until polling data suggest that voters decisively favor or oppose the new policy. The health care debate during the Clinton administration and the social security debate during the Bush administration are prominent examples of this pattern.

In this paper, we analyze a model of competitive advocacy that captures salient features of such political campaigns. We assume that advocates provide hard and unbiased information. Our analysis focuses on the trade-off between information costs and the probability of convincing the median voter.

Advocates often distort facts or try to present them in the most favorable light. Presumably, voters understand advocates incentives and interpret their information accordingly. We ignore the details of the inference problem confronting the median voter and take as our starting point the resulting unbiased information. Our aim is to study the strategic interaction between two competing advocates. The underlying uncertainty is about the median voter preferences. Specifically, we assume that there are two states, one in which the median voter prefers advocate 1's policy and one in which he prefers advocate 2's policy.

We first analyze the symmetric information case. All players are uncertain about the voter's preferences and learn as information about the policies is revealed. Hence, we assume that both advocates know the median voter's beliefs throughout the game. Underlying this assumption is the idea that advocates take frequent opinion polls that inform them of the median voter's beliefs. Alternatively, players may learn about an objective state that determines the median voter preferences.

We model the information flow as a continuous-time process. As long as one of the advocates provides information, all players observe a Brownian motion with unit variance and a state-dependent drift. The game stops when no advocate is willing to incur the cost of information provision. At that time, the median voter picks his preferred policy based on his final beliefs. We refer to this game as the "war of information."

The war of information differs from a war of attrition in two ways. First, in a war of information, players can temporarily quit providing information (for example, when they are ahead) but resume at a later date. In a war of attrition, both players incur costs as long as the game continues. Second, the resources spent during a war of information generate a payoff relevant signal. If the signal were uninformative and both players incurred costs for the entire game, then the war of information would become a war of attrition with a public randomization device.

We show that the war of information has a unique subgame perfect equilibrium. In that equilibrium, each advocate chooses a *belief threshold* and stops providing information if the the current belief is less favorable than this threshold. Let p_t denote the belief that advocate 1 offers the better policy. The median voter prefers advocate 1's policy if $p_t > 1/2$ and 2's policy if $p_t \leq 1/2$. Then, there are belief thresholds $r_1 < 1/2 < r_2$ such that advocate 1 provides information if $p_t \in [r_1, 1/2]$ and advocate 2 provides information if $p_t \in [1/2, r_2]$. The game ends at time t if $p_t = r_1$ or $p_t = r_2$. In the latter case, the voter chooses policy 1 (advocate 1 wins) and in the former case the voter chooses policy 2.

The belief thresholds can be determined as the equilibrium outcomes of a static game, the *simple war of information.* The simple war of information is formally equivalent to a Cournot duopoly game with a unique Nash equilibrium. Viewed as a game between two advocates, the simple war of information is a game of strategic substitutes. Increasing advocate i's cost make i's less aggressive and j more aggresive. An advocate with a low cost of information provision is more likely to win the political contest for two reasons. First, the lower cost implies that the advocate chooses a more aggressive belief threshold. Second, the advocate's opponent will choose a less aggressive belief threshold. Both of these effects will increase the probability that the low cost advocate wins the campaign.

The campaing's informativeness deteremines the voter's equilibrium utility. A very informative campaign increases the voter's accuracy while an uninformative campaign forces the voter to make a decision with little information. From the voter's perspective, the advocates's thresholds are complements: a more aggressive advocate raises the voter's marginal benefit from a more aggressive threshold of his opponent. Hence, voters are best served by balanced campaigns. If one advocate's information cost is very high, lowering the other advocate's cost does not help the voter.

Propositions 3 and 4 describe how changing advocates' costs affects voter utility. We interpret these flows as the cost of continuing the information campaign. For example, these could be the cost of raising funds during an election campaign. US Election laws limit the amount of money an individual donor can give. Hence, such laws raise the cost of campaigning. Consider the case of two advocates with an equally large group of supporters. If advocate 1's supporters are wealthier than advocate 2's, then advocate 1 has a lower cost of campaigning. Moreover, limiting the maximum donation will disproportionably affect advocate 1. Hence, we can interpret US campaign finance regulations as raising the cost of the low-cost advocate.

Propositions 3 and 4 identify situations where raising advocates' costs can increase the median voter's utility. Raising advocate 1's cost raises both advocates' thresholds. Advocate 1 becomes less aggressive (r_1 moves closer to 1/2) and advocate 2 becomes more aggressive (r_2 moves away from 1/2). For any advocate 2 cost k_2, there is a $f(k_2)$ such that when 1's cost k_1 is less than $f(k_1)$, increasing k_1 benefits the median voter, while if $k_1 > f(k_1)$, increasing k_1 hurts the median voter. Increasing the high cost advocate's cost is never benefical (i.e., $f(k_2) < k_2$). Hence, when costs are sufficiently asymmetric, taxing the low cost advocate increases voter utilty.

It may not always be feasible to discriminate between advocates. Proposition 5 asks whether taxing both advocates can be beneficial. We show that when the asymmetry between candidates is sufficiently large, raising both advocates' costs benefits the voter.

These results provide a rationale for limiting campaign spending even when campaigns offer undistorted hard information that is useful to voters. In particular, such regulations can increase voters' utility when one advocate has a substantial advantage.

The war of information can also be used to analyze court proceedings. In this interpretation, player 3 is a trier of fact (judge or jury) and the advocates seek a favorable verdict. During a trial, advocates expend resources to generate information. In this context, we may interpret the (median) juror's decision rule as one advocate's *burden of proof.* We interpret the burden of proof as a policy variable. We ask how the burden of proof should to maximize a trial's informativeness given a symmetric objective function. Suppose the payoff is 1 if the advocate with the correct position wins and zero otherwise. The burden of proof specifies a threshold π such that advocate 1 wins if the juror's final belief is above π and advocate 2 wins if this belief is below π. We compute the optimal burden of proof and show that it favors the high cost advocate. Moreover, if both advocates are equally likely to be correct ex ante, then the optimal burden of proof renders a verdict for the high cost advocate more likely than a verdict for his opponent. Hence, the optimal burden of proof more than offsets the disadvantage of the high cost candidate.

Section 5 considers the war of information with asymmetric information. We assume that advocate 1 knows the state while advocate 2 does not. For example, suppose that the informed advocate is a defendant who knows whether he is liable or not. The jury rules for the plaintiff if the probability that the defendant is liable is greater than 1/2.

We focus on a particular equilibrium in which the innocent defendant never gives up. In that equilibrium, the prosecutor behaves as in the symmetric information case: he sets a threshold $r_2 > 1/2$ and quits when the belief (of the jury) reaches r_2. The guilty defendant also sets a threshold $r_1 < 1/2$. When the belief reaches the threshold r_1, the guilty defendant randomizes. He drops out at a rate that exactly compensates for any further evidence of guilt. Hence r_2 acts as a *signaling barrier*, i.e., beliefs never drop below r_2. A consequence of the signaling barrier is that even after a long trial that reveals strong evidence of liability the jurors' posterior remain favorable to the defendant as long as he continues with the trial.

In this equilibrium, the defendant gets acquitted with probability 1 whenver he is not liable irrespective of the players' costs. The probability of a wrong verdict favoring the defendant depends on the on the plaintiff's cost but not on the defendant cost. If the defendant's cost goes up, the trial ends quicker but is equally informative.

1.1 Related Literature

The war of information is similar to models of contests (Dixit (1987), and rent seeking games (Tullock (1980)). The key difference is that in a war of information resources generate decision-relevant information for the voter/juror.

The literature on strategic experimentation (Harris and Bolton (1999, 2000), Cripps, Keller and Rady (2005)) analyzes situations where agents incur costs to learn the true state but can also learn from the behavior of others. This literature focuses on the resulting free-rider problem. In our paper, as in Harris and Bolton (1999), the signal is a Brownian motion with unknown drift.[1] However, the war of information induces different incentives. In the war of information, advocates benefit from a low cost beyond the direct cost saving because a low cost deters the opponent from experimenting. In the strategic experimentation literature, lower costs facilitate opponents' free-riding.

Yilankaya (2002) also provides an analysis of the optimal burden of proof. His model assumes an informed defendant, an uninformed prosecutor, and an uninformed judge. Yilankaya's model is static; that is, advocates commit to a fixed expenditure at the beginning of the game. Yilankaya explores the trade-off between an increased burden of proof and increased penalties for convicted defendants. He shows that an increased penalty may lead to larger errors, i.e., a larger probability of convicting innocent defendants or acquitting guilty defendants. In our model, an increased penalty is equivalent to a lower cost for the defendant. Our analysis shows that if the defendent is informed, chaning the defendent's cost does not affect the trial's accuracy.

2. The Simple War of Information

The *War of Information* is a three-person, continuous-time game. We refer to players 1 and 2 as *advocates* and player 3 as the *voter*. Nature endows one of the advocates with the correct position. Then, the advocates decide whether or not to provide information about their positions. Once the flow of information stops, the voter chooses an advocate. The voter's payoff is 1 if he chooses the advocate with the correct position and 0 otherwise. An advocate receives a payoff of 1 if his policy is chosen and 0 otherwise. Advocate i incurs a flow cost $k_i/4$ while providing information.

Let p_t denote the probability that the voter assigns at time t to advocate i having the correct position and let T denote the time at which the flow of information stops. Hence, choosing player 1's is optimal if and only if $p_T \geq 1/2$ is optimal for the voter. Conversely, choosing player 2's is optimal if and only if $p_T \leq 1/2$. Define functions $I_i : [0,1] \to [0,1], i = 1,2$ as follows: $I_1(x) = x$ and $I_2(x) = 1 - x$. We say that player i is

[1] Moscarini and Smith (2001) analyze optimal experimentation in a decision problem with Brownian information.

trailing at time t if ruling for his opponent would be the unique optimal action for player 3 if the game were to end at time t. Hence, i is trailing at t if and only if

$$I_i(p_t) < 1/2 \tag{1}$$

We assume that only the player who is trailing at time t provides information.[2] Hence, the game stops whenever the trailing player quits. We say that the game is running at time t, if at no $\tau \leq t$ a trailing player has quit. As long as the game is running, all three player observe the process X, where

$$X_t = \mu t + Z_t \tag{2}$$

and Z is a Wiener process. Hence, X is a Brownian motion with drift μ and variance 1. We set $X_0 = 0$ and assume that no player knows μ and all three players assign probability 1/2 to each of the two outcomes $\mu = 1/2$ and $\mu = -1/2$. We identify $\mu = 1/2$ with advocate 1 holding the correct position, while $\mu = -1/2$ means that advocate 2 holds the correct position. Let

$$p(x) = \frac{1}{1 + e^{-x}} \tag{3}$$

for all $x \in I\!R$; for $x = -\infty$, we set $p(x) = 0$ and for $x = \infty$, we set $p(x) = 1$. A straightforward application of Bayes' Law yields

$$p_t := \text{ Prob}\{\mu = 1/2 \,|\, X_t\} = p(X_t)$$

and therefore, i is trailing if and only if

$$(-1)^{i-1} X_t < 0 \tag{4}$$

Hence, providing costly information gives the trailing advocate a chance to catch up.

In this section, we restrict both advocates to stationary, pure strategies. We call the resulting game the *simple war of information*. In the next section, we will show that this restriction is without loss of generality. A stationary pure strategy for player 1 is a number

[2] This assumption is discussed at the end of section 3. The equilibria analyzed below remain equilibria when players are allowed to provide information when ahead.

