Endogenous Market Power*

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Job Market Paper
This draft: November 2005

Abstract

In this paper, I propose a framework for studying market interactions in economies without assuming that traders are price-takers or Nash players. The model endogenously derives market power from primitives: endowments, preferences, and cost functions. The novelty of the approach lies in the treatment of the off-equilibrium behavior of the economy. A trader correctly anticipates that any deviation from equilibrium quantity will be followed by a price change sufficient to encourage all other traders to absorb the deviation. This price response defines a downward sloping demand curve facing each trader, and traders take into account the ability to affect prices in equilibrum, but also when absorbing unilateral deviations. I show that equilibrium defined in this way exists in economies with smooth utility and cost functions, is generically locally unique and generically Pareto inefficient. The framework suggests that trader's market power depends positively on the convexity of preferences and cost functions of the trading partners. Consequently industries with nearly constant marginal costs are fairly competitive, even if the number of firms is small. In addition market power of different traders reinforce each other. The model predicts the following effects of non-competitive trading: the volume of trade is reduced and price bias can be positive or negative depending of the third derivatives.

Unlike the Marshallian approach, the framework makes it possible to determine market structure in a coherent way even if the number of operating firms is small. It also defines equilibrium prices and quantities when there are increasing returns to scale or a bilateral monopoly.

JEL clasification: D43, D52, L13, L14
Keywords: Market Power, Consistent Conjectures, Off-equilibrium Behavior, General Equilibrium.

*I am indebted to Truman Bewley, John Geanakoplos, and Donald Brown for their guidance and support. I would also like to thank Rudiger Bachmann, Jinhui Bai, Andres Caravajal, Herakles Polemarchakis, Sergio Turner, participants of the 1st Annual Caress-Cowels conference, and NSF/CEME conference at Berkeley and especially Marzena Rostek for helpful comments and suggestions. Of course all errors are my own.

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1 Introduction

The problem of the exchange of goods between rational traders is at the heart of economics. The study of this problem, originating with the work of Walras, and later refined by Fisher, Hicks, Samuelson, Arrow and Debreu provides a well-established methodology for analyzing market interactions in an exchange economy. The central concept of this approach is *competitive equilibrium,* in which it is assumed that individual traders cannot affect prices. Price taking behavior is often justified by the informal argument that the economy is so large that each individual trader is negligible and hence has no impact on price. This argument does not apply to economies in which the number of traders is small or when the traders are heterogeneous. A vast amount of empirical evidence shows that a significant fraction of total trade in financial markets is accounted for by a relatively small group of large institutional traders who *do* recognize their price impact[1]. The goal of this paper is to propose a general equilibrium framework that models economies so small or heterogeneous that it does not make sense to assume that all traders are price takers.

Economic thinking about market power has long been shaped by the game-theoretic tradition that started with the work of Cournot and Bertrand. Their models treat only some of the traders as strategic players, and therefore *a priori* assume away the market power of some of the market participants. My approach does not impose any *a priori* restriction on traders' price impacts, but derives them from preferences and endowments. Neither does it discriminate between consumers and producers or more generally buyers and sellers. In addition, the game-theoretic models are highly sensitive to the behavioral assumptions. Depending on the choice of a strategy space (prices, quantities or supply functions), such models predict outcomes with prices varying from competitive to monopolistic. The indeterminacy is associated with the fact that Nash equilibrium does not require that off-equilibrium behavior be rational. In my model, all traders are assumed to react optimally to prices in and *out* of equilibrium, which allows me to pin down market power for traders.

There are three building blocks in the proposed framework: the rational choice of a consumer and a firm with price impact; the off-equilibrium pricing mechanism; and the concept of equilibrium. I introduce these concepts now.

