# Framing Contingencies[*]

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**Abstract**

We introduce a model of decision making under uncertainty which incorporates framing effects for how contingencies are described. The primitive is a family of preferences, indexed by partitions of the state space. Each partition corresponds to a description of the state space. We axiomatically characterize the following partition-dependent expected utility representation. The decision maker has a nonadditive set function over events. She then computes expected utility with respect to her partition-dependent belief, which weights explicitly listed events. One interpretation of the model is in terms of unforeseen contingencies. We propose definitions for the events which are completely foreseen or unforeseen by decision maker and study their properties.

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## 1 Introduction

Beginning with Fischoff, Slovic, and Lichtenstein (1978), numerous psychological experiments suggest that the judged likelihood of an event depends on the manner in which it is described. The domains of standard economic models of uncertainty do not distinguish different descriptions, therefore preclude such framing effects. This paper introduces a novel methodology for formally incorporating the framing of the state space into decision making under uncertainty. Its primitives are distinct descriptions of acts. For example, consider the following health insurance contract, which associates deductibles on the left with contingencies on the right:

$$\begin{pmatrix} \$500 & \text{surgery} \\ \$100 & \text{prenatal care} \\ \vdots & \vdots \end{pmatrix}.$$

Compare this to the following contract, which includes some redundancies:

$$\begin{pmatrix} \$500 & \text{laminotomy} \\ \$500 & \text{other surgeries} \\ \$100 & \text{prenatal care} \\ \vdots & \vdots \end{pmatrix}.$$

Both contracts provide effectively identical levels of coverage. Nonetheless, a consumer might evaluate these lists differently, because the second formulation explicitly mentions laminotomies. She may have never heard of laminotomies or, if she had heard of them, she may have failed to fully consider them when evaluating the first contract. This oversight could manifest itself behaviorally if the consumer was willing to pay a higher premium for the second contract, reflecting an increased personal belief of the likelihood of surgery after laminotomies are mentioned.

Our general model expands the standard subjective model of decision making under uncertainty and introduces a richer set of primitives which distinguishes the different expressions for an act as distinct choice objects. In particular, lists of contingencies with associated outcomes are the primitive objects of choice. The following list

$$\begin{pmatrix} x_1 & E_1 \\ x_2 & E_2 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix},$$

denotes an act which delivers the outcome $x_i$ if the state of the world is in $E_i$. While acts are often notated as lists for ease of exposition, we take this notation quite literally. For example, if $E_1' \cup E_1'' = E_1$, then the following list

$$\begin{pmatrix} x_1 & E_1' \\ x_1 & E_1'' \\ x_2 & E_2 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix},$$

denotes the same act, but is here modeled as a distinct object. The decision maker might have different attitudes about the two presentations, because the second explicitly mentions the specific contingencies $E_1'$ and $E_1''$. This discrimination between presentations is the primary methodological innovation of the paper. The model is, to our knowledge, the first axiomatic attempt to incorporate framing of contingencies as an explicit consideration in decision making.

Aside from theoretical concerns, many real contracts are presented as such lists. Insurance plans are often described by a table of contingencies associated with coverage or liability amounts. Table 1 is a partial verbatim copy of a Blue Cross medical plan available to University of California employees expressed as procedures and deductibles. The other available plans are similarly described.

| | |
|---|---|
| Office visit | \$20 |
| Hospital visit | no charge |
| Preventive physical exam | \$20 |
| Maternity outpatient care | \$20 |
| Maternity inpatient care | \$250 |

Table 1: Blue Cross health insurance plan

We axiomatically characterize the following utility representation. The decision maker acts as if she places a weight $\nu(E)$ on each event $E$. When presented a description $E_1, \ldots, E_n$ of the possibilities, she judges the probability of $E_i$ to be $\nu(E_i)/\sum_j \nu(E_j)$. Since the weighting function $\nu$ is not necessarily additive, her probability of $E_1$ can depend on whether it is expressed as $E_1$ or expressed as $E_1' \cup E_1''$. Her utility for a list

$$\begin{pmatrix} x_1 & E_1 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix}$$

is obtained by aggregating her cardinal utilities $u(x_i)$ over the consequences $x_i$ by the normalized weights $\nu(E_i)/\sum_j \nu(E_j)$ on their corresponding events $E_i$. While more general functional forms are certainly imaginable and perhaps compelling, this particular form departs modestly and parsimoniously from standard subjected expected utility by a simple relaxation of additivity of $\nu$. The nonadditivity of $\nu$ can be used to detect and measure the effects of framing on decision making. Although the primitives are richer, the representation maintains the essential notions of expected utility and probabilistic sophistication from the standard model.

The following example illustrates the relationship between judged likelihood and framing more sharply. The mathematician Jean d'Alembert argued that "the probability of observing at least one head in two tosses of a fair coin is 2/3 rather than 3/4. Heads, as he said, might appear on the first toss, or, failing that, it might appear on the second, or, finally, might not appear on either. D'Alembert considered the three possibilities equally likely (Savage 1954, p. 65)." D'Alembert's fundamental mistake was in his framing of the states. He failed to split the first event into its two atoms: heads then tails, and heads then heads. He mistakenly viewed the world as three events: $\{HH, HT\}$, $\{TH\}$, and $\{TT\}$. Had he explicitly separated the two subevents comprising $\{HH, HT\}$ and framed the possible tosses appropriately, he may have avoided the error.

Such framing effects are finessed in the standard models of decision making under uncertainty introduced by Savage (1954) and by Anscombe and Aumann (1963). These models do not distinguish between different presentations of the same act, implicitly assuming the psychological principle of extensionality, that the framing of an event is inconsequential. Despite its obvious normative appeal, extensionality is descriptively questionable and repeatedly violated in experiments. Unpacking a contingency into finer components affects its perceived likelihood. For example, in a now classic experiment, Fischoff, Slovic, and Lichtenstein (1978) found that car mechanics' diagnostic assessments of whether a car fails to start because of a specific part will depend on whether this part's subcomponents are explicitly listed. Tversky and Koehler (1994) proposed an explanation, which they called support theory, which has since enjoyed considerable success among behavioral decision theorists and psychologists. One contribution of this paper is to provide an axiomatic foundation for a generalized version of support theory. But, we should immediately note that many of the general behavioral intuitions of the model here should be credited to this psychological literature.

Economists now appreciate the sensitivity of decision making to how consequences are framed, especially as formalized by prospect theory (Kahneman and Tversky 1979). In contrast, the effects of how states are framed are surprisingly obscure. We hope this paper helps direct more economic attention to how the framing of contingencies influences the judgment of likelihood, which we think is a significant psychological finding with potentially important economic consequences.

One interpretation of framing effects is as a consequence of unforeseen contingencies. The general idea of a decision maker with a coarse understanding of the state space appears in papers by Ghirardato (2001) and Mukerji (1997). Our model's contribution is in comparing preferences across descriptions to identify which contingencies had been unforeseen. This basic insight of using the explicit expression of unforeseen contingencies as a foundation for their identification is not entirely novel, in either psychology or economics. Tversky and Koehler (1994, p. 565) point out the connection between nonextensional judgment and unforeseen contingencies:

> The failures of extensionality . . . highlight what is perhaps the fundamental problem of probability assessment, namely the need to consider unavailable possibilities. . . . The extensionality principle, we argue, is normatively unassailable but practically unachievable . . . . People . . . cannot be expected to think of all relevant conjunctive unpackings or to generate all relevant future scenarios.

The relationship between awareness of the domain and sensitivity to framing is suggested in experiments by Fox and Clemen (2005), who asked MBA students at Duke University to assess the probabilities of various salary ranges of recent Duke MBA graduates and of recent Harvard Law graduates. Their assessments for Harvard Law salaries were very sensitive to how finely each range was subdivided, while their assessments for their own salaries were much less susceptible to framing effects. In economics, the connection to description was anticipated by Dekel, Lipman, and Rustichini (1998a, p. 524), who distinguish unforeseen contingencies from null events, because "an 'uninformative' statement – such as 'event $x$ might or might not happen' – can change the agent's decision." Our model provides a formal mechanism to precisely execute their suggested test.

Beside unforeseen contingencies, there are other compelling psychological explanations for nonextensional judgment. For example, the car mechanics surveyed by Fischoff, Slovic, and Lichtenstein (1978) had surely heard of the mechanical failures before. Rather, the mechanics more likely forgot or overlooked certain possibilities, which were made available once explicitly mentioned. To explain nonextensionality, Tversky and Koehler (1994)[p. 549] appeal to "memory and attention . . . . Unpacking a category . . . into its components . . . might remind people of possibilities that would not have been considered otherwise. Moreover, the explicit mention of an outcome tends to enhance its salience hence its support." It seems difficult to completely disentangle the behavioral implications of unforeseen and overlooked contingencies. The behavioral connection between availability and unforeseen contingencies has already been noted by Dekel, Lipman, and Rustichini (1998a)[p.

524], who write that “an unforeseen contingency is not necessarily one the agent *could not* conceive of, just one he *doesn't* think of at the time he makes his choice.”

A more striking set of experimental findings seems less related to availability or unforeseen contingencies. Different expressions for temperatures (Tversky and Fox 1995) and days of the week (Fox and Rottenstreich 2003) elicited different judgments of likelihood. For example, Fox and Rottenstreich (2003) find that the subjects' judgments of whether Sunday will be the hottest day of the coming week depend significantly on whether they were primed to think of the other days a single event or to differentiate the complement as Monday, Tuesday, and so on. It seems implausible that the other days of the week were unavailable or unforeseen to the subjects. These findings seem more directly related to salience.

Beyond the judgment of likelihood, the sensitivity of behavior to how different categories are framed appears to be a general and robust psychological phenomenon. A number of studies document excessive diversification of future consumption over different varieties (Simonson 1990, Ratner, Kahn, and Kahneman 1999), a tendency sometimes called diversification bias (Read and Loewenstein 1995). Fox, Ratner, and Lieb (2005) point out that diversification bias implies that portfolio allocation will be sensitive to the manner in which the options are partitioned and present evidence corroborating this implication. In general, when there is some quantity that needs to be divided, whether it is wealth or probability, its allocation can depend on how the bins are constructed.

The next section introduces the primitives of our theory. Section 3 formally defines the suggested partition-dependent expected utility representation for the model. Section 4 provides axiomatic characterizations for the representation and discusses the uniqueness of its identification. Finally, Section 5 exploits the structure of the representation. It connects properties of the weighting function to behavioral patterns. It also defines and examines two families of events under the interpretation of unforeseen contingencies, those events which are foreseen to the decision maker, and those which are completely unforeseen.

## 2 A nonextensional model of decision making

We introduce our general model of nonextensional decision making. Let $S$ denote an arbitrary state space, capturing all relevant uncertainty. Let $\Pi^*$ denote the collection of all finite partitions of $S$. We interpret a particular partition $\pi \in \Pi^*$ as a description of $S$: it explicitly mentions categories of possible states and these categories are comprehensive. For any partition $\pi \in \Pi^*$, let $\sigma(\pi)$ denote the algebra induced by $\pi$. Since $\pi$ is finite, its induced algebra $\sigma(\pi)$ is the family of unions of cells in $\pi$ and the empty set. Define the binary relation $\geq$ on $\Pi^*$ by $\pi' \geq \pi$ if $\sigma(\pi') \supset \sigma(\pi)$, i.e. if $\pi'$ is finer than $\pi$. In words, $\pi'$

is a richer description of the state space than under $\pi$. The relation $\geq$ is transitive, but generally incomplete. The meet $\pi \wedge \pi'$ denotes the finest common coarsening of $\pi$ and $\pi'$, while the join $\pi \vee \pi'$ denotes the coarsest common refinement of $\pi$ and $\pi'$.

The model considers some set of these descriptions $\Pi \subset \Pi^*$. We assume that $\Pi$ includes the vacuous description $\{S\}$ and is closed under the operations $\wedge$ and $\vee$. Two collections of events will be of particular importance: $\mathcal{C} = \cup_{\pi \in \Pi} \pi$, which denotes the collection of all events which are cells of partitions in $\Pi$, and $\mathcal{E} = \bigcup_{\pi \in \Pi} \sigma(\pi)$, which denotes the collection of all events explicitly described in some partition in $\Pi$. Clearly, $\mathcal{E}$ is the algebra generated by $\mathcal{C}$. Most of our results focus on two special cases of $\Pi$. In the first, descriptions become progressively finer, in which case $\Pi$ is a filtration. In the second, all possible descriptions are included in the model, in which case $\Pi = \Pi^*$. We will discuss the distinction shortly.

Let $X$ denote a finite set of consequences or prizes. Invoking the Anscombe–Aumann structure, let $\Delta X$ denote the set of all lotteries on $X$. An Anscombe–Aumann act $f : S \to \Delta X$ maps each state $s \in S$ to an an objective lottery over consequences $f(s) \in \Delta X$. An act $f$ is simple if it takes finitely many values, $|\{f(s) : s \in S\}| < \infty$. Slightly abusing notation, let $p \in \Delta X$ also denote the corresponding constant act which maps every state to the lottery $p$. Let $\mathcal{F}_\pi$ denote the family of simple Anscombe–Aumann acts which respect the partition $\pi$, i.e. $f^{-1}(p) \in \sigma(\pi)$ for all $p \in \Delta X$. In words, the act $f$ if $\sigma(\pi)$-measurable if it assigns a constant lottery to all states in a particular cell of the partition: if $s, s' \in E \in \pi$, then $f(s) = f(s')$. Let $\mathcal{F} = \bigcup_{\pi \in \Pi} \mathcal{F}_\pi$ denote the universe of acts under consideration. For any act $f \in \mathcal{F}$, let $\pi(f)$ denote the coarsest available partition $\pi \in \Pi$ such that $f \in \mathcal{F}_\pi$.[1] Note that when $\Pi \neq \Pi^*$, $\pi(f)$ could be strictly finer than the partition induced by $f$ if this coarsest possible description is not available as an element of $\Pi$. Similarly for any pair of acts $f, g \in \mathcal{F}$, let $\pi(f, g)$ be the coarsest available partition $\pi \in \Pi$ such that $f, g \in \mathcal{F}_\pi$.

Our primitive is a *family* of preferences $\{\succsim_\pi\}_{\pi \in \Pi}$ indexed by partitions $\pi$, where each preference relation $\succsim_\pi$ is defined over the family $\mathcal{F}_\pi$ of $\pi$-measurable acts. The strict and symmetric components $\succ_\pi$ and $\sim_\pi$ carry their standard meanings. Our interpretation of $f \succsim_\pi g$ is that $f$ is weakly preferred to $g$ when the state space $S$ is described as the partition $\pi$. If $f \notin \mathcal{F}_\pi$, then the description $\pi$ is too coarse to express the conditional payment schedule implied by the act $f$. If either $f$ or $g$ is not $\pi$-measurable, then the statement $f \succsim_\pi g$ is nonsensical.

Starting with a family of preferences might not immediately appear to be related to our original motivation of studying lists. But, in fact, this family provides a parsimonious prim-

[1] The existence of $\pi(f)$ is guaranteed by our assumption that $\Pi$ is closed under the operation $\wedge$. To see this, let $\pi \in \Pi$ be any partition according to which $f$ is measurable. Since $\pi$ is a finite partition, there are finitely many partitions that are (weakly) coarser than $\pi$, hence the set $\Pi' = \{\pi \in \Pi \mid \pi' \in \Pi, \pi' \leq \pi, \& f \in \mathcal{F}_{\pi'}\}$ is finite and nonempty since $\pi \in \Pi'$. Let $\pi(f) := \wedge_{\pi' \in \Pi'} \pi'$. Since $f \in \mathcal{F}_{\pi'}$ for each $\pi' \in \Pi$, we have that $f \in \mathcal{F}_{\pi(f)}$. Let $\rho \in \Pi$ be such that $f \in \mathcal{F}_\rho$. Since $f \in \mathcal{F}_\pi$ we conclude that $f \in \mathcal{F}_{\rho \wedge \pi}$. Hence $\rho \wedge \pi \in \Pi'$, implying $\pi(f) \leq \rho \wedge \pi \leq \rho$, as desired.

itive which loses little descriptive power relative to a model which begins with preferences over lists. Suppose we started with a list

$$\begin{pmatrix} x_1 & E_1 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix}$$

which is a particular expression of the act $f$. This list is more compactly represented as a pair $(f, \pi)$, where the partition $\pi = \{E_1, \ldots, E_n\}$ denotes the list of explicit contingencies on the right. This description $\pi$ is necessarily richer than the coarsest expression of $f$, so assume $f \in \mathcal{F}_\pi$. Now suppose the decision maker is deciding between two lists, which are represented as $(f, \pi_1)$ and $(g, \pi_2)$. Then the events in both $\pi_1$ and $\pi_2$ are explicitly mentioned. So the set of delineated events is the coarsest common refinement of $\pi_1$ and $\pi_2$, their join $\pi = \pi_1 \vee \pi_2$. Then $(f, \pi_1)$ is preferred to $(g, \pi_2)$ if and only if $(f, \pi)$ is preferred to $(g, \pi)$. We can therefore restrict attention to the preferences over pairs $(f, \pi)$ and $(g, \pi)$ where $f, g \in \mathcal{F}_\pi$. Moving the partition from being carried by the acts to being carried as an index of the preference relation arrives at exactly the model studied here.

We stress that the model is really that of a decision maker deciding between lists. The lists are expressed through indexed preference relations for the resulting economy of notation, which will simplify understanding the technical mechanics of the model. The partition $\pi$ indexing the preference of $f \succsim_\pi g$ is the coarsest refinement of the observable descriptions in the lists being compared by the decision maker, and is not meant to be interpreted as anything more. We certainly do not mean for $\pi$ to reflect the events which are foreseen to the decision maker or the way she conceptualizes the state space in her mind. In fact, in Section 5, we suggest a method for inferring her subjective understanding of the state space from her preferences over lists.

For example, suppose the decision maker is deciding between the Blue Cross health plan described on Table 1 and the health plan available from Kaiser Permanente and depicted in Table 2. The partitioning of the Kaiser Permanente plan differs from the partitioning of the

| | |
|---|---|
| Primary and specialty care visits | \$50 |
| Well-child visits to age two | \$15 |
| Family planning visits | \$50 |
| Scheduled prenatal care and first postmartum visit | \$50 |
| Maternity inpatient care | \$250 |

Table 2: Kaiser Permanente health insurance plan

Blue Cross plan; some contingencies are explicitly listed on one list but not on the other.

A newly hired and naive assistant professor, in the process of comparing health insurance options, has the possibility of maternity outpatient care in mind from the Blue Cross plan, and the possibility of family planning visits in mind from the Kaiser Permanente plan. This is a consequence of simply reading the lists and being exposed to both policies. Since the decision maker is reading both, we set $\pi$ to be the coarsest refinement of the two lists when making a comparison. The model therefore assumes that the decision maker can reason about the intersections and complements of described events.