$y_1 < 0$ ($y_1 = -\infty$ is allowed) such that player 1 quits providing information as soon as X reaches y_1. That is, player 1 provides information as long as $X_t > y_1$ and stops at $\inf\{t \,|\, X_t = y_1\}$. Similarly, a stationary pure strategy for player 2 is an extended real number $y_2 > 0$ such that player 2 provides information if and only if $0 < X_t < y_2$ and stops as soon as $X_t = y_2(t)$. Let

$$T = \inf\{t > 0 \,|\, X_t - y_i = 0 \text{ for some } i = 1, 2\} \tag{5}$$

if $\{t \,|\, X_t = y_i \text{ for some } i = 1, 2\} \neq \emptyset$ and $T = \infty$ otherwise. Observe that the game runs until time T. At time $T < \infty$, player 3 chooses player i if and only if $X_T = y_j$ for $j \neq i$. If $T = \infty$, we let $p_T = 1$ and assume that both players win.[3] Let $y = (y_1, y_2)$ and let $v(y)$ denote the probability that player 1 wins given the strategy profile y; that is,

$$v(y) = \text{Prob}\{p_T > 1/2\}$$

More generally, the probability of player i winning is:

$$v_i(y) = I_i(v(y)) \tag{6}$$

To compute the advocates' cost given the strategy profile y, define $C : [0, 1] \to \{0, 1\}$ such that

$$C(s) = \begin{cases} 1 & \text{if } s < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

then, the expected information cost of player i given the strategy profile y is

$$c_i(y) = \frac{k_i}{4}\, E \int_0^T I_i(C(p_t))dt \tag{7}$$

Note that the expectation is taken both over the possible realizations of μ and the possible realizations of W. Then, the advocates' expected utilities are

$$U_i(y) = v_i(y) - c_i(y) \tag{8}$$

[3] This specification of payoffs for $T = \infty$ has no effect on the equilibrium outcome since staying in the game forever is not a best responses to any opponent strategy for any probability of winning. We chose this particular specification to simplify the notation and exposition.

while the voter's expected utility, (i.e.,, the accuracy of the campaign) is:

$$U_3(y) = E[\max\{p_T, 1 - p_T\}] \tag{9}$$

It is more convenient to describe behavior and payoffs as functions of the following transformations of strategies. Let

$$\alpha_i = (-1)^{i-1}\big(1 - 2p(y_i)\big)$$

Hence, $\alpha_1 = 1 - 2p(y_1) \in [0,1]$ and $\alpha_2 = 2p(y_2) - 1 \in [0,1]$. For both players, higher values of α_i indicate a greater willingness to bear the cost of information provision. If α_i is close to 0, then player i is not willing to provide much information; he quits at y_i close to 0. Conversely, if $\alpha_i = 1$, then player i does not quit no matter how far behind he is (i.e., $y_1 = -\infty$ or $y_2 = \infty$). Without risk of confusion, we write $U_i(\alpha)$ in place of $U_i(y)$, where $\alpha = (\alpha_1, \alpha_2) \in (0,1]^2$. Lemma 1 below describes the payoffs associated with a stationary, pure strategy profile given optimal voter behavior:

Lemma 1: *For any $\alpha = (\alpha_1, \alpha_2)$, the payoffs for the three players are as follows:*

$$U_i(\alpha) = \frac{\alpha_i}{\alpha_1 + \alpha_2}\Big(1 - k_i \alpha_j \ln \frac{1 + \alpha_i}{1 - \alpha_i}\Big)$$
$$U_3(\alpha) = \frac{1}{2} + \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}$$

where $i, j \in \{1, 2\}$, $j \neq i$. If $\alpha_i = 1$, then $U_i(\alpha) = -\infty$.

Lemma 2 below utilizes Lemma 1 to establish that player i's best response to α_j is well-defined, single valued, and differentiable. Furthermore, the simple war of information is dominance solvable. In section 3, we use this last fact to show that the war of information has a unique subgame perfect Nash equilibrium even if nonstationary and mixed strategies are permitted.

The function $B_i : (0,1] \to (0,1]$ is advocate 1's best response function if

$$U_1(B_1(\alpha_2), \alpha_2) > U_1(\alpha_1, \alpha_2)$$

for all $\alpha_2 \in (0,1]$ and $\alpha_1 \neq B_1(\alpha_2)$. Advocate 2's best response function is defined in an analogous manner. Lemma 2 below establishes that best response functions are well-defined. Then, α_1 is a Nash equilibrium strategy for advocate 1 if and only if it is a fixed-point of the mapping ϕ, where $\phi(\alpha_1) = B_1(B_2(\alpha_1))$. Lemma 2 below ensures that ϕ has a unique fixed-point.

Lemma 2: *There exists differentiable, strictly decreasing best response functions $B_i : (0,1] \to (0,1]$ for both advocates. Furthermore, if $\alpha^1 \in (0,1)$ is a fixed-point of ϕ, then $0 < \phi'(\alpha^1) < 1$.*

Proposition 1: *The simple war of information has a unique Nash equilibrium.*

Our first comparative statics result, Proposition 2, describes how equilibium behavior changes as costs change. Part (i) establishes that increasing an advocate's cost makes the advocate less aggressive and his opponent more aggressive. It follows from (i) that an advocate's own equilibrium payoff is decreasing in his own cost. Part (ii) observes that as an advocate cost approaches 0, he becomes infinitely aggressive, that is, he is willing to provide information no matter how unfavorable X_t becomes. Conversely, part (iii) shows that as an advocates cost approaches ∞, he is willing to provide no information.

Proposition 2: *Let $\alpha = (\alpha_1, \alpha_2)$ be the unique equilibrium of the simple war of information. Then,*

(i) $\dfrac{\partial \alpha_i}{\partial k_i} < 0$ and $\dfrac{\partial \alpha_i}{\partial k_j} > 0$

(ii) $\lim_{k^n} \alpha_i = 1$ *whenever* $k_i^n \to 0$

(iii) $\lim_{k^n} \alpha_i = 0$ *whenever* $k_i^n \to \infty$

for $i = 1,2$ and $j \neq 1$.

A corollary of Propositions 1 and 2 is that every interior strategy profile $\alpha \in (0,1) \times (0,1)$ is the equilibrium for some cost parameters (k_1, k_2), $k_i > 0$.

Corollary 2: *Let $\alpha = (\alpha_1, \alpha_2) \in (0,1) \times (0,1)$. There exist (k_1, k_2) such that α is the equilibrium of the simple war of information with costs (k_1, k_2).*

2.1 Taxing Advocates

Our next result (Proposition 3) provides conditions under which campaign accuracy improves as costs increase. Call i the better advocate if $k_i < k_j$ for $j \neq i$ and call i the worse advocate if $k_i > k_j$ for $j \neq i$. Proposition 3 shows that an increasing the worse advocate's cost always reduces accuracy. By contrast, increasing the better advocate's cost increases accuracy provided the costs are sufficiently different. When the the advocates's costs are close, increasing the better advocate's cost reduces accuracy.

Let $U_3^*(k_1, k_2)$ be player 3's equlibrium payoff if the costs are (k_1, k_2).

Proposition 3: *There exists a continuous, function $f : I\!R \to I\!R$ with $f(r) = 0$ for $r \leq \bar{r} < \infty$, f strictly increasing at $r > \bar{r}$, $f(r) < r$, and $f \to \infty$ as $r \to \infty$ such that*

$$(k_1 - f(k_2)) \frac{dU_3^*(k_1, k_2)}{dk_1} < 0$$

for $k_1 \neq f(k_2)$.

The corollary below restates Proposition 3 as follows: when the advocates' equilibrium strategies are similar, increasing either cost hurts the voter. Conversely, when one advocate is much more aggressive than the other, increasing the more aggressive advocate's cost benefits the voter while increasing the less aggressive candidate's cost hurts the voter.

Let $g : (0, 1] \to (0, 1]$ be such that $g(z)$ satisfies

$$g(z) = \frac{z^2}{2(z + g(z))} \cdot \left[1 - g(z)^2 + \frac{(1 - g(z)^2)^2}{2g(z)} \ln \left(\frac{1 + g(z)}{1 - g(z)} \right) \right]$$

It is straightforward to show that g is well defined, continuous, and strictly increasing with $z > g(z)$ and $g(z) \to 0$ as $z \to 0$. Moreover, $g(1)$ is approximately 0.48. Figure 1 below depicts the graph of the function g.

—- Insert Figure 1 here ———

Corollary 3 below follows from Propositions 2 and 3: Let (α_1, α_2) be the unique equilibrium stategy profile given the costs (k_1, k_2). Then, increasing advocate 1's cost benefits the voter if $\alpha_2 < g(\alpha_1)$ and hurts the voter if $\alpha_2 > g(\alpha_1)$.

Corollary 3: *If $g(\alpha_1) \neq \alpha_2$, then*

$$(g(\alpha_1) - \alpha_2)\frac{dU_3^*(k_1, k_2)}{dk_1} > 0$$

Propositions 3 and 4 examine the case where only one advocate is taxed. In some cases, such a discriminatory tax may be infeasible. The next proposition shows that even a uniform tax on information provision can be beneficial provided that the high cost advocate has sufficiently high cost. Let $\alpha = (\alpha_1, \alpha_2)$ be the equilibrium of the simple war of information with costs $(k_1 + t, k_2 + t)$.

Proposition 4: *For every k_1, there is $\bar{k}_2$ such that for $k_2 > \bar{k}_2$*

$$\left.\frac{dU_3^*(k_1 + t, k_2 + t)}{dt}\right|_{t=0} > 0$$

Propositions 4 follows form Proposition 3 and Corollary 3: observe that a small increase in k_2 has negligible effect compared to a small increase in k_1 when k_2 is large.

2.2 The Value of Campaigns

We have assumed that the states have equal prior probability. To deal with an arbitrary prior π, we can choose the initial state $X_0 = x_0$ so that $p(x_0) = \pi$. The equilibrium strategies are unaffected by the choice of the initial state and hence if (α_1, α_2) is the equilibrium for $X_0 = 0$, then (α_1, α_2) is also an equilibrium for $X_0 = x_0$.

If $\pi \neq 1/2$ (the threshold for player 3), then one of the advocates may quit immediately. In particular, let $\alpha = (\alpha_1, \alpha_2)$ denote the equilibrium strategies. If

$$\pi \leq \frac{1 - \alpha_1}{2}$$

then player 1 gives up immediately and player 3's payoff is $1 - \pi > 1/2$. Similarly, if

$$\pi \geq \frac{1 + \alpha_2}{2}$$

then player 2 gives up immediately and player 3's payoff is $\pi > 1/2$.

Without the campaign, player 3's payoff is $\max\{\pi, 1-\pi\}$. Therefore, the *value of the campaign* for the voter is

$$V = U_3 - \max\{\pi, 1-\pi\}$$

Proposition 5 describes the value of a campaign V as a function of the strategies. Its corollary shows that V goes to 0 as k_i goes to infinity.

Proposition 5: *Let (α_1, α_2) be the equilibrium of the simple war of information.*

(i) If $\pi \notin \left[\frac{1-\alpha_1}{2}, \frac{1+\alpha_2}{2}\right]$, then $V = 0$.

(ii) If $\pi \in \left(\frac{1-\alpha_1}{2}, \frac{1+\alpha_2}{2}\right)$ and $\pi \geq 1/2$, then $V = \frac{a_1 a_2}{a_1+a_2} + \frac{a_1}{a_1+a_2}(1-2\pi)$.

Proof: Part (i) is obvious. For part (ii) let v be the probability that player 1 wins. Then,

$$v\frac{1+a_2}{2} + (1-v)\frac{1-\alpha_1}{2} = \pi$$

To see this, note that player 1 wins if $p(X_T) = \frac{1-\alpha_1}{2}$ and loses if $p(X_T) = \frac{1+\alpha_2}{2}$. The claim then follows from the fact that the stochastic process $p(X_t)$ is a martingale. Substituting for v in

$$V = v\frac{1+a_2}{2} + (1-v)\frac{1+\alpha_1}{2} - \pi$$

then yields the result $\square$

Corollary 5: *If $k_i \to \infty$ for some i, then $V \to 0$.*

Note that if $\pi > 1/2$ and k_2 is sufficiently large, then player 2 will quit immediately and hence we are in the case of Proposition 5(i) so that $V = 0$. On the other hand, if $k_1 \to \infty$, then Proposition 2(iii) implies that $V \to 0$.

Proposition 5 and the Corollary illustrate the complementary value of the advocates actions for voters. If one advocate gives up very quickly, then campaigns have no social value. This is true, even if the campaign is informative, i.e., even if $p_T \neq \pi$.

2.3 Arbitrary Drift and Variance

In the simple war of information, the drift of X_t is $\mu \in \{-1/2, 1/2\}$ and the variance is $\sigma^2 = 1$. Also, the prior probability that $\mu = 1/2$ is $1/2$. We show in this section that these assumptions are normalizations and hence are without loss of generality.

First, we consider an σ^2. We can rescale time so that each new unit corresponds to $\frac{1}{\sigma^2}$ old units. Hence, the cost structure with the new time units is $k_* = \sigma^2 k$, where $k = (k_1, k_2)$ is the cost structure with the old time units. Note also that the variance of X_t with the new time units is 1. Hence, the analysis of section 1 applies after replacing k with k_*.

Next, we consider arbitrary drifts but maintain the other assumptions. Consider values of μ_1, μ_2 such that $\mu_1 - \mu_2 > 0$. By Bayes' Law the conditional probability of advocate 1 holding the correct position given X_t is:

$$p_t = \frac{1}{1 + e^{A(t)}}$$

where $A(t) = -(\mu_1 - \mu_2) + \frac{(\mu_1 - \mu_2)(\mu_1 + \mu_2)t}{2}$. The voter rules in favor of player 1 if $p_t \geq 1/2$; that is, if

$$X_t \geq \frac{\mu_1 + \mu_2}{2} \cdot t$$

Hence, player i is trailing whenever $I_i(X_t - \frac{\mu_1 + \mu_2}{2} \cdot t) < 0$. When $\mu_1 + \mu_2 \neq 0$, the voter's and hence the advocates optimal strategies will be time-dependent. Suppose player i quits when $X_t = Y_t^i$ for Y_t^i defined by

$$Y_t^i = y_i + \frac{\mu_1 + \mu_2}{2} \cdot t \tag{10}$$

for y_i such that $I_i(y_i) > 0$. Hence, the moment player i quits, (i.e., $X_t = Y_t^i$) we have

$$p_t = \frac{1}{1 + e^{-y_i}}$$

Thus, strategies (y_1, y_2) described in (10) are stationary in the sense that they are time-independent functions of p_t. Moreover, player i is trailing whenever $I_i(X_t - \frac{\mu_1 + \mu_2}{2} \cdot t) < 0$. Hence, the winning probabilities and the expected costs for the strategy profile (y_1, y_2) in this game are the same as the winning probabilities and the expected costs associated with $y = (y_1, y_2)$ in the simple war of information and the analysis of section 1 applies.