First, in the proposed framework, traders recognize the impact of their trades on prices and take this into consideration when maximizing preferences or profits. The novel feature of the consumer's problem is that due to the price impact, the set of all affordable bundles, a budget set, is strictly convex. More importantly, the budget set depends not only on observed prices but also on the demand chosen. Similarly, firms' iso-profit curves are strictly convex and depend on the selected trade. Because of such endogeneity, I cannot use Walrasian choice theory. In its stead, I propose an alternative concept of demand, *stable demand*, defined as an optimal choice of trade given the budget set or profit function defined by this choice.

The second and key ingredient of the model is the behavior of the economy after a unilateral deviation by a trader. The need for defining such behavior was recognized by Shapley and Shubik [1977] in their game-theoretic model of exchange. In order to use a market clearing condition as a principle determining disequilibrium prices, they let traders

[1] For the evidence on price impacts of institutional traders, see Kraus and Stoll [1972], Holthousen Leftwich and Mayers [1987, 1990], Hausman, Lo and McKinlay [1992], Chan and Lakonishok [1993, 1995] and Keim and Madhavan [1996].

make decisions about nominal rather than real levels of consumption. A drawback of their approach is that it requires that consumers choose how much to spend in nominal terms *before* they learn the prices, so that they make suboptimal decisions off equilibrium. In this paper, a deviation from equilibrium consumption is followed by a change of prices which is sufficient to encourage other households to absorb the excess supply (or demand). The off-equilibrium market response defines an inverse demand function for each trader and thus her market power.

Finally I redefine the concept of equilibrium so as to incorporate traders' beliefs about their market power. It can be summarized as follows: i) an equilibrium allocation is such that all markets clear; ii) agents chose optimally their trades, given their beliefs about the effects on prices; and iii) in their conjectures, traders correctly recognize these impacts, given that prices tend to clear the market off-equilibrium. The proposed concept of equilibrium is conceptually similar to a consistent conjectural variation equilibrium introduced by Bresnahan [1981] in a duopoly model[2] also studied by Perry [1982] and Boyer and Moreaux [1983], or to a rational conjectural equilibrium in Hart [1985]. In equilibrium from this paper, however, traders have conjectures about the price impacts—price changes as a function of off-equilibrium *deviations* rather then absolute levels of trade. In addition unlike in Bresnahan [1981] traders make consistent conjectures not about the response of one rational opponent who maximizes preferences, but about the outcome of the interactions in an economic system with many heterogenous traders. The concept of equilibrium used in this paper is defined for an abstract game in Weretka [2005c] where I call it a "perfect conjectural equilibrium". In the existing literature on conjectural equilibrium, the equilibria may not exist or be locally unique. Perfect conjectural equilibrium does always exist in oligopolistic settings with differentiable utility and convex cost functions, and it is generically determinate.

I use the model to study the following phenomena in a general equilibrium framework:

• The determinants of market power.

The model predicts that the price impact captured by the slope of a demand faced by a trader, is directly related to the convexity of the preferences or cost functions of the trading partners. The traders' price impacts mutually reinforce each other and they depend negatively on the number of traders. As do duopoly models with consistent conjectures, my model predicts outcomes that are more competitive than those of the Cournot model, and therefore it suggests that industries with very high Herfindahl-Hirschman Index (HHI) may be fairly competitive when the convexity of the technology is low.

• The effects of non-competitive trading on the allocation, prices and welfare.

Market power provides incentives strategically to reduce trade in order to affect price, so that non-competitive trading is associated with a smaller then competitive volume of trade. The model does not give sharp predictions about the effects of non-competitive trading on prices. The prices can be above, below or equal to the competitive ones, depending on the specification of the economy. It is shown, however, that in a pure exchange economy with identical utility functions, the sign of the price bias equals to the sign of the third derivative of the assumed utility functions. The welfare cost is not shared uniformly by the traders.

[2] The concept of conjectural variation was first introduced by Bowley [1924] in the model of the Cournot duopoly and Frish [1933]. The idea of *consistent* conjectural variation equilibrium was studied by Laitner [1980], Bresnahan [1981], Perry [1982], Kamien and Schwartz [1983] and Boyer and Moreaux [1983]. For a recent survey of the theory, see Charles Fiuguers at al [2004].

