# How large is a policy's greatest Pareto improvement when asset markets are incomplete?*

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**Abstract**

Research shows that absent all frictions but incomplete asset markets, at almost every equilibrium a Pareto improvement is supported by many types of intervention, financial, monetary, and fiscal.

Surprisingly, little is known about the size of these Pareto improvements, or even how to define their "size."

We provide a measure of the maximal Pareto improvement, as the largest fraction of current resources a society is willing to pay for the improvement. It can evaluate global policy changes, not just local.

This measure admits an exact formula in the quasilinear case, an upper bound in the general case, and it obeys the law of diminishing returns. We show that local information already captures the benefits of global policies, thus supporting the literature's focus on local Pareto improvements despite global ones being greater.

We define and calibrate the insurance deficit in future income, and then estimate the maximal Pareto improvement in the US to be one third of one percent. We justify this calibration by proving a novel correspondence between insurance deficit and equilibrium consumption: equilibrium consumption is the maximum of a social welfare function, whose parameters are the insurance deficit as well as individual weights, extending a classical result of Lange (1942) to incomplete markets.

**Keywords**: incomplete markets, Pareto improvement, social welfare function, social risk, policy

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# 1 Introduction

When asset markets provide only incomplete insurance against uncertainty in future income, it is well known that for almost all economies, all its equilibria are Pareto inefficient, Magill and Quinzii (1996). The generic existence of Pareto improvements, via lump sum changes in future income, is robust in several senses. It does not require creating new markets, the existing ones suffice; nor optimizing agents, they may be behavioral, Nagata (2005). Further, if there are multiple commodities per state, it is robust even to the flexibility of lump sum changes in income, restrictive policies suffice: lump sum changes in portfolios, Geanakoplos and Polemarchakis (1986); lump sum changes in current income plus a mild policy instrument, Citanna, Kajii, and Villanacci (1998); taxation of asset trades, Citanna, Polemarchakis, and Tirelli (2006); anonymous income taxes, Tirelli (2002); excise taxes or capital gains taxes, Turner (2005).

Though the existence of Pareto improvements is by now robustly established, there is little on the how "large" these Pareto improvements are. It is important that they be larger than the policy maker's ex ante cost of realizing them, whether it be the cost of information or of implementation. In particular, computing them seems to involve fine information. For example, preferences are recoverable if one can know every agent's entire excess demand function, including out of equilibrium, Geanakoplos and Polemarchakis (1990); or know the equilibrium prices that would prevail if the endowments were perturbed in an arbitrary but small direction, Kübler et al. (2002).

We define the "size" of a Pareto improvement to be the fraction of current total income agents are willing to pay for it. This measure is in real terms, for ordinal preferences. To be more explicit, assume agents' preferences are over income $y^h = (y_0^h, y_{\mathbf{1}}^h)$, meaning a current amount and an uncertain, state-contingent future amount. If the income redistribution $y = (y^h)_h$ weakly Pareto improves on the income distribution $x = (x^h)_h$, then each agent has a willingness to pay $p^h$ out of current income, defining a total willingness to pay $p(y) := \Sigma p^h$–here $p^h$ is the maximum value making the agent still weakly prefer $y^h - (p^h, 0)$ to $x^h$. We define the **size** of a weak Pareto improvement to $x$ to be this, normalized by current total income:

$$\rho_x(y) := \frac{p(y)}{\Sigma x_0^h}$$

Size is by construction between 0 and 1, an interval along which all Pareto improvements become ordered.

A policy intervention, whose goal is to support a Pareto improving reallocation $y$ and whose cost is $c$ as a fraction of current total income, is not worthwhile if $c > \rho_x(y)$, that is, if agents are not willing to pay for it. The Pareto improvement it supports is "too small." A basic problem is settling when the above Pareto improvements, proved to exist generically, are more than "too small." What is the quantitative significance of the qualitative inefficiency?

We focus on the following question: Quantitatively, how inefficient is an allocation? Say, one arising from equilibrium trade of incomplete assets? We define the **size** of the inefficiency of $x$ to be the maximum size of any weak Pareto improvement to $x$ :

$$\rho_x := \max_{y \succsim x} \rho_x(y)$$

This size is between 0 and 1 still, and is 0 if and only if the allocation is Pareto efficient, under mild conditions. We deem the inefficiency "too small" if $c > \rho_x$, that is, if agents are not willing to pay for any weak Pareto improvement whatsoever.

Our measure of inefficiency involves a restriction on the timing of payment, that it be out of current income, not future as well. This restriction is natural because the underlying reallocations are to be Pareto improving ex ante the resolution of uncertainty. How they handle the uncertainty is precisely what makes them improving. The restriction is also crucial because such reallocations need not be Pareto improving ex post the resolution of uncertainty. Thus the willingness to pay ex post may be different–perhaps even zero due to moral hazard. Strategic reasons motivate not just our restriction on timing tacit in $\rho_x$, they motivate the very incompleteness of assets. In this regard, our measure of inefficiency $\rho_x$ is an updated version of Debreu's (1951). Debreu's is the maximum fraction of current *and future* total income agents are willing to pay for a Pareto improvement. (His coefficient of resource utilization is one minus this.) With dated commodities, his measure hinges on ex post payments for an ex ante willingness that may vanish ex post. (In any case, to the extent our measure is small, Debreu's is even smaller.)

The size of inefficiency $\rho_x$ admits an explicit formula in case preferences have a utility representation that is quasilinear in current consumption: $u^h(x) = x_0 + u^h_{\mathbf{1}}(x_{\mathbf{1}})$. Our measure requires maximizing $p(y)$ subject to $y$ being both a reallocation of and weakly Pareto improving on $x$, i.e. to $\Sigma y^h = \Sigma x^h$ and $y^h_0 \geq x^h_0 + u^h_{\mathbf{1}}(x_{\mathbf{1}}) - u^h_{\mathbf{1}}(y^h_{\mathbf{1}})$. If $y$ is a reallocation of $x$–and noting the willingness to pay $p^h$ solves $(y^h_0 - p^h) + u^h_{\mathbf{1}}(y^h_{\mathbf{1}}) = x^h_0 + u^h_{\mathbf{1}}(x_{\mathbf{1}})$–the total willingness to pay $p(y) = \Sigma p^h$ is $p(y) = \Sigma u^h_{\mathbf{1}}(y^h_{\mathbf{1}}) - \Sigma u^h_{\mathbf{1}}(x_{\mathbf{1}})$. To maximize this subject to $y$ being weakly Pareto improving is to maximize $\Sigma u^h_{\mathbf{1}}(y^h_{\mathbf{1}})$ subject to $y^h_0 := x^h_0 + u^h_{\mathbf{1}}(x_{\mathbf{1}}) - u^h_{\mathbf{1}}(y^h_{\mathbf{1}}) \geq 0$. Let $y_{\mathbf{1}}(x)$ be a solution, so

$$\rho_x = \frac{\Sigma u^h_{\mathbf{1}}(y^h_{\mathbf{1}}(x)) - \Sigma u^h_{\mathbf{1}}(x^h_{\mathbf{1}})}{\Sigma x^h_0}$$

That this is the inefficiency reflects the classical understanding that with quasilinearity the appropriate measure of "aggregate welfare" is the sum of utilities. Here, inefficiency is the amount by which future "aggregate welfare" $w_{\mathbf{1}}(x) := \Sigma u^h_{\mathbf{1}}(x^h_{\mathbf{1}})$ fails to be maximized, over current "aggregate welfare" $w_0(x) := \Sigma x^h_0$.[1]

Knowing how large inefficiency $\rho_x$ can be depends on knowing the income distribution $x$.

If the income distribution $x$ is known, the size of inefficiency is a specific number, given preferences. We compute the inefficiency for an estimated US economy to be very small, $\rho_{US} = .0027$, based on data on households' disposable income. The economy would not even pay one third of one percent of current total income for any Pareto improvement whatsoever; it is nearly Pareto optimal. To estimate the economy, we must specify the households and future states, first, and second (a) the income distribution, both current and over these states, (b) these households' preferences for income over these states. With all this freedom for specification, any particular one is ad hoc. Ours is in addition very crude. First, we specify three agents, representatives of the three income terciles, defined from data on the distribution of current disposable income; and we imagine three future states, the status quo terciles, a rich-middle class switch in terciles, and a middle class-poor switch in terciles–thus automatically specifying the full income distribution. Second, we specify preferences to be von Neumann -Morgenstern utilities whose Bernoulli utility is CRRA with parameter 1/4, as in Kocherlakota (1996). The body of the paper details how we extract the common state

[1] There are many ways to specify $y_0$ from the maximand $y_{\mathbf{1}}$ and associated value $p$. For any nonnegative sequence $(p^h)$ such that $p = \Sigma p^h$, let $y^h_0 := x^h_0 + u^h_{\mathbf{1}}(x_{\mathbf{1}}) - u_{\mathbf{1}}(y^h_{\mathbf{1}}) + p^h$. This weakly improves $h$ and makes his willingness to pay precisely $p^h$.

probabilities; broadly, they appear as unknowns in the first order conditions for optimal asset trades, which we make incomplete by specifying only the riskless bond as tradeable.

On the other hand, if the income distribution $x$ is arbitrary, the size of inefficiency can be a whole subinterval $[0, k]$; further, $k = 1$ if utilities are quasilinear in current consumption. The intuition is simple. Given total income, take any Pareto efficient distribution, where $\rho$ is $0$, and take any Pareto inefficient distribution, where $\rho$ is some $k > 0$. The segment connecting these two distributions must, by the intermediate value and maximum theorems, span all intermediate values of $\rho, [0, k]$. To see how to arrange $k = 1$ with quasilinearity, observe from the above formula that $\rho_x = 1$ if and only if all $y_0^h - p^h = 0$, if and only if all $u_{\mathbf{1}}^h(x_{\mathbf{1}}) = u_{\mathbf{1}}(y_{\mathbf{1}}^h) - x_0^h$. This lower bound for $u_{\mathbf{1}}^h(x_{\mathbf{1}})$ is achievable by changing $x_{\mathbf{1}}$ as follows. Take two states and assign some but not all households to one state and all the others to the other state, then make all "state one" households donate nearly all their income in that state to the "state two" households, and vice versa; an Inada condition on utility for future income then guarantees that $u_{\mathbf{1}}^h(x_{\mathbf{1}}) \to -\infty$ as these donations drive $x_{\mathbf{1}}$ toward the boundary. Since $u_{\mathbf{1}}^h(x_{\mathbf{1}}) \to -\infty$, to undo this change agents are willing to give up most of their current income-utility $x_0^h \to 0$; that is, $\rho_x \to 1$. This argument for the arbitrariness of $\rho_x$ remains valid with various modifications, in case $x$ must not be arbitrary but an equilibrium allocation with incomplete asset markets; for example, in the absence of any assets, any allocation is an equilibrium allocation.