Combining the arguments of this subsection establishes that Propositions 1-3 generalize to the case of arbitrary μ_1, μ_2, and σ^2 provided we replace k_i with $\frac{\sigma^2 k_i}{\mu_1 - \mu_2}$ for $i = 1, 2$.

Let $\delta = \frac{\sigma^2}{\mu_1 - \mu_2}$; hence $1/\delta$ is the precision of the campaign. Then, we can state payoffs associated with the profile $\alpha = (\alpha_1, \alpha_2)$ in the game with arbitrary σ^2, μ_1, μ_2 as follows:

$$\begin{aligned} U_i(\alpha) &= \frac{\alpha_i}{\alpha_1 + \alpha_2}\Big(1 - k_i \alpha_j \delta \ln \frac{1+\alpha_i}{1-\alpha_i}\Big) \\ U_3(\alpha) &= \frac{1}{2} + \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \end{aligned} \tag{11}$$

where $i, j \in \{1, 2\}$, $j \neq i$. If $\alpha_i = 1$, then $U_i(\alpha) = -\infty$ for $i \neq 3$. Comparing (11) with the payoffs for the simple war of information described in Lemma 1 reveals that the analysis in the previous sections extends immediately to the case of general μ_1, μ_2, and σ^2.

The parameter $1/\delta$ measures the information's precision. Proposition 6 below utilizes (11) to establish the limits as δ converges to zero or infinity. Define $h : \mathbb{R}_+ \to [0, 1]$ as follows:

$$h(x) = \frac{1}{3x}\left(x + 2\sqrt{1 - x + x^2} - 2\right)$$

and note that $h(1) = 1/3, h(0) = 0$ and $h \to 1$ as $r \to \infty$.

Proposition 6: *Let $\alpha = (\alpha_1, \alpha_2)$ be the unique equilibrium of the war of information with the cost structure (k_1, k_2) and precision $1/\delta$. Then,*

$$\begin{aligned} &(i) \lim_{\delta_n \to 0} U_j(\alpha) = 1/2; && \lim_{\delta_n \to 0} U_3(\alpha) = 1 \\ &(ii) \lim_{\delta_n \to \infty} U_j(\alpha) = h(k_j/k_i); && \lim_{\delta_n \to \infty} U_3(\alpha) = 0 \end{aligned}$$

for $i, j = 1, 2, j \neq i$.

Proposition 6 states that as the trial becomes very precise, the juror always makes the correct decision and the advocates information costs vanish. As the trial becomes very imprecise, no information will be revealed - hence the juror's payoff converges to 1/2 - but advocates receive a positive payoff that depends on the ratio of their costs. If the costs are equal, then this payoff is 1/3. If one advocate has a large cost advantage, then this advocate will receive a payoff of 1 (and his opponent receives a payoff of zero.)

As the trial becomes infinitely precise, an advocate can nearly guarantee that he wins whenever he holds the correct position. Hence, for any $\epsilon > 0$, he can make sure that his payoff is within ϵ of 1/2, yielding (i).

3. Nonstationary Strategies and Subgame Perfection

In this section, we relax the restriction to stationary strategies. We will show that the unique equilibrium of the simple war of information is also the unique subgame perfect equilibrium of the dynamic game.

With nonstationary strategies, it is possible to have Nash equilibria that fail subgame perfection. To see this, let $\hat{\alpha}_2 = B_2(1)$ and $\hat{\alpha}_1 = B_1(\hat{\alpha}_2)$, where B_i are the stationary best response functions analyzed in Section 2. Hence, $\hat{\alpha}_2$ is advocate 2's best response to an opponent who never quits and $\hat{\alpha}_1$ is advocate 1's best response to an opponent who quits at $\hat{\alpha}_2$.

Define the function $a_i : I\!R \to [0, 1]$ as

$$a_i(x) = (-1)^{-i}(1 - 2p(x))$$

where p is as defined in (3). Consider the following strategy profile: $\alpha_2 = \hat{\alpha}_2$ and $\alpha_1 = \hat{\alpha}_1$ if $a_2(X_\tau) < \hat{\alpha}_1$ for all $\tau < t$ and $\alpha_1 = 1$ otherwise. Hence, advocate 2 plays the stationary strategy $\hat{\alpha}_2$ while advocate 1 plays the strategy $\hat{\alpha}_1$ along any history that does not require advocate 2 to quit. But if 2 deviates and does not quit when he is supposed to, then advocate 1 switches to the strategy of never quitting.

To see why this is a Nash equilibrium, note that 1's strategy is optimal by construction. For player 2, quitting before X_t reaches x is clearly suboptimal. Not quitting at $X_t = x$ is also suboptimal since such a deviation triggers $\alpha_1 = 1$. However, the strategy profile is not subgame perfect because the 1's behavior after a deviation by 2 is suboptimal: at any X_t such that $a_1(X_t) < \hat{\alpha}_1$, advocate 1 would be better off quitting.

To simplify the analysis, we will utilize a discrete version of the war of information. Advocates choose their action and observe the stochastic process X_t only at times $t \in \{0, \Delta, 2\Delta, \ldots\}$. The initial state is x_0, i.e., $X_0 = x_0$. We refer to $t = n\Delta$ as period n. Each period n, player i chooses $\alpha_i \in [0, 1]$. The game ends at $t \in [(n-1)\Delta, n\Delta]$ if

$$t = \inf\{\tau \in [n\Delta, (n+1)\Delta] \,|\, a_i(X_\tau) \le \alpha_i \text{ for some } i = 1, 2\}$$

If

$$\{\tau \in [(n-1)\Delta, n\Delta] \,|\, a_i(X_\tau) \le \alpha_i \text{ for some } i = 1, 2\} = \emptyset$$

the game continues and the players choose new α_i's in period $n+1$. Note that $\alpha_i \leq a_i(x_0)$ means that player i quits immediately.

A pure strategy for player i in period n associates with every history $(X_0, \ldots, X_{(n-1)\Delta})$ an action:

Definition: *A pure strategy for player i is a sequence $f^i = (f_1^i, f_2^i, \ldots)$ such that $f_n^i : \mathbb{R}^n \to [0,1]$ is a measurable function for all n.*

Let n^* be the smallest integer n such that for some $t \in [(n-1)\Delta, n\Delta]$ and some $i = 1, 2$

$$a_i(X_t) \leq f_n^i(X_0, \ldots, X_{(n-1)\Delta})$$

If $n^* = \infty$, set $T = \infty$. If $n^* < \infty$ let

$$T = \inf\{t \in [(n^*-1)\Delta, n^*\Delta] \,|\, a_i(X_t) \leq f_{n^*}^i(X_0, \ldots, X_{(n^*-1)\Delta}) \text{ for some } i = 1, 2\}$$

The game ends at time T. Given the definition of T, the payoffs of the game are defined as in the previous section (Equations (6)-(8)).

The advocate's payoffs following a history $\zeta = (x_0, x_1, \ldots, x_{k-1})$ are defined as follows: Let $\hat{f} = (\hat{f}^1, \hat{f}^2)$ where $\hat{f}_n^i(\hat{x}_0, \ldots, \hat{x}_{n-1}) = f_{n+k}^i(\zeta, \hat{x}_0, \ldots \hat{x}_{n-1})$ for all $n \geq 2$. Hence, we refer to $\zeta \in \mathbb{R}^n$ as a subgame and let $U^i_{(\zeta, \hat{x}_0)}(f) = U^i_{\hat{x}_0}(\hat{f})$.

Definition: *The strategy profile f is a subgame perfect Nash equilibrium if and only if*

$$U^1_{(\zeta,\hat{x}_0)}(f) \geq U^1_{(\zeta,\hat{x}_0)}(\tilde{f}^1, f^2)$$
$$U^2_{(\zeta,\hat{x}_0)}(f) \geq U^2_{(\zeta,\hat{x}_0)}(f^1, \tilde{f}^2)$$

for all $\tilde{f}^1, \tilde{f}^2$.

Let E be the set of all subgame perfect Nash equilibria and let E_i be the set of all subgame perfect Nash equilibrium strategies of player i; that is,

$$E^i = \{f^i \,|\, (f^1, f^2) \in E \text{ for some } f^j, j \neq i\}$$

Let $\alpha = (\alpha_1, \alpha_2)$ be the unique equilibrium of the simple war of information studied in the previous section. Without risk of confusion, we identify $\alpha' \in [0,1]$ with the constant function $f_n^i = \alpha'$ and the stationary strategy $f^i = (\alpha', \alpha', \ldots)$. The proposition below establishes that the stationary strategy profile α is the only subgame perfect Nash equilibrium of the game.

Proposition 7: *The strategy profile α is the unique subgame perfect Nash equilibrium of the discrete war of information.*

Proof: See Appendix. □

Next, we provide intuition for Proposition 7. Let $\bar{\alpha}_i$ be the supremum of i's strategy and let $\underline{\alpha}_i$ be the infimum. We show in the proof of Proposition 5

$$B_2(\underline{\alpha}_1) \geq \bar{\alpha}_2$$

and

$$B_1(\bar{\alpha}_2) \leq \underline{\alpha}_1$$

and therefore

$$B_1(B_2(\underline{\alpha}_1)) \equiv \phi(\underline{\alpha}_1) \leq B_1(\bar{\alpha}_2) \leq \underline{\alpha}_1$$

The stationary equilibrium is a fixed point of ϕ and, moreover, ϕ has slope less than 1 (Proposition 1). Therefore,

$$\phi(\underline{\alpha}_1) \leq \underline{\alpha}_1$$

which implies that

$$\underline{\alpha}_1 \geq \alpha_1$$

A symmetric argument shows that $\bar{\alpha}_1 \leq \alpha_1$ and hence Proposition 5 follows.

3.1 Both Advocates Buy Information

So far, we have assumed that only the trailing advocate buys information. We made this assumption to simplify the analysis, in particular, to avoid having to specify the information flow when both advocates are expanding resources.

A general model would allow the advocate that is ahead to increase the information flow, if he chooses. Note that the equilibrium of the simple war of information characterized above would remain an equilibrium even in such a general model. Given a stationary strategy for i, advocate $j \neq i$ has no incentive to provide additional information when he is ahead. To see this, note that p_t is a martingale and therefore a player cannot increase his probability of winning by increasing the information flow. Since information provision is costly it follows that such a deviation can never be profitable.

Consider the following simple extension of our model: both agents can incur information costs at the same time but the added effort of the second advocate does not change the information flow. In that case, if players cannot observe *who* provides information, it is a dominant strategy for the leading advocate not to provide information. It can be shown that even if the identity of the information provider is observable, the equilibrium of the simple war of information remains the unique subgame perfect equilibrium.

If simultaneous purchase of information by both advocates leads to faster learning, there may be subgame perfect equilibria other than the unique equilibrium of the simple war of information.

4. Asymmetric Standards of Proof

So far, we have assumed that player 3 chooses player 1 if and only if the probability that 1 has the correct position is greater than 1/2. In this section, we relax this symmetry and consider an arbitrary threshold $\gamma \in [0, 1]$ such that player 3 chooses advocate 1 if $p_T > \gamma$ and chooses advocate 2 if $p_T < \gamma$.

The purpose of this extension is to examine the effect of different standards of proof on information provision. Suppose players 1 and 2 are litigants and player 3 is the juror. Suppose further, that the jury is committed (by law) to a particular standard of proof. Proposition 9 below characterizes the optimal γ, i.e., the optimal standard of proof.

Let W^γ denote the simple war of information with threshold (standard of proof) γ. As before, let p_t denote the probability that player 3 assigns at time T to player i having the correct position. Player 1 is trailing if $p_t < \gamma$ and player 2 is trailing if $p_t > \gamma$. As before, we assume that only the trailing player can provide information. A stationary strategy for player 1 is denoted $z_1 \in [0, \gamma]$ and a strategy for player 2 is a $z_2 \in [\gamma, 1]$. The interpretation is that player 1 quits when $p_t \leq z_1$ and player 2 quits when $p_t > z_2$.

As we have argued in the previous section, the equilibrium of the war of information stays unchanged if we change the prior from $1/2$ to π since a different prior can be thought of as a different starting point of the stochastic process X. Let

$$\pi = p(x_0) = \frac{1}{1+e^{-x_0}} = \gamma$$

where $x_0 = X_0$. Hence π is the starting point of the stochastic process p_t.

Let $z = (z_1, z_2)$ and let $v_i(z)$ denote the probability that player i wins given the strategy profile z; that is,

$$v_i^\gamma(z) = \frac{\pi - z_i}{z_j - z_i}$$

for $j \neq i$. To compute the advocates' cost associated with the strategy profile y, define $C : [0,1] \to \{0,1\}$ such that

$$C^\gamma(s) = \begin{cases} 1 & \text{if } s < \gamma \\ 0 & \text{otherwise} \end{cases}$$

then, the expected information cost of player i given the strategy profile z is

$$c_i^\gamma(z) = \frac{k_i}{4} E \int_0^T I_i(C(p_t))dt$$

Then, the advocates' expected utilities are

$$U_i^\gamma(z) = v_i^\gamma(z) - c_i^\gamma(z)$$

while the juror's expected utility, (i.e.,, the accuracy of the trial) is:

$$U_3^\gamma(y) = E[\max\{p_T, 1-p_T\}]$$

The following proposition shows that the results for the simple war of information carry over to W^γ. Note that the equilibrium strategy profile does not depend on the initial belief π. However, if this belief is not in the interval (z_1, z_2) the game ends immediately.