• The impact of ownership structure on the equilibrium outcome.

Contrary to the Walrasian model, maximizing profit production is not optimal from the perspective of each individual shareholder. This is because apart from receiving dividends, owners would like to use the firm as a device to affect market prices. In other words, having control over a firm is associated with the advantage of having additional market power. The complete definition of a firm must therefore include a description of how managerial decisions are made. In this paper I study the implications of two extreme types of ownership: sole proprietorship, and for-profit corporations.

• Endogenous market structures.

The Marshallian model of long run equilibrium assumes price taking behavior by all market participants. Consequently, that model is not suitable for studying market structures in economies with a small number of firms. My model does not define traders as competitive or monopolistic and consequently it nests various types of market interactions, and allows diverge market structures to arise endogenously within the same framework. The other advantage of the framework over the Marshallian approach is that it defines equilibrium prices and quantities when producers have increasing returns to scale and there is a bilateral monopoly.

A number of papers have studied market power in a general equilibrium framework. In pioneering work, Negishi [1960][3] introduced monopolistically competitive firms into a general equilibrium model. In his setup, the demands faced by producers are downward sloping but exogenous. Dixit and Stiglitz [1977] and Hahn [1977] made the demands endogenous by including competitive consumers in a simple model of monopolistic competition. These three models make *a priori* assumptions about the market power of the traders and assume that the consumers have no market power power. This criticism extends to models studied by Gabszewicz and Vial [1972], Roberts and Sonnenschein [1977] and Hart [1979]. Finally, the theoretical approach is most closely related to the literature on rational conjectures in a static framework, among others Hahn [1978], Bresnahan [1981], Perry [1983], and Hart [1983], [1985].

The rest of this paper is organized as follows. In Section 2, I define the economy and discuss its assumptions. In Section 4, I model the choices of a consumer and a firm with an exogenously given price impact. For this model, I first define a budget set and characterize its properties. Next, I analyze a choice of a rational consumer. In Section 5, I discuss a firm's rational choice. In Section 6, I endogenize price impacts, using the principle of off-equilibrium market clearing. Section 7 is the heart of the paper. In this section, I formally define a concept of an equilibrium in an exchange sub-economy and give the theoretical results on the existence, generic local uniqueness, (in)efficiency, convergence and testability of an equilibrium. Section 8 studies the effects of non-competitive trading on equilibrium allocation, prices and welfare in several economies and in Section 9 I discuss the determinants of market power. Section 10 shows how other models of non-competitive trade relate to the proposed framework and Section 12 discusses how tax systems can re-establish Pareto efficiency of the equilibrium allocation (analog of the Second Welfare Theorem).

I adopt the following notation: $x^i \in \mathbb{R}^L$ is an L dimensional column vector. The notation $x^i \geq 0$ means that each element of the vector is non-negative, $x^i > 0$ denotes a non-negative,

[3] A similar model of monopolistic competition can be found in Arrow and Hahn [1971].

non-zero vector, and $x^i >> 0$ is a vector with all strictly positive elements. $\mathbb{R}_+^L$ ($\mathbb{R}_{++}^L$) denotes the set of non-negative (strictly positive) $L-$vectors. For any smooth function $u : \mathbb{R}^L \to \mathbb{R}$, Du^i denotes the gradient (or Jacobian) of u, and D^2u^i is the Hessian of this function. The proofs of all results are in the Appendix.