There is a potential criticism of the prevailing analysis of Pareto suboptimality of equilibria: it is local. Such is the analysis in all the works first cited above, for example. The prevailing analysis focuses only on when and which local interventions are Pareto improving, and is mute on whether global interventions are more dramatically Pareto improving and, if so, whether they point in the same direction as the local ones. A natural lens to scrutinize this potential criticism is the measure of inefficiency $\rho_x$, since it accounts for global interventions as well as local. We find that this criticism is unjustified in the following two senses.

First, the willingness to pay for "policy activation" is relatively larger than for "policy continuation." Let $y$ be an alternative allocation to $x$ for which there is some willingness to pay $p(y) \geq 0$, and $y(t) := x + t(y - x)$ the path leading to this alternative $y(1) = y$ from the status quo $y(0) = x, t \in [0, 1]$. We show $p(y(t)) \geq tp(y)$, so that the worth $p(y(t))$ of activating only a $t$-fraction $t(y - x)$ of the full policy exceeds a $t$-fraction of the worth $p(y)$ of completing the full policy $(y - x)$. In the same spirit, we show $p(y(t))$ is concave in $t$. Lastly, $p(y(t))$ is increasing, if $y$ is the weakly Pareto improving reallocation that maximizes $p(y)$; this is a direction of change where a partial implementation is dominated by any fuller implementation.[2] See proposition **??**.

Second, though local Pareto improvements clearly are bounded above by more global Pareto improvements in terms of willingness to pay (thanks to continuity), the converse is true in an informational sense: global Pareto improvements are bounded above by local Pareto information, that is, by the information typical in analyses of local Pareto improvements. Specifically, an upper bound for size of local Pareto improvements is also an upper bound for the size of global Pareto improvements, upon formally replacing the local by the global in the expression. To clarify this, let $\nabla_s^h(x^h)$ be the marginal rate of substitution of income in future state $s$ for current income. Locally, the total willingness to pay for having agent $h$ donate to $i$ the infinitesimal income $\dot{x}_s^h$ in state $s$ is $(\nabla_s^i - \nabla_s^h)\dot{x}_s^h$, in terms of current income. To maximize

[2] Increasingness of $p(y(t))$ does not imply that $y(t)$ is weakly Pareto superior to $x$.

the total willingness to pay for an infinitesimal transfer $\dot{x}_s$ in state $s$, everyone must donate to that state's most deprived agent $i(s)$–meaning $i(s)$ has the largest $\nabla^i_s(x^i)$–thus *exactly* achieving the infinitesimal willingness

$$\frac{\Sigma_s \Sigma_h (\nabla^{i(s)}_s - \nabla^h_s)\dot{x}^h_s}{\Sigma x^h_0} \tag{1}$$

as a fraction of total current income. The maximum subject to $\dot{x}$ being in a given ball is an upper bound for the maximum subject to the further constraint of $\dot{x}$ being weakly Pareto improving. This latter maximum is the infinitesimal analogue of the inefficiency $\rho_x$, and as noted is bounded above by (1). Globally, consider the total willingness to pay for such donations to the deprived, but now involving the full income $x$ instead of the infinitesimal income $\dot{x}$. Since willingness to pay is concave, what was an upper bound for the infinitesimal analogue of $\rho_x$ should certainly remain an upper bound for $\rho_x$, on expanding the infinitesimal donation $\dot{x}$ to the greatest donation $x$ :

$$R_x := \frac{\Sigma_s \Sigma_h (\nabla^{i(s)}_s - \nabla^h_s)x^h_s}{\Sigma x^h_0} \tag{2}$$

That $\rho_x \leq R_x$ is indeed true; proposition 3. In conclusion, the expression (1) for an upper bound for the local Pareto improvements is, remarkably, also an upper bound for all Pareto improvements: one need only replace the infinitesimal $\dot{x}$ by the global $x$–the income distribution in question. In particular, the information required to compute the *direction of some local* Pareto improvement $y = x + \epsilon\dot{x}$, namely $\nabla(x)$, is enough to compute an upper bound $R_x$ for the *size of all global* Pareto improvements $y = x + \Delta x$, namely $p_x(y) \leq \rho_x \leq R_x$. Incidentally, this upper bound is zero at an interior Pareto optimum, where all the marginal rates of substitution are equal.

## 1.1 Calibration strategy

We define the insurance deficit of an allocation, in terms of the marginal rates of substitution $\nabla^h(x)$ as

$$\mu^h := \nabla^h - \overline{\nabla}$$

where $\overline{\nabla} := \frac{1}{H}\Sigma\nabla^h$ is the average. For any allocation $x$, clearly $\Sigma\mu^h = 0$. For an equilibrium allocation $x$ relative to the asset structure $a \in \mathbb{R}^{S\times J}$ with an incomplete $J \leq S$ number of assets, the first order conditions for agent's optimal portfolios imply a key property:

$$\mu^h a = 0, \text{ all } h \tag{3}$$

Note, $x$ is a complete asset markets allocation if and only if it is Pareto optimal iff $\mu(x) = 0$. For this reason we call $\mu$ the **insurance deficit**.

The insurance deficit is a function of the utilities defining the marginal rate of substitution, and of the allocation. If utilities are von Neumann Morgenstern, and the Bernoulli utilities given, then we may view the insurance deficit just as a function of the state probabilities and of the allocation, $\mu = \mu(\pi, x)$. Specifying the state space, we take data on consumption $x$ and seek probabilities that solve equation (3), which is necessary to rationalize the data as equilibrium consumption. It turns out that these equations are linear in $\pi$, and number precisely $\dim \pi = S$ for our choice of the state space, so there is a unique solution $\pi$

if the data are in general position. This is an atypically favorable situation for calibration. With the state probabilities recovered, so are the preferences, allowing a numerical computation of the inefficiency.

A theoretical question that plagues calibrations is whether the calibrating equations, here (3), contain all the model's restrictions. We show this is not the case here. That is, given preferences and a vector $\mu$ satisfying (3), there is a unique equilibrium consumption $x$ whose insurance deficit $\mu(x)$ is $\mu$. This correspondence between insurance deficits and equilibrium consumptions is an extension along the lines traced in Tirelli (2005) of a classical result of Lange (1942) to incomplete markets result: equilibrium consumptions are the maxima of a social welfare function whose parameters are the *insurance deficits* as well as *individual weights*, both of which lie is linear spaces.

We proceed as follows. Section 2 defines the model. Section 3 defines the measure of inefficiency. Section 4 characterizes it for special utilities, bounds it for general utilities, and shows it obeys the law of diminishing marginal returns to interventions. Section 5 connects our measure of inefficiency with the traditional local calculus analysis, and provides a justification for this tradition. Section 6 calibrates the inefficiencies in several regions. Section 7 describes the global parameterization of equilibrium allocations. An appendix contains the longer or less important proofs.

## 1.2 Related literature on equilibrium welfare in the absence of complete asset markets

On equilibrium welfare in the absence of complete asset markets, a focal contribution is Levine and Zame (2002), who show that the incomplete assets still allow full risk sharing, if (1) agents are infinitely patient, (2) there is a single commodity per state, (3) idiosyncratic income risk is transitory, (4) a one-period bond is tradeable. The underlying idea is in Bewley (1980), that the bond allows for an eventual build up of a buffer stock of savings, unlikely to ever be depleted by a long sequence of transitory shocks, and therefore likely to always help finance constant consumption; with nearly infinite patience, the consumption foregone while initially saving affects only a vanishing fraction of total utility. The idea that a risk averse agent would optimally smooth consumption over time appears in Friedman's (1957) permanent income hypothesis. Levine and Zame do not quantify the smallness of the individual welfare loss, when patience is large but merely finite, but any Pareto inefficiency is clearly vanishing in the sense of our willingness to pay ex ante. We may view the broader literature on the topic in relation to how it relaxes assumptions (2-4), with relaxation of (1) being standard.

Lucas (1987) relaxes (4) alone, in an extreme way, positing a representative agent who in equilibrium cannot trade anything, in particular a bond. Equilibrium is necessarily Pareto optimal, and the interesting welfare question is as to how large the risk premium of idiosyncratic income is. That is, what is the representative agent's willingness to pay for replacing risky future income by riskless expected income, for smoothing the business cycle? Lucas calibrates it to be about one half of one percent of expected income. The question we ask, instead, when allowing asset markets a role, how well do they enable agents to allocate future income, risky as it is? A literature has followed, enriching Lucas' model with heterogeneous agents, assets, and trade frictions, Deborah Lucas (1994), Krusell and Smith (1998, 2002) and Rios-Rull (1994) being representative, which still find the risk premium to be small. Other attempts such as the introduction in these models of preferences with habit formation had also have little impact on this measure, see Diaz, Pijoan-Mas, Rios-Rull (2003).

Kurz (2005) challenges Lucas' small number as evidence that the business cycle is negligible for welfare, noting that it is calibrated with data that may already reflect substantial and successful smoothing of the business cycle; rather, it is at best interpretable as the worth of *marginal* smoothing to existing smoothing. Further, he builds a model where the smoothing arises not exogenously, but endogenously as firms and consumers build buffer stocks, and estimates the implied cost to be much higher, 4% of total income.

Relaxation of (3) alone is the domain of our two-period model here, where the utility for the second period income shock is interpretable as the discounted utility of a permanent income shock. In this interpretation of the model, although a one-period bond is not explicited for future trade, it would not be traded if available, since those with a negative permanent shock cannot borrow forever, and those with a positive permanent do not want to save forever. Keeping to an infinite-period model, Krebs (2003) allows for two-dimensional, persistent shocks to idiosyncratic income and finds a cost of 9-11% of expected income, much larger than Lucas'.

Geanakoplos and Polemarchakis (1986) relax (2) and show that Pareto inefficiency obtains even relative to reallocations that must be financed by existing assets. Subsequent research in this line, as cited above, reaffirms the strong sense of Pareto inefficiency, relative to various other restrictions on reallocations, such as supportability by particular fiscal, monetary, or financial policies. Levine and Zame (2002) themselves relax (3).

The literature, broadly split, either focuses on Pareto inefficiency, but then only qualitatively, or else on quantitative measures, but then only by eliminating the risk in future incomes. We focus on quantifying Pareto inefficiency of equilibrium, and in particular on the success of incomplete assets in allocating risky future incomes; our question requires us not to touch the risk in future incomes, not to smooth the business cycle.

Our paper is organized as follows. **Section 2** defines the model. **Section 3** defines the measure of inefficiency. **Section 4** characterizes it for special utilities, bounds it for general utilities, and shows it obeys the law of diminishing marginal returns to interventions. **Section 5** connects our measure of inefficiency with the traditional local calculus analysis, and provides a justification for this tradition. **Section 6** calibrates the inefficiencies for the US. **Section 7** describes the global parameterization of equilibrium allocations. An appendix contains the longer or less important proofs.

## 2 Economy and equilibria

**Primitives** *Households* $h = 1, ..., H$ know the present *state* of *nature*, denoted 0, but are uncertain as to which state among $s = 1, ..., S$ nature will reveal next. In each state there is a nonstorable *commodity* available for *consumption*, and in state 0 there are *assets* $j = 1, ..., J$ available for *trade*.