Proposition 8: *The simple war of information with threshold γ has a unique equilibrium (z_1, z_2). If $I_i(\pi) \leq z_i$, then player i quits at time 0.*

Proof: See Appendix.

Consider the situation where player 3 places the same weight on both mistakes, i.e.,

$$U_3 = \max\{p_T, 1 - p_T\}$$

where T is the time when the process of information provision stops. Assume the threshold γ is chosen independently of player 3's utility function to maximize U_3. In other words, player 3 commits to a threshold γ prior to the game.

If player 3 commits to $\gamma \in (0,1)$ then advocates 1 and 2 play the game W^γ. The results below analyzes how a change in γ affects player 3's payoff. For the remainder of this section, we assume that in the initial state x_0, the belief is 1/2, i.e., $\pi = 1/2$. Note that the equilibrium strategies characterized in Proposition 8 remain equilibrium strategies for $p_0 = 1/2$. However, for $z_1 \geq 1/2$ or $z_2 \leq 1/2$ the game ends immediately.

If the game ends immediately, then the payoff player 3's is 1/2 and hence the value of the information generated by the war of information is zero. The lemma below describes how changing γ affects player 3's payoff if $z_1 < 1/2 < z_2$, i.e., the game does not end at time 0.

Lemma 3: *Let (z_1, z_2) be the equilibrium of W^γ and assume that $z_1 < 1/2 < z_2$. Then, U_3 is increasing in γ if*

$$\frac{(z_2 - \gamma)(2z_2 - 1)^2}{z_2^2(1 - z_2)^2} < \frac{(\gamma - z_1)(1 - 2z_1)^2}{z_1^2(1 - z_1)^2}$$

Proof: See Appendix.

Increasing γ implies that player 1 incurs a greater share of the cost of the war of information. The optimal choice of γ depends on the costs of players 1 and 2. Assume that player 1 is the low cost advocate, i.e., $k_1 < k_2$ and Proposition 2 then implies that for $\gamma = 1/2$ player 1 wins with greater probability than player 2. This follows because, in equilibrium

$$1/2 - z_1 > z_2 - 1/2$$

and the win probability of advocate i is equal to

$$v_i(z) = \frac{1/2 - z_i}{z_j - z_i}$$

The next proposition shows that at the optimal γ it must be the case the player 2 wins with greater probability than player 1. Hence, the optimal γ implies that the high cost advocate wins with greater probability than the low cost advocate.

Recall that $\pi = 1/2$ and therefore both candidates are equally likely to be holding the correct position. Let $U_3^*(\gamma)$ denote the player 3's equilibrium utility when the burden of proof is $\gamma \in (0,1)$, let γ^* be a maximizer of U_3^*, and finally, let(z_1^*, z_2^*) be the equilibrium of W^{γ^*}.

Proposition 9: *If $k_1 < k_2$, then $\gamma^* > 1/2$ and $v_2(z^*) > v_1(z^*)$.*

Proof: See Appendix.

Proposition 9 shows that it is optimal to shift costs to the low cost advocate. Moreover, at the optimal threshold the low cost advocate wins with lower probability than the high cost advocate. Hence, the shift in the threshold more than compensates for the initial cost advantage.

5. Asymmetric Information

In this section, we study a war of information with asymmetrically information. We assume advocate 1 knows the state while players 2 and 3 have the same information as in section 2. Hence, there are two types of advocate 1: type 0 knows that $\mu = -1/2$ and type 1 knows that $\mu = 1/2$. As in section 2, we assume that $\gamma = \pi = 1/2$ and $\sigma^2 = 1$.

5.1 Mixed Strategies

Let $(\Omega^i, \mathcal{F}^i, \mathcal{P}^i)$ be probability spaces for $i \in \{0,1\}$. Let X^i be a Brownian motion on $(\Omega^i, \mathcal{F}^i, \mathcal{P}^i)$ with mean $(-1)^{i-1}/2$, variance 1, and let $\mathcal{F}_t^i$ be the filtration generated by X^i. It will be convenient to treat player 1 types as separate players (0 and 1) with identical costs (i.e., $k_0 = k_1$). Player 0's knows that the signal is X^0 and player 1 knows that the signal is X^1. Consider the measurable space $(\Omega^2, \mathcal{F}^2)$ where

$$\Omega^2 = \Omega^0 \cup \Omega^1 \text{ and } \mathcal{F}^2 = \{A^0 \cup A^1, \mid A^i \in \mathcal{F}^i, i = 0, 1\}$$

Since player 2 does not know μ, his beliefs about the signal are described the probability P^2 on $(\Omega^2, \mathcal{F}^2)$ such that for all $A^1 \in \mathcal{F}^1, A^2 \in \mathcal{F}^2$,

$$P(A^1 \cup A^2) = \frac{1}{2}(P^1(A^1) + P^2(A^2))$$

Define player 2's filtration on $(\Omega^2, \mathcal{F}^2)$ as follows:

$$\mathcal{F}_t^2 = \{A^0 \cup A^1 \mid A^i \in \mathcal{F}_t^i \text{ and } X_\tau^0(A^0) = X_\tau^1(A^1) \text{ for all } \tau \leq t\}$$

A (general) strategy for player $i = 0, 1, 2$ is a probability measure η^i on $(I\!R_+^e \times \Omega^i, \sigma(\mathcal{B}^e \times \mathcal{F}^i))$, where $I\!R_+^e = I\!R_+ \cup \{\infty\}$, $\mathcal{B}^e = \{E \subset I\!R_+^e \mid E \cap I\!R \in \mathcal{B}\}$ and $\mathcal{B}$ are the Borel sets of $I\!R$. For any $A \in I\!R_+^e$, let $\eta^i[A|\mathcal{F}^i]$ be a version of the conditional probability of A given $\mathcal{F}^i$. In particular, let

$$Q_t^i(\omega) = \eta^i[A|\mathcal{F}^i]_\omega$$

for all $\omega \in \Omega$, $A = [0, t]$, $t \in I\!R_+$. Hence, $Q_t^i(\omega)$ is non-decreasing and right-continuous in t for every ω. To be consistent with informational and strategic requirements of the war of information, η^i must satisfy the following additional properties: for all $A \in \mathcal{F}$, ω, $t \geq s$,

$$\text{(i)} \;\; \eta^i(I\!R_+^e \times A) = P^i(A)$$

$$\text{(ii)} \;\; Q_t^i \text{ is } \mathcal{F}_t^i \text{ measurable}$$

Condition (i) captures the fact that P^i describes players' beliefs over Ω^i; condition (ii) ensures that players' strategies are feasible given their information. Henceforth, we identify player i's mixed strategies with Q^i (or the corresponding to η^i) satisfying the two conditions above. We interpret $Q_t^i(\omega)$ as the probability with which advocate i quits by time t given the sample point ω, conditional on advocate $j \neq i$ continuing until time t.

5.2 Payoffs

Let $p_t(\omega)$ denote the probability assessment of player 2 at time t that his opponent is type 1 at the state ω. Hence,

$$p_t = \frac{1 - Q_t^1}{1 - Q_t^1 + (1 - Q_t^0)e^{-X_t}} \tag{12}$$

If both Q_t^0, Q_1^1 are 1, let $p_t = 1/2$. Then, let

$$Q_t^{01}(\omega) = (1 - p_t(\omega))Q_t^0(\omega) + p_t(\omega)Q_t^1(\omega)$$

Hence, $Q_t^{01}(\omega)$ is player 2's belief that his opponent will quit by time t at state ω.

For $q = (Q^0, Q^1, Q^2)$, we can compute, $D_t^i(\omega)$, player i assessment of the probability that the war of information ends by time t at a particular ω as follows:

$$D_t^i(\omega) = \begin{cases} Q_t^0(\omega) + Q_t^2(\omega) - Q_t^0(\omega)Q_t^2(\omega) & \text{for } i = 0 \\ Q_t^1(\omega) + Q_t^2(\omega) - Q_t^1(\omega)Q_t^2(\omega) & \text{for } i = 1 \\ Q_t^{01}(\omega) + Q_t^2(\omega) - Q_t^{01}(\omega)Q_t^2(\omega) & \text{for } i = 2, 3 \end{cases} \tag{13}$$

Equation (13) follows from the fact that conditional on what the players know at time t, their quitting decisions are independent. Since advocates can only quit when trailing, for all ω, $Q_t^i(\omega)$ and $Q_t^2(\omega)$ have no common points of discontinuity for $i = 0, 1$. Hence, we can restate $D_t^i(\omega)$ as follows:

$$D_t^i(\omega) = \int_{\tau=0}^{t} (1 - Q_\tau^{j(i)}(\omega))dQ_\tau^i(\omega) + \int_{\tau=0}^{t} (1 - Q_\tau^i(\omega))dQ_\tau^{j(i)}(\omega) \tag{14}$$

where $j(0) = j(1) = 2$ and $j(2) = 01$. Note that the first term on the right-hand side of (14) is the probability that advocate 1 ends the game at some time $\tau \leq t$. The second term is the corresponding expression for player 2. Hence, the probability that advocate i wins is

$$v_i(q) = 1 - E \int_{t=0}^{\infty} (1 - Q_t^{j(i)}(\omega))dQ_t^i(\omega)$$

Expectations are taken with the probability P^i.

For an arbitrary probability process $\tilde{p}$ and strategy profile q define advocate i's expected cost as follows:

$$c_i(\tilde{p}, q) = \frac{k_i}{4} E\Big\{ \int_{t=0}^{\infty} \int_{\tau=0}^{t} I_i(C(\tilde{p}_\tau))d\tau dD_t^i(\omega) \Big\}$$

for $i = 0, 1, 2$.

Define the advocates' utilities as

$$U_i(\tilde{p}, q) = v_i(q) - c_i(\tilde{p}, q)$$

Note that conditional on ω and the game ending at time t, the payoff to the juror is $p(X_t(\omega))$ if $C_t(\omega) = 1$ and $1 - p(X_t(\omega))$ if $C_t(\omega) = 0$. Hence, the expected payoff of player 3 is:

$$U_3(\tilde{p}, q) = E\Big\{ \int_{t=0}^{\infty} I_{2-C_t(\omega)}\big(\tilde{p}_t(\omega)\big) dD_t^3(\omega) \Big\}$$

Let $q = (Q^0, Q^1, Q^2)$ and $\hat{q} = (\hat{Q}^0, \hat{Q}^1, \hat{Q}^2)$ be two strategy profiles and let p be the belief process induces by (Q^0, Q^1) (hence, p is as defined in equation (12) above). Finally, let $(q, \hat{q}, i)$ denote the strategy profile obtained by replacing Q^i in q with $\hat{Q}^i$. The profile q is a Nash equilibrium if we have $U_i(p, q) \geq U_i(p, (q, \hat{q}, i))$ for all $\hat{q}$ and all i.

5.3 Equilibrium Strategies

In this section, we demonstrate that the following strategies constitute an equilibrium in the war of information with asymmetric information. Let $Y_t = \inf_{\tau < t} X_\tau$ and for any $z > 0$, let $Y_t^z = \min\{0, Y_t - z\}$. The following strategy profile $q(x, y) = (Q^0(x), Q^1, Q^2(x, y))$ depends on two parameters, $x, y > 0$. Let

$$\begin{aligned}
Q_t^0(x) &= 1 - e^{Y_t^x} \\
Q_t^1 &\equiv 0 \\
Q_t^2(x, y) &= \begin{cases} 1 & \text{if } X_t \geq Y_t^x + y \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

In this strategy profile type 1 never quits and type 2 randomizes when $X_t = Y_t^x$. Therefore, if player 1 quits it must be type 0 and hence $p_t = 0$ conditional on player 1 quitting. By equation (12), along any path where player 1 does not quit, we have

$$\begin{aligned}
p_t &= \frac{1 - Q_t^1}{1 - Q_t^1 + (1 - Q_t^0)e^{-X_t}} \\
&= \frac{1}{1 + e^{Y_t^y - X_t}}
\end{aligned}$$

Note that

$$p_t \geq \frac{1}{1 + e^{-x}} \equiv \underline{p}$$

and therefore, conditional on player 1 not quitting beliefs are bounded below by $\bar{p}$. We refer to $\bar{p}$ as a *signaling barrier*.

The strategy profile $q(x, y)$ has the property that (1) player 1 never quits; (2) player 0 does not quit for $p_t > \underline{p}$ and randomizes when $p_t = \underline{p}$. Player 2 quits when $p_t = \bar{p}$ where

$$\bar{p} := \frac{1}{1 + e^{-y}}$$

Let y^* satisfy

$$\frac{1 + B_2(1)}{2} = \frac{1}{1 + e^{-y^*}}$$

Hence, y^* corresponds to the optimal threshold of player 2 against an opponent who never quits. (Recall that $\frac{1+\alpha_2}{2}$ is the belief threshold corresponding to the action α_2 in the simple war of information).

Lemma 4: *Player 2's unique best response to* $(Q_0(x), Q_1)$ *is* $Q_2(x, y^*)$.