2 A Private Ownership Economy

I study market interactions in a small group of traders that form a private ownership economy $\mathcal{E}$, defined as follows: $\mathcal{E}$ is a standard, 1 -period, L - good, exchange economy, composed of I consumers and J producers. $\mathcal{L}$ is the set of all types of traded commodities, $\mathcal{I}$ is the set of all consumers and $\mathcal{J}$ is a set of firms. Superscript $i \in \mathcal{I}$, e.g. x^i indicates that the variable refers to the i^{th} consumer, j^{th} superscript denotes variables of j^{th} firm and subscript $l \in \mathcal{L}$ refers to the l^{th} commodity. In particular, $x_l^i \in \mathbb{R}_+$ $\left(e_l^i \in \mathbb{R}_+\right)$ denotes a consumption (endowment) of the l^{th} good by i^{th} consumer, $x^i \in \mathbb{R}_+^L$ ($e^i \in \mathbb{R}_{++}^L$) is a consumption (endowment) vector of all goods, and $x \in \mathbb{R}_+^{L\times I}$ $\left(e \in \mathbb{R}_{++}^{L\times I}\right)$ is an allocation (initial allocation) of goods across consumers. Each consumer is endowed with the claims (shares) to the profits of the firms. Vector $\theta^i \in [0,1]^J$ denotes the share portfolio of consumer i, and $\theta = \left\{\theta^i\right\}_{i\in\mathcal{I}}$ specifies portfolios for all consumers. For consumer i, her trade is defined as a net demand $t_l^i \equiv x_l^i - e_l^i$, and for firm j, it is a negative of a supply $t^j = -y^j$. In the proposed framework, "markets" do not discriminate among the consumers and firms and both types of market participants are modelled as *traders.* Therefore it is convenient to introduce the following notation: The set of all traders in the economy is denoted by $\mathcal{N} \equiv \mathcal{I} \cup \mathcal{J}$, the number of traders is $N = I + J$ and the typical trader is indexed by $n = i, j$.

Consumer i has rational preferences over consumption of goods given by a utility function $u^i : \mathbb{R}_+^L \to \mathbb{R}$. The vector function u specifies utility functions for all consumers, $u = \left\{u^i\right\}_{i\in\mathcal{I}}$. Another commodity is a numeraire or a unit of account. If numeraire $m^i \in \mathbb{R}$ is spent optimally outside of $\mathcal{E}$, it gives a positive constant marginal utility[4]. In other words, the permanent income hypothesis holds. Without loss of generality, marginal utility of numeraire is normalized to 1, and trader's endowment of numeraire good is equal to zero[5].

Next I give the assumptions on $\mathcal{E}$:

A1) Household i maximizes a quasi-linear, separable[6] utility function

$$U^i\left(x^i, m^i\right) = u^i\left(x^i\right) + m^i = \sum_{l\in\mathcal{L}} u_l^i\left(x_l^i\right) + m^i. \qquad (1)$$

A2) Differentiability: u^i is twice continuously differentiable, for all $i \in \mathcal{I}$

A3) Strict monotonicity: $D_x u^i >> 0$, for all $x^i \in \mathbb{R}_{++}^L$ and $i \in \mathcal{I}$

[4] Numeraire consumption m^i can be positive or negative. Its negative value could be interpreted as obtaining the numeraire from selling goods outside of the economy and spending it in the economy.

[5] Roberts and Sonnenchein [1977] demonstrated that equilibrium with monopolistic firms may fail to exist for more general utility functions in the Cournot-Chamberlain-Walras model. This is also the case in the presented framework. The quasi-linearity assumption sets income effects to be equal to zero and hence guarantees a well-behaved demand functions.

[6] An example of a natural application of the framework with separable preferences is a trade under risk, with von Neumann-Morgenstern expected utility function. The model then becomes the financial one.

A4) Concavity: $D_x^2 u^i$ is negative definite for all $x^i \in \mathbb{R}^L_{++}$ and $i \in \mathcal{I}$

In the proposed model, the concept of equilibrium relies critically on the differentiability of the demands. Therefore I include a condition assuring the interiority of potential equilibria. The condition is as follows:

A5) Interiority: for any good $l \in \mathcal{L}$, and $i, i' \in \mathcal{I}$

$$\lim_{x_l^i \to 0} \frac{\partial u_l^i\left(x_l^i\right)}{\partial x_l} > \frac{\partial u_l^{i'}\left(e_l^{i'}\right)}{\partial x_l^{i'}}. \tag{2}$$

The interiority condition A5 gives a common lower bound for the marginal utility at zero. Note that such a condition is weaker than the standard Inada condition: $\lim_{x_l^i \to 0} \partial u_l^i\left(x_l^i\right)/\partial x_l = \infty$.