**Economy** Given limited resources $r \in \mathbb{R}^{S+1}_{++}$ of the commodities and limited number $J \leq S$ of assets, an economy $(e, a) \in \mathbb{R}^{H(S+1)}_{+} \times \mathbb{R}^{S \times J}$ specifies to each household $h$ an endowment of $e^h \in \mathbb{R}^{S+1}_{++}$ units and to each asset $j$ a claim of $a^j \in \mathbb{R}^S$ units of the commodity across states, constrained so that $\Sigma e^h = r$ and $a$ has rank $J$. We write $E(r) := \left\{ e \in \mathbb{R}^{H(S+1)}_{++} : \Sigma e^h = r \right\}$.

**Markets** Markets specify that each asset $j$ is tradeable at a price of $q^j$ units of the state 0 commodity, by specifying $q = p'a$ for some $p \in \mathbb{R}^S_{++}$. $Q \subset \mathbb{R}^J$ denotes these asset prices, which arise as the cost of their claims relative to some future *state prices*. Viewing asset prices as a negative claim, asset claims become $W := \begin{pmatrix} -q \\ a \end{pmatrix} \in \mathbb{R}^{S+1\times J}$. Households are free to trade any amount $\theta^h_j \in \mathbb{R}$ of any asset: buy $\theta^h_j > 0$, sell $\theta^h_j < 0$, or neither $\theta^h_j = 0$. Trades of asset $j$ **clear** if $\Sigma\theta^h_j = 0$.

**Consumption** Fixing asset claims throughout, each household trades assets as a function of asset prices and its endowment, $\theta^h : Q \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^J$, consuming $x^h\left(q, e^h\right) := e^h + W\ \theta^h\left(q, e^h\right)$. Trades are **optimal** for the utility function $u^h : \mathbb{R}^{S+1}_{+} \to \mathbb{R}$ if

$$u^h(x^h(q, e^h)) = \sup u^h(e^h + W\mathbb{R}^J)$$

**Definition 1** *$(q, e) \in Q \times E(r)$ is an **equilibrium** if trades of all assets clear, $\Sigma\theta^h(q, e^h) = 0$. It is a **no-trade equilibrium** if $\theta^h(q, e^h) = 0$ for every $h$.*

The sets of equilibria, no-trade equilibria are denoted by $\mathbb{E}(r), \mathbb{T}(r)$.

**Definition 2** *$x \in \mathbb{R}^{H(S+1)}_{+}$ is **Pareto optimal** $\leftrightarrow$ $y \in \mathbb{R}^{H(S+1)}_{+}, \Sigma y^h = \Sigma x^h$ imply $(u^h(y^h))_h \geq (u^h(x^h))_h$ can only hold with equality.*

## 2.1 Assumptions

**Assumption 1** *In the economy, $e^h \gg 0$ is strictly positive and $a \in \mathbb{R}^{S\times J}$ has rank $J$.*

**Assumption 2 (smooth preferences)** *$\forall h$, $u^h$ is continuous, $C^2$ in $\mathbb{R}^{S+1}_{++}$, strictly increasing ($\forall x \in \mathbb{R}^{S+1}_{++}, Du^h(x) \gg 0$), strictly concave ($\forall x \in \mathbb{R}^{S+1}_{++}, D^2u^h(x)$ is negative definite), and boundary averse ($\forall x' \in \mathbb{R}^{S+1}_{++}, u^h(x) \geq u^h(x') \Rightarrow x \in \mathbb{R}^{S+1}_{++}$).*

**Assumption 3** *Trades by $h$ are optimal for $u^h$.*

# 3 Measure of inefficiency

We propose a variation of Debreu's coefficient of resource utilization (1951).

**Definition 3** *The **risk sharing inefficiency** of $x \in E(r)$ is the value $\rho_x$ of*

$$\max_y \rho \qquad s.t. \qquad u^h(y^h) \geq u^h(x^h), \Sigma y^h_{\mathbf{1}} = r_{\mathbf{1}}, \Sigma y^h_0 = (1-\rho)r_0 \tag{4}$$

It is the maximum fraction of current resources $r_0$ that society is willing to pay for an ex ante Pareto improving reallocation of tomorrow's risky resources $r_{\mathbf{1}}$. Thus a solution $y$ is an **optimal arbitrage** allocation.

The only, but major, difference with Debreu's definition is his constraint on future resources, $\Sigma y_{\mathbf{1}} = (1-\rho)r_{\mathbf{1}}$, which reduces them. Since the role of assets is to allocate risky future resources $r_{\mathbf{1}}$, changing them is to deflect our question: how well do assets fulfill this role? (To the extent our coefficient is small, Debreu's is even smaller.)

Rewriting the last constraint, $\rho = 1 - \frac{\Sigma y_0^h}{r_0}$, renders the problem more convenient for analysis:

$$\max_{y_1} 1 - \frac{\Sigma y_0^h}{r_0} \qquad s.t. \qquad u^h(y^h) \geq u^h(x^h), \Sigma y_{\mathbf{1}}^h = r_{\mathbf{1}} \tag{5}$$

**Proposition 1** *Suppose $y \gg 0$ is feasible for (4)=(5). Then it is a solution iff is Pareto optimal and makes all welfare constraints bind.*

**Proof.** Necessity. Let $y$ be a solution and $\rho$ its value. Suppose $z$ is a counterexample, i.e. $\Sigma z_0^h = (1-\rho)r_0, \Sigma z_{\mathbf{1}}^h = r_{\mathbf{1}}$ and $u^h(z^h) \geq u^h(y^h)$ with strict inequality for some $i$. Since $y^h \gg 0$ by assumption, boundary aversion implies $z^h \gg 0$; reduce $z_0^i$ to $z_0^i - \epsilon r_0$ by some $\epsilon > 0$. By continuity, a small enough $\epsilon$ is feasible still, in that $u^i(z_0^i - \epsilon r, z_{\mathbf{1}}^i) > u^i(y^i) \geq u^i(x^i)$. But now this modified $z_0$ sums to $(1-(\rho+\epsilon))r_0$ so that $\rho + \epsilon > \rho$ is feasibly supported, contradicting the maximality of $\rho$. By a similar argument, a solution gives welfare slack $u^i(y^i) = u^i(y_0^i, y_{\mathbf{1}}^i) > u^i(x^i)$ to no $i$.

Sufficiency. If not, let $z$ be feasible for (5) and imply greater destruction of 0 resources than does $y$. Since $z$ is feasible and $y$ is welfare binding, $u^h(z^h) \geq u^h(x^h) = u^h(y^h)$. Thus $z$ weakly Pareto dominates $y$, and the $z^+$ gotten from $z$ by restoring the extra 0 resource not already destroyed by $y$, by monotonicity, Pareto dominates $y$, contradicting its Pareto optimality. ■

Tacitly, we allow the reallocation $y$ in problem (4) to reflect any lump-sum transfer, even state contingent. If we were to constrain further the reallocation to arise from a particular policy–fiscal, monetary, or financial– then problem (4) would have a no greater feasible set, hence define a no greater "constrained inefficiency." (For a formalization of policy, see the appendix.) So the inefficiency as defined by (4) is an upper bound, uniformly over all policies, for such an alternate constrained inefficiency.

**Corollary 1** *Suppose $x \gg 0$. Then $\rho_x = 0$ iff it is Pareto optimal.*

## 3.1 The greatest Pareto improvement

We define the size of a Pareto improving reallocation, and show that its maximum value is precisely $r_0 \rho_x$. Thus $\rho$ gives meaning to the phrase "the greatest Pareto improvement."

**Definition 4** *Fix $x \in E(r), x \gg 0$. If $\tilde{y} \in E(r)$ is a weak Pareto improving reallocation of $x$, its **size** is $p(\tilde{y}) := \Sigma p^h$, where for every $h$ $p^h$ is the maximum real number such that $u^h(\tilde{y}_0^h - p^h, \tilde{y}_{\mathbf{1}}^h) \geq u^h(x^h)$.*[3]

We interpret $p^h$ for each agent $h$ as his maximum willingness to pay for $\tilde{y}$, and $p(\tilde{y})$ is the maximum total willingness to pay for $\tilde{y}$, out of current income. Incidentally, continuity implies the equality $u^h(\tilde{y}_0^h - p^h, \tilde{y}_{\mathbf{1}}^h) = u^h(x^h)$.

[3]Note, the asserted maximum $p^h$ exists. It suffices that the set of $q^h$ satisfying $u^h(\tilde{y}_0^h - q^h, \tilde{y}_{\mathbf{1}}^h) \geq u^h(x^h)$ be nonempty, closed, and bounded above, by the completeness of reals. It is nonempty: $q^h = 0$ works since $u^h(\tilde{y}^h) \geq u^h(x^h)$ by assumption. It is closed since $u^h$ is continuous. It is bounded above since boundary aversion and $x \gg 0$ imply that $u^h(\tilde{y}_0^h - q^h, \tilde{y}_{\mathbf{1}}^h) \geq u^h(x^h)$ only if $(\tilde{y}_0^h - q^h, \tilde{y}_{\mathbf{1}}^h) \gg 0$, so $q^h \leq \tilde{y}_0^h$.

Thanks to the theorem of the maximum and our assumptions on utility, the size $p = p(\tilde{y})$ is continuous, as a function of the weak Pareto improvement $\tilde{y}$, so by Weierstrass' theorem it attains its maximum value, as the set of weak Pareto improvements in $E(r)$ is closed and $E(r)$ is compact.

**Remark 1** *Fix $x \in E(r), x \gg 0$. Then among the weakly Pareto improving reallocation of $x$, the size attains its maximum value, termed* ***the greatest Pareto improvement.***

The following simple fact is proved in the appendix.

**Proposition 2 (inefficiency as the greatest Pareto improvement)** *Fix $x \in E(r), x \gg 0$. Then the greatest Pareto improvement is $r_0 \rho_x$.*

### 3.2 On the arbitrariness of equilibrium inefficiency

Research shows that absent all frictions but incomplete asset markets, at almost every equilibrium a Pareto improvement is supported by many types of intervention, financial, monetary, and fiscal. How large are these Pareto improvements in the sense of $\rho$? We would like to decide whether $\rho$ is "large" or "small." This is impossible since the variety of income distributions, even if restricted to be equilibrium allocations, supports any value of inefficiency in the interval $[0,1]$.