In the proof of Lemma 4, we show that the uninformed player's (i.e., player 2) payoff is the payoff he would receive in the symmetric information game against an opponent who never quits. Therefore, player 2's best response is a belief-threshold $\bar{p}^*$ that corresponds to the symmetric information game strategy $\alpha_2 = B_2(1)$. Recall that the strategy α_2 corresponds to the belief threshold $\frac{1+\alpha_2}{2}$. Define the following maximization problem:

$$\max_{x<0} \Pi(x, y^*) := \frac{1}{1 - e^{y^* - x}} \left(1 - e^{-x} - 2k_1(1 - e^{y^*})(1 - e^{-x}(1 + x))\right) \qquad (*)$$

Lemma 5: *(i) The maximization problem (*) has a unique solution* x^**. (ii) The strategies* $Q^1, Q^0(x^*)$ *are a best response to* $Q(x^*, y^*)$*.*

Lemmas 2 and 3 imply that $q(x^*, y^*) = Q_0(x^*), Q_1, Q_2(x^*, y^*)$ is a Nash equilibrium of the war of information with asymmetric information.

Proposition 10: *The strategy profile* $q(x^*, y^*)$ *is an equilibrium of the war of information with asymmetric information.*

Note that since player 1 never quits, p_t is well-defined after every history. Hence, the equilibrium in Proposition 10 is also a sequential equilibrium.

In the asymmetric information, player 1 signals his strength by not quitting. As long as $p_t > \underline{p}$ for all $t \in (0, \tau)$, the equilibrium $q(x^*, y^*)$ is like the equilibrium of the symmetric information game; the signal X_τ alone determine beliefs. Once p_t hits $\underline{p}$, player 0's quit decision also affects beliefs. In fact, type 0 quits at a rate that exactly offsets any negative information revealed by the signal X_t so that $p_t = \underline{p} := \frac{1}{1+e^{-y}}$ ends: beliefs are bounded below by $\underline{p}$, the signaling barrier. Of course, once player 0 quits $p_t = 0$.

Let T be the time at which the war of information with an informed player 1 ends. Note that in equilibrium X_T can be arbitrarily small; that is, for all $x \in I\!\!R$, the probability that $X_T \leq x$ is strictly positive. Yet, p_T will never be less than $p(y)$ as long as the game continues. This occurs because of signalling. Note that if the informed player has not quit at t such that $X_t < y^*$, then one of the following must be true: either he knows that $\mu = 1/2$ (i.e., he is player 1) or he knows he is player 0 but by chance his random quitting strategy had him continue until time t. The probability of player 0 quitting by time t given that player 2 has not quit before t is $1 - e^{x-y}$. Hence, for high x, the probability of 0 quitting by time t will be high, which enables p_t to stay above $p(y^*)$. Hence, for x high, the informed player counters the public information $X_t = x$ with his private information. This means that with positive probability advocate 2 will lose the trial even though the physical indicates that he holds the correct position (i.e., $X_T < 0$).

Since the informed player only quits when he is type 0, the exact location of the signaling barrier is payoff irrelevant for players 2 and 3. All that matters is that advocate 1 never gives up unless he has the wrong position. This implies that changing advocate 1's costs has no effect on the payoff of players 2 and 3. If player 1 has very high costs of information provision, then the signaling barrier $\underline{p} < 1/2$ will be close to 1/2 but the information revealed by the war of information will be unchanged. On the other hand, advocate 2's cost affects his oppnent's payoff; lowering advocate 2's always increases player 3's payoff but advocate 1's payoff.

Signaling creates multiple equilibria in the war of information with asymmetric information. For example, consider strategies that "punish" advocate 1 for *not* quitting: If advocate 1 does not quit at a particular point he is believed to be type 0. Such strategies can sustain many equilibria in which even the strong type of advocate 1 quits with positive probability. However, these equilibria will not satisfy standard signalling game refinements such as universal divinity (Banks and Sobel (1987)). The signaling component also gives rise to equilibria in which the informed player buys information even when he is leading, i.e., when $p_t > 1/2$. Such equilibria can be sustained by the belief that the decision to quit indicates advoate 1 is type 0. We have ruled out this latter equilibria by assumption - i.e., by assuming that only the trailing advocate pays for information. As we explain in section 3.1, this assumption is easily in the symmetric information game. It is harder to justify when there is asymmetric information.

6. Appendix

6.1 Proof of Lemma 1

Let $E\big[C(X_t)|\,\mu = r\big]$ be the expected cost incurred by player 1 given the strategy profile $y = (y_1, y_2)$ and $\mu = r$. (Recall that $\sigma^2 = 1$.) Hence, the expected delay cost of player 1 is:

$$E\big[C(X_t)\big] = 1/2\, E\big[C(X_t)\big|1/2\big] + 1/2\, E\big[C(X_t)\big| - 1/2\big] \tag{A1}$$

First, we will show that

$$E\big[C(X_t)|\mu\big] = \frac{1}{2\mu^2}\left(\frac{1 - e^{-2\mu y_2}}{1 - e^{-2\mu(y_2 - y_1)}}\right)\big(1 - e^{2\mu y_1}(1 + 2\mu y_1)\big) \tag{A2}$$

For $z_1 \leq 0 \leq z_2$, let $P(z_1, z_2)$ be the probability that a Brownian motion X_t with drift μ and variance 1 hits z_2 before it hits z_1 given that $X_0 = 0$. Harrison (1985) shows (p. 43) shows that

$$P(z_1, z_2) = \frac{1 - e^{2\mu z_1}}{1 - e^{-2\mu(z_2 - z_1)}} \tag{A3}$$

For $z_1 \leq 0 \leq z_2$, let $T(z_1, z_2)$ be the expected time a Brownian motion with drift μ spends until it hits either z_1 or z_2 given that $X_t = 0$ (where $z_1 \leq 0 \leq z_2$). Harrison (1985) shows (p. 53) that

$$T(z_1, z_2) = \frac{z_2 - z_1}{r} P(z_1, z_2) + \frac{z_1}{r}$$

To compute $E\big[C(X_t)|\mu\big]$, let $\epsilon \in (0, y_2]$ and assume that player 1 bears the cost until $X_t \in \{-y_1, \epsilon\}$. If $X_t = \epsilon$ then player 2 bears the cost until $X_{t+\tau} \in \{0, y_2\}$. If $X_{t+\tau} = 0$

then process repeats with player 1 bearing the cost until $X_{t+\tau+\tau'} \in \{-y_1, \epsilon\}$ and so on. Clearly, this yields an upper bound to $E\big[C(X_t)|\mu\big]$. Let T^ϵ denote that upper bound and note that

$$T^\epsilon = T(y_1,\epsilon) + P(y_1,\epsilon)(1 - P(-\epsilon, y_2 - \epsilon))T^\epsilon$$

Substituting for $T(y_1,\epsilon)$ and $P(y_1,\epsilon)$ we get

$$\mu T^\epsilon = \left(\frac{(\epsilon - y_1)(1 - e^{2\mu y_1})}{1 - e^{-2\mu(\epsilon - y_1)}} + y_1\right)\left(1 - \frac{(1 - e^{2\mu y_1})(e^{-2\mu\epsilon} - e^{-2\mu y_2})}{(1 - e^{-2\mu(\epsilon - y_1)})(1 - e^{-2\mu y_2})}\right)$$

and therefore

$$E\big[C(X_t)|\mu\big] \le \lim_{\epsilon \to 0} T^\epsilon = \frac{1}{2\mu^2}\left(\frac{1 - e^{-2\mu y_2}}{1 - e^{-2\mu(y_2 - y_1)}}\right)\left(1 - e^{2\mu y_1}(1 + 2\mu y_1)\right)$$

Choosing $\epsilon < 0$ we can compute an analogous lower bound which converges to the right hand side of (A2) as $\epsilon \to 0$. This establishes (A2).

Recall that $p(y_i) = \frac{1}{1+e^{-y_i}}$ and $\alpha_1 = 1 - 2p(y_1), \alpha_2 = 2p(y_2) - 1$. Then, $(A1), (A2)$ yield

$$E\big[C(X_t)\big] = \frac{4\alpha_1 \cdot \alpha_2}{\alpha_1 + \alpha_2} \ln \frac{1 + \alpha_1}{1 - \alpha_1}$$

Let v be the probability that player 1 wins. Since P_T is a martingale and $T < \infty$

$$vp(y_2) + (1 - v)p(y_1) = p_T = 1/2$$

Hence,

$$v = \frac{\alpha_1}{\alpha_1 + \alpha_2}$$

The last two display equations yield

$$U_1(\alpha) = \frac{\alpha_1}{\alpha_1 + \alpha_2}\left(1 - k_1\alpha_2 \ln \frac{1 + \alpha_1}{1 - \alpha_1}\right) \qquad (A5)$$

A symmetric argument establishes yields the desired result of U_2. □

6.2 Proof of Lemma 2

By Lemma 1, advocate i's utility is strictly positive if and only if

$$\alpha_i \in \left(0, \frac{e^{\frac{1}{k_i \alpha_j}} - 1}{e^{\frac{1}{k_i \alpha_j}} + 1}\right)$$

Furthermore, throughout this range, $U_i(\cdot, \alpha_j)$ is twice continuously differentiable and strictly concave in α_i. To verify strict concavity, note that U_i can be expressed as the product of two concave functions f, g that take values in $I\!R_+$, where one function is strictly increasing and the other strictly decreasing. Hence, $(f \cdot g)^{''} = f^{''}g + 2f^{'}g^{'} + fg^{''} < 0$. Therefore, the first order condition characterizes the unique best response of player i to α_j. Player i's first order condition is:

$$U_i = \frac{2\alpha_i^2 k_i}{1 - \alpha_i^2} \tag{A6}$$

Note that $(A6)$ implicitly defines the best response functions B_i. Equation $(A6)$ together with the implicit function and the envelop theorems yield

$$\frac{dB_i}{d\alpha_j} = \frac{\partial U_i}{\partial \alpha_j} \cdot \frac{(1 - \alpha_i^2)^2}{4\alpha_i k_i} \tag{A7}$$

Equation $(A5)$ implies

$$\frac{\partial U_i}{\partial \alpha_j} = -\frac{1}{\alpha_1 + \alpha_2}\left(U_i + \alpha_j k_i \ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)\right) \tag{A8}$$

Note that $(A8)$ implies $\frac{\partial U_i}{\partial \alpha_j} < 0$. The three equations $(A6)$, $(A7)$, and $(A8)$ yield

$$\frac{dB_i}{d\alpha_j} = -\frac{\alpha_i(1 - \alpha_i^2)}{2(\alpha_1 + \alpha_2)} \cdot \left(1 + \frac{1 - \alpha_i^2}{2\alpha_i} \ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)\right) \tag{A9}$$

Then using the fact that $\ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right) \leq \frac{2\alpha_i}{1-\alpha_i}$ yields

$$\frac{dB_i}{d\alpha_j} \geq -\frac{\alpha_i(1 - \alpha_i^2)(2 + \alpha_i)}{2(\alpha_1 + \alpha_2)} \tag{A10}$$

Hence, since $\phi' = \frac{dB_1}{d\alpha_2}\frac{dB_2}{d\alpha_1}$ we have

$$0 < \phi(\alpha_1) \leq \frac{\alpha_1(1 - \alpha_1^2)(2 + \alpha_1)\alpha_2(1 - \alpha_2^2)(2 + \alpha_2)}{4(\alpha_1 + \alpha_2)^2} \tag{A11}$$

Note that the $\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2} \leq 1/2$ and hence, $\phi'(\alpha_1) < 1$ if

$$(1 - \alpha_i^2)(2 + \alpha_i) < 2\sqrt{2}$$

It is easy to verify that the left-hand side of the equation above reaches its maximum at $\alpha_i < 1/2$. At such α_i, the left-hand side is no greater than $5/2 < 2\sqrt{2}$, proving that $0 < \phi'(\alpha_1) < 1$. $\square$

6.3 Proof of Proposition 1

By Lemma 2, B_i are decreasing, continuous functions. It is easy to see such that $B_i(1) > 0$ and $\lim_{r \to 0} B_i(r) = \sqrt{\frac{1}{1+2k_i}}$ (Note that $U_i \to 1$ if $\alpha_j \to 0$ for $j \neq i$). Hence, we can continuously extend B_i and hence, ϕ to the compact interval $[0,1]$, so that ϕ much have a fixed-point. Since B_is are strictly increasing, $B_i(0) < 1$ implies that neither 0 nor 1 is a fixed-point. Hence, every fixed-point of ϕ must be in the interior of $[0,1]$. Let r be the infimum of all fixed-points of ϕ. Clearly, r itself is a fixed-point and hence $r \in (0,1)$. Since $\phi'(r) < 1$, there exists $\varepsilon > 0$ such that $\phi(s) > s$ for all $s \in (r, r+\varepsilon)$. Let $s^* = \inf\{s \in (0,1) \,|\, \phi(s) = s\}$. If the latter set in nonempty, s^* is well-defined, a fixed-point of ϕ, and not equal to r. Since $\phi(s) < s$ for all $s \in (r, s^*)$, we must have $\phi(s^*) \geq 1$, contradicting Lemma 2. Hence, $\{s \in (0,1) \,|\, \phi(s) = s\} = \emptyset$ proving that r is the unique fixed-point of ϕ and hence the unique equilibrium of the simple war of information. $\square$

6.4 Proof of Proposition 2

(i) Consider advocate 1's best response as a function of both α_2 and k^1. The analysis in Lemma 2 ensures that $B_1 : (0,1] \times I\!R^+ \backslash \{0\} \to (0,1]$ is differentiable. Hence, the unique equilibrium of the simple war of information is characterized by

$$B_1(B_2(\alpha_1), k_1) = \alpha_1$$

Taking a total derivative and rearranging terms yields

$$\frac{d\alpha_1}{dk_1} = \frac{\frac{\partial B_1}{\partial k_1}}{1 - \frac{d\phi}{d\alpha_1}}$$

where $\frac{d\phi}{d\alpha_1} = \frac{\partial B_1}{\partial \alpha_2} \cdot \frac{dB_2}{d\alpha_1}$. By Lemma 1, $\phi' < 1$. Taking a total derivative of $(A6)$ (for fixed α_2 establishes that $\frac{\partial B_1}{\partial k_1} < 0$ and hence $\frac{d\alpha_1}{dk_1} < 0$ as desired. Then, note that k_1 does not appear in $(A6)$ for player 2. Hence, a change in k_1 affects α_2 only through its effect on α_1 and therefore has the same sign as

$$\frac{dB_2}{dk_1} = \frac{dB_2}{d\alpha_1} \cdot \frac{d\alpha_1}{dk_1} > 0 \qquad (A12)$$

By symmetry, we also have $\frac{d\alpha_2}{dk_2} < 0$ and $\frac{d\alpha_1}{dk_2} > 0$

(ii) By (i), as k_i goes to 0, the left-hand side of $(A6)$ is bounded away from 0. Hence, $\frac{2\alpha_i^2}{1-\alpha_i^2}$ must go to infinity and therefore α^i must go to 1.