In this paper, I restrict attention to firms producing non-numeraire commodities using a numeraire as an input. (This assumption, along with the separability of the utility functions, is relaxed in a more technical paper, Weretka [2005a]). Vector $y_l^j \in \mathbb{R}^L_+$ denotes the level of production of good l, and the profit of this firm is given by π^j. A technology of producer j is given by a separable cost function

$$f^j\left(y^j\right) = \sum_{l \in \mathcal{L}} f_l^j\left(y_l^j\right), \tag{3}$$

where each $f_l^j\left(y_l^j\right)$ specifies the cost of production y_l^j in terms of a numeraire. Function f^j satisfies the following assumptions:

A6) Differentiability: f^j is twice continuously differentiable

A7) Zero inaction cost: $f_l^j(0) = 0$

A8) Strict monotonicity: $\partial f_l^j\left(y_l^j\right)/\partial y_l^j > 0$ (and $\lim_{y_l^j \to 0} \partial f_l^j\left(y_l^j\right)/\partial y_l^j = 0$)

A9) Convexity: $\partial^2 f_l^j()/\partial\left(y_l^j\right)^2 > 0$ (and $0 < \lim_{y_l^j \to 0} \partial^2 f_l^j\left(y_l^j\right)/\partial\left(y_l^j\right)^2 < \infty$

Vector function $f = \left\{f^j\right\}_{j \in \mathcal{J}}$ specifies the cost functions for all firms $j \in \mathcal{J}$. Formally, a private ownership economy $\mathcal{E}$ is given by

$$\mathcal{E} = (u, e, \theta, f), \tag{4}$$

where there are two or more traders, $N = I + J \geq 2$, with at least one consumer $I \geq 1$. An economy $\mathcal{E}$ satisfying assumptions A1-A9 is called *smooth and separable.* Now I introduce a concept of market power of a trader.

3 Market Power

Market power of a trader $n = i,\ j$, is defined as her ability to affect prices by varying the demand. In order to make this idea more precise, suppose that in equilibrium, a trader observes a non-numeraire price vector $\bar{p} \in \mathbb{R}^L_+$, while demanding $\bar{t}^n \in \mathbb{R}^L$. The price impact of trader n is a relation between the price change of non-numeraire goods, $\Delta p = p - \bar{p}$, and the trade deviation, $\Delta t^n = t^n - \bar{t}^n$. Formally this relation is denoted by $\Delta p^n\left(\Delta t^n\right)$, a map from $\mathbb{R}^L$ to $\mathbb{R}^L$. Provided that $\Delta p^n(t^n - \bar{t}^n) \cdot t^n$ is concave, I can restrict attention to models of price impact that are linear. I may do so, because even if traders face a nonlinear demand, the only information they use is the first-order derivative of this function evaluated at the equilibrium trade, which is, of course, the slope of a linearized demand[7]. Linear $\Delta p^n(\Delta t^n)$ can be written as

$$\Delta p^n(\Delta t^n) = \bar{M}^n \Delta t^n = \bar{M}^n(t^n - \bar{t}^n), \tag{5}$$

where $\bar{M}^n$ is an $L \times L$ *price impact matrix,* which will be positive definite and symmetric. By $\bar{M} = \{\bar{M}^n\}_{n \in \mathcal{N}}$, I denote the vector of price impact matrices for all traders, and $\bar{t} = \{\bar{t}^n\}_{n \in \mathcal{N}}$ are their equilibrium trades. A typical entry of $\bar{M}^n$, $\bar{M}^n_{k,l}$ is a change of the price of good k, caused by an increase of the demand for good l by one unit. When making decisions, traders take their price impact matrices $\bar{M}^n$ as given. With equilibrium values of $\bar{p}$, $\bar{t}^n$ and $\bar{M}^n$, trader n faces a linear inverse demand function

$$p_{\bar{p},\bar{t}^n,\bar{M}^n}(t^n) = \bar{p} + \bar{M}^n(t^n - \bar{t}^n). \tag{6}$$