**Example 1** *$H = 2 = S$, common utility $u(x) = x_0 + \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2 = x_0 + \frac{1}{2}\ln x_1 x_2$. Total resources are $\overline{r}$ in every state. Only a bond exists. The allocation $x(\epsilon)$ is $x^1 = (\frac{\overline{r}}{2}, \epsilon, \overline{r} - \epsilon), x^2 = (\frac{\overline{r}}{2}, \overline{r} - \epsilon, \epsilon)$, a no-trade GEI due to its symmetry. Clearly $\epsilon_0 = \frac{\overline{r}}{2}$ makes $x(\epsilon_0)$ Pareto optimal, so by corollary 1 $\rho_{x(\epsilon_0)} = 0$. We now find an $\epsilon_1$ with $\rho_{x(\epsilon_1)} = 1$. It will then follow, by the continuity of $\rho \circ x$ and the intermediate value theorem, that any value in $[0,1] = [\rho_{x(\epsilon_0)}, \rho_{x(\epsilon_1)}]$ is the inefficiency of the allocation $x(\epsilon)$ for some $\epsilon$ between $\epsilon_0, \epsilon_1$.*

*We compute $\rho_{x(\epsilon)}$. Since utilities are common and status quo utility levels $u(x^1) = u(x^2)$ too, the optimal arbitrage $y$ treats them equally. Since it is Pareto optimal also, it must have $y^h_{\mathbf{1}} = (\frac{\overline{r}}{2}, \frac{\overline{r}}{2})$. The question is now to find the smallest $y^h_0 = c$ satisfying $u(y) \geq u(x)$ :*

$$c + \frac{1}{2}\ln\frac{\overline{r}}{2}\frac{\overline{r}}{2} \geq \frac{\overline{r}}{2} + \frac{1}{2}\ln\epsilon(\overline{r} - \epsilon)$$

*Simplifying and rearranging,*

$$2c = \overline{r} + \ln\epsilon(\overline{r} - \epsilon) - \ln\frac{\overline{r}}{2}\frac{\overline{r}}{2} = \overline{r} + \ln\frac{4\epsilon(\overline{r} - \epsilon)}{\overline{r}^2}$$

*By definition,*

$$\rho_x = 1 - \frac{\Sigma y^h_0}{r_0} = 1 - \frac{2c}{\overline{r}} = -\frac{1}{\overline{r}}\ln\frac{4\epsilon(\overline{r} - \epsilon)}{\overline{r}^2}$$

*Choosing $\epsilon$ such that $\ln\frac{4\epsilon(\overline{r}-\epsilon)}{\overline{r}^2} = -\overline{r}$ implies $\rho_x = -\frac{1}{\overline{r}}(-\overline{r}) = 1$, as desired. One can check such $\epsilon$ is $\frac{\overline{r}}{2}(1 - \sqrt{1 - e^{-\overline{r}}})$.*

In view of this arbitrariness, the rest of the paper assumes the income distribution is known. In the theoretical part, the income distribution appears in parametric form; in the empirical part, it appears as extrapolated data.

# 4 Welfare analysis of global interventions

By definition, the inefficiency $\rho$ measures the maximum total willingness to pay for a Pareto improvement, and here we provide three results about it. First, we provide a closed expression for inefficiency as a function of the income distribution and of 0-quasilinear utilities. By **0-quasilinear** utilities we mean with respect to current consumption: $u(x) = x_0 + u_{\mathbf{1}}(x_{\mathbf{1}})$. As it turns out, it reaffirms the classical understanding that with quasilinearity an appropriate measure of "aggregate welfare" is the sum of utilities. Second, for general utilities, we provide an upper bound for inefficiency as a function of the income distribution and the marginal rates of substitution. The derivation makes clear that the error in the upper bound is increasing in concavity of the utilities, as it effectively linearizes them. In addition, knowledge that the allocation is an equilibrium allocation provides makes the upper bound more explicit. Third, we prove that the law of diminishing marginal returns is valid for global interventions, whatever the direction.

Both the upper bound on inefficiency and the law of diminishing marginal returns to intervention are based on the following basic fact about concavity, which makes a global statement based on infinitesimal information:

**Lemma 1** *Suppose* $f : A \to R$ *is* $C^1$. *Then it is concave iff for all* $a^*$, *interior* $a \in A$

$$f(a^*) - f(a) \leq Df(a)(a^* - a) \tag{6}$$

## 4.1 Quasilinear case: exact formula for inefficiency

We provide a closed expression for inefficiency as a function of the income distribution and of 0-quasilinear utilities, $u(x) = x_0 + u_{\mathbf{1}}(x_{\mathbf{1}})$.

Let $\tilde{x}$ be a weakly Pareto improving redistribution of $x \in E(r)$. Then an agent's maximum willingness to pay $p^h$ out of current income clearly satisfies $u(\tilde{x}^h) - p^h = u^h(x^h)$, which rearranged and summed gives the total willingness to pay

$$\begin{aligned} p(\tilde{x}) &= \Sigma u(\tilde{x}^h) - \Sigma u^h(x^h) = \Sigma u_{\mathbf{1}}(\tilde{x}^h_{\mathbf{1}}) - \Sigma u^h_{\mathbf{1}}(x^h_{\mathbf{1}}) + \Sigma \tilde{x}^h_0 - \Sigma x^h_0 \\ &= \Sigma u_{\mathbf{1}}(\tilde{x}^h_{\mathbf{1}}) - \Sigma u^h_{\mathbf{1}}(x^h_{\mathbf{1}}) \end{aligned}$$

since the redistribution satisfies $\Sigma \tilde{x}^h_0 - \Sigma x^h_0 = 0$. By proposition 2 $\rho_x = \frac{\max p(\tilde{x})}{r_0}$, so

$$\rho_x = \frac{1}{r_0}\left[\max_{y \in E(r)} \Sigma u_{\mathbf{1}}(y^h_{\mathbf{1}}) - \Sigma u^h_{\mathbf{1}}(x^h_{\mathbf{1}})\right]$$

That this is the greatest Pareto improvement is consonant with the classical understanding that with quasilinearity the appropriate measure of "aggregate welfare" is the sum of utilities $\Sigma u^h_{\mathbf{1}}(x^h_{\mathbf{1}})$. Here, here inefficiency of an allocation specializes to the amount by which "aggregate welfare" fails to be maximized.

## 4.2 General case: upper bound for inefficiency

Ideally, there would exist a formula for the inefficiency $\rho$ explicitly in terms of fundamentals. Unfortunately, such a formula is elusive, and that it is should not be surprising, given that such a formula for Debreu's coefficient is missing in the fifty years since its definition. Nonetheless, for some questions, an exact formula is superfluous if an inexact formula can answer them. The question we have in mind is when incompleteness of asset markets by itself is a rationale for lump-sum intervention, granting full knowledge of the state-contingent allocation. The answer would be no, if the upper bound –the inexact formula– were tiny and the intervention constrained to be Pareto improving. For a tiny upper bound implies that the maximum willingness to pay for a lump-sum intervention is tiny too. The advantage of the upper bound is that it requires knowledge not of the whole utilities but of their marginal rates of substitution. The idea of the upper bound is to remove all concavity (diminishing marginal utility) for an arbitrageur to extract an even larger total willingness to pay his proposed Pareto improving reallocations.

Denote the marginal rate of substitution of future income for current one by

$$\bigtriangledown := \frac{(D_{x_s}u)_s}{D_{x_0}u} \in \mathbb{R}^S_{++}$$

and for an allocation $x \gg 0$ the state-by-state maximum over households by

$$\bigtriangledown^* := (\max_h \bigtriangledown^h_s)_s$$

We show next that an upper bound is[4]

$$R_x := \frac{1}{r_0}\Sigma\left(\bigtriangledown^*(x) - \bigtriangledown^h(x^h)\right)x^h_{\mathbf{1}} \tag{7}$$

The idea is that $\bigtriangledown^*$ indexes the most deprived household in each state, so that the total willingness to pay is maximized by having, in each state, all households donate to the most deprived. If the donation is the infinitesimal amount $\dot{x}^h_{\mathbf{1}}$, calculus says that the net total infinitesimal willingness to pay is *exactly* $\Sigma\left(\bigtriangledown^*(x) - \bigtriangledown^h(x^h)\right)\dot{x}^h_{\mathbf{1}}$, out of current income. Because of diminishing returns, if we were to increase the donation from the infinitesimal to the full income $x^h_{\mathbf{1}}$, the total willingness to pay should be *at most* this expression with $x^h_{\mathbf{1}}$ instead of $\dot{x}^h_{\mathbf{1}}$.

**Proposition 3** *The inefficiency of* $x \in E(r), x \gg 0$ *is bounded above by*

$$\rho_x \leq R_x \tag{8}$$

**Proof.** The lemma implies that if $y$ satisfies $0 \leq u^h(y^h) - u^h(x^h)$ then it satisfies $0 \leq Du^h(x^h)(y^h - x^h)$. Therefore an upper bound for the value $\rho$ of problem (5) is the value of the relaxed problem

$$\rho \leq \max_{y_{\mathbf{1}}} 1 - \frac{\Sigma y^h_0}{r_0} \qquad s.t. \qquad \begin{array}{c} \Sigma y^h_{\mathbf{1}} = r_{\mathbf{1}}, y_{\mathbf{1}} \geq 0 \\ 0 \leq Du^h(x^h)(y^h - x^h) \end{array}$$

We recast this problem. Rewrite the welfare constraint as $0 \leq \lambda^h_0(y^h_0 - x^h_0) + D_{\mathbf{1}}u^h(x^h)(y^h_{\mathbf{1}} - x^h_{\mathbf{1}})$ where $\lambda^h_0 := D_0u^h(x^h)$, divide by $\lambda^h_0$ and rearrange to get

$$y^h_0 \geq x^h_0 - \nabla^h(y^h_{\mathbf{1}} - x^h_{\mathbf{1}}) := \underline{y}^h_0$$

[4]For a vector $v \in \mathbb{R}^{S+1}, v_{\mathbf{1}} \in \mathbb{R}^S$ denotes omission of the first coordinate, $v_0$ its first coordinate. So $\bigtriangledown^h$ is $\frac{(Du^h)_{\mathbf{1}}}{D_{x_0}u^h}$

Since the objective $1-\frac{\Sigma y_0^h}{r_0}$ is decreasing in $y_0$, this inequality implies that at the optimal arbitrage $y_{\mathbf{1}}$ the value of the objective, which is $\rho_x$, is at most

$$1-\frac{\Sigma\left(x_0^h-\nabla^h(y_{\mathbf{1}}^h-x_{\mathbf{1}}^h)\right)}{r_0}=\frac{1}{r_0}\Sigma\nabla^h(y_{\mathbf{1}}^h-x_{\mathbf{1}}^h)=:R(x,y_{\mathbf{1}})$$

It suffices to show $R(x,y_{\mathbf{1}})\leq R_x$. Since the optimal arbitrage $y_{\mathbf{1}}$ satisfies $\Sigma y_{\mathbf{1}}^h=r_{\mathbf{1}},y_{\mathbf{1}}\geq 0$, clearly $R(x,y_{\mathbf{1}})$ is at most

$$\max_{y_{\mathbf{1}}} R(x,y_{\mathbf{1}})\qquad s.t.\qquad \Sigma y_{\mathbf{1}}^h=r_{\mathbf{1}},y_{\mathbf{1}}\geq 0$$

We show this value is $R_x$. A feasible $y_{\mathbf{1}}$ for this problem is, for each $s>0$, to allocate everything to a household $i=i(s)$ with the highest $\nabla_s^i(x^i):\ y_s^{*i}=r_s$ and $y_s^{*h\neq i}=0$. To see it is maximizing, note that the $R(x,y_{\mathbf{1}})$ objective's controllable part is $\Sigma\nabla^h y_{\mathbf{1}}^h$ since $-\Sigma\nabla^h x_{\mathbf{1}}^h$ is given. Since $\Sigma\nabla^h y_{\mathbf{1}}^h$ is to be maximized, note for any feasible $y_{\mathbf{1}}$ that

$$\begin{aligned}\Sigma\nabla^h y_{\mathbf{1}}^h &= \Sigma_s\left(\Sigma_h\nabla_s^h y_s^h\right)\leq\Sigma_s\left(\Sigma_h\nabla_s^{h(s)}y_s^h\right)\\ &= \Sigma_s\left(\nabla_s^{h(s)}\Sigma_h y_s^h\right)=\Sigma_s\left(\nabla_s^{h(s)}r_s\right)\\ &= \Sigma_s\left(\Sigma_h\nabla_s^h y_s^{*h}\right)=\Sigma\nabla^h y_{\mathbf{1}}^{*h}\end{aligned}$$

revealing $y_{\mathbf{1}}^*$ as maximizing. ■

Thus we bound above a global object $\rho$ by local information $\left(\nabla^h(x^h)\right)_h$. The argument used to derive the bound implies that it will be tighter the less concave are utilities.