The restriction that $f$ is $\pi$-measurable entails some loss. For example, consider a contract which covers eighty percent of the cost of surgery. The exact benefit from the contract depends on the exact procedure required, so the monetary transfer implied by the insurance contract varies with which surgery is required, while the contract is described without explicitly mentioning every possible surgery. Hence the contract is arguably not measurable with respect to its description. However, it is measurable with respect to events of the form "surgeries which cost $x$ dollars," about which the decision maker might form likelihood assessments even without understanding the exact surgeries which comprise each event. Hence one facile modeling device would be to consider an augmented state space consisting of pairs of surgery types and surgery costs, over which the act $f$ is measurable.[2] While non-measurability is an interesting and important issue, we will proceed here by abstracting away from this consideration. Since we can identify the weighting function from the restricted information in the model, we leave open how preferences over non-measurable acts are resolved.

An important consideration is exactly which preferences are available or observable to the analyst. How rich are the preferences which can be sensibly elicited from the decision maker? This question speaks directly to the structure of the collection $\Pi$. Consider the interpretation of framing in terms of availability or recall. Once an event is explicitly mentioned to the decision maker, this pronouncement cannot be reversed. She cannot consequently be made to forget it. In particular, suppose the state space partitions into three disjoint events $S = E \cup F \cup G$. Once the decision maker is presented a bet on $E$, which must be framed as $\{E, F \cup G\}$, she is immediately reminded of the event $E$. If she is then immediately presented a bet on $G$, she is reminded the event $G$. But, since she was already reminded of $E$, the observed choices at this second stage will be made as if $E$, $F$, and $G$ had been described to the decision maker. The decision maker is effectively acting as if she had read the finest partition $\{E, F, G\}$. More generally, after being presented with prior

[2]We thank Todd Sarver for suggesting this. More formally, the augmented state space $S' = S \times \mathbb{R}_+$ consists of pairs of the form ($s$, cost of surgery in state $s$). This extension is arguably undesirable because it artificially increases the primitive state space, hence the acts and the choice data. However this extension is feasible as long as costs of surgeries is contractible or observable, in which case contracts on the augmented space could be explicitly offered to a decision maker.

partitions $\pi_1, \ldots, \pi_{t-1}$, the relevant behavior after also being told $\pi_t$ is with respect to the refinement of the prior presentations $\pi_1, \ldots, \pi_{t-1}$ and the current $\pi_t$. So the appropriate assumption is that $\Pi$ is a filtration, where the decision maker is implicitly or explicitly presented with progressively finer descriptions.

On the other hand, under certain motivations for framing, it seems more reasonable to allow the analyst to access the family of all descriptions. For example, if framing effects are driven by the salience of an event, then this salience could depend on the description at hand, independently of prior descriptions. A similar argument can be made for the representativeness heuristic.[3] Recall the experiment by Fox and Rottenstreich (2003), mentioned in the introduction, where framing Sunday in isolation increases the judged probability that Sunday will be the hottest day of next week. This suggests that the experimenter can manipulate the relevant description quite powerfully. Even for motivations where preferences under the full set of descriptions cannot be elicited for a single subject, the analyst could believe there is enough uniformity in the population to elicit preferences across subjects, in which case a particular description could be given to one subject while alternative descriptions are given to others. Similarly, the analyst might find it useful to consider counterfactual assessments about what a particular decision maker would or should have done if she had been presented alternative sequences of descriptions.

We therefore consider two canonical cases. In the first, $\Pi$ is a filtration. In the second, $\Pi$ is the family of all finite partitions. The appropriateness of either case depends on the application being considered. From the standpoint of constructing a representation, neither case is obviously more technically challenging. When $\Pi$ is larger, the analyst has access to more information about the decision maker, but also must rationalize more of the decision maker's choices.

We conclude the description of the model by introducing some basic definitions which will be useful in the sequel. Given a partition $\pi = \{E_1, \ldots, E_n\} \subset \mathcal{E}$ and acts $f_1, \ldots, f_n \in \mathcal{F}$ define a new act by:

$$\begin{pmatrix} f_1 & E_1 \\ \vdots & \vdots \\ f_n & E_n \end{pmatrix}(s) = \begin{cases} f_1(s) & \text{if } s \in E_1 \\ \vdots & \vdots \\ f_n(s) & \text{if } s \in E_n \end{cases}.^4$$

[3]The most famous example of this is the Linda problem, where subjects are told that "Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations." The subjects believe the event "Linda is a bank teller" is less probable than the event "Linda is a bank teller and is active in the feminist movement" (Tversky and Kahneman 1983, p. 297).

[4]Note that the partition $\pi$ does not necessarily belong to $\Pi$. However the assumption that $\pi \subset \mathcal{E}$ guarantees that $\pi$ is coarser than some partition $\pi' \in \Pi$. To see this, let $\pi_i \in \Pi$ be such that $E_i \in \sigma(\pi_i)$ for each $i = 1, \ldots, n$ and let $\pi' = \pi_1 \vee \pi_2 \vee \ldots \vee \pi_n \in \Pi$. Then $\pi \leq \pi'$ and the new act defined above belongs

Also, let $pEq$ denote the act which assigns the lottery $p$ to the event $E$ and the lottery $q$ to its complement $E^{\complement}$.

The following definition adapts the standard concept of null events for our setting with a family of preferences.

**Definition 1.** Given $\pi \in \Pi$, an event $E \in \sigma(\pi)$ is $\boldsymbol{\pi}$**-null** if either $E = \emptyset$ or

$$\begin{pmatrix} p & E \\ f & E^{\complement} \end{pmatrix} \sim_{\pi} \begin{pmatrix} q & E \\ f & E^{\complement} \end{pmatrix},$$

for all $f \in \mathcal{F}_{\pi}$ and $p, q \in \Delta X$. $E \in \sigma(\pi)$ is $\boldsymbol{\pi}$**-nonnull** if it is not $\pi$-null. The event $E$ is **null** if $E$ is $\pi$-null for any $\pi$ such that $E \in \pi$. $E$ is **nonnull** if it is not null.

Note that for an event to be nonnull, it only needs to be nonnull for some partition $\pi$, but not necessarily for all partitions whose algebras include it.

Any expression of a contract $f$ must include at least the variation in payments which is necessary for its description. At a minimum, the events in $\pi(f)$ must be explicitly mentioned, recalling that $\pi(f)$ is the coarsest available partition $\pi \in \Pi$ which adapts $f$. Similarly, when comparing two acts $f$ and $g$, then the coarsest description available to express both $f$ and $g$ is $\pi(f, g)$, where none of the payoff-relevant contingencies are unpacked into finer subevents. This motivates the following binary relation $\succsim$ on $\mathcal{F}$.

**Definition 2.** For all $f, g \in \mathcal{F}$ define $f \succsim g$ if $f \succsim_{\pi(f,g)} g$.[5]

Certain conditions will be more compactly defined on the global relation $\succsim$, without referencing the entire family of preferences. The single relation $\succsim$ is theoretically powerful because it carries all the essential information about the family of relations $\{\succsim_{\pi}\}_{\pi\in\Pi}$. Suppose the analyst wanted to understand the decision maker's response to an act $f$ where some contingencies are unpacked, so the corresponding list includes a description of the state space which is strictly finer than $\pi(f)$. Then the finer description $\pi$ must entail some redundancies. For example, perhaps $f^{-1}(p) = E_1 \cup E_2$, but the description separately lists $E_1$ and $E_2$, even though they return the same lottery. Now consider the following act $f'$ which is very similar to $f$, but whose minimal expression does require separate expressions for $E_1$ and $E_2$: $f'$ assigns a very close but different lottery $p' \neq p$ to $E_2$ and is equal to $f$ everywhere else. Then $f'$ is very close to $f$ in terms of effective payments, but very different in terms of its implied minimal description.

Assuming $f, g \in \mathcal{F}_{\pi}$ assures that $\pi$ is at least as fine as $\pi(f, g)$. This highlights an important interpretive difference between standard theories of Bayesian updating and our

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to $\mathcal{F}_{\pi'\vee\pi(f_1)\vee\ldots\vee\pi(f_n)} \subset \mathcal{F}$.

[5]An alternative global relation is the comparison of certainty equivalents. Define $f \succsim^* g$ if $\mathrm{CE}(f) \succsim \mathrm{CE}(g)$ whenever $f \sim \mathrm{CE}(f) \in \Delta X$ and $g \sim \mathrm{CE}(g) \in \Delta X$. The examination of $\succsim^*$ is part of ongoing work.

theory of framing contingencies. In the former, acts must be restricted to respect the information, or lack thereof, embodied in an algebra on the state space. In our model, it is the algebra that must be expanded to reflect the contingencies explicitly mentioned in the description of an act. Partitions or subalgebras are often used to model the arrival of information about the actual state of the world, where each cell of a partition represents an updated restriction on the truth. Our interpretation is quite different. We take each partition $\pi$ a description of the entire state space. Each cell represents an event that is explicitly mentioned. In our model, the decision maker does *not* learn at some ex interim stage which particular cell actually obtains. Any differences in behavior across partitions are attributable only to the different descriptions of the entire state space, not to different arrival of information regarding the veracity of particular events.

## 3 Partition-dependent expected utility

We propose the following utility representation for every $\succsim_\pi$. The decision maker has a nonnegative set function $\nu : \mathcal{C} \to \mathbb{R}_+$ over relevant contingencies. When she is presented with a description $\pi = \{E_1, E_2, \ldots, E_n\}$ of the state space, she places a weight $\nu(E_k)$ on each described event. Normalizing these weights by their sum, $\mu_\pi(E_k) = \nu(E_k)/\sum_i \nu(E_i)$ defines a probability measure $\mu_\pi$ over $\sigma(\pi)$, the algebra induced by $\pi$. Then, her utility for the act $f$ expressed as:

$$f = \begin{pmatrix} p_1 & E_1 \\ p_2 & E_2 \\ \vdots & \vdots \\ p_n & E_n \end{pmatrix},$$

is $\sum_{i=1}^n u(p_i)\mu_\pi(E_i)$, where $u : \Delta X \to \mathbb{R}$ is an affine von Neumann–Morgenstern utility function on objective lotteries over consequences.

The following restriction avoids division by zero when normalizing the set function.

**Definition 3.** A set function $\nu : \mathcal{C} \to \mathbb{R}_+$ is *nondegenerate* if $\sum_{E \in \pi} \nu(E) > 0$ for all $\pi \in \Pi$.

We can now formally define our suggested utility representation.

**Definition 4.** $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a **partition-dependent expected utility** representation if there exist a nonconstant affine vNM utility function $u : \Delta X \to \mathbb{R}$ and a nondegenerate positive set function $\nu : \mathcal{C} \to \mathbb{R}_+$ such that for all $\pi \in \Pi$ and $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \Longleftrightarrow \int_S u \circ f \, d\mu_\pi \geq \int_S u \circ g \, d\mu_\pi,$$

where $\mu_\pi$ is the unique probability measure on $(S, \sigma(\pi))$ such that, for all $E \in \pi$:

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F \in \pi} \nu(F)}. \tag{1}$$

When such a pair $(u, \nu)$ exists, we call it a partition-dependent expected utility representation.

The set function $\nu$ in the representation is not necessarily additive, and its nonadditivity provides a channel for detecting the sensitivity of decision making to the description of the act. The structure of $\nu$ is quite general: $\nu$ is not necessarily monotone nor convex-ranged. It can also be strictly bounded away from zero for nonempty events, in which case there are no null events, even if the state space is uncountably rich. While it involves nonadditive set functions, the representation is only superficially similar to Choquet expected utility (Schmeidler 1989). In fact, the decision maker acts as if she is probabilistically sophisticated for any fixed description $\pi$.

A partition-dependent expected utility representation provides the following guidelines for the decision maker's response to framing. Each event $E \subsetneq S$ carries a value $\nu(E)$, which corresponds to its relative weight in lists where $E$, but not its subevents, are explicitly mentioned. The nonadditivity of $\nu$ captures the effects of framing: $E$ and $F$ can be disjoint yet $\nu(E)+\nu(F) \neq \nu(E \cup F)$. The normalization of dividing by $\sum_{E \in \pi} \nu(E)$ is also significant. It implies that if the complement of $E$ is unpacked and a partition $\pi_{E^\complement}$ of $E^\complement$ is explicitly mentioned, then the assessed likelihood of $E$ will be indirectly affected. So, the judged probability of $E$ will depend directly on its description $\pi_E$ and will depend indirectly on the partitioned description $\pi_{E^\complement}$ of its complement.

The utility function is obviously related to support theory, introduced by Tversky and Koehler (1994) and extended by Rottenstreich and Tversky (1997) as a treatment of nonextensional judgment. The primitives of support theory are descriptions of events, called hypotheses. Tversky and Koehler (1994) analyze binary comparisons of likelihood between pairs of hypotheses which they call evaluation frames, which consist of a focal hypothesis and an alternative hypothesis. The probability judgment of the focal hypothesis $A$ relative to the alternative $B$ in the evaluation frame $(A, B)$ is proposed to be $P(A, B) = s(A)/[s(A)+s(B)]$, where $s(A)$ is the support assigned to hypothesis $A$, based on the strength of its evidence. They offer a technical characterization of such judgments based on functional equations taking $P$ as primitive, but not one founded on preference (Tversky and Koehler 1994, Theorem 1). Our theory translates support theory from one of judgment to one of decision making, extends its scope beyond binary evaluation frames, and provides an axiomatic foundation from preference.

While the classic economic interpretation of $\nu$ is in terms of nonadditive likelihood, psychologists have a richer view of the support function and its sources. Tversky and Koehler (1994) suggest two other interpretations. The first is as a gauge of representativeness: "the hypothesis 'Bill is an accountant' may be evaluated by the degree to which Bill's personality matches the stereotype of an accountant." A second suggestion is in terms of representativeness: "the prediction 'an oil spill along the eastern coast before the end of next year' may be assessed by the ease with which similar accidents come to mind." Then $\nu$ has a quite literal interpretation as the number of examples or cases which the decision maker can think of. Both interpretations make violations of monotonicity possible. For example, if Bill's personality seems closer to that of an stereotypical accountant than to that of a stereotypical business executive, the support for the strict subset will be larger. Or, one might be able to recall more words that end with "ing" than words that end with "g." Such violations of monotonicity make a straight interpretation of $\nu$ in terms of likelihood less compelling.

Beside its experimental and psychological pedigree, there are sound methodological arguments for the suggested representation. These points will become more transparent as they develop in the sequel, but we summarize a few here. While the beliefs $\mu_\pi$ could be left unconnected across partitions, the consequent lack of basic structure would not be amenable to applications or comparative statics. Instead, we relate beliefs across descriptions with the single set function $\nu$. For example, an immediate implication of the representation is that the relative likelihood of disjoint events $A$ to $B$ is independent of how finely their complement is expressed. This form of consistency is assumed in some applications incorporating awareness, where the announcement of previously unforeseen contingencies does not alter the relative weight of already understood possibilities (Filiz 2006, Ozbay 2006).

One of the attractive features of the proposed representation is its compact form. Like standard Anscombe–Aumann expected utility, preference is summarized by two functions, one for utility and another for likelihood. It conservatively dispenses with the additivity of the standard model to allow for a rich set of framing effects. A virtue of the standard model is that a large number of implied preferences can be determined from a small number of choice observations. Under partition-dependent expected utility, once the weights of specific events are fixed, the weights of many others can be computed by comparing likelihood ratios. This tractably generates counterfactual predictions about behavior under alternative descriptions of the state space, an exercise that would be difficult without any structure across partitions.

A related benefit is that in some cases $\nu$ yields a clear interpretation which can be separately measured, e.g. in the availability heuristic as the number of cases. Then, as Tversky and Koehler (1994) point out, the analysis presented here could be reversed to move from measured support to relative likelihood: "in some cases, probability judgment may be predicted from independent assessments of support." Tversky and Koehler (1994, Study 3) conduct an experiment asking subjects to rate the strength to different NBA teams from 1 to 100 and to guess the probability that a particular team would win a home game against a specific opponent. The announced probabilities were strikingly close to the implied likelihoods under support theory using the strength ratings as a direct measurement of support, with an aggregate $R^2$ of 0.97. When possible, by measuring the degree of representativeness or availability in an independent experiment, one could infer which heuristic mediates judgment in settings with framing effects or unforeseen contingencies by comparing these measurements to the implied $\nu$ function.

Finally, the parametrization allows us to associate interesting classes of behavior with functional characteristics of $\nu$. This theoretical artifact of the representation therefore provides behavioral insights. For example, we can characterize specific kinds of framing effects by the subadditivity of $\nu$. The representation also guarantees natural structure on special fields of events, in particular those which are foreseen or completely unforeseen. These results are presented in Section 5.

In the special case that the set function $\nu$ is additive, the probabilities of events do not depend on their expressions, and the model reduces to standard subjective expected utility.

**Definition 5.** $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a **partition-independent expected utility** representation if there exist a nonconstant affine vNM utility function $u : \Delta X \to \mathbb{R}$ and a finitely additive probability measure $\mu : \mathcal{E} \to [0,1]$ such that for all $\pi \in \Pi$ and $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \Longleftrightarrow \int_S u \circ f \, d\mu \geq \int_S u \circ g \, d\mu.$$

The following examples provide some intuition for partition-dependent expected utility.

**Example 1** (Probability weighting)**.** Suppose $\nu(E) = \varphi(\mu(E))$, where $\mu : \mathcal{E} \to \mathbb{R}$ is a finitely additive probability measure and $\varphi : [0,1] \to [0,1]$ is a strictly increasing transformation with $\varphi(0) = 0$ and $\varphi(1) = 1$. In the theory of Choquet integration and nonadditive beliefs, such transformations are sometimes called probability distortion functions. In theories of non-expected utility over objective lotteries, like prospect theory (Kahneman and Tversky 1979) and anticipated utility theory (Quiggin 1982), such transformations are called probability weighting functions. Their application in our model is closest to the subjectively weighted utility theory of Karmarkar (1978). As Quiggin (1982) points out, because it is independent of the consequences tied to the lottery, the weighting function $\varphi$ in subjectively weighted utility must be linear, otherwise the preference violates stochastic dominance. In fact, the weighting function is applied in a rank-dependent fashion in anticipated utility and cumulative prospect theory (Tversky and Kahneman 1992) to maintain stochastic dominance. Here, because the manner in which $\varphi$ is applied depends on the framing of the act, we provide a rank-independent expression of subjectively weighted utility satisfying stochastic dominance.