Given the demand function $p^n_{\bar{p},\bar{t}^n,\bar{M}^n}(\cdot)$, the total value of non-numeraire trade t^n is equal to $p^n_{\bar{p},\bar{t}^n,\bar{M}^n}(t^n) \cdot t^n$. The derivative of $p^n_{\bar{p},\bar{t}^n,\bar{M}^n}(t^n) \cdot t^n$ says how much the cost of trade would increase if the demand went up marginally, or equivalently, what is the benefit in terms of a numeraire from the marginal reduction of a non-numeraire demand. Consequently, the derivative is a marginal revenue from selling the goods on a market. If $\bar{M}^i$ is symmetric, the vector of marginal revenues[8] is given by

$$\begin{aligned} MR(t^n) &\equiv D_{t^n}\left(p^n_{\bar{p},\bar{t}^n,\bar{M}^n}(t^n) \cdot t^n\right) \\ &= \bar{p} + \bar{M}^n(2t^n - \bar{t}^n). \end{aligned} \tag{7}$$

In particular, at the equilibrium trade $\bar{t}^n$, such vector becomes

$$MR(\bar{t}^n) = \bar{p} + \bar{M}^n \bar{t}^n. \tag{8}$$

(Observe that $\bar{t}^n$ is defined as a demand and hence $MR(\bar{t}^n)$ is increasing in $\bar{t}^n$.) It is convenient to define a trade, $\hat{t}^n$, for which marginal revenue is equal to zero, $MR(\hat{t}^n) = 0$. If $\bar{M}^n$ is positive definite, then solving (7) gives

$$\hat{t}^i = \frac{\left[\bar{t}^n - \left(\bar{M}^n\right)^{-1} \bar{p}\right]}{2}. \tag{9}$$

Trade $\hat{t}^n$ maximizes total revenue from the non-numeraire trade $p_{\bar{p},\bar{t}^n,\bar{M}^n}(\hat{t}^n) \cdot \hat{t}^n$ (or minimizes the total cost).

Informally the equilibrium in the economy is defined as a triple $(\bar{p}, \bar{t}, \bar{M})$ that satisfies the following four conditions: 1) for each consumer i, trade $\bar{t}^i$ maximizes utility, among all affordable trades, given the inverse demand function $p^i_{\bar{p},\bar{t}^i,\bar{M}^i}(\cdot)$; 2) for any firm j, its trade

[7] When $\Delta p^n(t^n - \bar{t}^n) \cdot t^n$ is not concave, then the linearization is not without loss of generality.

[8] I call it *marginal revenue* and the name *marginal cost* I reserve for the marginal cost of production of non-numeraire goods by firms.

maximizes profit, given $p^j_{\bar{p},\bar{t}^j,\bar{M}^j}(\cdot)$; 3) prices are such that all markets clear; and 4) the conjectured price impacts $\bar{M}$ are consistent with the *true* price impacts for all traders. In sections 4 and 5, I introduce a model of a rational choice of a consumer and a firm with market power.

4 Consumer's Choice

The section on consumer choice consists of two parts: In the first part I discuss the geometry and the properties of a budget set and in the second part I define a stable choice of a consumer on this set.

4.1 The Geometry of a Budget Set

In this section, all three elements of an equilibrium $(\bar{p}, \bar{t}, \bar{M})$ are considered as exogenous and predetermined. A budget set $B(\bar{p}, \bar{t}^i, \bar{M}^i)$ of consumer i is a set of all affordable trades t^i, given a downward sloping demand function (6), or alternatively, given $\bar{p}$, $\bar{t}^i$ and $\bar{M}^i$. The set is determined by the budget constraint

$$p^i_{\bar{p},\bar{t}^i,\bar{M}^i}\left(t^i\right) \cdot t^i + m^i \leq \theta^i \cdot \pi. \tag{10}$$