Note from the middle of the proof that $\frac{1}{r_0}\Sigma\nabla^h(y_{\mathbf{1}}^h-x_{\mathbf{1}}^h)$ is an even better upper bound, where $y$ is the optimal arbitrage. Of course, this is useless without knowledge of $y_{\mathbf{1}}$, which of course would allow us to compute the exact inefficiency $\rho_x=1-\frac{\Sigma y_0}{r_0}$ and make the upper bound superfluous. Yet it sheds light in two ways. First, if $x_{\mathbf{1}}^h=e_{\mathbf{1}}^h+a\theta^h$ is from an equilibrium allocation, then $\frac{1}{r_0}\Sigma\nabla^h(y_{\mathbf{1}}^h-x_{\mathbf{1}}^h)=\frac{1}{r_0}\Sigma\nabla^h(y_{\mathbf{1}}^h-e_{\mathbf{1}}^h)$ since $\Sigma\nabla^h a\theta^h=q\Sigma\theta^h=0$ thanks to the portfolio optimality equation $\nabla^h a=q$ and asset clearing $\Sigma\theta^h$. It follows that imitating the rest of the proof gives

**Corollary 2** *Suppose $x^h=e^h+W\theta^h\gg 0$ is an equilibrium allocation for the economy $e\in E(r)$. Then its inefficiency is bounded above by*

$$\rho_x\leq\frac{1}{r_0}\Sigma\left(\bigtriangledown^*(x)-\bigtriangledown^h(x^h)\right)e_{\mathbf{1}}^h$$

Second, writing $z:=y_{\mathbf{1}}^h-x_{\mathbf{1}}^h$ for the net trade to optimal arbitrage, note $\Sigma\nabla^h z^h=HCov(\nabla,z)$ since $\Sigma z^h=0$. So that $\rho_x\leq\frac{1}{r_0}\Sigma\nabla^h z^h$ we may interpret as stating that, to first order, an optimal arbitrage seeks net trades that maximize their covariance with the marginal rates of substitution.

## 4.3 The law of diminishing marginal returns to interventions

We show the law of diminishing marginal returns to interventions: for sequence of two interventions in the same direction, the first part is worth proportionally more than the second part, in terms of the total willingness to pay. In this sense also, small interventions capture a significant portion of the Pareto improvements from large interventions.

Let $x$ be a status quo income distribution and $y_{\mathbf{1}}$ a reallocation of the future income. As in definition 4, we ask what is household $h's$ willingness to pay for the option to consume the future alternative $y_{\mathbf{1}}^h$, when he can consume $x^h = (x_0^h, x_{\mathbf{1}}^h)$? To how little consumption $y_0^h$ would she be willing to deprive herself for this option? The answer is the smallest solution $y_0^h$ of

$$u^h(y_0^h, y_{\mathbf{1}}^h) \geq u^h(x^h) \tag{9}$$

**Assumption 4 (temporary)** *A smallest solution* $\tilde{y}_0^h = \tilde{y}_0^h(y_{\mathbf{1}}^h)$ *of (9) exists and is unique, whenever* $0 \leq y_{\mathbf{1}}^h \leq r_{\mathbf{1}}$.

As a fraction of current resources $r_0$, the **willingness to pay for** $y_{\mathbf{1}} = (y_{\mathbf{1}}^h)_h$ at $x$ is

$$\rho(x, y_{\mathbf{1}}) := 1 - \frac{\Sigma \tilde{y}_0^h(y_{\mathbf{1}}^h)}{r_0}$$

The connection with definition (5) of inefficiency is simple; if $y$ is the optimal arbitrage at $x$, then $\rho(x, y_{\mathbf{1}}) = \rho_x$, that is, the inefficiency is just the maximum willingness to pay.

The following establishes how a partial intervention compares with an intervention, in terms of willingness to pay. A **partial intervention** is $y_{\mathbf{1}}(t) =: (1-t)x_{\mathbf{1}} + ty_{\mathbf{1}}$, $t \in [0,1]$. Clearly it is a null reallocation for $t = 0$ and the original intervention for $t = 1$.

**Proposition 4** *Suppose assumption 4 and each* $u^h$ *is concave. Then* $\rho(x, y_{\mathbf{1}}(t))$ *is concave in* $t$ *and* $\rho(x, y_{\mathbf{1}}(t)) \geq t\rho(x, y_{\mathbf{1}})$*–with strict inequality if in addition* $t \in (0,1)$, $z_{\mathbf{1}} \neq 0$, *and each* $u^h$ *is strictly concave and continuous in 0-consumption.*

**Corollary 3 (monotonicity)** *Let* $y_{\mathbf{1}}$ *be the optimal arbitrage at* $x$*. Then the maximum willingness to pay for the partial arbitrage* $y_{\mathbf{1}}(t) := (1-t)x_{\mathbf{1}} + ty_{\mathbf{1}}, t \in [0,1]$ *is monotonically **increasing** from the minimum* $0$ *at* $t = 0$ *to the maximum* $\rho_x$ *at* $t = 1$.

**Proof.** As noted, $\rho(x, y_{\mathbf{1}}) = \rho_x$ when $y_{\mathbf{1}}$ is an optimal arbitrage, and $\rho(x, x_{\mathbf{1}}) = 0$ if $u^h$ is increasing in 0 consumption. That $\rho(x, y_{\mathbf{1}}(t))$ is concave and maximized at $t = 1$ implies it is monotonically increasing. ■

# 5 Welfare analysis of local interventions

There is a potentially serious criticism of the typical analysis of the inefficiency in GEI: it is local and, most importantly, local improvements has not quantified their size, or even defined the notion of size (see Magill and Quinzii (1996)). The main focus of this local analysis is to establish the generic existence of a direction of "local improvements," which could be small compared with "global improvements." By the size of the improvement, we mean the measure $\rho$, which is interpretable as the size of the greatest Pareto improvement. If this measure is small, local analysis does not miss much of the potential Pareto improvements. On the other hand, if $\rho$ is not small, local analysis would seem to miss most of the potential Pareto improvements.

We find that this criticism, although plausible, is unjustified. There is some evidence in favor of this thesis, already: in every direction of intervention, *i*) its welfare effect is diminishing, and *ii*) the diminution is greater the more concave the preferences are. We now show that, formally, the local information on which local interventions are based already captures the benefits of global interventions. Roughly put, an upper bound for the best Pareto improvement from local reallocations is also an upper bound for the best Pareto improvement from global reallocations, upon formally replacing the local by the global in the expression. Indeed, locally, the infinitesimal net welfare of having agent $h$ donate to $i$ the infinitesimal income $\dot{x}_s^h$ in state $s$ is $(\nabla_s^i - \nabla_s^h)\dot{x}_s^h$. Let $i(s)$ be the most deprived agent in this state, in that he has the largest $\nabla_s^h(x^h)$. To maximize the total willingness to pay, everyone should donate to the most deprived, state by state, and then the infinitesimal net welfare change of the infinitesimal reallocation $\dot{x}$ is *exactly*

$$\frac{1}{r_0}\Sigma\left(\nabla^*(x) - \nabla^h(x^h)\right)\dot{x}_{\mathbf{1}}^h$$

as a fraction of total current income. Globally, on the other hand, proposition 3 shows that an upper bound for $\rho_x$, the maximum total willingness to pay as a fraction of total current income, is

$$R_x := \frac{\Sigma_s\Sigma_h(\nabla_s^{i(s)} - \nabla_s^h)x_s^h}{\Sigma_h x_0^h}$$

Thus the expression for an upper bound for the best Pareto improvement from local reallocations is, remarkably, already an upper bound for the best Pareto improvement from global reallocations: one need only formally replace the infinitesimal $\dot{x}$ by the global $x$–the income distribution in question. In this formal sense, information relevant to local welfare already captures the global welfare.

## 5.1 Size of local Pareto improvement

To define a local measure from our global one, we ask how inefficiency $\rho_x$ changes as $x$ changes in the direction $z$, while fixing resources $\Sigma z^h = 0$ :

$$x(\lambda) := x + \lambda z, \lambda \in \mathbb{R} \tag{10}$$

**Proposition 5** *Suppose each* $u^h$ *is* $C^1$ *and* $x \gg 0$. *Fix* $z \in R^{H(S+1)}$ *with* $\Sigma z^h = 0$. *Assume a unique solution* $y = y(\lambda)$ *of problem (4) at* $x(\lambda)$ *exists and is differentiable at* $\lambda = 0$. *Then the following is well defined,*

$$\frac{d\rho(x(\lambda))}{d\lambda} = -\frac{1}{r_0}\Sigma\frac{Du^h(x^h)z^h}{D_0u^h(y^h)}$$

Proof is in the appendix.