**Example 2** (Principle of insufficient reason)**.** Suppose $\nu$ is a constant function, for example $\nu(E) = 1$ for each every nonempty $E$. Then the decision maker puts equal probability on all described contingencies. Such a criterion for cases of extreme ignorance or unawareness was advocated by Laplace and Leibnitz as the principle of insufficient reason. This principle is sensitive to the framing of the states. Consider the error of d'Alembert mentioned in the introduction, who attributed a probability of 2/3 to seeing at least one head among two tosses of a fair coin. The fundamental error was his framing of the state space as $H\text{-}, TH, TT$, or partitioned as $\{\{HT, HH\}; \{TH\}; \{TT\}\}$. If $\nu$ is constant, d'Alembert would have realized his error had he been presented a bet which pays only on a head followed by a tail, $HT$. On the other hand, he would have made a similar error had he reasoned that there can be either 0, 1, or 2 heads, partitioning the states into $\{\{TT\}; \{HT, TH\}; \{HH\}\}$. This criticism of the principle of insufficient reason is difficult to even formalize in a standard decision model; ours is specifically designed to capture such framing effects. In fact, we will identify the principle of insufficient reason as a special case of the model, corresponding to all events being completely unforeseen, in Section 5.

A more tempered resolution of unawareness is a convex combination of a probability measure $\mu$ and the ignorance prior: $\nu(E) = \alpha\mu(E) + (1 - \alpha)$ for some $\alpha \in [0, 1]$. Fox and Rottenstreich (2003) report experimental evidence which suggests that judgment is partially biased towards the ignorance prior. Fox and Clemen (2005) argue that this bias becomes more pronounced if the decision maker is more ignorant of the domain. For example, they asked American business students to assess the probabilities of different intervals of closing values for the Jakarta Stock Exchange, and students' judgments were extremely sensitive to the number of intervals that were explicitly mentioned.

# 4 Axioms and representations

## 4.1 Basic axioms

We first present axioms which will be required in all our results. The first five conditions essentially apply the standard Anscombe–Aumann axioms to each $\succsim_\pi$. We will refer to Axioms 1–5 collectively as the Anscombe–Aumann axioms.

**Axiom 1** (Weak Order)**.** $\succsim_\pi$ is complete and transitive for all $\pi \in \Pi$.

**Axiom 2** (Independence)**.** For all $\pi \in \Pi$, $f, g, h \in \mathcal{F}_\pi$ and $\alpha \in (0, 1)$: if $f \succ_\pi g$, then $\alpha f + (1 - \alpha)h \succ_\pi \alpha g + (1 - \alpha)h$.

**Axiom 3** (Archimedean Continuity)**.** For all $\pi \in \Pi$ and $f, g, h \in \mathcal{F}_\pi$: if $f \succ_\pi g \succ_\pi h$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ_\pi g \succ_\pi \beta f + (1-\beta)h$.

**Axiom 4** (Nondegeneracy)**.** For all $\pi \in \Pi$, there exist $f, g \in \mathcal{F}_\pi$ such that $f \succ_\pi g$.

**Axiom 5** (State Independence)**.** For all $\pi \in \Pi$, $\pi$-nonnull $E \in \sigma(\pi)$, $p, q \in \Delta X$, and $f \in \mathcal{F}_\pi$:

$$p \succsim_{\{S\}} q \Longleftrightarrow \begin{pmatrix} p & E \\ f & E^{\complement} \end{pmatrix} \succsim_\pi \begin{pmatrix} q & E \\ f & E^{\complement} \end{pmatrix}.$$

State Independence has some additional content in our model. Not only is the cardinal utility for a consequence invariant to the event $E$ in which it obtains, but also invariant to the way that event $E$ is expressed.

These familiar axioms guarantee an Anscombe–Aumann expected utility representation for each $\succsim_\pi$, consisting of a probability measure $\mu_\pi : \sigma(\pi) \to [0,1]$ and an affine function $u : \Delta X \to \mathbb{R}$ such that $\int_S u \circ f \, d\mu_\pi$ is a utility representation of $\succsim_\pi$. Given a fixed partition $\pi$, the decision maker's preferences $\succsim_\pi$ are completely standard. She is probabilistically sophisticated on $\sigma(\pi)$ and evaluates objective lotteries linearly. The model's interest derives from the relationship between preferences across different descriptions of the state space.

The sharpest assumption is that the preference for one act over another is insensitive to how they are described.

**Axiom 6** (Partition Independence)**.** For all $\pi, \pi' \in \Pi$ and $f, g \in \mathcal{F}_\pi \cap \mathcal{F}_{\pi'}$,

$$f \succsim_\pi g \Longleftrightarrow f \succsim_{\pi'} g.$$

Quite naturally, Partition Independence guarantees that subjective probability is extensional. We state the following variation of the Anscombe–Aumann theorem as a benchmark case, omitting the straightforward proof.

**Proposition 1.** *$\{\succsim_\pi\}_{\pi \in \Pi}$ admits a partition-independent expected utility representation if and only it satisfies the Anscombe–Aumann axioms and Partition Independence.*

Recall that the global relation $\succsim$ on $\mathcal{F}$ is defined by $f \succsim g$ if $f \succsim_{\pi(f,g)} g$, where $\pi(f,g)$ is the coarsest available partition $\pi \in \Pi$ which makes both acts measurable $f, g \in \mathcal{F}_\pi$. The following is a verbatim application of the classic axiom of Savage (1954) to the defined relation $\succsim$.

**Axiom 7** (Sure-Thing Principle)**.** For all events $E \in \mathcal{E}$ and acts $f, g, h, h' \in \mathcal{F}$,

$$\begin{pmatrix} f & E \\ h & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} g & E \\ h & E^{\complement} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} f & E \\ h' & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} g & E \\ h' & E^{\complement} \end{pmatrix}.$$

The standard justification for the Sure-Thing Principle is in establishing coherent conditional preferences. The relative likelihood of subevents of $E$ should be independent of the prizes associated with $E^{\complement}$. Because the description of $E^{\complement}$ is implicitly encoded in $h$ and $h'$, this axiom has additional content here. When the range of $h$ is disjoint from the ranges of $f$ and $g$, the contingencies needed to make the comparison in the left hand side can be divided into two parts: the description of $E$ implied by $f$ and $g$, and description of $E^{\complement}$ implied by $h$. The description of the state space in the right hand side can be similarly divided: the *same* description of $E$ generated by $f$ and $g$, and the possibly different description of $E^{\complement}$ generated by $h'$. Since the description of $E$ is similar for both comparisons and the acts being compared agree on $E^{\complement}$, the Sure-Thing Principle requires that the preferences are determined by where the acts differ on $E$. In other words, the effect of a particular expression of an event $E$ is independent of how its complement $E^{\complement}$ is expressed. The normative appeal of this axiom is similar to its implicitly dynamic justification in the standard setting. If $E$ was known to be true, then the description of the impossible event $E^{\complement}$ should not change preference over acts conditional on $E$.

### 4.2 $\Pi$ is a filtration

We write $\Pi$ is a filtration if the refinement relation $\geq$ is complete on $\Pi$. Given the restriction to finite partitions, $\Pi$ can then be indexed by a finite or countably infinite sequence $\mathcal{T}$ as $\Pi = \{\pi_t\}_{t\in\mathcal{T}}$ with $\pi_0 = \{S\}$ and $\pi_{t+1} > \pi_t$. When $\mathcal{T}$ is finite, there exists a finest partition $\pi_T$ which adapts the space of all relevant acts and events, $\mathcal{F} = \bigcup_{\pi\in\Pi}\mathcal{F}_\pi = \mathcal{F}_{\pi_T}$ and $\mathcal{E} = \bigcup_{\pi\in\Pi}\sigma(\pi) = \sigma(\pi_T)$. As a reminder, for any expressible act $f \in \mathcal{F}$, in this setting $\pi(f)$ refers to the first partition in $\{\pi_t\}_{t\in\mathcal{T}}$ which adapts $f$, but $\pi(f)$ could be strictly finer than the algebra induced by $f$. Similarly $\pi(f, g)$ need not be the minimal expression of the pair $f$ and $g$, but refers only the first partition in the filtration where $f$ and $g$ become describable.

**Theorem 1.** $\{\succsim_{\pi_t}\}_{t\in\mathcal{T}}$ *admits a partition-dependent expected utility representation if and only if it satisfies the Anscombe–Aumann axioms and the Sure-Thing Principle.*

*Proof.* See Appendix B.1.

While the utility function $u$ over lotteries is unique up to positive affine transformations, the uniqueness of the weighting function $\nu$ in the representation is surprisingly delicate. This delicacy also provides some intuition for how $\nu$ is constructed. Our general strategy is to use an appropriate chain of available partitions and betting preferences to calibrate the likelihood ratio $\nu(E)/\nu(F)$. For example, suppose $S = \{a, b, c, d\}$, $\mathcal{T} = \{1, 2\}$, $\pi_1 = \{\{a, b\}; \{c, d\}\}$, and $\pi_2 = \{\{a\}; \{b\}; \{c, d\}\}$. Consider the ratio $\nu(\{a, b\})/\nu(\{a\})$. First, we examine preferences when the states are described as the partition $\pi_1$ to identify the likelihood ratio of $\{a, b\}$ to $\{c, d\}$. Next, eliciting the preferences when the states are described as $\pi_2$ reveals the ratio of $\{c, d\}$ to $\{a\}$. The Sure-Thing Principle suggests the following argument: the ratio of $\{a, b\}$ to $\{a\}$ is equal to the ratio of $\{a, b\}$ to $\{c, d\}$ times the ratio of $\{c, d\}$ to $\{a\}$, i.e. "the $\{c, d\}$'s cancel" and the revealed likelihood ratios multiply out.

However, this strategy could fail for two reasons. First, if $\{c, d\}$ is $\pi_1$-null, the ratio is undefined. Second, if the filtration specifies $\pi_2 = \{\{a\}; \{b\}; \{c\}; \{d\}\}$, we lose the comparison cell $\{c, d\}$ and there is no cell common to $\pi_1$ and $\pi_2$ with which to execute the comparison. Consequently, we cannot link the likelihood of $\{a, b\}$ to $\{a\}$ across the two descriptions. In general, instead of achieving total uniqueness, the collection of cells $\mathcal{C}$ segregates into equivalence classes of cells which can be linked together in the described fashion. Without further restrictions, $\nu$ is unique only up to scale transformations for all such equivalence classes. This motivates the following definition, which guarantees that a nonnull cell is available to connect likelihoods across partitions.

**Definition 6.** A filtration $\Pi = \{\pi_t\}_{t \in \mathcal{T}}$ is **gradual with respect to** $\{\succsim_{\pi_t}\}_{t \in \mathcal{T}}$ if there exists a $\pi_t$-nonnull event $E \in \pi_t \cap \pi_{t+1}$ for all $t \geq 1$.

In words, $\Pi$ is gradual if it never splits all of the $\pi_t$-nonnull events into finer descriptions. For example, the example just given where $\pi_1 = \{\{a, b\}; \{c, d\}\}$ and $\pi_2 = \{\{a\}; \{b\}; \{c\}; \{d\}\}$ is not gradual. On the other hand, an alternative elicitation could gradually describe the state space as $\pi'_2 = \{\{a\}; \{b\}; \{c, d\}\}$ and then as $\pi'_3 = \{\{a\}; \{b\}; \{c\}; \{d\}\}$. This collects a strictly richer set of preferences while maintaining a filtration structure. So, another interpretation of the condition is that the experiment implied by the filtration $\{\pi_t\}_{t \in \mathcal{T}}$ does not throw away any information.

**Theorem 2.** *Suppose $\{\succsim_{\pi_t}\}_{t \in \mathcal{T}}$ admits a partition-dependent expected utility representation $(u, \nu)$. The following are equivalent:*

(i) *$\{\pi_t\}_{t \in \mathcal{T}}$ is gradual with respect to $\{\succsim_{\pi_t}\}_{t \in \mathcal{T}}$.*

(ii) *If $(u', \nu')$ also represents $\{\succsim_\pi\}_{\pi \in \Pi}$, then there exist numbers $a, c > 0$ and $b \in \mathbb{R}$ such that $u'(p) = au(p) + b$ for all $p \in \Delta X$ and $\nu'(E) = c\nu(E)$ for all $E \in \mathcal{C} \setminus \{S\}$.*

*Proof.* See Appendix B.2.

This is the sharpest identification to be expected. First, $\nu(E)$ is relevant for preference only when $E$ is a cell of some nontrivial description of the states. The value $\nu(F)$ is not identified when $F$ is never coarsely described; if $\{a, b\}$ is immediately described in $\pi_1$ as $\{a\} \cup \{b\}$, we can elicit the sum $\nu(\{a\}) + \nu(\{b\})$, but not $\nu(\{a, b\})$. Second, the only relevant feature for preference is the ratio of cell weights, which identifies $\nu$ up to a constant multiple which cancels itself when the weights are normalized. Finally, the value $\nu(S)$ at the vacuous description $\{S\}$ is unidentified because this quantity will always divide itself to unity.

### 4.3 $\Pi$ is the collection of all finite partitions

We now consider the case where $\Pi$ is the collection $\Pi^*$ of all finite partitions of $S$. In this case $\mathcal{E} = 2^S$ and $\mathcal{C} = 2^S \setminus \{\emptyset\}$. Unlike the previous case where $\Pi$ was a filtration, the Sure-Thing Principle is now insufficient to guarantee a partition-dependent expected utility representation. The problem is that the inferred likelihood ratio of different events can depend on the sequence of comparisons used to make the inference. In the prior filtration setting, there was only one sequence available, so this was not a concern. The next example demonstrates the issue concretely.

**Example 3.** Let $S = \{a, b, c, d\}$ and $\Delta X = [0, 1]$. Let $\pi^* = \{\{a, b\}; \{c, d\}\}$ with $\mu_{\pi^*}(\{a, b\}) = \frac{2}{3}$ and $\mu_{\pi^*}(\{c, d\}) = \frac{1}{3}$. For any $\pi \neq \pi^*$, let $\mu_\pi(C) = \frac{1}{|\pi|}$ for all cells $C \in \pi$. Suppose $u(p) = p$, so $\succsim_\pi$ is represented by $\int_S f \, d\mu_\pi$. These preferences satisfy the Anscombe–Aumann axioms and the Sure-Thing Principle, but cannot be rationalized by any partition-dependent expected utility with a fixed set function $\nu$. To the contrary, suppose $(u, \nu)$ was a partition-dependent expected utility representation. Let $\pi_1 = \{\{a, b\}; \{c\}; \{d\}\}$, $\pi_2 = \{\{a, d\}; \{b\}; \{c\}\}$, and $\pi_3 = \{\{a\}; \{b\}; \{c, d\}\}$. Then, by multiplying relevant likelihood ratios, we obtain:

$$\frac{\nu(\{a,b\})}{\nu(\{c,d\})} = \frac{\nu(\{a,b\})}{\nu(\{c\})} \times \frac{\nu(\{c\})}{\nu(\{b\})} \times \frac{\nu(\{b\})}{\nu(\{c,d\})} = \frac{\mu_{\pi_1}(\{a,b\})}{\mu_{\pi_1}(\{c\})} \times \frac{\mu_{\pi_2}(\{c\})}{\mu_{\pi_2}(\{b\})} \times \frac{\mu_{\pi_3}(\{b\})}{\mu_{\pi_3}(\{c,d\})} = 1.$$

We can directly obtain a contradictory conclusion:

$$\frac{\nu(\{a,b\})}{\nu(\{c,d\})} = \frac{\mu_{\pi^*}(\{a,b\})}{\mu_{\pi^*}(\{c,d\})} = 2.$$

So, we require additional restrictions on the preferences across descriptions. The example suggests that some control over sequences of implied likelihood ratios is required, which is where we will eventually arrive. Preferences across partitions are summarized in the generated relation $\succsim$, which ranks a pair of acts according to their coarsest available joint description. A conspicuous feature of the relation $\succsim$ is its intransitivity, since the implicit partitions $\pi(f, g)$, $\pi(g, h)$, and $\pi(f, h)$ required for pairwise comparisons of $f$, $g$, and $h$ are generally distinct. One common relaxation of transitivity is acyclicity. A preference relation $\succsim$ is acyclic if its strict component $\succ$ does not admit any cycles. Given that $\succsim$ is complete, this is equivalent to the following definition.

**Axiom 8** (Acyclicity)**.** For all acts $f_1, \ldots, f_n \in \mathcal{F}$,

$$f_1 \succ f_2, \ldots, f_{n-1} \succ f_n \Longrightarrow f_1 \succsim f_n.$$

In conjunction with the Anscombe–Aumann axioms, Acyclicity precludes framing effects and forces partition-independence.

**Proposition 2.** $\{\succsim_\pi\}_{\pi \in \Pi^*}$ *admits a partition-independent expected utility representation if and only if it satisfies the Ancsombe–Aumann axioms and Acyclicity.*[6]

*Proof.* See Appendix C.1.

So, to allow for nontrivial framing effects, Acyclicity must be further generalized. In particular, some types of cycles must be admitted. The problem with general Acyclicity is that it precludes cycles which may have no implications for likelihood across disjoint events. Yet, in a setting where framing and descriptions matter, the notion of relative likelihood is meaningful only across disjoint events, since two events with nonempty intersection cannot be cells of the same description. To specify exactly which cycles implicate likelihood, we first need to introduce some notation.

**Definition 7.** A sequence of events $E_1, E_2, \ldots$ is **sequentially disjoint** if $E_i \cap E_{i+1} = \emptyset$ for all $i$.

In particular, cycles involving simple binary bets across sequentially disjoint events are not allowed, since these are precisely the ones which have meaningful likelihood interpretations, even in the presence of framing. Any cycle must be more complicated than simple comparisons of bets on disjoint events.