(iii) Since $U_i \leq 1$ it follows from (A6) that $k_i \to \infty$ implies $\alpha_i \to 0$. □

6.5 Proof of Propositions 3 and 4

From Lemma 1, we have:

$$\frac{dU_3}{dk_1} = \frac{\alpha_2^2}{(\alpha_1 + \alpha_2)^2} \frac{d\alpha_1}{dk_1} + \frac{\alpha_1^2}{(\alpha_1 + \alpha_2)^2} \frac{d\alpha_2}{dk_1}$$

Since $\alpha_2 = B_2(\alpha_1)$, (A9) and (A12) imply $\frac{dU_3}{dk_1} < 0$ if and only if

$$\frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{2(\alpha_1 + \alpha_2)} \cdot \left[1 - \alpha_2^2 + \frac{(1 - \alpha_2^2)^2}{2\alpha_2} \ln \left(\frac{1 + \alpha_2}{1 - \alpha_2} \right) \right] > 0 \qquad (A13)$$

Define $g : (0, 1] \to (0, 1]$ by $g(\alpha_1) = \alpha_2$ where

$$\frac{\alpha_2}{\alpha_1} - \frac{\alpha_1}{2(\alpha_1 + \alpha_2)} \cdot \left[1 - \alpha_2^2 + \frac{(1 - \alpha_2^2)^2}{2\alpha_2} \ln \left(\frac{1 + \alpha_2}{1 - \alpha_2} \right) \right] = 0$$

Proof of Proposition 4: First, note that g is well-defined. For any fixed α_1 the right had side of (A13) is negative for α_2 sufficiently close to zero and strictly positive for $\alpha_2 = \alpha_1$. Note that $\frac{\alpha_1}{2(\alpha_1+\alpha_2)}$, $1 - \alpha_2^2$, and the last term inside the square bracket are all decreasing in α_2. Hence g is well defined.

Note also that the right hand side of (A13) is decreasing in α_1. Hence, g must be increasing.

Since the term in the brackets adds up to less than 1 it follows that $g(z) < z$. Setting $\alpha_1 = 1$, define $\hat{\alpha}_2$ such that the left hand side of (A13) is zero. (Note that $\hat{\alpha}_2$ is approximately 0.48.) By the monotonicity of the right hand side of (A13) in α_1 it follows that $g \leq \hat{\alpha}_2$. □

Proof of Proposition 3 By part (i) of Lemma 2, the first term on the left-hand side of $(A13)$ is increasing in k_1. Similarly, $\frac{\alpha_1}{2(\alpha_1+\alpha_2)}$, $1 - \alpha_2^2$, and the last term inside the square bracket are all decreasing in k_1. Furthermore, the terms inside the square bracket add up to a quantity between 0 and 1. Part (i) then follows because $g(z) < z$ and $\alpha_1 \leq \alpha_2$ for $k_1 \geq k_2$.

For part (ii) note that as k_1 goes to 0, α_1 goes to 1 (by Proposition 2(ii) above). Setting $\alpha_1 = 1$, define $\hat{\alpha}_2$ such that the left hand side of (A13) is zero. (Note that $\hat{\alpha}_2$ is approximately 0.48.) Let $\bar{r}$ be such that $B_2(1) = \hat{\alpha}_2$. By the monotonicity of the right hand side of (A13) in α_1 it follows that (A13) is greater than zero for all $\alpha_2 > \hat{\alpha}_2$. Conversely, for $\alpha_2 < \hat{\alpha}_2$ the proof of Proposition 4 above implies that there is a unique $\alpha_1 \in (0, 1)$ such that the right hand side of (A13) is zero. It follows that there is $f(k_2) > 0$ such that at $k_1 = f(k_2)$, $\frac{dU_3}{dk_1} = 0$. Clearly, there can be at most one such $f(k_2)$. The monotonicity of g and the monotonicity of α_i in k_i implies that $\frac{dU_3}{dk_1} < 0$ for $k_1 > f(k_2)$, $\frac{dU_3}{dk_1} > 0$ for $k_1 < f(k_2)$, and $\frac{dU_3}{dk_1} = 0$ at $f(k_2)$.

Since $f(k_2)$ is well-defined, since (α_1, α_2) are continuous functions of (k_1, k_2), and since g is continuous, the function f is also continuous. That f is strictly increasing follows from part (i) of Proposition 2 and the strict monotonicity of g.

That $f \to \infty$ as $r \to \infty$ follows from the fact that for every $\alpha_1 > 0$ the right hand side of (A13) is strictly negative for α_2 sufficiently small. □

Proof of Proposition 5 Note that since α_1 is increasing in k_2 it follows that

$$\left.\frac{dU_3^*}{dt}\right|_{t=0} \geq \frac{\alpha_2^2}{(\alpha_1 + \alpha_2)^2} \frac{d\alpha_1}{dk_1} + \frac{\alpha_1^2}{(\alpha_1 + \alpha_2)^2} \left(\frac{d\alpha_2}{dk_1} + \frac{d\alpha_2}{dk_2} \right)$$

From Proposition 4 we know that

$$\left.\frac{dU_3^*}{dt}\right|_{t=0} \geq \frac{\alpha_2^2}{(\alpha_1 + \alpha_2)^2} \frac{d\alpha_1}{dk_1} + \frac{\alpha_1^2}{(\alpha_1 + \alpha_2)^2} \frac{d\alpha_2}{dk_1} > 0$$

for k_2 sufficiently large. Since $\frac{d\alpha_1}{dk_1}$ is bounded away from zero for all k_2 it suffices to show that

$$\left(\frac{d\alpha_2}{dk_2} \right) / \left(\frac{d\alpha_2}{dk_1} \right) \to 0$$

as $k_2 \to \infty$ and since $\frac{d\alpha_2}{dk_1} = \frac{d\alpha_2}{d\alpha_1} \frac{d\alpha_1}{dk_1}$ with $\frac{d\alpha_1}{dk_1}$ bounded away from zero for all k_2 it suffices to show that

$$\left(\frac{d\alpha_2}{dk_2} \right) / \left(\frac{d\alpha_2}{d\alpha_1} \right) \to 0$$

Recall that the first order condition is

$$\frac{\alpha_2}{\alpha_1+\alpha_2}(1-k_2\alpha_1\ln\frac{1+\alpha_2}{1-\alpha_2})=\frac{2\alpha_2^2k_2}{1-\alpha_2^2}$$

and therefore:

$$\Big|\left(\frac{d\alpha_2}{dk_2}\right)/\left(\frac{d\alpha_2}{d\alpha_1}\right)\Big|=\Big|(\alpha_2+\alpha_1)\frac{-2\alpha_2(\alpha_1+\alpha_2)-\alpha_1\ln\frac{1+\alpha_2}{1-\alpha_2}+\alpha_2^2\alpha_1\ln\frac{1+\alpha_2}{1-\alpha_2}}{(1-\alpha_2^2)(k_2\alpha_2\ln(\frac{1+\alpha_2}{1-\alpha_2}+1)}\Big|$$

Note that $\alpha_2\to 0$ as $k_2\to\infty$ and hence the right hand side of the above expression goes to zero as $k_2\to\infty$ as desired. □

6.6 Proof of Proposition 6

For part (i) suppose i chooses the strategy $a_i=1-\epsilon$. Then, for δ sufficiently small we have $U_i\geq\frac{1-2\epsilon}{2-2\epsilon}-\epsilon$ for $i=1,2$. Since ϵ can be chosen arbitrarily small, the it follows that $U_i\to 1/2$ as $\delta\to 0$. The first order condition $(A6)$ implies that $\alpha_i\to 1$ which in turn implies that $U_3\to 1$.

For part (ii) note that $\alpha_i\to 0$ as $\delta\to\infty$. Let $k_1=1,k_2=k$ and define $r=\alpha_1/\alpha_2$ and $z=\alpha_1^2\delta$. Then, the first order condition $(A6)$ can be re-written as

$$\frac{1}{1+r}\left(1-zr\frac{\ln\left(\frac{1+\alpha_1}{1-\alpha_1}\right)}{\alpha_1}\right)=2z$$
$$\frac{1}{1+r}\left(1-zkr\frac{\ln\left(\frac{1+\alpha_2}{1-\alpha_2}\right)}{\alpha_2}\right)=2zkr$$

These two equations imply that z,r must be bounded away from zero and infinity for small δ. Moreover as $\delta\to\infty$ it must be that $\alpha_i\to 0$ for $i=1,2$. Therefore,

$$\frac{\ln\left(\frac{1+\alpha_i}{1-\alpha_i}\right)}{\alpha_i}\to 2$$

And therefore, the limit solution to the above equations must satisfy

$$\frac{1}{1+r}(1-2zr)=2z$$
$$\frac{1}{1+r}(1-2zkr)=2zkr$$

We can solve the two equations for r, z and find U_i from the first order condition $U_i = 2z = \frac{1}{3k}\left(k + 2\sqrt{1-k+k^2} - 2\right)$. □

6.7 Proof of Proposition 7

Lemma 3: *Let $f^i = (f_1^i, \ldots)$, $f^j = (f_1^j, \ldots)$, and $\tilde{f}^j = (\tilde{f}_1^j, \ldots)$ for $i = 1, 2$ and $j \neq i$ and let $\tilde{f}_n^j(\zeta) \geq f_n^j(\zeta)$ for every n and $\zeta \in I\!R^n$. Then, $U_{x_0}^i(f^1, f^2) \geq U_{x_0}^i(f^i, \tilde{f}^j)$.*

Proof: Consider any sample path $X(\omega)$. Let $T(\omega), \tilde{T}(\omega)$ denote the termination date corresponding to the strategy profile (f^i, f^j) and $(f^i, \tilde{f}^j)$ respectively. Note that $T(\omega) \leq \tilde{T}(\omega)$ and therefore the cost of player i is larger if the opponent chooses $\tilde{f}^j$. Furthermore, if $T(\omega) < \tilde{T}(\omega)$ then player i wins along the sample path $X(\omega)$ when the strategy profile is (f^i, f^j). Therefore, the probability of winning is higher under (f^i, f^j) than under $(f^i, \tilde{f}^j)$.
□

Lemma 4: *Let $f^i = (a_i, a_i, \ldots)$ be a stationary strategy and for $j \neq i$, let $f^j = (f_1^j, B_j(a_i), B_j(a_i), \ldots)$. (i) If $B_j(a_i) < \alpha_j(x_0) < f_1^j(x_0)$, then $U_{x_0}^j(f^1, f^2) < 0$ and (ii) If $B_j(a_i) \geq f_1^j(x_0) > \alpha_j(x_0)$, then $U_{x_0}^j(f^1, f^2) > 0$.*

Proof: Let $V(a, b)$ denote the payoff of player j if $f_1^j = b$ and $f_n^j, n \geq 2$ are chosen optimally, $f^i = (a_i, \ldots)$ and the initial state is $\alpha^{-1}(a)$. Let $b^* = \arg\max_{b \in [0,1]} V(1/2, b)$. It is easy to see that V is continuous and hence b^* is well defined. Next, we show that $V(a, b^*) \geq V(a, b)$ for all $a \in I\!R$ and for all $b \in [0, 1]$. To prove this, assume that $b > b^*$ and note that $V(a, b) - V(a, b^*) = cV(b^*, b)$ where c is the probability that $X(t)$ reaches the state $y = \alpha^{-1}(b^*)$ for some $t \in [0, \Delta]$. Since a is arbitrary it follows from the optimality of b^* that $V(b^*, b) \leq 0$. Since the decision problem is stationary, it follows that $f_n^j = b^*$ is a best response to $f^i = (a_i, \ldots)$. This in turn implies that $b^* = B_j(a_i)$ and $U_{x_0}^j(f^1, f^2) \leq 0$ if $B_j(a_i) < \alpha_j(x_0) < f_1^j(x_0)$. Let $b = f_1^j(x_0)$. If $U_{x_0}^j(f^1, f^2) = 0$ then by the argument above $f^j = (b, b, \cdot)$ is also a best response to $(a_i, a_i, \ldots)$. But this contradicts the fact that $B(a_i)$ is unique. Hence, a strict inequality must hold and part (i) of the Lemma follows. Part (ii) follows from a symmetric argument. □

Proof of Proposition 7: Let

$$\bar{a}_i = \sup\{\alpha_i(x) \,|\, f^i = (f_1^i, \ldots) \in E^i, f^i(x) > \alpha_i(x) \text{ for some } x\}$$
$$\underline{a}_i = \inf\{\alpha_i(x) \,|\, f^i = (f_1^i, \ldots) \in E^i, f^i(x) \leq \alpha_i(x) \text{ for some } x\}$$

Hence, $\underline{a}_i, \bar{a}_i$ are respectively, the least and most patient actions for i observed in any subgame perfect Nash equilibrium. Clearly, $\underline{a}_i \leq a_i^* \leq \bar{a}_i$.