**Corollary 4** *If in addition each* $u^h$ *is 0-quasilinear, then at* $\lambda = 0$ :

$\frac{d\rho}{d\lambda} = -\frac{1}{r_0}\Sigma\mu^h(x)z_{\mathbf{1}}^h$

inefficiency falls the fastest in the direction $z_{\mathbf{1}} = (z_{\mathbf{1}}^h) := (\mu^h)$

the norm of $\frac{d\rho}{d\lambda}$ as a functional of $z_{\mathbf{1}}$ is $\frac{\|\mu\|}{r_0}$

**Proof.** The assumption implies $D_0 u^h = 1$ and $D_{\mathbf{1}} u^h(x^h) = \nabla^h(x^h)$, so $\frac{d\rho}{d\lambda} = -\frac{1}{r_0}\Sigma(z_0^h + \nabla^h z_{\mathbf{1}}^h) = -\frac{1}{r_0}\Sigma\mu^h(x) z_{\mathbf{1}}^h$, using $\Sigma z_{\mathbf{1}}^h = 0$ for the latter. Next, the Cauchy-Schwarz inequality implies that among all $z_{\mathbf{1}}$ in a sphere, the inner product $\Sigma\mu^h(x) z_{\mathbf{1}}^h$ is maximized uniquely in the direction $z_{\mathbf{1}}^h = \mu^h$. So the maximum of $\frac{d\rho}{d\lambda}$ on the sphere $\|z_{\mathbf{1}}\|^2 = \Sigma z_{\mathbf{1}}^{h\prime} z_{\mathbf{1}}^h = 1$ is at $z_{\mathbf{1}}^h = c\mu^h$ where $c$ solves $c = \frac{1}{\|\mu\|}$ : $\frac{d\rho}{d\lambda} = -\frac{1}{r_0}\Sigma\mu^h z_{\mathbf{1}}^h = -\frac{1}{r_0} c \|\mu\|^2 = -\frac{\|\mu\|}{r_0}$.

**Remark 2 (quadratic case)** *Suppose identical utilities* $u(x) = x_0 + \Sigma\pi_s(x_s - \frac{1}{2}x_s^2)$. *Then the covariance between consumption and policy direction is a sufficient statistic:*

$$\frac{d\rho}{d\lambda} = -\frac{1}{r_0}\Sigma_h D_{\mathbf{1}} u^h z_{\mathbf{1}}^h = -\frac{1}{r_0}\Sigma_s \pi_s \Sigma_h (1 - x_s^h) z_s^h = \frac{1}{r_0}\Sigma_h \Sigma_s \pi_s x_s^h z_s^h = \frac{1}{r_0}\Sigma_h Cov(x_{\mathbf{1}}^h, z_{\mathbf{1}}^h)$$

■

# 6 Computing inefficiencies in the US

The inefficiency depends on the allocation and on the preferences. Accordingly, a presumption that inefficiency is large (or small) may find support in a clever choice of allocation and preferences. The objection to such a choice is that its empirical relevance is accidental at best. In contrast, our choice highlights the data. The choice of allocation follows an ostensibly neutral conversion of data into allocations; the choice of preferences defers to evidence reported in Kocherlakota (1996). As it turns out, the inefficiency implied by US income data of 2004 is exceedingly small–at most $0.27\%, \rho_{US} \leq .0027$.

We do not claim that actual inefficiency is tiny, for several reasons: (1) the interpretation of the primitives of the model (households, states, assets, preferences) is not unique, (2) the conversion of data into allocations is not unique, and (3) the model is simple relative to other models. Yet, we do think that these numbers deserve attention for various reasons: (1) they are rooted in the data, (2) they are small in every geographical region we report, (3) the implied state transition probabilities are unique, given the Bernoulli preferences and the states. The next section illustrate how we bring our model to data.

## 6.1 The insurance deficit equations

We revisit the calculus characterization of inefficiency, because it plays three key roles: in estimating $\rho$ for the US, in an upper bound for $\rho$, and in a global parameterization of the equilibrium set.

The assumptions imply that the optimal trade function $\theta^h(q, e^h)$ is $C^1$ and characterized as the unique solution of[5]

$$\nabla^h a - q = 0 \qquad (\theta)$$

while evaluating $\nabla^h$ at $e^h + W\theta^h$. Relevant are the differences

$$\mu^h := \nabla^h - \overline{\nabla}_t$$

[5] For a vector $v \in \mathbb{R}^{S+1}, v_{\mathbf{1}} \in \mathbb{R}^S$ denotes omission of the first coordinate, $v_0$ its first coordinate. $\bigtriangledown^h$ means $\frac{(Du^h)_{\mathbf{1}}}{D_0 u^h}$. For a matrix $m, \langle m \rangle$ denotes the real span of its columns.

where $\overline{\nabla}_t := \Sigma t^h \nabla^h$ is the average according to given weights $t \in R^H_{++}, \Sigma t^h = 1$, say, $t^h = \frac{1}{H}$. A classical result is that $\mu(x) = 0$ if and only if $x$ is Pareto optimal if and only if $x$ arises as an equilibrium with *complete insurance* markets–that is, markets with zero *insurance deficit.* So we call $\mu = \mu(x)$ the **insurance deficit**.

Two basic properties of the insurance deficit at equilibrium–the *only* ones, per theorem 1–are

$$\begin{array}{ll} a) & \Sigma t^h \mu^h = 0 \\ b) & \mu^h \in \langle a \rangle^{\perp} \end{array} \tag{11}$$

The first one follows from the definition of $\overline{\nabla}_t$, the second one from the optimality equation ($\theta$): averaging implies $\overline{\nabla}_t a - q = 0$ and subtracting implies $\mu^h a = (\nabla^h - \overline{\nabla}_t)a = q - q = 0$.

## 6.2 Choosing the state space

We report the current disposable income distribution $x_0 \in \mathbb{R}^3_+$ by terciles, for the US. These terciles suggest three households: of high, middle, and low ability. The assumption is that tercile incomes match abilities. These terciles also suggest three future states: in $s = 1$ incomes still match abilities, $s = 2$ high and middle ability switch terciles, $s = 3$ middle and low ability switch terciles. These states convert the data $x_0 \in \mathbb{R}^3_+$ to an interim allocation $x \in \mathbb{R}^{4\times 3}_+$. The allocation adds risk in future resources $r_{\mathbf{1}}$, rescaling the interim allocation by $1 + f, 1 - f, 1 + f$ in states $1, 2, 3$; for every $i$,

$$\begin{array}{ccl} & & x^i_1 =: x^i_0(1+f) \\ & \diagup & \\ x^i_0 & — & x^i_2 =: x^i_0 \\ & \diagdown & \\ & & x^i_3 =: x^i_0(1+f) \end{array}$$

The idea behind this risk pattern is that the economy expands by $f\%$ when the high ability household is in charge of the economy ($s = 1, 3$) and does not grow otherwise. Here we assume a time horizon of one year so we let $f = 4\%$ and $a = 1 \in \mathbb{R}^S$ is just the riskless bond.

## 6.3 Calibrating the insurance deficit

To specify each economy, we must assign it the missing object: preferences.

**Definition 5** *Preferences* $u = (u^h)_h$ ***rationalize*** $x \in \mathbb{R}^{H(S+1)}_+ \leftrightarrow x$ *is a* $u$*-no trade equilibrium allocation.*

Necessary equations for $u$ to rationalize $x$ are:

$$\mu^h(x, u)a = 0 \tag{12}$$

Indeed, point (11) has an equilibrium allocation $x^*$ satisfying $\mu^h(x^*)a = 0$ for every $h$. (We take $t^h = \frac{1}{H}$.)

**Remark 3 (sufficiency)** *System (12) is merely necessary for* $u$ *to rationalize* $x$*. The appendix proves the sufficiency, appealing to a global parameterization of the equilibrium set.*

We analyze system (12). If $\mu^{1,2}$ satisfy the equation, so does $\mu^3$. For the definition of $\mu^h(u,x) := \nabla^h(u,x^h) - \overline{\nabla}(u,x)$ implies $\Sigma\mu^h = 0$, as noted in (11): $\mu^3 a = -(\mu^1+\mu^2)a = -\mu^1 a - \mu^2 a = -0-0 = 0$. So system (12) reduces to two equations, $\mu^{1,2}1 = 0$. Accordingly, we make $u$ two-dimensional; each $u^h$ is von-Neumann Morgenstern CRRA,

$$u^h(y) := \frac{\beta^h}{\beta^h-1} y_0^{\frac{\beta^h-1}{\beta^h}} + \Sigma_{s=1}^S \pi_s \frac{\beta^h}{\beta^h-1} y_s^{\frac{\beta^h-1}{\beta^h}}$$

and the probability $\pi$ on $1,...,S$ is the two-dimensional variable, since $S=3$.

The allocation and CRRA parameters determine the Bernoulli insurance deficit $\frac{\mu_s^h}{\pi_s}$, so that the system $\mu^{1,2}a = 0$ is a linear system in the probability $\pi$. Unless it is singular, the probability solution is unique.[6]

## 6.4 Estimates

The US data on the US income distribution by terciles (normalized so that $r_0 = 1$) for the year 2004 are:[7]

| income / ability h | money | market | disposable |
|---|---|---|---|
| high=1 | .654 | .692 | .609 |
| middle=2 | .254 | .244 | .275 |
| low=3 | .092 | .064 | .116 |

Income aggregates follows the definitions given by US Census. *Money income* includes all money income received by individuals who are 15 years or older. It consists of income before deductions for taxes and other expenses and does not include lump-sum payments or capital gains. It also does not include the value of noncash benefits such as food stamps. This income concept is the basis for the official poverty measure. *Market income* includes money income except government cash transfers, imputed realized capital gains and losses, and an imputed rate of return on home equity. It subtracts imputed work expenses. Market income is mainly used as a starting point for examining the effect of government activity on income and poverty estimates. *Disposable income* includes money income; includes the value of noncash transfers (food stamps, public or subsidized housing, and free or reduced-price school lunches); it includes imputed realized capital gains and losses, and an imputed rate of return on home equity; it subtracts imputed work expenses, federal payroll taxes, federal and state income taxes, and property taxes on owner-occupied homes. A comparison of market income and disposable income captures the net impact of government transfers and taxes on income and poverty estimates.

Since households' disposable income is the one that better capture the pattern of final consumption allocation, we use its data to specify the date 0 consumption allocations, $x_0^{h=1,2,3} = (.609, .275, .116)$. Moreover, we let the CRRA parameters be empirically reasonable: $\beta^{1,2,3} = 1.3, 2.3, 3.3$, Kocherlakota (1996). Since resources expand and contract by $f = .04$, $r_0 = 1$, $r_{\mathbf{1}} = (1.04, 0, 1.04)$. Calibration yields a common probability $\pi = (.968, .017, .015)$, and an equilibrium interest rate of 3%. As anticipated, the inefficiency

[6] The Mathematica code is available on request.

[7] Terziles are computed using the data of aggregate household income distribution by quintile, US CENSUS (US income data table 2 - 1/26/06).

of this economy at the specified allocation is $\rho_{US} = 0.0035$; households would pay between three and four tenths of one percent of current resources, at most, for a Pareto improving reallocation of future resources. This measure is tiny because in our model the economy starts out at an allocation that is already closed to a Pareto optimum. Even accepting the idea that the model is representative of the real world, one may still argue that this measures is tiny because the allocation already benefit from government transfers, and income taxes. Fixed probabilities, these policies smooth the income distribution in every state, therefore reducing the idiosyncratic risk of moving to a less favorable income tercile. A more correct procedure should be that of measuring the "ex-ante" inefficiency, by using consumption allocations that would arise in absence of social insurance, or redistributive, policies. With such data, households consumption would be reduced by the costs of all those market activities that agents carry out to self-insure. [8] How significant is the effect of (at least) some of these policies can be estimated by comparing $\rho_{US}$ with the inefficiency computed using market income at the same probabilities, $\rho_{US}^M$. In our example, $\rho_{US}^M = 0.0055$, which is roughly 71% larger than $\rho_{US}$. Again, the absolute measure of inefficiency at market income is small, but in relative terms it is significantly different. Clearly, this is not it, since with our data $\rho_{US}^M$ and the relative measure capture the willingness to pay for only some of the policy interventions implemented and specifically addressed to households.