**Axiom 9** (Binary Bet Acyclicity)**.** For any sequentially disjoint cycle of sets $E_1, \ldots, E_n, E_1$ and lotteries $p_1, \ldots, p_n; q \in \Delta X$,

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\mathsf{c}} \end{pmatrix} \succ \begin{pmatrix} p_2 & E_2 \\ q & E_2^{\mathsf{c}} \end{pmatrix}, \ldots, \begin{pmatrix} p_{n-1} & E_{n-1} \\ q & E_{n-1}^{\mathsf{c}} \end{pmatrix} \succ \begin{pmatrix} p_n & E_n \\ q & E_n^{\mathsf{c}} \end{pmatrix} \Longrightarrow \begin{pmatrix} p_1 & E_1 \\ q & E_1^{\mathsf{c}} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\mathsf{c}} \end{pmatrix}.$$

The privileged status of binary or simple bets in the measurement of belief dates back to at least Ramsey (1931), who argued such bets provide the cleanest elicitation of the judged relative likelihood across events. Binary acts are especially important when preferences are intransitive. Like nearly all models of subjective uncertainty, partition-dependent expected

[6] Proposition 2 remains true if Acyclicity is replaced with transitivity of $\succsim$. Also, recall the certainty equivalent relation $\succsim^*$ defined in Footnote 5: $f \succsim^* g$ if any certainty equivalent of $f$ is weakly preferred to any certainty equivalent of $g$. Acyclicity of the defined relation $\succsim$ can also be replaced with monotonicity or weak admissibility of the certainty equivalent relation $\succsim^*$: $f \succsim^* g$ whenever $f(s) \succsim^* g(s)$ for all $s \in S$.

utility implies a form of consistency on judgment across events; Binary Bet Acyclicity characterizes this consistency exactly. For example, Savage suggests eliciting a subjective likelihood ordering $\geq$ over events by defining $A \geq B$ if $xAy \succsim xBy$ whenever $x \succ y$. This implied qualitative probability inherits completeness and transitivity from the preference relation $\succsim$. Fishburn (1989) relaxes transitivity of the relation $\succsim$ over acts in a Savage setting, but maintains the transitivity of the likelihood relation $\geq$ over events. This is expressed as a behavioral condition that the relation $\succsim$ is transitive over a class of binary acts (Fishburn 1989, Axiom P.1*).[7]

Savage and Fishburn both invoke fine partitions to calibrate the quantitative probability of an event from this implied likelihood ordering. This calibration would fail in our model: finer partitions alter the framing of the state space, which is exactly the main object of study. Instead, we invoke the Anscombe–Aumann structure to directly calibrate quantitative likelihood ratios for disjoint events using objective randomization over prizes. Therefore, the appropriate consistency condition is not only qualitative, but quantitative as well. Hence the acyclicity condition is applied even as the better outcome $p_i$ varies for each act. The restricted notion of transitivity used in Fishburn is identical if the hypothesis further assumed that the prizes $p_i$ are identical.

This consistency on likelihoods is only applicable across comparisons of disjoint events, a sensible restriction given our model of framing. If $A$ and $B$ intersect, then the statement $A$ is more likely than $B$ is delicate because we cannot directly elicit the likelihood of the holistic expression of "$A$" versus the holistic expression of "$B$," because no partition allows a comparison of $A$ to $B$. The best we can do is assess the subjective likelihood of "$[A \setminus B] \cup [A \cap B]$" versus "$[B \setminus A] \cup [A \cap B]$." But once framing effects are allowed and judgment is nonextensional, this is a conceptually distinct exercise.

Binary Bet Acyclicity can be viewed as a behavioral generalization of an implication of support theory called the product rule, which is well-known in the psychological literature. Roughly speaking, if $R(A, B)$ denotes the relative likelihood of hypothesis $A$ to a mutually exclusive hypothesis $B$, support theory implies the following product rule: $R(A, C)R(C, B) = R(A, D)R(D, B)$. This can be rewritten as $R(A, C)R(C, B)R(B, D) = R(A, D)$, which is a particular case of the kind of likelihood consistency implied by Binary Bet Acyclicity. The intuitions for the product rule and for Binary Bet Acyclicity are similar: the particular comparison event, $C$ or $D$, used to calibrate the quantitative likelihood ratio of $A$ to $B$ is irrelevant. In an experiment involving judging the likelihoods that professional

[7]Formally, Fishburn's Axiom P.1* reads: if $x \succ y$, then

$$\begin{pmatrix} x & E_1 \\ y & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} x & E_2 \\ y & E_2^{\complement} \end{pmatrix}, \begin{pmatrix} x & E_2 \\ y & E_2^{\complement} \end{pmatrix} \succsim \begin{pmatrix} x & E_3 \\ y & E_3^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} x & E_1 \\ y & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} x & E_3 \\ y & E_3^{\complement} \end{pmatrix}.$$

basketball teams would defeat others, Fox (1999) elicits ratios of support values and finds an "excellent fit of the product rule for these data" at both the aggregate and individual subject level.

The observable content of Acyclicity and Binary Bet Acyclicity is not uncontroversial. The preference relations that comprise the sequences in the axioms' hypotheses are indexed by distinct minimal partitions. This might be problematic if, for example, we interpreted the model in terms of the recall of contingencies which are not immediately available. If we present the decision maker with the first, the second, and then the third binary bets, she has been immediately reminded of more events than had she been presented the second and third bets in isolation. On the other hand, the preference implied in the axioms is really the latter. These concerns are fundamentally related to the observability of choices under all descriptions. But, we only require these axioms when these choices are explicitly observable. In Theorem 1, we showed that if these comparisons are counterfactual, then the Sure-Thing Principle is sufficient. Our position is that either the implied preferences are observable, in which case Acyclicity and Binary Bet Acyclicity can be falsified, or they are not observable, in which case these axioms can be dispensed. The interpretations and situations where Binary Bet Acylicity is controversial are exactly those where the axiom is not required.

We can now characterize partition-dependent expected utility when $\Pi$ is rich.

**Theorem 3.** $\{\succsim_\pi\}_{\pi\in\Pi^*}$ *admits a partition-dependent expected utility representation if and only if it satisfies the Anscombe–Aumann axioms, the Sure-Thing Principle, and Binary Bet Acyclicity.*

*Proof.* See Appendix C.2

Turning to uniqueness, the following condition translates Definition 6 of a gradual filtration for the setting where all events are cells of some available partition.

**Axiom 10** (Event Reachability)**.** For any distinct nonnull events $E, F \subsetneq S$, there exists a sequentially disjoint sequence of nonnull events $E_1, \dots, E_n$ such that $E_1 = E$ and $E_n = F$.

Event Reachability is immediately satisfied if all nonempty events are nonnull. To compare, the axiom of Strict Admissibility is sometimes normatively invoked as a strong form of monotonicity or dominance.

**Axiom 11** (Strict Admissibility)**.** If $f(s) \succsim g(s)$ for all $s \in S$ and $f(s') \succ g(s')$ for some $s' \in S$, then $f \succ g$.

Strict Admissibility implies that all events are nonnull, hence implies Event Reachability. Strict Admissibility is not inconsistent with our representation, even if the state space is very rich. For example, if the set function is uniformly bounded away from zero, i.e. $\nu(E) > \alpha > 0$ for any nonempty event $E$, then there are no nonempty null events and Strict Admissibility is satisfied. Such a uniform bound suggests a decision maker who always put some small positive probability on any explicitly mentioned contingency.

The converse is false: Event Reachability is strictly weaker than Strict Admissibility.

**Example 4** (Event Reachability $\not\Rightarrow$ Strict Admissibility)**.** Let $S = \{s_1, s_2, s_3\}$ and suppose that $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a partition-dependent expected utility representation, where only the events $\{s_1\}$, $\{s_2\}$, $\{s_3\}$, and $\{s_1, s_2\}$ have strictly positive $\nu$-weight. The specified $\nu$ is nondegenerate. Strict Admissibility fails since some nonempty events are null. Event Reachability is satisfied: any two singletons are immediately comparable, and and a sequentially disjoint path from $\{s_1, s_2\}$ to either $\{s_1\}$ or $\{s_2\}$ can be constructed through $\{s_3\}$.

So, while Strict Admissibility suffices to identify $\nu$ uniquely, it is a touch stronger than required. Rather, Event Reachability is the weakest assumption which guarantees uniqueness.

**Theorem 4.** *Assume that $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a partition-dependent expected utility representation $(u, \nu)$. The following are equivalent:*

(i) *$\{\succsim_\pi\}_{\pi\in\Pi^*}$ satisfies Event Reachability.*

(ii) *If $(u', \nu')$ also represents $\{\succsim_\pi\}_{\pi\in\Pi^*}$, then there exist numbers $a, c > 0$ and $b \in \mathbb{R}$ such that $u'(p) = au(p) + b$ for all $p \in \Delta X$ and $\nu'(E) = c\nu(E)$ for all $E \subsetneq S$.*

*Proof.* Follows from Lemma 4 in Appendix A.

If the partition-dependent expected utility representation $(u, \nu)$ for $\{\succsim_\pi\}_{\pi\in\Pi^*}$ is uniquely determined in the sense of part (ii) of Theorem 4, we write that $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a unique partition-dependent expected utility representation by $(u, \nu)$.

## 4.4 Monotonicity

In this section, let $\Pi = \Pi^*$. We now consider preferences which correspond to the restriction that $\nu$ is monotone with respect to set inclusion: $\nu(E) \leq \nu(F)$ whenever $E \subset F \subsetneq S$. While it seems natural that subsets should carry less weight, psychological experiments repeatedly demonstrate otherwise. For example, Tversky and Kahneman (1983) document numerous examples of the conjunction fallacy, where subjects judge the intersection of different events to be strictly more likely than its components. One reason may be an availability heuristic. When estimating the frequency of seven-letter words ending with "ing" versus seven-letter words with “n” as the sixth letter, subjects report a higher frequency for the former set, even though it is a strict subset of the latter. Violations of monotonicity due to the representativeness heuristic, as famously demonstrated by the Linda problem, are also remarkably robust, despite “a series of increasingly desperate manipulations designed to induce subjects to obey the conjunction rule” (Tversky and Kahneman 1983, p. 299). In our setting, a violation of monotonicity suggests that a particularly likely or salient subcontingency is overlooked unless explicitly mentioned.

Monotonicity can be behaviorally identified. When the set function $\nu$ is unique up to a scalar multiple, as characterized in Theorem 4, the following condition guarantees that $\nu$ is monotone.

**Axiom 12** (Monotonicity)**.** For all $E \subset F$ and $p, q, r, s \in \Delta X$ such that $p \succ q$,

$$s \succsim \begin{pmatrix} p & F \\ q & F^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} r & E \\ s & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} r & E \\ p & F \setminus E \\ q & F^{\complement} \end{pmatrix}.$$

The assumed preference in the left reflects the relative likelihood of $F$ versus $F^{\complement}$. In particular, the decision maker is willing to pay $s$ for the bet on $F$. The implied preference on the right reveals that the relative likelihood of $F \setminus E$ versus $F^{\complement}$ (conditional on $E^{\complement} = [F \setminus E] \cup F^{\complement}$ obtaining) must be smaller, since the decision maker is still willing to pay $s$ (again conditional on $E^{\complement}$ obtaining).[8] So, the relative likelihoods of $F$ and $F \setminus E$ against the same event $F^{\complement}$ are required to be ordered in the appropriate manner.

**Proposition 3.** *Assume that $\{\succsim_\pi\}_{\pi \in \Pi^*}$ admits a unique partition-dependent expected utility representation $(u, \nu)$. Then $\{\succsim_\pi\}_{\pi \in \Pi^*}$ satisfies Monotonicity if and only if $E \subset F \subsetneq S$ implies $\nu(E) \leq \nu(F)$.*

*Proof.* See Appendix D.1.

Event Reachability is indispensable in Proposition 3. In general, there could exist one representation where $\nu$ is monotone, but another where $\nu'$ is not. Example 6 of Appendix D.1 demonstrates this explicitly. Without Event Reachability, we can only conclude that the family of null sets is an ideal with all subevents of null events remaining null, i.e. if $F$ is null and $E \subset F$, then $E$ is also null.

One interesting consequence of the Monotonicity axiom is that, once imposed, Event Reachability and Strict Admissibility are equivalent. Therefore, if one begins with the hypothesis of Monotonicity, uniqueness can be falsified by finding a nonempty null set.

[8] Recall the Linda example, where subjects thought Linda was more likely to be a feminist librarian than a librarian. This behavior is excluded because the likelihood ratio of librarian to non-librarian must be larger than the likelihood ratio of feminist librarian to non-librarian.

**Proposition 4.** *Assume that* $|S| \geq 3$ *and* $\{\succsim_\pi\}_{\pi \in \Pi^*}$ *admits a partition-dependent expected utility representation and satisfies Monotonicity. Then* $\{\succsim_\pi\}_{\pi \in \Pi^*}$ *satisfies Event Reachability if and only if it satisfies Strict Admissibility.*

*Proof.* See Appendix D.2.

# 5 Framing and unforeseen contingencies

In this section, let $\Pi = \Pi^*$. Here, we introduce behavioral definitions of the decision maker's sensitivity to framing. We also discuss two interesting families of events, which are especially relevant when the model is interpreted as one of unforeseen contingencies.

For any nonempty event $E$, let $\Pi^*_E$ denote the set of all finite partitions of $E$. If $E \in \pi \in \Pi^*$ and $\pi'_E \in \Pi^*_E$, slightly abusing notation let $\pi \vee \pi'_E$ denote $\pi \vee [\pi'_E \cup \{E^{\complement}\}]$. For any partition $\pi \in \Pi^*$ we adopt the convention that $\pi \cup \{\emptyset\}$ denotes the partition $\pi$. Although $\emptyset$ is not in the domain of $\nu$, we adopt the convention that $\nu(\emptyset) = 0$.

## 5.1 Underpacking and overpacking

The following notions of underpacking and overpacking are defined directly on preference, without reference to any particular utility function. The decision maker underpacks if she is more willing to bet on an event as it is described in more detail. Conversely, she underpacks if she is less willing to bet on an event as it is described more finely. In the Anscombe–Aumann setting here, this willingness to pay can be directly calibrated from certainty equivalents.

For the interpretation in terms of unforeseen contigencies, we stress that these notions really identify the correction for potentially partially unforeseen contingencies. This is because the decision maker might understand that the description is incomplete and consequently attempt to compensate for the lack of finer detail. In doing so, she may undershoot or overshoot what her willingness to pay would be under the finer description. For example, consider a car owner who understands that her engine includes a transmission. She would pay more for a warranty which covers her entire engine than she would for one which only covers her transmission, because she understands that there are other components in the engine. But, she might pay yet even more for a warranty covering her entire engine when these other components are unpacked and listed explicitly, because she relatively underestimated their likelihood when she had only a vague sense of their existence.

**Definition 8.** Given a nonempty event $E$ and $\pi'_E \in \Pi^*_E$, $\{\succsim_\pi\}_{\pi \in \Pi^*}$ **underpacks** $\boldsymbol{\pi'_E}$ if, for any $p, q, r \in \Delta X$ such that $q \succ r$ and for any $\pi$ such that $E \in \pi$:

$$\begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix} \succsim_{\pi} p \Longrightarrow \begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix} \succsim_{\pi \vee \pi'_E} p.$$

$\{\succsim_\pi\}_{\pi \in \Pi^*}$ **underpacks** if $\{\succsim_\pi\}_{\pi \in \Pi^*}$ underpacks $\pi'_E$ for all nonempty events $E$ and all $\pi'_E \in \Pi^*_E$.

In words, if the decision maker's certainty equivalent for a bet on the event $E$ increases when it is described more finely as $\pi'_E$. This corresponds to the intuition that more vividly described events carry more weight. One way to consider the definition is that she is willing to pay more to insure against contingency $E$ as more of its subevents are explicitly described. So, finer descriptions of $E$ increase its weight in decision making. When different contingencies are implicitly grouped into a coarse description $E$, this grouping decreases the assess likelihood, so the contingencies are "underpacked." Consequently, to make the decision maker put more weight on $E$, it should be described in more detail. Violations of monotonicity entail severe underpacking, since a packed group will have less weight than one of its particular subcontingencies.

The opposite tendency, to put more weight on coarse descriptions, is defined symmetrically.

**Definition 9.** Given a nonempty event $E$ and $\pi'_E \in \Pi^*_E$, $\{\succsim_\pi\}_{\pi \in \Pi^*}$ **overpacks $\boldsymbol{\pi'_E}$** if, for any $p, q, r \in \Delta X$ such that $q \succ r$ and for any $\pi$ such that $E \in \pi$:

$$p \succsim_{\pi} \begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix} \Longrightarrow p \succsim_{\pi \vee \pi'_E} \begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix}.$$

$\{\succsim_\pi\}_{\pi \in \Pi^*}$ **overpacks** if $\{\succsim_\pi\}_{\pi \in \Pi^*}$ overpacks $\pi'_E$ for all nonempty events $E$ and all $\pi'_E \in \Pi^*_E$.

For example, a consumer who is mechanically ignorant might purchase more warranty protection when the engine's components are not explicitly described, because she overestimates the number of unknown components which could break down. Such a consumer overpacks.

When preferences admit a unique partition-dependent expected utility representation, subadditivity or superadditivity of the set function $\nu$ determines whether revealed likelihood increases or decreases as an event becomes more finely described.

**Definition 10.** A set function $\nu$ is **subadditive** [**superadditive**] if $\nu(E \cup F) \leq [\geq] \nu(E) + \nu(F)$ whenever $E \cap F = \emptyset$ and $E \cup F \neq S$.

Note that superadditivity is strictly weaker than convexity.[9] Convexity is commonly assumed for value functions in cooperative games or for capacities in Choquet integration,

[9] A set function $\nu$ is convex if $\nu(E \cup F) + \nu(E \cap F) \geq \nu(E) + \nu(F)$ for all $E, F \subset S$.

but carries little behavioral significance beyond the implied superadditivity in our model of framing.

In their original paper, Tversky and Koehler (1994) argue for and provide evidence suggesting subadditivity of the support function across disjunctions of hypotheses. More recently, Sloman, Rottenstreich, Wisniewski, Hadjichristidis, and Fox (2004) present experimental cases of superadditivity, where explicitly mentioning atypical or unlikely contingencies decreased the subjective probability of an event. Consequently, they suggest that subadditivity "should be jettisoned from future formulations of the theory (p. 581)." Without taking a prior position, subadditivity or superadditivity can be verified through the underpacking or overpacking of descriptions. The straightforward proof of the following observation is omitted.

**Proposition 5.** *Suppose $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a partition-dependent expected utility representation $(u,\nu)$ and satisfies Strict Admissibility. Then $\{\succsim_\pi\}_{\pi\in\Pi^*}$ underpacks* [overpacks] *if and only if $\nu$ is subadditive* [superadditive]*.*

Example 7 in Appendix E.1 demonstrates that strict admissibility is indispensable in Proposition 5.

## 5.2 Foreseen and completely unforeseen contingencies

We now identify the events which the decision maker understands completely. These are the events whose explicit descriptions have no effect on choice. For example, if a consumer had a possibility $E$ already in mind when deciding between contracts $f$ and $g$, then mentioning this possibility in the contract should have no bearing on her preference. Conversely, if explicitly describing $E$ reverses her preference, she must not have completely considered $E$ before it was described.

**Definition 11.** Fix $\{\succsim_\pi\}_{\pi\in\Pi^*}$. An event $E$ is **foreseen** if for any $\pi \in \Pi^*$ and for any $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \Longleftrightarrow f \succsim_{\pi\vee\{E,E^{\complement}\}} g.$$

Let $\mathcal{A}$ denote the family of all foreseen events.