First, we show that (i) $B_2(\underline{a}_1) \geq \bar{a}_2$. To see this note that if the assertion is false, then there exists x_0,$(f^1, f^2) \in E$ such that $f_1^2(x_0) > \alpha_2(x_0) > B_2(\underline{a}_1)$. By Lemma 3 and part (i) of Lemma 4, $U_{x_0}^2(f^1, f^2) \leq U_{x_0}(\underline{a}^1, f^2) < 0$, contradicting the fact that $(f^1, f^2) \in E$.

Next, we prove that (ii) $B_1(\bar{a}_2) \leq \underline{a}_1$. If the assertion is false, then there exists x_0,$(f^1, f^2) \in E$ such that $f_1^1(x_0) \leq \alpha_1(x_0) < B_1(\bar{a}_2)$. Then, $0 = U_{x_0}^1(f^1, f^2)$ and by Lemma 3 $U_{x_0}^1(f^1, f^2) \geq U_{x_0}^1(\tilde{f}^1, f^2) \geq U_{x_0}^1(\tilde{f}^1, \bar{a}_2)$ for all $\tilde{f}^1$. By Lemma 4 part (ii), there exists $\tilde{f}^1$ such that $U_{x_0}^1(\tilde{f}^1, \bar{a}_2) > 0$ and hence $U_{x_0}^1(f^1, f^2) > 0$, a contradiction.

The two assertions (i) and (ii) above together with the fact that B_i is nonincreasing yield $\phi(\underline{a}_1) = B_1(B_2(\underline{a}_1)) \leq B_1(\bar{a}_2) \leq \underline{a}_1$. Since the slope of ϕ is always less than 1 (Lemma 2), we conclude that $a_1^* \leq \underline{a}_1$ and therefore $a_1^* = \underline{a}_1$. Symmetric arguments to the ones used to establish (i) and (ii) above yield $B_2(\bar{a}_1) \leq \underline{a}_2$ and $B_1(\underline{a}_2) \geq \bar{a}_1$. Hence, $\phi(\bar{a}_1) = B_1(B_2(\bar{a}_1)) \geq B_1(\underline{a}_2) \geq \bar{a}_1$ and hence $a_1^* \geq \bar{a}_1$ and therefore $a_1^* = \underline{a}_1 = \bar{a}_1$ proving that a_1^* is the only action that 1 uses in a subgame perfect Nash equilibrium. Hence, $E_1 = \{a_1^*\}$ and therefore $E = \{(a_1^*, a_2^*)\}$ as desired. □

6.8 Proof of Proposition 8

If (z_1, z_2) is an equilibrium for $\pi = \gamma$ then (z_1, z_2) is also an equilibrium for $\pi = 1/2$. If $z_1 < 1/2 < z_2$ then the converse is also true. In the following, we assume that $\pi = \gamma$.

If $X_t = x$ then the belief of player 3 that 1 holds the correct position is

$$p_\gamma(x) = \frac{\gamma}{\gamma + (1-\gamma)e^{-x}}$$

Let $y_i = p_\gamma^{-1}(z_i)$ denote the strategy expressed in terms of the realization of X.

As in the proof of Lemma 1, let $E\big[C(X_t)|\,\mu = r\big]$ be the expected cost incurred by player 1 given the strategy profile $y = (y_1, y_2)$ and $\mu = r$. (Recall that $\sigma^2 = 1$.) Recall that

$$E\big[C(X_t)|\mu = r\big] = \frac{1}{2r^2}\left(\frac{1 - e^{2y_2}}{1 - e^{-2r(y_1 - y_2)}}\right)\big(1 - e^{-2ry_1}(1 + 2ry_1)\big) \qquad (A2)$$

The expected delay cost of player 1 is:

$$E\big[C(X_t)\big] = \gamma\, E\big[C(X_t)\big|\mu = 1/2\big] + (1-\gamma)\, E\big[C(X_t)\big|\mu = -1/2\big] \qquad (A1)$$

Let v be the probability that player 1 wins. Since P_T is a martingale and $T < \infty$, $vz_1 + (1-v)z_2 = Ep_T = \gamma$, Hence,

$$v = \frac{\gamma - z_1}{z_2 - z_1}$$

Substituting for y_i we find that $U_i, i = 1, 2$ is given by

$$U_1(z_1, z_2) = \frac{\gamma - z_1}{z_2 - z_1}\left(1 - k_1(z_2 - \gamma)\left(\frac{2\gamma - 1}{\gamma(1-\gamma)} + \frac{1 - 2z_1}{\gamma - z_1}\ln\frac{(1 - z_1)/z_1}{(1-\gamma)/\gamma}\right)\right)$$

and

$$U_2(z_1, z_2) = \frac{z_2 - \gamma}{z_2 - z_1}\left(1 - k_2(\gamma - z_1)\left(\frac{1 - 2\gamma}{\gamma(1-\gamma)} + \frac{2z_2 - 1}{z_2 - \gamma}\ln\frac{z_2/(1 - z_2)}{\gamma/(1-\gamma)}\right)\right)$$

The first order condition for this maximization problem yields:

$$(z_2 - z_1)^2(z_2 - \gamma)g(z_1, z_2, \gamma) = 0$$

where

$$g(z_1, z_2, \gamma) = -1/k_1 + (1 - 2z_2)\ln\frac{z_1/(1 - z_1)}{\gamma/(1-\gamma)} - \frac{(\gamma - z_1)(z_2 z_1(1 - 2\gamma) - (1-\gamma)z_2 + \gamma z_1)}{z_1(1 - z_1)\gamma(1-\gamma)}$$

Note that the second order condition for a maximum is satisfied at z_1 if and only if $\partial g(z_1, z_2, \gamma)/\partial z_1 < 0$. A direct calculation yields

$$\frac{\partial g(z_1, z_2, \gamma)}{\partial z_1} = -\frac{z_2 - z_1}{(z_1(1 - z_1))^2} < 0$$

and hence the second order condition is satisfied. Note that $g(\gamma, z_2, \gamma) = -1/k_1$ and $\lim_{z_1 \to 0} g(z_1, z_2, \gamma) \to \infty$. Hence, any solution to the maximization problem is interior. Moreover, since the second order condition holds for any solution of the first order condition, it follows that the maximum is unique. This in turn implies that $B_1(z_2)$ is a continuous function. To see that B_1 is strictly decreasing, note that $g(z_1, z_2, \gamma) = a + bz_2$ for some constants $a < 0, b > 0$. Since $z_2 > 0$ and $g(z_1, z_2, \gamma) = 0$ at an optimum, it follows that $b > 0$. Therefore,

$$\frac{\partial B_1(z_2)}{\partial z_2} = \frac{-b}{\partial g(z_1, z_2, \gamma)/\partial r_1} > 0$$

(Differentiability follows from the fact that $\frac{\partial g(z_1,z_2,\gamma)}{\partial z_1} \neq 0$.)

By the above argument, B_1 is continuous and therefore the analogous function B_2 is also continuous. Moreover, $B_1 \in [0, \gamma]$ and analogously $B_2 \in [\gamma, 1]$. Therefore, $(B_1, B_2) : [0, \gamma] \times [\gamma, 1] \to [\gamma, 1] \times [0, \gamma]$ has a fixed point.

To show uniqueness, define

$$\phi(z_1) = B_1(B_2(z_1))$$

We will show that $|\phi' < 1|$ if $z_1 = \phi(z_1)$. Let z_1 be a fixed point of ϕ and let $z_2 = B_2(z_1)$. Then,

$$|\phi'(z_1)| = |\frac{dB_1}{z_2}\frac{dB_2}{z_1}| = |\frac{\partial g(z_1, z_2, \gamma)/\partial z_1}{\partial g(z_1, z_2, \gamma)/\partial z_2} \cdot \frac{\partial g(1 - z_2, 1 - z_1, 1 - \gamma)/\partial z_2}{\partial g(1 - z_2, 1 - z_1, 1 - \gamma)/\partial z_1}| := h$$

A direct calculation shows that

$$h = \frac{z_1^2(1 - z_1)^2 z_2^2(1 - z_2)^2}{(z_2 - z_1)^2} \left(2 \ln \kappa(1 - z_2, 1 - \gamma) - \lambda(1 - z_2, 1 - \gamma)\right) \left(2 \ln \kappa(z_1, \gamma) - \lambda(z_1, \gamma)\right)$$

where $\kappa(a, b) = \frac{a/(1-a)}{b/(1-b)}, \lambda(a, b) = \frac{(ab-(1-a)(1-b))(b-a)}{b(1-b)a(1-a)}$. Note that $-\ln x \leq -1 + 1/x$ for $x \leq 1$. Further note that $\ln \kappa(1-z_2, 1-\gamma) < 0, \ln \kappa(z_1, \gamma) < 0, \lambda(z_1, \gamma) > 0, \lambda(1-z_2, 1-\gamma) > 0$. Therefore, substituting $-1 + 1/\kappa(z_1, \gamma)$ for $\ln \kappa(r_1, \gamma)$ and $-1 + 1/\kappa(1 - z_2, \gamma)$ for

$\ln \kappa(1 - z_2, \gamma)$ we get an upper bound for h. Following the substitution we are left with the following expression:

$$h \leq \frac{z_1(1-z_1)(\gamma - z_1)z_2(1-z_2)(z_2-\gamma)(1+\gamma-z_1)(1+z_2-\gamma)}{\gamma^2(1-\gamma)^2(z_2-z_1)^2}$$
$$\leq 9/4 \frac{z_1(\gamma - z_1)(1-z_1)z_2(1-z_2)(z_2-\gamma)}{\gamma^2(1-\gamma)^2(z_2-z_1)^2}$$

Note that $z_1 \leq \gamma, 1 - z_2 \leq 1 - \gamma$. Choose δ_1, δ_2 so that $z_1 = \delta_1\gamma, 1 - z_2 = \delta_2(1-\gamma)$ then the above expression simplifies to

$$9/4 \frac{(1-\delta_1\gamma)(1-\delta_2(1-\gamma))\delta_1(1-\delta_1)\delta_2(1-\delta_2)}{(1-\delta_1\gamma-\delta_2(1-\gamma))^2}$$

with $\delta \in [0,1], \gamma \in [0,1]$. The above expression is maximal at $\gamma = 1/2, \delta_1 = \delta_2 = \delta$ for some $\delta \in (0,1)$ and therefore it suffices to show that

$$9/4 \frac{\delta^2(1-\delta)^2(1-\delta/2)^2}{(1-\delta)^2} \leq 1$$

The last equality holds by a direct calculation. □

6.9 Proof of Lemma 3

Let g be as defined in the proof of Proposition 6. Note that

$$\frac{dz_1}{d\gamma} = -\frac{\partial g(z_1, z_2, \gamma)}{\partial \gamma}\left[\frac{\partial g(z_1, z_2, \gamma)}{\partial z_1}\right]^{-1}$$

Substituting for g and calculating the above partial derivatives yields

$$\frac{dz_1}{d\gamma} = \frac{z_2 - \gamma}{\gamma^2(1-\gamma)^2}\frac{z_1^2(1-z_1)^2}{z_2 - z_1}$$

and an analogous argument yields

$$\frac{dz_2}{d\gamma} = \frac{\gamma - z_1}{\gamma^2(1-\gamma)^2}\frac{z_2^2(1-z_2)^2}{z_2 - z_1}$$

Next, note that

$$\frac{\partial U_3}{z_i} = 2(-1)^i \frac{(z_i - 1/2)^2}{(z_2 - z_1)^2}$$

Combining the three displayed equations then yields Lemma 3. □

6.10 Proof of Proposition 9

Part (i): First, we show that $U(1-\gamma) > U(\gamma)$ for $\gamma < 1/2$. This will prove that $\gamma^* \geq 1/2$. Let $z_i(\gamma)$ denote the equilibrium strategy of advocate i in W^γ.