Continuing the US example, we can end this section illustrating how the upper bound of $\rho_{US}$, $R_{US}$, is computed. Here the household with the highest marginal rate of substitution $\nabla_s^h$ in state $1,2,3$ is $h = 3,1,2$,

$\left(\nabla_s^h\right) =$ (13)

| s \ h | 1 | 2 | 3 |
|---|---|---|---|
| 1 | .9401 | .9522 | .9571 |
| 2 | .0307 | .0119 | .0168 |
| 3 | .0143 | .0210 | .0112 |

so proposition 3 implies the upper bound is

$$\begin{aligned} R_{US} \quad : \quad &= \frac{\Sigma_s \Sigma_h (\nabla_s^{i(s)} - \nabla_s^h) x_s^h}{\Sigma_h x_0^h} \\ &= \left[\nabla_1^3 r_1 - \Sigma_h \nabla_1^h x_1^h\right] + \left[\nabla_2^1 r_2 - \Sigma_h \nabla_2^h x_2^h\right] + \left[\nabla_3^2 r_3 - \Sigma_h \nabla_3^h x_3^h\right] \\ &= \left[\nabla_1^3 - \Sigma_h \nabla_1^h x_0^h\right](1.04) + \left[\nabla_2^1 - \Sigma_h \nabla_2^h x_0^h\right] + \left[\nabla_3^2 - \Sigma_h \nabla_3^h x_3^h\right](1.04) \\ &= 0.031 \end{aligned}$$

which is roughly eight times the inefficiency $\rho_{US}$, the 3.1% of date 0 aggregate resources.

[8] This point of distinguishing between ex-ante and ex-post measures has been recently made by Kurz (2005), in the context of the debate on the relevance of stabilization policies and aggregate volatility.

# 7 Global parameterization of equilibria

First we intuit the parameterization, and only then formalize it.

With complete markets, a classical conclusion (Lange 1942) is that an $\overline{e}$-allocation $x \gg 0$ is an equilibrium allocation if and only if it maximizes

$$\Sigma \delta^h u^h(x^h) \qquad \text{subject to} \qquad \Sigma x^h = \overline{e}$$

with $\delta$ fixed by equation (14):

$$\boxed{\delta^h := \frac{t}{D_0 u^h(x^h)}} \tag{14}$$

where $t > 0$ is fixed. Thanks to concavity and linear constraints, maximization amounts to solving the equations

$$\delta^h D_{\mathbf{1}} u^h - \rho' = 0 \tag{15}$$

for some $\rho \in \mathbb{R}^S_{++}$, which (14) reveals to be $t\nabla^h - \rho' = 0$ and which averaged becomes $\rho' = t\overline{\nabla}$. So an $\overline{e}$-allocation $x \gg 0$ is an equilibrium allocation if and only if makes zero the differences

$$\boxed{\mu^h := t(\nabla^h - \overline{\nabla})} \tag{16}$$

With incomplete markets, in contrast, $\mu^h$ is possibly nonzero. Nonetheless, given a possibly nonzero $\mu$, we can rewrite equation (16) as

$$(t\nabla^h - \mu^h) - t\overline{\nabla} = 0$$

and paraphrase it as

$$(\delta^h D_{\mathbf{1}} u^h - \mu^h) - \rho' = 0 \tag{17}$$

for some $\rho \in \mathbb{R}^S_{++}$. Comparing equations (15), (17), a natural conjecture is whether an $\overline{e}$-allocation $x \gg 0$ is an equilibrium allocation if and only if it maximizes

$$\Sigma(\delta^h u^h(x^h) - \overline{\mu}^h x^h) \qquad \text{subject to} \qquad \Sigma x^h = \overline{e}$$

with $\delta$ fixed by equation (14) still, and $\mu$ by some others, where $\overline{\mu}^h := (0, \mu^h)$. We prove this conjecture, when $\mu$ is constrained according to point 11:

$$\mu \in \left(a^{\perp}\right)^H \quad \text{and} \quad \Sigma \mu^h = 0$$

Finally, we note the conjecture is true for some $t > 0$ if and only if it is true for any $t > 0$. We think of $t$ as a normalization of $\delta$, the most convenient being $t = 1$ for computation and amounting to $\Sigma \frac{1}{\delta^h} = \Sigma D_0 u^h$, and $t = \Sigma D_0 u^h$ for theory and amounting to $\Sigma \frac{1}{\delta^h} = 1$. (The latter has the virtue of making $\delta^h$ invariant to a common differentiably increasing transformation of utilities, as is consumption, hence making equation (14) invariant too.)

Formalizing this intuition[9] leads to the following, proved in the appendix:

[9] It is easy to extend our parameterization to multiple commodities per state, by reinterpreting our single commodity as the value of multiple commodity endowments.

**Theorem 1** *The no-trade* $\overline{e}$*-equilibria* $\mathbb{T}(\overline{e})$ *are globally parameterized via (14), (16) by the product of*

$$\mathbb{D} := \left\{ \delta \in \mathbb{R}^H_{++} \mid \Sigma \frac{1}{\delta^h} = 1 \right\} \qquad \mathbb{U} := \left\{ \mu \in \left(a^{\perp}\right)^H \mid \Sigma \mu^h = 0 \right\}$$

Recall, our goal is to show $\mu$ is small, with $\mu = \mu(\widehat{x})$ ranging as a function of allocations $\widehat{x} = x(q, e)$ arising from equilibria $(q, e) \in \mathbb{E}$. Since $\widehat{x} = x(q, \widehat{x})$ and $(q, \widehat{x}) \in \mathbb{T}$, this is the same as $\nabla = \nabla(\widehat{x})$ ranging as a function of allocations $\widehat{x}$ arising from no-trade equilibria $(q, e) \in \mathbb{T}$.

# 8 Appendix

## 8.1 Global parameterization of equilibria

The global parameters are the Cartesian product of

$$\begin{aligned} \mathbb{D} \; &: \; = \left\{ \delta \in \mathbb{R}^H_{++} \mid \Sigma \frac{1}{\delta^h} = 1 \right\} \; : \text{dimension } \; H-1 \\ \mathbb{U} \; &: \; = \left\{ \mu \in \left(a^{\perp}\right)^H \mid \Sigma \mu^h = 0 \right\} \; : \text{dimension } \; (H-1)(S-J) \end{aligned}$$

**Theorem 2** $\mathbb{T}(\overline{e})$ *is a smooth* $(H-1)(S-J+1)$*-manifold diffeomorphic to* $\mathbb{D} \times \mathbb{U}$ *via (14), (16)*

**Corollary 5** $\mathbb{E}(\overline{e})$ *is a* $(H-1)J-$*vector bundle on* $\mathbb{T}$, *hence a smooth* $(H-1)(S+1)-$*manifold.*

Starting from the well known fact[10] that

**Proposition 6** $\mathbb{E}$ *is a smooth manifold.*

our argument applies the very useful

**Lemma 2 (3.2.1 in Balasko (1988))** *Let* $\phi : X \to Y, \psi : Y \to X$ *be smooth maps between smooth manifolds making* $\phi \circ \psi$ *the identity. Then* $\psi(Y)$ *is a smooth submanifold of* $X$ *diffeomorphic to* $Y$.

where

$$\begin{aligned} X &\text{ is } \mathbb{E} \\ Y &\text{ is } \mathbb{D} \times \mathbb{U} \end{aligned}$$

The maps are the following. $\phi : \mathbb{E} \to \mathbb{D} \times \mathbb{U}$ is

$$\phi(q, e) = \begin{bmatrix} ..., \frac{\Sigma D_0 u^i}{D_0 u^h}, ... \\ ..., (\Sigma D_0 u^i)(\nabla^h - \nabla^1), ... \end{bmatrix} \tag{18}$$

evaluated at consumptions $e + W\theta(q, e)$. $\psi : \mathbb{D} \times \mathbb{U} \to \mathbb{E}$ is

$$\psi(\delta, \mu) = \left(\nabla^1 a, x\right) \tag{19}$$

where $\nabla^1$ is evaluated at $x^1, x := \arg\max_{x \in \overline{E}} \Sigma(\delta^h u^h(x^h) - \overline{\mu}^h x^h)$, and $\overline{\mu}^h := (0, \mu^h)$.[11]

[10] Geanakoplos and Polemarchakis (1986), section 6.

[11] $\overline{\Omega}$ is the closure of $\Omega$.

**Lemma 3** *$\phi$ and $\psi$ are well defined and satisfy the hypothesis in lemma 2.*

**Proof.** $\boxed{\phi \text{ is well-defined}}$, i.e. $\phi(q,e) \in \mathbb{D} \times \mathbb{U}$. Clearly $\left(..., \frac{\Sigma D_0 u^i}{D_0 u^h}, ...\right) \in \mathbb{D}$; since equilibrium trades are optimal, $\nabla^h a = q$, so subtracting $\left(\nabla^h - \nabla^1\right) a = q - q = 0$ or $\left(\Sigma D_0 u^i\right)\left(\nabla^h - \nabla^1\right) \in a^{\perp}$.

$\boxed{\psi \text{ is well-defined}}$, i.e. $\psi(\delta,\mu)$ exists, is unique, and in $\mathbb{E}$. $x$ exists because its objective is continuous and $\overline{E}$ compact. By boundary aversion in assumption 2, $x \gg 0$; moreover, it is unique because assumption 2 implies that the objective is strictly concave in the interior. We now show $\left(\nabla^1 a, x\right) \in \mathbb{E}$, by showing $\boxed{\left(\nabla^1 a, x\right) \in \mathbb{T}}$, i.e. $\theta^h\left(\nabla^1 a, x^h\right) = 0$. By $(\theta)$, we must show

$$\nabla^h a - \nabla^1 a = 0 \qquad (*)$$

while evaluating $\nabla^h$ at $x^h + W0 = x^h$. By Kuhn-Tucker, there exists $\rho_+ = (\rho_0, \rho) \in \mathbb{R}^{S+1}$ such that $x \gg 0$ maximizes

$$\Sigma(\delta^h u^h(x^h) - \overline{\mu}^h x^h) - \rho_+' \Sigma x^h$$

So

$$\delta^h D u^h - \overline{\mu}^h = \rho_+' \qquad (**)$$

This says about state 0 that

$$\delta^h = \frac{\rho_0}{D_0 u^h} = \frac{\Sigma D_0 u^i}{D_0 u^h} \qquad (20)$$

where $\delta \in \mathbb{D}$ implies $\rho_0 = \Sigma D_0 u^i$; and about states $\mathbf{1} = \{1, ..., S\}$ that

$$\mu^h = \delta^h D u^h - \rho' = \left(\Sigma D_0 u^i\right) \nabla^h - \rho'$$

Taking the average and using $\mu \in \mathbb{U}$ reveals $0 = \frac{1}{H}\Sigma\mu^h = \left(\Sigma D_0 u^i\right)\overline{\nabla} - \rho'$ or that $\rho' = \left(\Sigma D_0 u^i\right)\overline{\nabla}$, so that

$$\mu^h = \left(\Sigma D_0 u^i\right)\left(\nabla^h - \overline{\nabla}\right) \qquad (21)$$

Since $\mu \in \mathbb{U}$ and $\Sigma D_0 u^i \neq 0$, equation (21) verifies (*).