We will use $A, B$ to denote generic events in the collection $\mathcal{A}$. Let $\pi \in \Pi^*$ and $f, g \in \mathcal{F}_\pi$. If $\pi' \in \Pi^*$ is such that $\pi' \subset \mathcal{A}$, then iterated application of Definition 11 to the events in $\pi'$ obtains the following:

$$f \succsim_\pi g \Longleftrightarrow f \succsim_{\pi\vee\pi'} g.$$

In words, explicitly describing the already foreseen contingencies in $\pi'$ does not affect the decision maker's choices, which are made with an implicit understanding of the description $\pi'$.

**Proposition 6.** *Suppose* $\{\succsim_\pi\}_{\pi\in\Pi^*}$ *admits a partition-dependent expected utility representation* $(u,\nu)$ *and satisfies Strict Admissibility. Then*

(i) $A \in \mathcal{A}$ *if and only if* $\nu(E) = \nu(E \cap A) + \nu(E \cap A^{\complement})$ *for any event* $E \neq S$.

(ii) $\mathcal{A}$ *is an algebra.*

(iii) $\nu$ *is additive on* $\mathcal{A} \setminus \{S\}$, *i.e. for all disjoint* $A, B \in \mathcal{A}$ *such that* $A \cup B \neq S$:

$$\nu(A \cup B) = \nu(A) + \nu(B).$$

*Moreover,* $\nu(A) + \nu(A^{\complement}) = \nu(B) + \nu(B^{\complement})$ *for any* $A, B \in \mathcal{A} \setminus \{\emptyset, S\}$.

(iv) $\mathcal{A} = 2^S$ *if and only if* $\nu$ *is additive on* $2^S \setminus \{S\}$.

*Proof.* See Appendix E.2.

First, under the partition-dependent expected utility representation, the family $\mathcal{A}$ of completely foreseen events can be simply characterized through the set function $\nu$. Specifically, the weight of an arbitrary event $E$ is additive across its conjunctions with the events that the decision maker already has in mind. Second, such a representation guarantees that $\mathcal{A}$ is closed under intersection and unions, hence is an algebra. So, $(S, \mathcal{A})$ can be viewed as the subjective prior understanding of the decision maker of the state space, which can be coarser or finer across different agents. The decision maker behaves as if, before any explicit description of particular contingencies, she arrives with an implicit understanding of some subjective algebra $\mathcal{A}$ of the state space. The descriptions which might cause her to reconsider her preferences are those outside of $\mathcal{A}$, or those which she did not completely comprehend until they were described. Third, the decision maker's weights are additive on that subjective algebra. Also, complementary weights sum to a constant number, so we can define the value $\nu(S) = \nu(A) + \nu(A^{\complement})$ for an arbitrary $A \in \mathcal{A}$. Then $\nu|_{\mathcal{A}}$ is a finite measure, hence $(S, \mathcal{A}, \nu|_{\mathcal{A}})$ defines a probability space after appropriate normalization. Fourth, the case where the decision maker displays no framing effects is characterized by partition-independence.

**Example 5.** Suppose $\pi^* = \{A_1, \ldots, A_n\}$ be a partition of the state space. An adaptation of the principle of insufficient reason is to define $\nu(E) = \sum_{i=1}^n |E \cap A_i|$. Then the family of foreseen events $\mathcal{A}$ is the algebra generated by $\pi^*$. For example, a consumer might understand that requiring chemotherapy, surgery, drugs, and behavioral counseling are relevant contingencies in purchasing health insurance which she has in mind even if they are not specifically mentioned. But, when a specific disease is mentioned, she distributes the likelihood of its relevant treatments using the principle of insufficient reason.

We hope these results reinforce our prior caution that the partition $\pi$ should not be interpreted as the subjective understanding of the state space on behalf of the decision maker. Instead, speaking informally, the theoretical artifact $\mathcal{A}$ reflects the decision maker's prior understanding of the state space, $\pi$ reflects the contingencies which are observably mentioned by the lists being compared, and $\pi \vee \mathcal{A}$ reflects her resulting model of the world after reading the lists.

As a counterpoint to the events which are understood perfectly, we now discuss the events which are completely overlooked.

**Definition 12.** Fix $\{\succsim_\pi\}_{\pi \in \Pi^*}$. An event $E \subset S$ is **suppressed** if $E = \emptyset$ or if, for all three cell partitions $\{E, F, G\}$ of $S$ and $p, q, r \in \Delta X$:

$$\begin{pmatrix} p & E \cup F \\ q & G \end{pmatrix} \sim r \iff \begin{pmatrix} p & F \\ q & E \cup G \end{pmatrix} \sim r.$$

$E$ is **completely unforeseen** if $E$ is nonnull and suppressed.

In words, $E$ is suppressed if the decision maker never puts any weight on $E$ unless it is explicitly described to her. In the first comparison, she attributes all the likelihood of receiving $p$ to $F$, because $E$ carried no weight when it is not separately mentioned; in the second comparison, all the likelihood of $q$ is similarly attributed to $G$. Due to the framing of both acts, $E$ remains occluded and the certainty equivalents are equal because both appear to be bets on $F$ and $G$. It is important to notice that an event does not have to be either foreseen or completely unforeseen; the two conditions represent extreme cases admitting many intermediate possibilities.

Definition 12 distinguishes a completely unforeseen event from a null event. Whenever $E \cup F \neq S$, the following preference is consistent with $E$ being completely unforeseen:

$$\begin{pmatrix} p & E \cup F \\ q & G \end{pmatrix} \succ \begin{pmatrix} p' & E \\ p & F \\ q & G \end{pmatrix}.$$

Here, the presentation of the second act explicitly mentions $E$, at which point she assigns it some positive likelihood. In contrast, this strict preference is precluded whenever $E$ is null, because then the decision maker would be indifferent to whether $p'$ or $p$ was assigned to the impossible event $E$. One way to understand a completely unforeseen event is as one which is null only when it is not explicitly mentioned.

On the other hand, we cannot determine whether a null and suppressed event is completely unforeseen. There are two possible reasons why $E$ contributes no additional likelihood to $E \cup F$. First, the decision maker may have completely overlooked the event $E$

when it was grouped as $E \cup F$. Second, she may have actually considered its possibility, but concluded that $E$ was impossible. These reasons cannot be distinguished behaviorally.

**Proposition 7.** *Suppose $\{\succsim_\pi\}_{\pi \in \Pi^*}$ admits a unique partition-dependent expected utility representation $(u, \nu)$ where $\nu$ is monotone. Then*

(i) *$E$ is suppressed if and only if $\nu(E \cup F) = \nu(F)$ for any nonempty event $F$ disjoint from $E$ such that $E \cup F \neq S$.*

(ii) *If $E$ and $F$ are suppressed and $E \cup F \neq S$, then $E \cap F$ and $E \cup F$ are also suppressed.*

(iii) *If $|S| \geq 3$ and all nonempty events are completely unforeseen, then $\nu(E) = \nu(F)$ for all nonempty $E, F \neq S$.*

The first part of the proposition relates suppressed events with their marginal contribution to the weighting function $\nu$. The second part shows that the family of suppressed events has some desirable properties: closure under set operations is guaranteed when the sets do not cover all of $S$.

The third part characterizes the principle of insufficient reason. This extreme case where all nonempty events are completely unforeseen is represented by a constant capacity where $\nu(E) = 1$ for every nonempty $E$. The decision maker places a uniform distribution over the events which are explicitly mentioned in a description $\pi$. In fact, this statement can be strengthened to the following. If two disjoint sets $E$ and $F$, with $E \cup F \neq S$, are completely unforeseen, then the principle of insufficient reason is applied to subevents of their union: $\nu(D) = \nu(D')$ for all $D, D' \subset E \cup F$. Then $E \cup F$ can be considered an area of the state space of which the decision maker has no understanding.

We hope this interpretation of the model complements existing decision theoretic work on unforeseen contingencies.[10] Kreps (1979) introduced an axiomatic model of preference over menus of objects, where demand for flexibility is interpreted as a response to unforeseen contingencies which are captured in the proposed representation as subjective taste uncertainties. Dekel, Lipman, and Rustichini (2001), henceforth DLR, extend this approach to menus of lotteries, where the linear structure allows an essential identification of the subjective state space. Then the analyst can remarkably determine the space of uncertainty as a theoretical artifact of preference, rather than assume an exogenous state space.

The DLR methodology provides a powerfully unified treatment of states, beliefs, and utilities. On the other hand, because it depends on preferences to elicit the subjective states, the recovery of unforeseen contingencies is difficult. It encounters the basic conundrum that the decision maker cannot reveal something completely unforeseen to her at the point of

[10]There is also a growing literature on the epistemic foundations of awareness. A recent set of references can be found in Heifetz, Meier, and Schipper (2007).

choice. In fact, in DLR's main representations, the decision maker acts as if she has foresight of some subjective state space.

To address the conundrum, Epstein, Marinacci, and Seo (2007) suggest generalizations of the DLR model which incorporate minimization over sets of beliefs, delivering analogs to Gilboa and Schmeidler (1989) with subjective states and to Ghirardato (2001) and Mukerji (1997) with coarse subjective states. This connection between ambiguity aversion and unforeseen contingencies resonates with earlier work by Ghirardato (2001), Mukerji (1997), and Nehring (1999) who capture the decision maker's partial understanding of unforeseen contingencies through Choquet integration of belief functions or capacities, originally used to model ambiguity by Schmeidler (1989). One point of our model is to demonstrate another method of eliciting partially unforeseen contingencies without invoking ambiguity aversion. In fact, our representation satisfies the standard expected utility axioms for any particular description of the state space. This makes some inroads towards a charge forwarded by Dekel, Lipman, and Rustichini (1998a) to "distinguish between unforeseen contingencies and 'standard' uncertainty aversion."

Unlike DLR, we do not recover the state space as a component of the representation. We hope the specification of a state space as part of the model is justified by its conceptual dividends. It also seems more palatable when imagining applications. It strikes us as difficult to verify and enforce contracts which depend on DLR's subjective states, which are purely theoretical constructions. Insofar as unforeseen contingencies bear on contracting, assuming some sort of objective state space seems less heroic. In fact, one motivation for developing the model was to accommodate unforeseen contingencies in a subjective setting without invoking menus or multi-valued consequences, so the primitives bear as close a resemblance as possible to the way that actual contracts, like insurance policies or warranties, are presented.

# A Preliminary observations

In this section we will state and prove a set of preliminary lemmas and a uniqueness result for general $\Pi$. In particular, the results in this section will apply to both the case where $\Pi$ is a filtration and the case where $\Pi$ is the set of all finite partitions. We start by noting that the first five axioms provide a simple generalization of the Anscombe–Aumann Expected Utility Theorem.

**Lemma 1.** *The collection $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies the Anscombe–Aumann axioms if and only if there exist an affine utility function $u : \Delta X \to \mathbb{R}$ with $[-1,1] \subset u(\Delta X)$ and a family of probability measures $\{\mu_\pi\}_{\pi\in\Pi}$ with $\mu_\pi : \sigma(\pi) \to [0,1]$ such that*

$$f \succsim_\pi g \iff \int_S u \circ f \, d\mu_\pi \geq \int_S u \circ g \, d\mu_\pi.$$

*Proof.* For each $\pi \in \Pi$, the necessity of the Anscombe–Aumann axioms follows immediately from the standard Anscombe–Aumann Expected Utility Theorem. To prove sufficiency note that for each $\pi \in \Pi$, Axioms 1–5 guarantee a probability measure $\mu_\pi$ on $(S, \sigma(\pi))$ and a non-constant affine vNM utility function $u_\pi : \Delta X \to \mathbb{R}$ such that $f \succsim_\pi g$ if and only if $\int_S u_\pi \circ f \, d\mu_\pi \geq \int_S u_\pi \circ g \, d\mu_\pi$, for all $f, g \in \mathcal{F}_\pi$. For all $\pi, \pi' \in \Pi$, State Independence implies that $p \succsim_\pi q$ if and only if $p \succsim_{\pi'} q$, therefore $u_\pi(p) \geq u_\pi(q)$ if and only if $u_{\pi'}(p) \geq u_{\pi'}(q)$. Then the uniqueness component of the standard Anscombe–Aumann Expected Utility Theorem implies that $u_{\pi'}$ is a positive affine transformation of $u_\pi$. By appropriately normalizing, we lose no generality by assuming $u_\pi = u_{\pi'} = u$. Nondegeneracy ensures that $u$ is not constant, so we may further assume that its image contains the interval $[-1, 1]$, again by appropriately normalizing. $\square$

The next Lemma states that the Sure-Thing principle is always a necessary condition for a partition dependent expected utility representation.

**Lemma 2.** *If $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a partition-dependent expected utility representation, then $\succsim$ satisfies the Sure-Thing Principle.*

*Proof.* For any $f, g \in \mathcal{F}$, note that $D(f,g) \equiv \{s \in S : f(s) \neq g(s)\} \in \sigma(\pi(f,g))$, hence:

$$\begin{aligned}
f \succsim g \quad &\iff \quad f \succsim_{\pi(f,g)} g \\
&\iff \quad \int_{D(f,g)} u \circ f \, d\mu_{\pi(f,g)} \geq \int_{D(f,g)} u \circ g \, d\mu_{\pi(f,g)} \\
&\iff \quad \sum_{\substack{F \in \pi(f,g): \\ F \subset D(f,g)}} u(f(F))\nu(F) \geq \sum_{\substack{F \in \pi(f,g): \\ F \subset D(f,g)}} u(g(F))\nu(F),
\end{aligned}$$

where the second equivalence follows from multiplying both sides by $\sum_{F' \in \pi(f,g)} \nu(F')$.

Now, to demonstrate the Sure-Thing Principle, let $E \in \mathcal{E}$ and $f, g, h, h' \in \mathcal{F}$. Let

$$\hat{f} = \begin{pmatrix} f & E \\ h & E^{\complement} \end{pmatrix}; \qquad \hat{g} = \begin{pmatrix} g & E \\ h & E^{\complement} \end{pmatrix};$$

$$\hat{f}' = \begin{pmatrix} f & E \\ h' & E^{\complement} \end{pmatrix}; \qquad \hat{g}' = \begin{pmatrix} g & E \\ h' & E^{\complement} \end{pmatrix}.$$

Note that $D \equiv D(\hat{f}, \hat{g}) = D(\hat{f}', \hat{g}') \subset E$ and $\pi_D \equiv \{F \in \pi(\hat{f}, \hat{g}) : F \subset D(\hat{f}, \hat{g})\} = \{F \in \pi(\hat{f}', \hat{g}') : F \subset D(\hat{f}', \hat{g}')\}$. Hence by the observation made in the first paragraph:

$$\begin{aligned}
\hat{f} \succsim \hat{g} \quad &\iff \quad \sum_{F \in \pi_D} u(\hat{f}(F))\nu(F) \geq \sum_{F \in \pi_D} u(\hat{g}(F))\nu(F) \\
&\iff \quad \sum_{F \in \pi_D} u(f(F))\nu(F) \geq \sum_{F \in \pi_D} u(g(F))\nu(F) \\
&\iff \quad \sum_{F \in \pi_D} u(\hat{f}'(F))\nu(F) \geq \sum_{F \in \pi_D} u(\hat{g}'(F))\nu(F) \\
&\iff \quad \hat{f}' \succsim \hat{g}'.
\end{aligned}$$

$\square$

The next Lemma summarizes the general implications of the Anscombe-Aumann axioms and the Sure-Thing Principle.

**Lemma 3.** *Assume that* $\{\succsim_\pi\}_{\pi\in\Pi}$ *satisfies the Anscombe–Aumann axioms and the Sure-Thing Principle. Then* $\{\succsim_\pi\}_{\pi\in\Pi}$ *admits a representation* $(u, \{\mu_\pi\}_{\pi\in\Pi})$ *as in Lemma 1. For any events* $E, F \in \mathcal{C}$ *and partitions* $\pi, \pi' \in \Pi$*:*

(i) *If* $E \in \pi, \pi'$*, then* $\mu_\pi(E) = 0 \Leftrightarrow \mu_{\pi'}(E) = 0$*.*

(ii) *If* $E, F \in \pi, \pi'$ *and* $E \cap F = \emptyset$*, then* $\mu_\pi(E)\mu_{\pi'}(F) = \mu_\pi(F)\mu_{\pi'}(E)$

*Proof.* To prove part (i), it is enough to show that if $E \in \pi, \pi'$, then $\mu_\pi(E) = 0 \Rightarrow \mu_{\pi'}(E) = 0$. Suppose that $\mu_\pi(E) = 0$. Select any two lotteries $p, q \in \Delta X$ satisfying $u(p) > u(q)$ and any two acts $h, h' \in \mathcal{F}$ such that $\pi(h) = \pi$, and $\pi(h') = \pi'$. Then

$$\begin{pmatrix} p & E \\ h & E^{\complement} \end{pmatrix} \sim \begin{pmatrix} q & E \\ h & E^{\complement} \end{pmatrix}$$

by Lemma 1. Hence

$$\begin{pmatrix} p & E \\ h' & E^{\complement} \end{pmatrix} \sim \begin{pmatrix} q & E \\ h' & E^{\complement} \end{pmatrix}$$

by the Sure-Thing Principle. Since $u(p) > u(q)$, the last indifference can hold only if $\mu_{\pi'}(E) = 0$ by Lemma 1.

To prove part (ii), observe that if either side of the desired equality is zero, then part (ii) is immediately implied by part (i). So we may proceed assuming that both sides are strictly positive. Then all of the terms $\mu_\pi(E)$, $\mu_{\pi'}(F)$, $\mu_\pi(F)$, and $\mu_{\pi'}(E)$ are strictly positive. As before, select any two lotteries $p, q \in \Delta X$ such that $u(p) > u(q)$, and define a new lottery $r$ by

$$r = \frac{\mu_\pi(E)}{\mu_\pi(E) + \mu_\pi(F)} p + \frac{\mu_\pi(F)}{\mu_\pi(E) + \mu_\pi(F)} q.$$

Select any two acts $h, h' \in \mathcal{F}$ such that $p, q, r \notin h(S) \cup h'(S)$, $\pi(h) = \pi$, and $\pi(h') = \pi'$. By the choice of $r$ and the expected utility representation of $\succsim_\pi$, we have:

$$\begin{pmatrix} p & E \\ q & F \\ h & (E \cup F)^{\complement} \end{pmatrix} \sim \begin{pmatrix} r & E \cup F \\ h & (E \cup F)^{\complement} \end{pmatrix}$$

Hence by the Sure-Thing Principle,

$$\begin{pmatrix} p & E \\ q & F \\ h' & (E \cup F)^{\complement} \end{pmatrix} \sim \begin{pmatrix} r & E \cup F \\ h' & (E \cup F)^{\complement} \end{pmatrix}.$$

This indifference, in conjunction with the expected utility representation of $\succsim_{\pi'}$, implies that

$$u(r) = \frac{\mu_{\pi'}(E)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} u(p) + \frac{\mu_{\pi'}(F)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} u(q).$$

We also have

$$u(r) = \frac{\mu_\pi(E)}{\mu_\pi(E) + \mu_\pi(F)} u(p) + \frac{\mu_\pi(F)}{\mu_\pi(E) + \mu_\pi(F)} u(q),$$

by the definition of $r$. Subtracting $u(q)$ from each side of the two expressions for $u(r)$ above, we obtain

$$\frac{\mu_{\pi'}(E)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} [u(p) - u(q)] = \frac{\mu_\pi(E)}{\mu_\pi(E) + \mu_\pi(F)} [u(p) - u(q)],$$

which further simplifies to $\frac{\mu_{\pi'}(F)}{\mu_{\pi'}(E)} = \frac{\mu_\pi(F)}{\mu_\pi(E)}$ since both sides of the previous equality are strictly positive. □

By part (i) of Lemma 3, for any $\pi, \pi' \in \Pi$, any event $E \in \pi, \pi'$ is $\pi$-null if and only if it is $\pi'$-null. Hence under the Anscombe-Aumann axioms and the Sure-Thing Principle, we can change quantifiers in the definitions of null and nonnull events in $\mathcal{C}$. An event $E \in \mathcal{C}$ is null if and only if $E$ is $\pi$-null for *some* partition $\pi \in \Pi$ with $E \in \pi$. Dually, an event $E \in \mathcal{C}$ is nonnull if and only if $E$ is $\pi$-nonnull for *every* partition $\pi \in \Pi$ with $E \in \pi$.[11]

We will next state and prove a general uniqueness result which will imply the uniqueness Theorems 2 and 4. To do so, we first need to generalize the Event Reachability condition so that it applies to our general model.