Inequality (10) implies that a non-numeraire demand is financed either by income from shares or a negative consumption of numeraire. The only departure from the Walrasian approach is that in (10), prices are not constant, but depend linearly on trade, and therefore the budget constraint is *quadratic* in t^i and *linear* in m^i. Given the strictly monotone preferences, the equilibrium consumption of numeraire is given by

$$\bar{m}^i = \theta^i \cdot \pi - \bar{t}^i \cdot \bar{p}. \tag{11}$$

There are three important trade points that satisfy (10) with equality, namely: the autarky trade, $(t^i, m^i) = (0, \theta^i \cdot \pi)$; the equilibrium trade, $(\bar{t}^i, \bar{m}^i)$; and what I call an *anchor* trade, $\left(-\left(\bar{M}^i\right)^{-1}\bar{p}, \bar{m}^i\right)$[9]. The anchor trade has the property that it does not depend on $\bar{t}^i$, and it will serve as an "anchor" for the family of budget sets defined by various $\bar{t}^i$, but fixed $\bar{p}$ and $\bar{M}^i$. Example 1 illustrates the properties of a budget set.

Example 1 *Suppose in an economy there are two non-numeraire goods, and equilibrium values of $\bar{p}$, $\bar{t}^i$ and $\bar{M}^i$ are*

$$\bar{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \bar{t}^i = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \bar{M}^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pi = 0. \tag{12}$$

In equilibrium, consumer i exchanges one unit of good 1 for one unit of good 2. With prices of both goods equal to one, such trade involves no monetary payment, $\bar{m}^i = -\bar{p} \cdot \bar{t}^i = 0$. The inverse demand, faced by i

$$p^i_{\bar{p},\bar{t}^i,\bar{M}^i}\left(t^i\right) = \begin{bmatrix} t^i_1 \\ t^i_2 + 2 \end{bmatrix}, \tag{13}$$

[9] To see it, observe that for symmetric and positive definite $\bar{M}^i$, the left hand side of a budget constraint (10) can be reduced to $\bar{p}\bar{t}^i + \bar{m}^i$ which, by definition of $\bar{m}^i$, is equal to $\theta^i \cdot \pi$.

implies the following budget constraint (see (10)*)*

$$(14) \qquad \left(t_1^i\right)^2 + \left(t_2^i + 1\right)^2 \leq 1 - m^i.$$

What is the set of all trades that i can afford, given zero consumption of numeraire ($m^i = \bar{m}^i = 0$)*? In a two dimensional space* $\left(t_1^i, t_2^i\right)$, *equation* (14) *defines a circle with the center* $(0, -1)$ *and the radius* $r = \sqrt{1 - m^i} = 1$ *(see Figure 1.a). The autarky point* $(0, 0)$*, the equilibrium trade* $(1, -1)$ *and the anchor trade* $(-1, -1)$ *are on the boundary of the circle. Unlike the Walrasian budget set, this set is strictly convex. This occurs for two reasons. First, by decreasing the consumption of good 1 (increasing market supply), i adversely affects the marginal revenue from this good. As a result, i receives less and less numeraire per each additional unit sold. Second, as the consumption/demand of the other good, t_2, goes up, its marginal cost also increases. This implies that fewer units of the second good can be purchased for each additional unit of numeraire. Both effects imply that, contrary to the Walras case, the slope of the budget constraint is not constant.*

At the trade $\hat{t}_1^i = 0$*, the marginal revenue from selling good 1 becomes zero, and the further increase of its supply results in a reduction of total revenue. Similarly, for* $\hat{t}_2^i = -1$*, the marginal revenue from selling good 2 is equal to zero. In fact,* $\hat{t}^i \equiv (0, -1)$*, the trade for which marginal revenue from selling each of the two goods is zero, is located exactly in between the equilibrium trade* $\bar{t}^i$ *and the anchor trade* $-\left(\bar{M}^i\right)^{-1} \bar{p} = (-1, -1)$*, and it is the center of the circle. The four quadrants of the circle are associated with different combinations of "signs" of marginal revenue, and the northeast quadrant is characterized by a positive marginal revenue for each of the goods.*