$\boxed{\phi \circ \psi = id}$ Given $(\delta,\mu)$, define $(q,x) := \psi(\delta,\mu)$; we want $\phi(q,x) = (\delta,\mu)$. We have seen that $(\delta,\mu)$ is the right side of (20), (21), evaluated at this $x$. Also, the right side of (20), (21), evaluated at $x + W\theta(q,x)$ is the definition of $\phi(q,x)$, for any $(q,x)$. So the right sides agree, and therefore $(\delta,\mu) = \phi(q,x)$, if both consumption agree, i.e. if $\phi(q,x) = 0$, which holds since $(q,x) = \psi(\delta,\mu) \in \mathbb{T}$, as just argued above.

$\boxed{\text{Smoothness}}$ That $\phi$ is $C^1$ follows from its definition and that $u^h$ is $C^2$, $\theta^h$ is $C^1$. That $\psi$ is $C^1$ follows from the implicit function theorem by a standard argument that we omit. ■

Now we provide the theorem's

**Proof.** Lemma 3 verifies the hypothesis of lemma 2, which concludes $im\psi$ is a smooth submanifold of $\mathbb{E}$ diffeomorphic to $\mathbb{D} \times \mathbb{U}$. It remains to show $im\psi = \mathbb{T}$. The proof of lemma 3 shows $im\psi \subset \mathbb{T}$ (where $\psi$ is shown well-defined), so we show $\mathbb{T} \subset im\psi$, by showing $id_{\mathbb{T}} = \psi \circ \phi\,|_{\mathbb{T}}$. Fix $(q,e) \in \mathbb{T}$. Write $(\delta,\mu) := \phi(q,e)$, which is (18) evaluated at the no-trade consumption $e + W0 = e$, so that $q = \nabla^1(e^1)a$ by equation $(\theta)$. If $e = \arg\max_{x \in \overline{E}} \Sigma(\delta^h u^h(x^h) - \overline{\mu}^h x^h)$, then by definition $\psi(\delta,\mu) = (q,e)$, as desired. To verify $e$ is a maximand, rearrange $(\delta, \mu) = \phi(q, e)$ to get (**). Defining $\rho'_+ = \left(\Sigma D_0 u^i\right)(1, \overline{\nabla}) \gg 0$, (**) reads

$$(\delta^h Du^h - \overline{\mu}^h) - \rho'_+ = 0$$

which are the necessary conditions for $e$ maximizing $\Sigma(\delta^h u^h(x^h) - \overline{\mu}^h x^h) - \rho'_+ \Sigma x^h$. Since the latter is concave and its maximum is interior, these conditions are sufficient. Since $e \in E$ (because asset markets clear), the easy half of Kuhn-Tucker implies that this $e$ is the constrained arg max in (19). ■

Finally, we provide the corollary's

**Proof.** The projection $\mathbb{E} \to \mathbb{T}, \pi(q, e) = (q, x(q, e))$ is well defined; its fibers $\pi^{-1}(q, x)$ are clearly

$$\pi^{-1}(q, x) = \{q\} \times \left\{e \in E : \forall h, e^h = x^h - \Delta^h \text{ for some } \Delta^h \in \langle W \rangle \right\}$$

So fibers are parameterized by an open set of $\Delta^{h>1}$ in $\langle W \rangle^{H-1}$ –here $e^1 = x^1 + \Sigma_{h>1} \Delta^h$–which is a convex set of dimension $(H-1)J$, depending smoothly on $(q, e)$. ■

## 8.2 Proof of proposition 2

**Proof.** Let $\tilde{y}$ be the greatest Pareto improvement and $p(\tilde{y}) = \Sigma p^h(\tilde{y})$ its size; it suffices to show $y := ((\tilde{y}_0^h - p^h, \tilde{y}_{\mathbf{1}}^h))_h$ is an optimal arbitrage, for then

$$\rho_x =_{opt.arb.} 1 - \frac{\Sigma y_0^h}{r_0} =_{def} 1 - \frac{\Sigma \tilde{y}_0^h - \Sigma p^h}{r_0} = 1 - \frac{r_0 - \Sigma p^h}{r_0} = \frac{p(\tilde{y})}{r_0}$$

we must show $p(\tilde{y}) = r_0 \rho_x$.

To see it is an optimal arbitrage, we apply the sufficiency part of proposition 1, which does not require $y \gg 0$ (but $y \gg 0$ is easy to show too): by continuity of utility $u^h(y^h) =_{def} u^h(\tilde{y}_0^h - p^h, \tilde{y}_{\mathbf{1}}^h) = u^h(x^h)$ is binding, and clearly $y$ must be Pareto optimal, or else by continuity again $\tilde{y}$ would not be the greatest Pareto improvement. ■

## 8.3 Proof of proposition 4

**Proof.** Define $y^h(t) := (y_0^h(t), y_{\mathbf{1}}^h(t))$ the linear combination of $x^h =: (x_0^h, x_{\mathbf{1}}^h)$ and $\tilde{y}^h := (\tilde{y}_0^h(y_{\mathbf{1}}^h), y_{\mathbf{1}}^h)$, $t \in [0, 1]$. By concavity, $u^h(y^h(t)) \geq (1-t)u^h(x^h) + tu^h(\tilde{y}^h) \geq_{assn.4} u^h(x^h)$. Therefore $\tilde{y}_0^h(y_{\mathbf{1}}^h(t)) \leq y_0^h(t)$; it follows that

$$\rho(x, y_{\mathbf{1}}(t)) \geq 1 - \frac{\Sigma y_0^h(t)}{r_0} = (1-t)(1 - \frac{\Sigma x_0^h}{r_0}) + t(1 - \frac{\Sigma \tilde{y}_0^h(y_{\mathbf{1}}^h)}{r_0}) = (1-t)(0) + t(1 - \frac{\Sigma \tilde{y}_0^h(y_{\mathbf{1}}^h)}{r_0}) = t\rho(x, y_{\mathbf{1}})$$

Under the addition assumption, $u^h(y^h(t)) > u^h(x^h)$, and by continuity in 0-consumption, $\tilde{y}_0^h(y_{\mathbf{1}}^h(t)) < y_0^h(t)$, so $\rho(x, y_{\mathbf{1}}(t)) > t\rho(x, y_{\mathbf{1}})$.

To establish concavity we can repeat the last argument for every pair of allocations. Equivalently, fix $\underline{t}, \overline{t} \in [0, 1]$, and define these allocations to be $y(\underline{t}), y(\overline{t})$; we want to show that for every $s \in [0, 1]$

$$* = \rho(y(\underline{t}), y_{\mathbf{1}}((1-s)\underline{t} + s\overline{t})) \geq (1-s)\ \rho(y(\underline{t}), y_{\mathbf{1}}(\underline{t})) + s\ \rho(y(\underline{t}), y_{\mathbf{1}}(\overline{t})) = **.$$

Since $u$ is concave, $u^h(y^h(t)) \geq (1-t)u^h(x^h) + tu^h(y^h) \geq_{assn.4} u^h(x^h)$ for every $t \in [0,1]$; and $u^h((1-s)y^h(\underline{t}) + sy^h(\bar{t})) \geq (1-s)u^h(y^h(\underline{t})) + su^h(y^h(\bar{t})) \geq_{assn.4} u^h(x^h)$. Therefore, as above,

$$\tilde{y}_0^h(y_{\mathbf{1}}^h((1-s)\underline{t} + s\bar{t})) \leq (1-s)\tilde{y}_0^h(y_{\mathbf{1}}(\underline{t})) + s\tilde{y}_0^h(y_{\mathbf{1}}(\bar{t}))$$

implying,

$$* \geq 1 - \frac{\Sigma((1-s)\tilde{y}_0^h(y_{\mathbf{1}}(\underline{t})) + s\tilde{y}_0^h(y_{\mathbf{1}}(\bar{t}))}{r_0} = (1-s)(1 - \frac{\Sigma\tilde{y}_0^h(y_{\mathbf{1}}(\underline{t}))}{r_0}) + s(1 - \frac{\Sigma\tilde{y}_0^h(y_{\mathbf{1}}(\bar{t}))}{r_0}) = **$$

■

## 8.4 Proof of proposition 5

**Proof.** By proposition 1, $u^h(y^h(\lambda)) = u^h(x^h(\lambda))$ is an identity. Differentiating it at $\lambda = 0$,

$$D_0 u^h(y^h)\dot{y}_0^h + D_{\mathbf{1}} u^h(y^h)\dot{y}_{\mathbf{1}}^h = Du^h(x^h)\dot{x}^h = Du^h(x^h)z^h$$

Dividing by $D_0 u^h(y^h)$, the $\nabla^h(y^h) = \frac{D_{\mathbf{1}} u^h(y^h)}{D_0 u^h(y^h)}$ is a common $\nabla$ for all $h$, since, by proposition 1, $y$ is Pareto optimal. Summing over households,

$$\Sigma\dot{y}_0^h + \nabla\left(\Sigma\dot{y}_{\mathbf{1}}^h\right) = \frac{Du^h(x^h)z^h}{D_0 u^h(y^h)}$$

Note $\Sigma\dot{y}_{\mathbf{1}}^h = 0$ : For $\Sigma y_{\mathbf{1}}^h(\lambda) = \Sigma x_{\mathbf{1}}^h(\lambda) = \Sigma x_{\mathbf{1}}^h + \Sigma z_{\mathbf{1}}^h = \Sigma x_{\mathbf{1}}^h$, by the assumption that $\Sigma z_{\mathbf{1}}^h = 0$; differentiating, $\Sigma\dot{y}_{\mathbf{1}}^h = 0$. So the above reduces to

$$\Sigma\dot{y}_0^h = \Sigma\frac{Du^h(x^h)z^h}{D_0 u^h(y^h)}$$

Finally, the identity $\rho(x(\lambda)) = 1 - \frac{\Sigma y_0^h(\lambda)}{\Sigma x_0^h(\lambda)} = 1 - \frac{\Sigma y_0^h(\lambda)}{r_0}$ holds (since $\Sigma z_0^h = 0$), showing the derivative of $\rho(x(\lambda))$ exists and equals the claim. ■

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