**Axiom 13** (Generalized Event Reachability)**.** For any distinct nonnull events $E, F \in \mathcal{C} \setminus \{S\}$, there exists a sequence of nonnull events $E_1, \dots, E_n \in \mathcal{C}$ such that $E = E_1$, $F = E_n$, and for each $i = 1, \dots, n-1$ there is $\pi \in \Pi$ such that $E_i, E_{i+1} \in \pi$.[12]

Note that when $\Pi$ is the set of all finite partitions, Generalized Event Reachability is equivalent to Event Reachability.

**Lemma 4.** *Assume that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation $(u, \nu)$. Then, the following are equivalent:*

(i) *$\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Generalized Event Reachability.*

(ii) *If $(u', \nu')$ also represents $\{\succsim_\pi\}_{\pi\in\Pi}$, then there exist numbers $a, c > 0$ and $b \in \mathbb{R}$ such that $u'(p) = au(p) + b$ for all $p \in \Delta X$ and $\nu'(E) = c\nu(E)$ for all $E \in \mathcal{C} \setminus \{S\}$.*

*Proof.* Assume that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits the partition-dependent expected utility representation $(u, \nu)$. Let $\mathcal{C}^*$ denote the set of nonnull events in $\mathcal{C}$. The collection $\mathcal{C}^*$ is nonempty since Nondegeneracy ensures that $S \in \mathcal{C}^*$. Define the binary relation $\approx$ on $\mathcal{C}^*$ by $E \approx F$ if there exist a sequence of events $E_1, \dots, E_n \in \mathcal{C}^*$ with $E = E_1$, $F = E_n$, and for each $i = 1, \dots, n-1$ there is $\pi \in \Pi$ such that $E_i, E_{i+1} \in \pi$. The relation $\approx$ is reflexive, symmetric, and transitive, defining an equivalence relation on $\mathcal{C}^*$. For any $E \in \mathcal{C}^*$, let $[E] = \{F \in \mathcal{C}^* : E \approx F\}$ denote the equivalence class of $E$ with respect to $\approx$. Let $\mathcal{C}^*/\approx\, = \{[E] : E \in \mathcal{C}^*\}$ denote the quotient set of all equivalence classes of $\mathcal{C}^*$ modulo

[11] Remember that events in $\mathcal{E} \setminus \mathcal{C}$ are all null by definition. Note that $\emptyset$ is null and $S$ is nonnull by Nondegeneracy. Also, there may exist a nonnull event $E \in \mathcal{C}$, which is $\pi$-null for some $\pi \in \Pi$ such that $E \in \sigma(\pi)$. From the above observation concerning the quantifiers, this can only be possible if $E$ is not a cell in $\pi$ but a union of its cells. This would correspond to a representation where, for example, $E$ is a disjoint union of two subevents $E = E_1 \cup E_2$ and $\nu(E) > 0$, yet $\nu(E_1) = \nu(E_2) = 0$.

[12] If $\Pi$ were a filtration, then either $E \subset F$, $F \subset E$, or $E \cap F = \emptyset$.

$\approx$, with a generic class $R \in \mathcal{C}^*/\approx$. Note that, given the above definitions, Event Reachability is equivalent to $\mathcal{C}^*/\approx$ consisting of two indifference classes $\{S\}$ and $\mathcal{C}^* \setminus \{S\}$.

We first show the "(i) $\Rightarrow$ (ii)" part. Suppose that $(u', \nu')$ is a partition-dependent expected utility representation of $\{\succsim_\pi\}_{\pi \in \Pi}$ and that Generalized Event Reachability is satisfied. For each $\pi \in \Pi$, let $\mu_\pi$ and $\mu'_\pi$ respectively denote the probability distributions derived from $\nu$ and $\nu'$ by Equation (1). Applying the uniqueness component of the Anscombe–Aumann Expected Utility Theorem to $\succsim_\pi$, we have $\mu_\pi = \mu'_\pi$ and $u' = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.

If $E \in \mathcal{C}$ is null, then $\nu(E) = \mu_\pi(E) = 0 = \mu'_\pi(E) = \nu'(E)$ for any $\pi \in \Pi$ with $E \in \pi$. Also note that if $E, F \in \mathcal{C}^*$ are such that there exists $\pi \in \Pi$ with $E, F \in \pi$, then

$$\frac{\nu(E)}{\nu(F)} = \frac{\mu_\pi(E)}{\mu_\pi(F)} = \frac{\mu'_\pi(E)}{\mu'_\pi(F)} = \frac{\nu'(E)}{\nu'(F)}.$$

We will next extend the equality $\frac{\nu(E)}{\nu(F)} = \frac{\nu'(E)}{\nu'(F)}$ to any pair of events $E, F \in \mathcal{C}^* \setminus \{S\}$, in order to conclude that there exists $c > 0$ such that $\nu'(E) = c\nu(E)$ for all $E \in \mathcal{C} \setminus \{S\}$. Let $E, F \in \mathcal{C}^* \setminus \{S\}$. By Generalized Event Reachability, there exist $E_1, \ldots, E_n \in \mathcal{C}^*$ such that $E = E_1$, $F = E_n$, and for each $i = 1, \ldots, n-1$ there is $\pi \in \Pi$ such that $E_i, E_{i+1} \in \pi$. Then:

$$\frac{\nu(E)}{\nu(F)} = \frac{\nu(E_1)}{\nu(E_2)} \times \ldots \times \frac{\nu(E_{n-1})}{\nu(E_n)} = \frac{\nu'(E_1)}{\nu'(E_2)} \times \ldots \times \frac{\nu'(E_{n-1})}{\nu'(E_n)} = \frac{\nu'(E)}{\nu'(F)}$$

where the middle equality follows from the existence of $\pi \in \Pi$ such that $E_i, E_{i+1} \in \pi$, for each $i = 1, \ldots, n-1$. Thus $\nu'$ is a scalar multiple of $\nu$ on $\mathcal{C}^* \setminus \{S\}$, determined by the constant $c = \nu'(E)/\nu(E)$ for any $E \in \mathcal{C}^* \setminus \{S\}$.

To see the "(i) $\Leftarrow$ (ii)" part, suppose that Generalized Event Reachability is not satisfied. Then the relation $\approx$ defined above has at least two distinct equivalence classes $R$ and $R'$ different from $\{S\}$. Define $\nu' : \mathcal{C} \to \mathbb{R}_+$ by:

$$\nu'(E) = \begin{cases} \nu(E) & \text{if } E \in R, \\ 2\nu(E) & \text{otherwise.} \end{cases}$$

for $E \in \mathcal{C}$. Take any $\pi \in \Pi$. If $E \in \pi \cap R \neq \emptyset$, then $\nu'(E) = \nu(E)$ for all $E \in \pi$. If $\pi \cap R = \emptyset$, then $\nu'(E) = 2\nu(E)$ for all $E \in \pi$. Hence $(u, \nu)$ and $(u, \nu')$ are two partition-dependent expected utility representations of $\{\succsim_\pi\}_{\pi \in \Pi}$ such that there does not exist a $c > 0$ with $\nu'(E) = c\nu(E)$ for all $E \in \mathcal{C} \setminus \{S\}$. $\square$

# B Proofs of Section 4.2: $\Pi$ is a filtration

## B.1 Proof of the representation result for filtrations (Theorem 1)

Necessity is implied by Lemmas 1 and 2. We next prove sufficiency. Let $u$ and $\{\mu_\pi\}_{\pi \in \Pi}$ be as guaranteed by Lemma 1. We will define $\nu$ on $\cup_{t=0}^{k} \pi_t$ inductively on $k \geq 0$, which will define $\nu$ on the whole $\mathcal{C} = \cup_{t \in \mathcal{T}} \pi_t$ by the principle of recursive definition. [13]

[13] The $c_k$ constants in the iterative definition show just how flexible we are in defining $\nu$, which also hints to the role of gradualness in guaranteeing uniqueness. In the iterative definition, step 1 is a subcase of the subsequent step, however we prefer to write it down explicitly because it is substantially simpler.

**Step 0:** Let $\nu(S) := c_0$ for an arbitrary constant $c_0 > 0$.

**Step 1:** For all $E \in \pi_1$, set $\nu(E) := c_1 \mu_{\pi_1}(E)$, for an arbitrary constant $c_1 > 0$.

**Step $\mathbf{k+1}$ ($\mathbf{k \geq 0}$):** Assume the following inductive assumptions:

(i) the nonnegative set function $\nu$ has already been defined on $\cup_{t=0}^{k} \pi_t$;

(ii) for all $t = 0, 1, \ldots, k$: $\sum_{E' \in \pi_t} \nu(E') > 0$ (i.e. nondegeneracy is satisfied);

(iii) for all $t = 0, 1, \ldots, k$ and for all $E \in \pi_t$: $\mu_{\pi_t}(E) = \nu(E) / \sum_{E' \in \pi_t} \nu(E')$.

*Case 1.* Assume that there exists $E^* \in \pi_k \cap \pi_{k+1}$ such that $\mu_{\pi_k}(E^*) > 0$. Then by Lemma 3 $\mu_{\pi_{k+1}}(E^*) > 0$ and by the inductive assumption $\nu(E^*) > 0$. For all $E \in \pi_{k+1} \setminus \pi_k = \pi_{k+1} \setminus (\cup_{t=1}^{k} \pi_t)$ (the equality is because we have a filtration) define $\nu(E)$ by

$$\nu(E) = \frac{\nu(E^*)}{\mu_{\pi_{k+1}}(E^*)} \mu_{\pi_{k+1}}(E) \tag{2}$$

Equation (2) also holds (as an equation rather than a definition) for $E \in \pi_{k+1} \cap \pi_k$, since

$$\frac{\nu(E)}{\nu(E^*)} = \frac{\mu_{\pi_k}(E)}{\mu_{\pi_k}(E^*)} = \frac{\mu_{\pi_{k+1}}(E)}{\mu_{\pi_{k+1}}(E^*)}.$$

where the first equality is by the inductive assumption and the second by Lemma 3. It is now easy to verify that $\nu$ satisfies (i), (ii), and (iii) on $\cup_{t=1}^{k+1} \pi_t$.

*Case 2.* Assume that for all $E \in \pi_k \cap \pi_{k+1}$: $\mu_{\pi_k}(E) = 0$. Let $c_{k+1} > 0$ be an arbitrary constant and for all $E \in \pi_{k+1} \setminus \pi_k = \pi_{k+1} \setminus (\cup_{t=1}^{k} \pi_t)$ define $\nu(E)$ by

$$\nu(E) = c_{k+1} \mu_{\pi_{k+1}}(E) \tag{3}$$

Equation (2) actually also holds (as an equation rather than a definition) for $E \in \pi_{k+1} \cap \pi_k$, since for all such $E$, $\mu_{\pi_k}(E) = 0$, hence by Lemma 3 $\mu_{\pi_{k+1}}(E) = 0$ and by the inductive assumption $\nu(E) = 0$. It is now easy to verify that $\nu$ satisfies (i), (ii), and (iii) on $\cup_{t=1}^{k+1} \pi_t$. □

## B.2 Proof of the uniqueness result for filtrations (Theorem 2)

In light of the general uniqueness result Lemma 4, we only need to prove that Generalized Event Reachability is equivalent to Gradualness for filtrations. Let $\Pi = \{\pi_t : t \in \mathcal{T}\}$ be the filtration representation of $\Pi$. Suppose that $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a partition-dependent expected utility representation $(u, \nu)$.

First assume that $\Pi$ is gradual with respect to $\{\succsim_\pi\}_{\pi \in \Pi}$. Let $E, F \in \mathcal{C} \setminus \{S\}$ be distinct nonnull events. Then there exist $\pi_i, \pi_j \in \Pi$ such that $i, j \in \mathcal{T} \setminus \{0\}$, $E \in \pi_i$, and $F \in \pi_j$. Without loss of generality let $i \leq j$, let $E_{i-1} := E$, $E_j := F$, and for each $t \in \{i, i+1, \ldots, j-1\}$ let $E_t \in \pi_t \cap \pi_{t+1}$ be a $\pi_t$-nonnull event as guaranteed by gradualness. Then $E_{i-1}, E_i, E_{i+1} \ldots, E_j \in \mathcal{C}$ is sequence of nonnull events such that $E = E_{i-1}$, $F = E_j$, and $E_t, E_{t+1} \in \pi_{t+1} \in \Pi$ for each $t = i-1, i, \ldots, j-1$. Hence Generalized Event Reachability is satisfied.

Now assume that Generalized Event Reachability is satisfied. Let $t^*, t^*+1 \in \mathcal{T} \setminus \{0\}$. By Non-degeneracy, there exist a $\pi_{t^*}$-nonnull event $E \in \pi_{t^*}$ and a $\pi_{t^*+1}$-nonnull event $F \in \pi_{t^*+1}$. Then

$E, F \in \mathcal{C} \setminus \{S\}$ are non-null, hence by Generalized Event Reachability, there exists a sequence of nonnull events $E_1, \ldots, E_n \in \mathcal{C}$ such that $E = E_1$, $F = E_n$, and for each $i = 1, \ldots, n-1$ there is $t \in \mathcal{T}$ such that $E_i, E_{i+1} \in \pi_t$. For each $i = 1, \ldots, n$, let $\underline{t}(i) = \min\{t \in \mathcal{T} : E_i \in \pi_t\}$ and $\bar{t}(i) = \sup\{t \in \mathcal{T} : E_i \in \pi_t\}$.[14] Then $E_i \in \pi_t$ if and only if $\underline{t}(i) \leq t \leq \bar{t}(i)$. Note that $\underline{t}(1) \leq t^* \leq \bar{t}(1)$, $\underline{t}(n) \leq t^* + 1 \leq \bar{t}(n)$, and $\underline{t}(i+1) \leq \bar{t}(i)$ for $i = 1, \ldots, n-1$. Hence $\underline{t}(i) \leq t^*$ and $t^* + 1 \leq \bar{t}(i)$ for some $i = 1, \ldots, n$. Then $E_i \in \pi_{t^*} \cap \pi_{t^*+1}$ and $E_i$ is non-null, hence $E_i$ is $\pi_{t^*}$-non null by Lemma 3. We conclude that $\Pi$ is gradual with respect to $\{\succsim_\pi\}_{\pi \in \Pi}$. $\square$

# C Proofs of Section 4.3: $\Pi$ is the Set of All Finite Partitions

## C.1 Proof of the partition-independent representation result for all finite partitions (Proposition 2)

Assume that $\Pi$ is the set of all finite partitions of $S$. For the necessity part, assume that $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a partition-independent expected utility representation $(u, \mu)$. Note that $f \succsim g$ if and only if $\int_S u \circ f \, d\mu \geq \int_S u \circ f \, d\mu$ for any $f, g \in \mathcal{F}$. Thus $\succsim$ is transitive, hence acyclic. The necessity of the Anscombe–Aumann axioms follows immediately from the standard Anscombe–Aumann Expected Utility Theorem.

Now turning to sufficiency, assume that $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies the Anscombe-Aumann Axioms and Acyclicity. Let $u$ and $\{\mu_\pi\}_{\pi \in \Pi}$ be as guaranteed by Lemma 1. We will first show that

$$\forall \pi \in \Pi \setminus \{\{S\}\} \text{ and } E \in \pi : \;\; \mu_\pi(E) = \mu_{\{E, E^\complement\}}(E) \tag{4}$$

Suppose for a contradiction that $\mu_\pi(E) > \mu_{\{E,E^\complement\}}(E)$ in (4). Let $\mu_\pi(E) > \alpha > \mu_{\{E,E^\complement\}}(E)$. Since the range of $u$ contains the interval $[-1, 1]$, there exist $p, q \in \Delta X$ such that $u(p) = 1$ and $u(q) = 0$. Define the act $h$ by

$$h = \begin{pmatrix} p & E \\ q & E^\complement \end{pmatrix}.$$

Note that $\alpha p + (1-\alpha)q \succ h$. Let $f \in \mathcal{F}$ be such that $\pi(f) = \pi$ and for all $s \in S$, $u(f(s)) < 0$. Then there exists $\varepsilon \in (0, 1)$ such that the act $h^\varepsilon \equiv (1-\varepsilon)h + \varepsilon f$ satisfies $\pi(h^\varepsilon) = \pi$ and $h^\varepsilon \succ_\pi \alpha p + (1-\alpha)q$. Then $h \succ h^\varepsilon \succ \alpha p + (1-\alpha)q \succ h$, a contradiction to $\succsim$ being acyclic. The argument for the case where $\mu_\pi(E) < \mu_{\{E,E^\complement\}}(E)$ is entirely symmetric, hence omitted.

Define $\mu \colon 2^S \to [0, 1]$ by $\mu(\emptyset) \equiv 0$, $\mu(S) \equiv 1$, and $\mu(E) \equiv \mu_{\{E,E^\complement\}}(E)$ for $E \neq \emptyset, S$. To see that $\mu$ is finitely additive, let $E, F$ be nonempty disjoint sets. If $E \cup F = S$, then $F = E^\complement$ so

$$\mu(E) + \mu(F) = \mu_{\{E,E^\complement\}}(E) + \mu_{\{E,E^\complement\}}(E^\complement) = 1 = \mu(E \cup F).$$

If $E \cup F \subsetneq S$, let $\pi = \{E, F, (E \cup F)^\complement\}$ and $\pi' = \{E \cup F, (E \cup F)^\complement\}$. Then by (4),

$$\mu(E) + \mu(F) = \mu_\pi(E) + \mu_\pi(F) = 1 - \mu_\pi((E \cup F)^\complement) = 1 - \mu_{\pi'}((E \cup F)^\complement) = \mu_{\pi'}(E \cup F) = \mu(E \cup F).$$

[14] We use supremum here since this value can be $+\infty$.