Fact 1: *Assume $\gamma < 1/2$ and $k_1 < k_2$. Then, $1 - z_1(\gamma) > \max\{z_2(1-\gamma), 1 - z_1(1-\gamma)\}$.*

Proof: Note that $1 - z_1^\gamma = z_2(1-\gamma)$ if $k_1 = k_2$. Since z_1 is increasing in k_1 and decreasing in k_2 it follows that $1 - z_1(\gamma) > z_2(1-\gamma)$. $1 - z_1(\gamma) > 1 - z_1(1-\gamma)$ follows from the fact that z_1 is increasing in γ. □

Fact 2: *Assume $\gamma < 1/2$, $k_1 < k_2$ and $z_2(\gamma) > 1/2$. Then, $z_2(1-\gamma) - z_1(1-\gamma) \geq z_2(\gamma) - z_1(\gamma)$.*

Proof: Let $\Delta^+(x) = z_2(1/2 + x) - z_1(1/2 + x)$ and let $\Delta^-(x) = z_2(1/2 - x) - z_1(1/2 - x)$ for $x \in [0, \bar{x}]$ where $\bar{x} = 1/2$ is the largest $x \leq 1/2$ such that $z_2(1/2 - x) \geq 1/2$. First, we establish that if $\Delta^+(x) \leq \Delta^-(x)$ then $\partial \Delta^+(x)/\partial x > \Delta^-(x)/\partial x$. To see this note that from the proof of Lemma 3)

$$\frac{dz_1}{d\gamma} = \frac{z_2 - \gamma}{\gamma^2(1-\gamma)^2} \frac{z_1^2(1-z_1)^2}{z_2 - z_1}$$

$$\frac{dz_2}{d\gamma} = \frac{\gamma - z_1}{\gamma^2(1-\gamma)^2} \frac{z_2^2(1-z_2)^2}{z_2 - z_1}$$

From Fact 1 and $z_1(\gamma) \leq 1/2, z_2(\gamma) > 1/2$ we conclude that for $\gamma < 1/2$

$$(z_2(\gamma) - \gamma) z_1(\gamma)^2 (1 - z_1(\gamma))^2 < (1 - \gamma - z_1(1-\gamma)) z_2(1-\gamma)^2 (1 - z_2(1-\gamma))^2$$

and

$$(z_2(1-\gamma) - (1-\gamma)) z_1(1-\gamma)^2 (1 - z_1(1-\gamma))^2 < (\gamma - z_1(\gamma)) z_2(\gamma)^2 (1 - z_2(\gamma))^2$$

and hence the assertion follows. Note that $\Delta^+(0) = \Delta^-(0)$. Since $\partial \Delta^+(x)/\partial x > \Delta^-(x)/\partial x$ when $\Delta^+(x) = \Delta^-(x)$ and Δ^+, Δ^- are continuously differentiable, we conclude that $\Delta^+(x) \geq \Delta^-(x)$. □

To complete the proof of part (i), let $a_1 := 1 - 2z_1; a_2 := 2z_2 - 1$. Then,

$$U_3(\gamma) - U_3(1-\gamma) = \frac{a_1(\gamma)a_2(\gamma)}{a_1(\gamma) + a_2(\gamma)} - \frac{a_1(1-\gamma)a_2(1-\gamma)}{a_1(1-\gamma) + a_2(1-\gamma)}$$

Fact 1 implies that

$$a_1(\gamma) > \max\{a_2(1-\gamma), a_1(1-\gamma)\}$$

and Fact 2 implies that

$$a_1(1-\gamma) + a_2(1-\gamma) > a_1(\gamma) + a_2(\gamma)$$

It is easy to check that these two inequalities imply that $U_3(\gamma) < U_3(1-\gamma)$ as desired. □

Part (ii): By part (i) we have $\gamma^* \geq 1/2$. If $1/2 - z_1^* > z_2^* - 1/2$ then $\gamma - z_1 > z_2 - \gamma$ and hence Lemma 3 implies that U_3 is strictly increasing in γ. This proves part (ii). □

6.11 Asymmetric Information

Proof of Lemma 4: (i) Let $\underline{p}, \bar{p}$ be the belief thresholds corresponding to the profile $Q(x,y)$. Note that player 1 type 1 never quits and therefore the probability player 2 wins is v where v satisfies

$$(1-v)\cdot q + v \cdot 1 = 1/2$$

This follows since p_t is a martingale and hence by the martingale stopping theorem $E(p_T) = 1/2 = p_0$. Note that v is independent of the strategy of player 0. In particular, we may assume that player 0 never quits and hence v corresponds to the win probability of player 2 if his opponent never quits.

Next, consider the cost of the strategy $Q^2(y)$ with $p = \frac{1}{1+e^{-y}}$. We can compute an upper bound to this cost as in Lemma 1. Let $p_t = 1/2$ and assume player 2 incurs the cost until time τ with $p_\tau \in \{1/2 - \epsilon, p\}$. Choose ϵ so that $1/2 - \epsilon < \underline{p} = \frac{1}{1+e^{-x}}$. Let c^ϵ denote this cost. Let π^ϵ denote the probability that $p_{t'}$ reaches $1/2$ at some time after τ given that $p_\tau = 1/2 - \epsilon$. This probability satisfies

$$(1/2)\cdot \pi^\epsilon + 1 \cdot (1-\pi^\epsilon) = 1/2 - \epsilon$$

Note that π^ϵ is independent of the strategy of player 0. An upper bound to player 2's cost is therefore given by

$$\bar{C}(q) = c^\epsilon(q) + \pi^\epsilon \bar{C}(q)$$

Note that this upper bound is independent of the strategy of player 0. We may therefore assume that player 0 never quits. For that case, Lemma 1 provides the limiting cost as $\epsilon \to 0$ by setting $\alpha_1 = 1$. An analogous argument can be used to derive a lower bound (as in Lemma 1). Hence, we can apply Lemma 1 to show that the utility of player 2 from strategy y is given by

$$U_2(p) = \frac{1}{2\bar{p}} \left(1 - k_w \ln \frac{\bar{p}}{1-\bar{p}}\right)$$

where $\frac{p=1}{1+e^{-y}}$.

By Lemma 2, U_2 is strictly concave with a maximum $\bar{p} = \frac{1+B(1)}{2}$. Hence, for $q < \bar{p}$, $U_2(\bar{p}) - U_2(q) > 0$. Assume player 2 quits if and only if $p_t \geq \bar{p}$. Let $V(q)$ denote the player's continuation payoff conditional on a history with $p_t(\omega) = q$. Note that

$$\pi^q V(q) = U_2(\bar{p}) - U_2(q)$$

where π^q is the probability that p_t reaches q before it terminates. Hence, for any ω, t with $p_t(\omega) < \bar{p}$ there is a strategy for player 2 that guarantees a strictly positive payoff. This implies that player 2 quits with probability zero for all t, ω with $p_t(\omega) < \bar{p}$.

Next, we show that player 2 must quit at $\bar{p}$. Suppose player 2 never quits but incurs the cost only if $p_t \leq \bar{p}$. For $q > \bar{p}$ the cost is zero. Clearly, this is an infeasible strategy (because 2 does not incur the cost of information provision for $p_t > \bar{p}$) that provides an upper bound to the utility of any strategy that does not quit at $\bar{p}$. Let W^* be the utility of player 2 at this strategy. We will show below that $W^* = U^2(\bar{p}^*)$. This implies that the overall cost incurred by player 2 cannot exceed the cost of the threshold strategy x^* and therefore $Q^2(x^*, y^*)$ is the unique best reply.

Claim: $W^* = U^2(\bar{p})$.

Proof: Let $V(q) = \frac{U_2(\bar{p}) - U_2(q)}{\pi^q}$ for $1/2 \leq q < \bar{p}$ where π^q is the probability that p_t reaches q. Note that $V(q)$ is the continuation value of the strategy $Q(x, y^*)$ at $p_t = q$. Furthermore, π^q is bounded away from zero for all $q \in [1/2, q^*]$.

Consider the following upper bound for W^*. If $p_t = \bar{p}$ then information is generated at no cost until either $p_\tau = \bar{p} - \delta$ (which occurs with probability r or $p_\tau = 1$ (which occurs with probability $1 - r$). In the latter case, the player 2 wins. In the former case, the agent proceeds with the threshold strategy $\bar{p}$ until either $\bar{p}$ is reached or the opponent quits. If $\bar{p}$ is reached then free information is generated again, as described above. By the martingale property of p_t we have

$$r(\bar{p}-\delta)+(1-x)1=\bar{p}$$

which implies that

$$r=\frac{1-\bar{p}}{1-\bar{p}+\delta}$$

and therefore

$$\begin{aligned}W^* &\leq U^2(\bar{p})+\frac{U^2(\bar{p})-U_2(q)}{\pi^q(1-r^\delta)}\\ &=U^2(\bar{p})+\pi^q(1-q+\delta)\frac{U^2(\bar{p})-U_2(\bar{p}-\delta)}{\delta}\end{aligned}$$

Note that as $\delta\to 0$ we have

$$\frac{U^2(\bar{p})-U^2(\bar{p}-\delta)}{\delta}\to U_2'(\bar{p})=0$$

by the optimality of the threshold $\bar{p}$ and hence the claim follows. It follows that the player must quit when $p_t>\bar{p}$. □

noindent**Proof of Lemma 5:** From the proof of Lemma 1 we can compute the payoff of player 0 if he uses a threshold $Q^0(y)$ and quits at y as follows. Let $P(x,y)$ denote the probability of reaching y before x when $\mu=-1/2$ and let $E[C(X_t)|-1/2]$ denote the cost when $y_1=x,y_2=y^*$. Lemma 1 provides expressions for $P(x,y)$ and $E[C(X_t)|-1/2]$. Substituting those we obtain

$$\Pi^0(x)=P(x,y^*)-E[C(X_t)|-1/2)]=\Pi(x,y^*)$$

It is straightforward to show that $\Pi(x,y^*)$ has a unique maximum x^*. Next, we show that $x^*<0$. To see that $x^*<0$ for all $k_1>0$ and all $y^*>0$ note that

$$\partial\frac{\tilde{\Pi}^0(x,y^*)}{\partial x}|_{x=0}=-\frac{1}{e^{y^*}-1}$$

and hence $x^*=0$ cannot be optimal for any k_1. Note that $\Pi^0(x,y)$ is concave in x. Hence, quitting before y^* cannot be optimal.

Next, we show that quitting at y^* is optimal. To demonstrate this, assume that player 0 never quits but incurs the cost of information only if $p_t \in (\underline{p}, 1/2]$ with $\underline{p} = \frac{1}{1+e^{x^*}}$. Clearly, this strategy provides an upper bound to the payoff of player 1. Denote the corresponding payoff by W^*. We can compute an upper bound to W^* by assuming that when y_0^* is reached then information is provided at no cost to the players until a $p_t = \frac{1}{1+e^{-(x^*+\delta)}}$ is reached. Let $Z_t = \sup_{0\leq s\leq t}(X_t - X_s)$ and let $p_0 = \frac{1}{1+e^{-x^*}}$. Then,

$$p_t \leq \frac{1}{1+e^{-y_0^*+Z_t}}$$

and the probability that $p_t > \frac{1}{1+e^{-(y_0^*+\delta)}}$ for all t is bounded below by $e^{-\delta}$ since $\Pr\{Z_t \leq z\} \to 1 - e^{-z}$ as $t \to \infty$ (See Harrison, page 15). Hence, the probability that $p_t = \frac{1}{1+e^{-y_0^*+\delta}}$ is at most $e^{-\delta}$. Let V^δ denote the continuation value of player 0 at $p_t = \frac{1}{1+e^{-(x^*+\delta)}}$ when the player quits at $\underline{p} = \frac{1}{1+e^{-x^*}}$ and note that

$$V^\delta = \frac{1}{1-P(x^*+\delta, y^*)}\left(\Pi^0(x^*, y^*) - \Pi^0(x^*+\delta, y^*)\right)$$

Then,

$$W^* = \Pi^0(x^*, y^*) + \frac{V^\delta}{1-e^{-\delta}}$$

Note that

$$\lim_{\delta\to 0} \frac{\Pi^0(x^*, y^*) - \Pi^0(x^*+\delta, y^*)}{1-e^{-\delta}} = 0$$

since $\partial\Pi^0(x^*, y^*)/\partial x = 0$ by the optimality of x^*. Therefore, $W^* \to \Pi^0 * (x^*, y^*)$ as $\delta \to 0$ This shows that quitting at x^* is optimal. The argument above also shows that $V^\delta \to 0$ and hence player 0 is indifferent between quitting and not quitting at $p_t = \underline{p}$.

It remains to show that the strategy Q^1 is optimal. Let

$$\Pi^1(x, y^*) := \frac{1}{1-e^{-y^*+x}}\left(1 - e^x - 2k_1(1-e^{-y^*})(1-e^x(1-x))\right)$$

and note that Π^1 is the payoff of type 1 if 1 quits at $\underline{p} = \frac{1}{1+e^{-x}}$. A straightforward calculation shows that

$$\frac{\partial\Pi^1(x,y)}{\partial x} - \frac{\partial\Pi^0(x,y)}{\partial x} < 0$$

for all $x^* \leq x < 0$ and therefore it is optimal for player 1 not to quit. □

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