Therefore $\mu$ is a probability measure. To conclude, note that for any $\pi \in \Pi$, the definition of $\mu$ and (4) imply that $\mu_\pi(E) = \mu(E)$ for all $E \in \pi$. Hence $(u, \mu)$ is a partition-independent representation of $\{\succsim_\pi\}_{\pi\in\Pi}$. $\square$

## C.2 Proof of the partition-dependent representation for all finite partitions (Theorem 3)

The necessity of the Anscombe–Aumann axioms follow from the standard Anscombe–Aumann Expected Utility Theorem. The necessity of the Sure-Thing Principle was established in Lemma 2. We next check the necessity of Binary Bet Acyclicity.

**Lemma 5.** *Assume that $\Pi$ is the set of all finite partitions of $S$ and that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation. Then $\succsim$ satisfies Binary Bet Acyclicity.*

*Proof.* First note that for any (possibly empty) disjoint events $E$ and $F$, and (not necessarily distinct) lotteries $p, q, r \in \Delta X$, we have:

$$\begin{pmatrix} p & E \\ q & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} r & F \\ q & F^{\complement} \end{pmatrix} \qquad \Longleftrightarrow \qquad [u(p) - u(q)]\nu(E) \geq [u(r) - u(q)]\nu(F).$$

To see the necessity of Binary Bet Acyclicity, let $E_1, \ldots E_n, E_1$ be a sequentially disjoint cycle of events and $p_1, p_2, \ldots, p_n; q \in \Delta X$ be such that

$$\forall i = 1, \ldots n-1: \quad \begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix}.$$

The observation made in the first paragraph implies that $[u(p_1) - u(q)]\nu(E_1) > [u(p_2) - u(q)]\nu(E_2) > \ldots > [u(p_n) - u(q)]\nu(E_n)$. Since $[u(p_1) - u(q)]\nu(E_1) > [u(p_n) - u(q)]\nu(E_n)$, we conclude that

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

$\square$

We next prove the sufficiency part. In the rest of the section (in particular in Lemmas 6 and Lemma 7) assume that $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies the Anscombe–Aumann axioms, the Sure-Thing Principle, and Binary Bet Acyclicity. Let $(u, \{\mu_\pi\}_{\pi\in\Pi})$ be a representation of $\{\succsim_\pi\}_{\pi\in\Pi}$ guaranteed by Lemma 1. For any two disjoint nonnull events $E, F$, define the ratio:

$$\frac{E}{F} \equiv \frac{\mu_\pi(E)}{\mu_\pi(F)}$$

where $\pi$ is a partition such that $E, F \in \pi$. The value of $\frac{E}{F}$ does not depend on the particular choice of $\pi$, by part (ii) of Lemma 3. Moreover, $\frac{E}{F}$ is well-defined and strictly positive since $E$ and $F$ are nonnull. Finally, $\frac{F}{E} \times \frac{E}{F} = 1$ by construction. The following appeals to Binary Bet Acyclicity in generalizing this equality.

**Lemma 6.** *For any sequentially disjoint cycle of nonnull events $E_1, \ldots, E_n, E_1 \in \mathcal{E}$ and lotteries $p_1, \ldots, p_n, q \in \Delta X$ such that $u(q) = 0$ and $u(p_i) \in (0,1)$ for $i = 1, \ldots, n$:*

$$(\forall i = 1, \ldots n-1) : \begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

*Proof.* It is enough to show that the hypothesis above implies

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Let $\bar{\varepsilon} \in (0,1)$ be such that $u(p_i) + \bar{\varepsilon} < 1$ for $i = 1, \ldots, n$. Since the range of the utility function $u$ over lotteries contains the unit interval $[-1,1]$, for each $\varepsilon \in (0, \bar{\varepsilon})$ and $i \in \{1, \ldots, n\}$, there exists $p_i(\varepsilon) \in \Delta X$ such that $u(p_i(\varepsilon)) = u(p_i) + \varepsilon^i$, where $\varepsilon^i$ refers to the $i$th power of $\varepsilon$.[15] The expected utility representation of Lemma 1 and the fact that $E_i$ is nonnull implies that for sufficiently small $\varepsilon \in (0, \bar{\varepsilon})$,

$$\begin{pmatrix} p_i(\varepsilon) & E_i \\ q & E_i^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_{i+1}(\varepsilon) & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix},$$

for $i = 1, \ldots, n-1$. By Binary Bet Acyclicity, this implies

$$\begin{pmatrix} p_1(\varepsilon) & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n(\varepsilon) & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Appealing to the continuity of the expected utility representation of Lemma 1 in the assigned lotteries $f(s)$ and taking $\varepsilon \to 0$ proves the desired conclusion. □

**Lemma 7.** *For any sequentially disjoint cycle of nonnull events $E_1, \ldots, E_n, E_1 \in \mathcal{E}$:*

$$\frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \cdots \times \frac{E_{n-1}}{E_n} \times \frac{E_n}{E_1} = 1.$$

*Proof.* The case where $n = 2$ immediately follows from our definition of event ratios, so assume that $n \geq 3$. Fix $t_1 > 0$, and recursively define

$$t_i = t_1 \times \frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \ldots \times \frac{E_{i-1}}{E_i}.$$

for $i = 2, \ldots, n$. By selecting a sufficiently small $t_1$, we may assume that $t_1, \ldots t_n \in (0,1)$. Also note that $\frac{t_{i+1}}{t_i} = \frac{E_i}{E_{i+1}}$ for $i = 1, \ldots, n-1$. Recall the range of the utility function $u$ over lotteries contains the unit interval $[-1,1]$, so there exist lotteries $p_1, \ldots, p_n, q \in \Delta X$ such that $u(p_i) = t_i$ for $i = 1, \ldots, n$ and $u(q) = 0$.

Fix any $i \in \{1, \ldots, n-1\}$. Let $\pi = \{E_i, E_{i+1}, (E_i \cup E_{i+1})^{\complement}\}$. Since $\frac{t_{i+1}}{t_i} = \frac{E_i}{E_{i+1}}$, we have

[15] We invoke the Axiom of Choice by assuming that we can fix a lottery $p_i(\varepsilon)$ for each $\varepsilon \in (0, \bar{\varepsilon})$. Lemma 6 remains true without invoking the Axiom of Choice, but requires a longer proof.

$\mu_\pi(E_{i+1})u(p_{i+1}) = \mu_\pi(E_i)u(p_i)$. Hence:

$$\begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix}$$

by the expected utility representation of Lemma 1. Since the above indifference holds for any $i \in \{1, \ldots, n-1\}$, by Lemma 6, we have

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Hence by the expected utility representation of $\succsim_\pi$ for $\pi = \{E_1, E_n, (E_1 \cup E_n)^{\complement}\}$, we have $\mu_\pi(E_1)u(p_1) = \mu_\pi(E_n)u(p_n)$. This implies $\frac{t_n}{t_1} = \frac{E_1}{E_n}$. Recalling the construction of $t_n$, we then have the desired conclusion:

$$\frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \ldots \times \frac{E_{n-1}}{E_n} = \frac{E_1}{E_n}. \qquad \square$$

We can now conclude the proof of sufficiency. We first define $\mathcal{C}^*$ and $\approx$ as we did in the proof of Lemma 4. Let $\mathcal{C}^*$ denote the set of nonnull events in $\mathcal{C}$. The collection $\mathcal{C}^*$ is nonempty since Nondegeneracy ensures that $S \in \mathcal{C}^*$. Define the binary relation $\approx$ on $\mathcal{C}^*$ by $E \approx F$ if there exist a sequentially disjoint sequence of nonnull events $E_1, \ldots, E_n \in \mathcal{C}^*$ with $E = E_1$ and $F = E_n$.[16] The relation $\approx$ is reflexive, symmetric, and transitive, defining an equivalence relation on $\mathcal{C}^*$. For any $E \in \mathcal{C}^*$, let $[E] = \{F \in \mathcal{C}^* : E \approx F\}$ denote the equivalence class of $E$ with respect to $\approx$. Let $\mathcal{C}^*/\approx\, = \{[E] : E \in \mathcal{C}^*\}$ denote the quotient set of all equivalence classes of $\mathcal{C}^*$ modulo $\approx$, with a generic class $R \in \mathcal{C}^*/\approx$.[17] Select a representative event $G_R \in R$ for every equivalence class $R \in \mathcal{C}^*/\approx$, invoking the Axiom of Choice if the quotient is uncountable.

We next define $\nu$. For all null $E \in \mathcal{C}$, let $\nu(E) = 0$. For every class $R \in \mathcal{C}^*/\approx$, *arbitrarily assign* a positive value $\nu(G_R) > 0$ for its representative. We will conclude by defining $\nu(E)$, for any $E \in \mathcal{C}^* \setminus \{S\}$. If $E = G_{[E]}$, then $E$ represents its equivalence class and $\nu(E)$ has been assigned. Otherwise, whenever $E \neq G_{[E]}$, since $E \approx G_{[E]}$, there exists a sequentially disjoint path of nonnull events $E_1, \ldots, E_n \in \mathcal{C}^*$ such that $E = E_1$, $G_{[E]} = E_n$. Then let:

$$\nu(E) = \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

Note that the definition of $\nu(E)$ above is independent of the particular choice of the path $E_1, \ldots, E_n$, because for any other such sequentially disjoint path of nonnull events $E = F_1, \ldots, F_m = G_{[E]}$:

$$\frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \frac{F_m}{F_{m-1}} \times \ldots \times \frac{F_2}{F_1} = 1$$

by Lemma 7.

[16] Note that this definition is slightly different than the one we gave in the general uniqueness result (Lemma 4). It can be checked that the two definitions are equivalent here, since $\Pi$ is the set of all finite partitions.

[17] Note that $[S] = \{S\}$ and $E \approx F$ for any disjoint nonnull $E$, $F$.

We will next verify that $\nu\colon \mathcal{C}\setminus\{S\} \to \mathbb{R}_+$ defined above is a nondegenerate set function satisfying

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F\in\pi}\nu(F)} \tag{5}$$

for any event $E \in \pi$ of any partition $\pi \in \Pi \setminus \{\{S\}\}$.

Let $\pi \in \Pi \setminus \{\{S\}\}$. By Nondegeneracy and the expected utility representation for $\succsim_\pi$, there exists a $\pi$-nonnull $F \in \pi$. Then, since Lemma 3 implies that $\pi$-nonnull events in $\mathcal{C}$ are nonnull, $F$ is nonnull so the denominator on the right hand side of Equation (5) is strictly positive, so the fraction is well-defined. This also implies that $\nu$ is a nondegenerate set function. Observe that Equation (5) immediately holds if $E$ is null, since then $\nu(E) = 0$ and $\mu_\pi(E) = 0$ follows from $E$ being $\pi$-null. Let $\mathcal{C}^*_\pi \subset \pi$ denote the nonnull cells of $\pi$. To finish the proof of the Theorem, we will show that $\frac{\mu_\pi(E)}{\mu_\pi(F)} = \frac{\nu(E)}{\nu(F)}$ for any distinct $E, F \in \mathcal{C}^*_\pi$. Along with the fact that $\sum_{E\in\mathcal{C}^*_\pi}\mu_\pi(E) = 1$, this will prove Equation (5).

Let $E, F \in \mathcal{C}^*_\pi$ be distinct. Note that $[E] = [F]$ since $E$ and $F$ are disjoint. Suppose first that neither $E$ nor $F$ is $G_{[E]}$. Then there exist a sequentially disjoint path of nonnull events $E_1, \ldots, E_n \in \mathcal{C}^*$ such that $E = E_1$, $G_{[E]} = E_n$, and:

$$\nu(E) = \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

But then $F, E_1, \ldots, E_n = G_{[E]}$ forms such a path from $F$ to $G_{[E]}$, hence we have:

$$\nu(F) = \frac{F}{E_1} \times \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

Dividing the term for $\nu(E)$ by the term for $\nu(F)$, we obtain $\frac{E}{F} = \frac{\nu(E)}{\nu(F)}$.

The other possibility is that exactly one of $E$ or $F$ (without loss of generality $E$) is $G_{[E]}$. Then the nonnull events $F = E_1, E_2 = E$, make up a path from $F$ to $E = G_{[E]}$. Then

$$\nu(F) = \frac{F}{E} \times \nu(E)$$

as desired. $\square$

# D Proofs of Section 4.4

## D.1 Proof of Proposition 3

Suppose $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a (not necessarily unique) partition-dependent expected utility representation $(u,\nu)$. Then for any events $E \subset F$ and $p,q,r,s \in \Delta X$:

$$s \succsim \begin{pmatrix} p & F \\ q & F^{\complement} \end{pmatrix} \qquad \iff \qquad u(s)[\nu(F)+\nu(F^{\complement})] \geq u(p)\nu(F) + u(q)\nu(F^{\complement})$$

$$\begin{pmatrix} r & E \\ s & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} r & E \\ p & F\setminus E \\ q & F^{\complement} \end{pmatrix} \qquad \iff \qquad u(s)[\nu(F\setminus E)+\nu(F^{\complement})] \geq u(p)\nu(F\setminus E) + u(q)\nu(F^{\complement}).$$

The next Lemma shows that the existence of a partition-dependent expected utility representation with a monotone set function $\nu$ implies Monotonicity. This is true even without Event Reachability, or without uniqueness of $\nu$, providing a stronger version of the necessity of the axiom required in Proposition 3.

**Lemma 8.** *If $\{\succsim_\pi\}_{\pi\in\Pi^*}$ admits a (not necessarily unique) partition-dependent expected utility representation by $(u,\nu)$ and $\nu$ is monotone, then $\{\succsim_\pi\}_{\pi\in\Pi^*}$ satisfies Monotonicity.*

*Proof.* Let $E \subset F$ and $p,q,r,s \in \Delta X$ such that $p \succ q$ and $u(s)[\nu(F)+\nu(F^{\complement})] \geq u(p)\nu(F) + u(q)\nu(F^{\complement})$. If $F = S$ or if $\nu(F\setminus E)+\nu(F^{\complement}) = 0$, then the desired conclusion holds. Otherwise $F \subsetneq S$ and $\nu(F\setminus E)+\nu(F^{\complement}) > 0$ and $\nu(F)+\nu(F^{\complement}) > 0$ by nondegeneracy of $\nu$. Since $F\setminus E \subset F \subsetneq S$, by monotonicity of $\nu$ we have $\nu(F) \geq \nu(F\setminus E)$. Hence

$$\frac{\nu(F)}{\nu(F)+\nu(F^{\complement})} \geq \frac{\nu(F\setminus E)}{\nu(F\setminus E)+\nu(F^{\complement})}.$$

But then since $u(p) > u(q)$, the inequality:

$$u(s) \geq \frac{\nu(F)}{\nu(F)+\nu(F^{\complement})}u(p) + \frac{\nu(F^{\complement})}{\nu(F)+\nu(F^{\complement})}u(q)$$

implies

$$u(s) \geq \frac{\nu(F\setminus E)}{\nu(F\setminus E)+\nu(F^{\complement})}u(p) + \frac{\nu(F^{\complement})}{\nu(F\setminus E)+\nu(F^{\complement})}u(q).$$

Therefore $\{\succsim_\pi\}_{\pi\in\Pi^*}$ satisfies Monotonicity. $\square$

The next example shows that, without Event Reachability, we cannot guarantee monotonicity of $\nu$ for *every* partition-dependent expected utility representation $(u,\nu)$ of $\{\succsim_\pi\}_{\pi\in\Pi^*}$. It requires a state space with at least three elements; otherwise, any $\nu$ is trivially monotone according to our definition.

**Example 6.** Consider an arbitrary state space $S$ with $|S| \geq 3$, and fix an nonempty event $A \subsetneq S$. Define $\nu$ by

$$\nu(E) = \begin{cases} 1 & \text{if } E \cap A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for any event $E \neq S$. Note that $\nu$ is nondegenerate. Let $\{\succsim_\pi\}_{\pi \in \Pi^*}$ be represented by $(u, \nu)$ for some non-constant $u$. Any event $B$ such that $A \subset B \subsetneq S$ has nonempty intersection with all nonnull events, hence it can not be linked to any other nonnull set through sequentially disjoint nonnull sets: In the notation of the proof of Theorem 3, such an event $B$'s reachability class $[B]$ consists of only $B$. Hence although $\nu$ itself is monotone, it is straightforward to verify that $\nu'$ obtained from $\nu$ by changing $\nu(B)$ to $\frac{1}{2}$ continues to represent the same preference. Moreover if we choose $B$ such that $|B| \geq 2$, then there exists a $C$ such that $C \subsetneq B$ and $\nu'(C) = 1 > \frac{1}{2}\nu'(B)$, so $\nu'$ is not monotone.

We show in the next lemma that it is possible to guarantee a weaker version of monotonicity of the set function from the Monotonicity of $\{\succsim_\pi\}_{\pi \in \Pi^*}$: subsets of null events are also null, so the family of null sets is an ideal.

**Lemma 9.** *Suppose $\{\succsim_\pi\}_{\pi \in \Pi^*}$ admits a (not necessarily unique) partition-dependent expected utility representation by $(u, \nu)$. If $\{\succsim_\pi\}_{\pi \in \Pi^*}$ satisfies Monotonicity, then:*

$$E \subset F \ \& \ \nu(F) = 0 \ \Rightarrow \ \nu(F \setminus E) = 0.$$

*Proof.* Suppose that there exist events $E, F$ such that $E \subset F$ and $\nu(F \setminus E) > \nu(F) = 0$. Since $\nu(F) = 0$, by non-degeneracy of $\nu$, we have $\nu(F^{\complement}) > 0$. Let $p, q, s \in \Delta X$ be such that $u(p) > u(q) = u(s)$. Then

$$u(s)[\nu(F) + \nu(F^{\complement})] = u(p)\nu(F) + u(q)\nu(F^{\complement}),$$

so by Monotonicity, we should have

$$u(s)[\nu(F \setminus E) + \nu(F^{\complement})] \geq u(p)\nu(F \setminus E) + u(q)\nu(F^{\complement}).$$

However the latter inequality is not possible, since $u(p) > u(q) = u(s)$ and $\nu(F \setminus E) > 0$, a contradiction. $\square$

In the next Lemma, we prove the sufficiency of the Monotonicity axiom for the existence of a monotone representation in Proposition 3, given Event Reachability and the uniqueness of the set function up to scalar multiples.

**Lemma 10.** *Suppose $\{\succsim_\pi\}_{\pi \in \Pi^*}$ admits a unique partition-dependent expected utility representation by $(u, \nu)$. If $\{\succsim_\pi\}_{\pi \in \Pi^*}$ satisfies Monotonicity, then $\nu$ is monotone.*

*Proof.* We prove by contraposition. Suppose that $\nu$ is not monotone. Then there exist events $E, F$ such that $E \subset F \subsetneq S$ and $\nu(F \setminus E) > \nu(F)$. By Lemma 9, we can assume that $\nu(F) > 0$. We also have that $\nu(F^{\complement}) > 0$, because otherwise by Lemma 9, any subevent of $F^{\complement}$ is null, hence $F$ and $F \setminus E$ are nonnull events that can not be linked by sequentially disjoint nonnull events, contradicting Event Reachability.

Let $p, q, s \in \Delta X$ be such that $u(p) > u(q)$ and

$$s = \frac{\nu(F)}{\nu(F) + \nu(F^{\complement})} p + \frac{\nu(F^{\complement})}{\nu(F) + \nu(F^{\complement})} q.$$

Then $u(s)[\nu(F) + \nu(F^{\complement})] = u(p)\nu(F) + u(q)\nu(F^{\complement})$, so by Monotonicity, we have $u(s)[\nu(F \setminus E) + \nu(F^{\complement})] \geq u(p)\nu(F \setminus E) + u(q)\nu(F^{\complement})$. Together with $u(p) > u(q)$, these imply:

$$\frac{\nu(F)}{\nu(F) + \nu(F^{\complement})} \geq \frac{\nu(F \setminus E)}{\nu(F \setminus E) + \nu(F^{\complement})},$$

a contradiction to $\nu(F \setminus E) > \nu(F)$ and $\nu(F^{\complement}) > 0$. $\square$

### D.2 Proof of Proposition 4

Given the existence of a partition-dependent expected utility representation, Strict Admissibility is equivalent to all nonempty events being nonnull. The "if" part is immediate. We proceed contrapositively to prove the "only if" part. Let $\{\succsim_\pi\}_{\pi \in \Pi^*}$ be represented by $(u, \nu)$. Now suppose that there is a nonempty null event $E$. By nondegeneracy of $\nu$, $E^{\complement}$ is nonnull. By Lemma 9, all subevents of $E$ are null. If there is an event $B$ such that $E^{\complement} \subset B \subsetneq S$, then $B$ is nonnull by Lemma 9. Hence $E^{\complement}$ and $B$ are two nonnull events that are not linked by sequentially disjoint nonnull sets, so Event Reachability fails. If there is no such event $B$, then since $|S| \geq 3$, $E^{\complement}$ must consist of at least two elements. In this case, let $E^{\complement} = E_1 \cup E_2$, where $E_1$ and $E_2$ are nonempty and disjoint. Then $\{E_1, E_2, E\}$ is a partition of $S$ where $E$ is null, so one of the other two events, say $E_i$, is nonnull by nondegeneracy of $\nu$. But then $E^{\complement}$ and $E_i$ are two nonnull events that are not linked by sequentially disjoint nonnull sets, so again Event Reachability fails.

## E Proofs of Section 5

### E.1 Indispensability of Strict Admissibility in Proposition 5

**Example 7.** Let $S = \{s_1, s_2, s_3, s_4\}$ and suppose that $\{\succsim_\pi\}_{\pi \in \Pi^*}$ admits a partition-dependent expected utility representation $(u, \nu)$, where $\nu(\{s\}) = 1$ for all $s \in S$, $\nu(\{s_1, s_2\}) = \nu(\{s_1, s_3\}) = \nu(\{s_2, s_3\}) = 3$, and $\nu(E) = 0$ for any other event $E \neq S$. Strict Admissibility fails since some nonempty events are null. It can be verified that $\nu$ is nondegenerate, Event Reachability is satisfied, and $\{\succsim_\pi\}_{\pi \in \Pi^*}$ underpacks. However, $\nu$ is not subadditive since $\nu(\{s_1, s_2\}) > \nu(\{s_1\}) + \nu(\{s_2\})$ and $\lambda(\{\{s_1\}, \{s_2\}\}) = \frac{2}{3} < 1$.

### E.2 Proof of Proposition 6

*Proof of (i).* To see the "$\Rightarrow$" part of (i), assume that $A \in \mathcal{A}$ and let $E$ be any event. Assume without loss of generality that $E \neq \emptyset$. Consider the partition $\pi = \{E, E^{\complement} \cap A, E^{\complement} \cap A^{\complement}\}$. Since $E \neq S$, the sets $E^{\complement} \cap A$ and $E^{\complement} \cap A^{\complement}$ can not both be empty. Hence by strict admissibility $\nu(E^{\complement} \cap A) + \nu(E^{\complement} \cap A^{\complement}) > 0$. Assume without loss of generality that $[0, 1] \subset u(\Delta X)$ and let $p, q, r \in \Delta X$ be such that $u(p) = 1$,

$u(q) = 0$, and

$$u(r) = \frac{\nu(E)}{\nu(E) + \nu(E^{\complement} \cap A) + \nu(E^{\complement} \cap A^{\complement})}. \tag{6}$$

Define the act $f$ by

$$f = \begin{pmatrix} p & E \\ q & E^{\complement} \end{pmatrix}.$$

Then $f \in \mathcal{F}_\pi$ and $f \sim_\pi r$. Hence by $A \in \mathcal{A}$ we have that $f \sim_{\pi \vee \{A, A^{\complement}\}} r$. Since $\pi \vee \{A, A^{\complement}\} = \{E \cap A, E \cap A^{\complement}, E^{\complement} \cap A, E^{\complement} \cap A^{\complement}\}$, the last indifference implies that

$$u(r) = \frac{\nu(E \cap A) + \nu(E \cap A^{\complement})}{\nu(E \cap A) + \nu(E \cap A^{\complement}) + \nu(E^{\complement} \cap A) + \nu(E^{\complement} \cap A^{\complement})}. \tag{7}$$

By Equations (6), (7), and $\nu(E^{\complement} \cap A) + \nu(E^{\complement} \cap A^{\complement}) > 0$, we conclude that $\nu(E) = \nu(E \cap A) + \nu(E \cap A^{\complement})$.

To see the "$\Leftarrow$" part of (i), assume that $\nu(E) = \nu(E \cap A) + \nu(E \cap A^{\complement})$ for any event $E \neq S$. Take any $\pi \in \Pi^*$. If $\pi$ is the trivial partition then the desired conclusion follows trivially from state independence. So assume without loss of generality that $\pi$ is nontrivial and let $\pi' = \pi \vee \{A, A^{\complement}\}$. It suffices to show that $\mu_\pi(F) = \mu_{\pi'}(F)$ for all $F \in \pi$. To see this, note that

$$\mu_\pi(F) = \frac{\nu(F)}{\sum_{E \in \pi} \nu(E)} = \frac{\nu(F \cap A) + \nu(F \cap A^{\complement})}{\sum_{E \in \pi} [\nu(E \cap A) + \nu(E \cap A^{\complement})]} = \mu_{\pi'}(F),$$

where the middle equality follows from our assumption and $F \neq S, E \neq S$ since $\pi$ is nontrivial. $\square$

*Proof of (ii).* By definition, $\mathcal{A}$ is closed under complements and $\emptyset, S \in \mathcal{A}$. It suffices to show that $\mathcal{A}$ is closed under intersections. Let $A, B \in \mathcal{A}$, and take any event $E \neq S$. We have that

$$\begin{aligned} \nu(E) &= \nu(E \cap A) + \nu(E \cap A^{\complement}) \\ &= \nu(E \cap A \cap B) + \nu(E \cap A \cap B^{\complement}) + \nu(E \cap A^{\complement}), \end{aligned}$$

by part (i), $A, B \in \mathcal{A}$, and $E, E \cap A, \neq S$. Similarly we have that

$$\begin{aligned} \nu(E \cap (A \cap B)^{\complement}) &= \nu(E \cap (A \cap B)^{\complement} \cap A) + \nu(E \cap (A \cap B)^{\complement} \cap A^{\complement}) \\ &= \nu(E \cap A \cap B^{\complement}) + \nu(E \cap A^{\complement}). \end{aligned}$$

The two equalities above imply that

$$\nu(E) = \nu(E \cap A \cap B) + \nu(E \cap (A \cap B)^{\complement}).$$

Therefore by part (i), $A \cap B \in \mathcal{A}$. $\square$

*Proof of (iii).* We next prove the first part of (iii). Let $A, B \in \mathcal{A}$ be disjoint events such that $A \cup B \neq S$. Since $A \in \mathcal{A}$, we have by part (i) that:

$$\nu(A \cup B) = \nu([A \cup B] \cap A) + \nu([A \cup B] \cap A^{\complement}) = \nu(A) + \nu(B).$$

Hence $\nu$ is additive on $\mathcal{A} \setminus \{S\}$.

To see the second part of (iii), let $A, B \in \mathcal{A} \setminus \{\emptyset, S\}$. Note that:

$$\nu(A) + \nu(A^{\complement}) = \nu(A \cap B) + \nu(A \cap B^{\complement}) + \nu(A^{\complement} \cap B) + \nu(A^{\complement} \cap B^{\complement}),$$

by part (i) applied twice to $B \in \mathcal{A}$ and to $A, A^{\complement} \neq S$. By the exact symmetric argument interchanging the roles of $A$ and $B$ we also have that

$$\nu(B) + \nu(B^{\complement}) = \nu(B \cap A) + \nu(B \cap A^{\complement}) + \nu(B^{\complement} \cap A) + \nu(B^{\complement} \cap A^{\complement})$$

Hence $\nu(A) + \nu(A^{\complement}) = \nu(B) + \nu(B^{\complement})$ as desired. $\square$

*Proof of (iv).* Immediately follows from parts (i) and (iii). $\square$

The next example demonstrates that $\mathcal{A}$ is not necessarily the maximal algebra on which $\nu$ is additive.

**Example 8.** Let $S = \{s_1, s_2, s_3, s_4\}$ and $\pi = \{\{s_1\}, \{s_2\}, \{s_3, s_4\}\}$. Assume that $\nu$ satisfies strict admissibility and assume that $\nu$ on $\sigma(\pi)$ is defined by $\nu(\{s_1\}) = \nu(\{s_2\}) = \nu(\{s_3, s_4\}) = 1$, $\nu(\{s_1, s_2\}) = \nu(\{s_1, s_3, s_4\}) = \nu(\{s_3, s_3, s_4\}) = 2$, and $\nu(S) = 3$. Note that $\nu$ is additive on $\sigma(\pi)$. However if $\nu(\{s_1, s_2, s_3\}) = 4$ and $\nu(\{s_3\}) = 1$ then $\{s_1, s_2\} \notin \mathcal{A}$ from part (i) of Proposition 6, since:

$$\nu(\{s_1, s_2, s_3\}) = 4 \neq 2 + 1 = \nu(\{s_1, s_2, s_3\} \cap \{s_1, s_2\}) + \nu(\{s_1, s_2, s_3\} \cap \{s_1, s_2\}^{\complement})$$

In this example $\sigma(\pi)$ is an algebra on which $\nu$ is additive, but $\sigma(\pi)$ is not a subset of $\mathcal{A}$. Hence $\mathcal{A}$ is not the maximal algebra on which $\nu$ is additive.

## E.3 Proof of Proposition 7

Note that the uniqueness of the partition-dependent representation implies, by Theorem 4, that event Reachability is satisfied. Therefore by monotonicity and Proposition 4, strict admissibility is also satisfied.

*Proof of (i).* The "$\Leftarrow$" part of (i) is easily seen to hold even without monotonicity of $\nu$. To see the "$\Rightarrow$" part, assume that $E$ is suppressed. If $E = \emptyset$ then the conclusion is immediate, so assume without loss of generality that $E \neq \emptyset$. Take any nonempty event $F$ disjoint from $E$ such that $E \cup F \neq S$. Let $G = S \setminus (E \cup F) \neq \emptyset$.

We first show that

$$\frac{\nu(E \cup F)}{\nu(G)} = \frac{\nu(F)}{\nu(E \cup G)}. \tag{8}$$

The fractions above are well defined since strict admissibility guarantees that the denominators do not vanish. To see (8), let $p, q, r \in \Delta X$ be such that $u(p) > u(q)$ and

$$\frac{\nu(E \cup F)}{\nu(E \cup F) + \nu(G)} u(p) + \frac{\nu(G)}{\nu(E \cup F) + \nu(G)} u(q) = u(r) \iff \begin{pmatrix} p & E \cup F \\ q & G \end{pmatrix} \sim r. \tag{9}$$

By $E$ being suppressed, we have

$$\frac{\nu(F)}{\nu(F)+\nu(E\cup G)}u(p)+\frac{\nu(E\cup G)}{\nu(F)+\nu(E\cup G)}u(q)=u(r) \iff \begin{pmatrix} p & F \\ q & E\cup G \end{pmatrix} \sim r. \tag{10}$$

Since $u(p) > u(q)$, (9) and (10) imply that

$$\frac{\nu(E\cup F)}{\nu(E\cup F)+\nu(G)}=\frac{\nu(F)}{\nu(F)+\nu(E\cup G)}$$

which is equivalent to (8).

By monotonicity of $\nu$, we have that

$$\frac{\nu(F)}{\nu(E\cup G)} \leq \frac{\nu(F)}{\nu(G)} \leq \frac{\nu(E\cup F)}{\nu(G)}$$

By Equation (8), all the weak equalities above are indeed equalities, hence in particular $\nu(F) = \nu(E\cup F)$ as desired. □

*Proof of (ii).* Assume that $E$ and $F$ are suppressed and $E\cup F \neq S$. To see that $E\cup F$ is suppressed, let $G$ be a nonempty event disjoint from $E\cup F$ such that $E\cup F\cup G \neq S$. Then $G$ is disjoint from $E$ and $E\cup G \neq S$. By part (i), we have $\nu(E\cup G) = \nu(G)$. Moreover, $E\cup G$ is disjoint from $F$ and $E\cup F\cup G \neq S$. Again by part (i) we have, $\nu(E\cup F\cup G) = \nu(E\cup G)$. Hence $\nu(E\cup F\cup G) = \nu(G)$, as desired.

To see that $E\cap F$ is suppressed, suppose that $G$ is a nonempty event disjoint from $E\cap F$ such that $[E\cap F]\cup G \neq S$. We will show that $\nu(G\cup[E\cap F]) = \nu(G)$ by consider three cases. this will imply by part (i) that $E\cap F$ is suppressed.

*Case 1: $G\subset E$.* In this case $G\backslash F \neq \emptyset$, for otherwise $G\subset E\cap F$ would not be disjoint from $E\cap F$. Moreover $(G\backslash F)\cup F = G\cup F \subset E\cup F \neq S$, hence by part (i) we have that $\nu([G\backslash F]\cup F) = \nu(G\backslash F)$. By monotonicity

$$\nu(G) \leq \nu(G\cup[E\cap F]) \leq \nu(G\cup F) = \nu([G\setminus F]\cup F) = \nu(G\setminus F) \leq \nu(G). \tag{11}$$

Hence $\nu(G\cup[E\cap F]) = \nu(G)$.

*Case 2: $G\subset F$.* We again have that $\nu(G\cup[E\cap F]) = \nu(G)$, by exactly the same argument as the one above, changing the roles of events $E$ and $F$.

*Case 3: $G\setminus E \neq \emptyset$ and $G\setminus F \neq \emptyset$.* It can not be that both $G\cup E$ and $G\cup F$ are equal to $S$, because otherwise $[G\cup E]\cap[G\cup F] = G\cup[E\cap F] = S$ contradicting the hypothesis. Assume without loss generality that $G\cup F \neq S$. Hence by part (i) we have that $\nu([G\setminus F]\cup F) = \nu(G\setminus F)$. By Equation (11) again, we conclude that $\nu(G\cup[E\cap F]) = \nu(G)$. □

*Proof of (iii).* The "$\Leftarrow$" part of (iii) is easily seen to hold even without monotonicity of $\nu$. We will only prove the "$\Rightarrow$" part. We first show that $\nu(G) = \nu(G^{\complement})$ if $G \neq \emptyset, S$. To see this, note that since there are at least three states $G$ or $G^{\complement}$ is not a singleton. Without loss of generality suppose that $G$

has at least two elements and let $\{G_1, G_2\}$ be a two element partition of $G$. Then by part (i),

$$\nu(G) = \nu(G_1 \cup G_2) = \nu(G_1) = \nu(G_1 \cup G^{\complement}) = \nu(G^{\complement}),$$

where the second equality follows because $G_2$ and $G_1 \cup G_2 \neq S$ are completely unforeseen; the third equality follows because $G^{\complement}$ and $G_1 \cup G^{\complement} \neq S$ are completely unforeseen; and the fourth equality follows because $G_1$ and $G_1 \cup G^{\complement} \neq S$ are completely unforeseen.

Take any distinct events $E, F \neq \emptyset, S$. If $E \setminus F \neq \emptyset$ then

$$\nu(E \setminus F) \leq \nu(E) = \nu(E^{\complement}) \leq \nu((E \setminus F)^{\complement}) = \nu(E \setminus F)$$

where the inequalities follow from monotonicity of $\nu$, hence $\nu(E) = \nu(E \setminus F)$. Similarly

$$\nu(E \setminus F) \leq \nu(F^{\complement}) = \nu(F) \leq \nu((E \setminus F)^{\complement}) = \nu(E \setminus F),$$

hence $\nu(F) = \nu(E \setminus F) = \nu(E)$ as desired. The case where $F \setminus E \neq \emptyset$ is symmetric, therefore omitted. □

To see that monotonicity is indispensable for the "⇒" parts of (i) and (iii) to hold, consider the following example.

**Example 9.** Let $S = \{s_1, s_2, s_3\}$ and suppose that $\{\succsim_\pi\}_{\pi \in \Pi^*}$ admits a partition-dependent expected utility representation $(u, \nu)$, where $\nu(\{s_1\}) = \nu(\{s_2, s_3\}) = 1$, $\nu(\{s_2\}) = \nu(\{s_1, s_3\}) = 2$, and $\nu(\{s_3\}) = \nu(\{s_1, s_2\}) = 3$. Strict Admissibility is satisfied therefore the partition-dependent expected utility representation is unique. The set function $\nu$ is not monotone since $\nu(\{s_2\}) > \nu(\{s_2, s_3\})$. It is easy to see that all nonempty events are completely unforeseen. Let $E = \{s_1\}$ and $F = \{s_2\}$, then $\nu(E \cup F) = 3 \neq 2 = \nu(F)$. Hence parts (i) and (iii) fail in the absence of monotonicity.

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