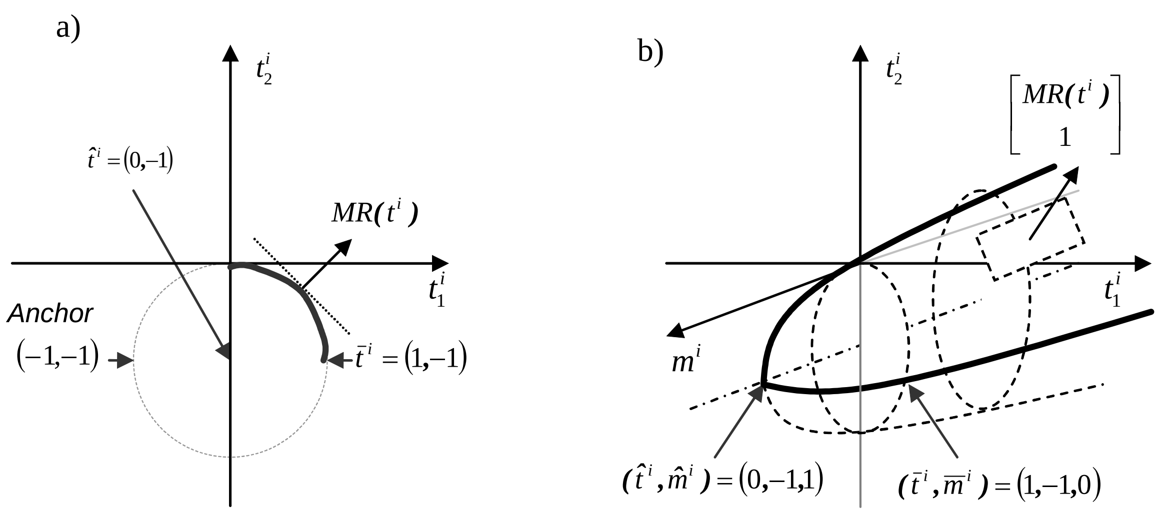

Figure 1. The budget set in (a) two and (b) three dimensional space.

How does the budget set look when $m^i \neq 0$*? Figure 1.b shows the set of all affordable trades in three dimensional space* $\left(t_1^i, t_2^i, m^i\right)$*. For an arbitrary but fixed level of numeraire,* m^i*, inequality* (14) *defines a circle with the center at* $(0, -1, m^i)$ *and the radius* $r = \sqrt{1 - m^i}$*. Note that r increases as m^i becomes more negative, but the rate of increase is decreasing. This shows that the budget set has the shape of a paraboloid around the axis* $(0, -1, m^i)$*.*

Interestingly, there exists no affordable trade t^i when $m^i > 1$. The intuition behind this fact is as follows: the total revenue (in terms of numeraire) is maximized when marginal revenues equal zero, which occurs when $\hat{t}^i = (0, -1)$ and $\hat{m}^i = 1$. Further increase in the supply of either of the goods reduces the received amount of numeraire. The trade $\left(\hat{t}^i, \hat{m}^i\right) = (0, -1, 1)$ is the base point of the budget set.

Now I discuss the properties of a budget set in a general case. A budget constraint is a quadratic form and therefore for any positive definite, diagonal matrix $\bar{M}^i$, a budget set is an elliptic paraboloid[10]: for any level of m^i, it is an ellipse with the center $\hat{t}^i$ and a semi-minor/semi-major radius $r_l = \sqrt{\left(\hat{m}^i - m^i\right) / \bar{M}^i_{l,l}}$, where $\bar{M}^i_{l,l}$ is the l^{th}diagonal entry of $\bar{M}^i$. (See Figure 2.) The maximal revenue from non-numeraire trade is attained at $\hat{t}^i$, which determines a base of the budget set. The numeraire consumption at this point is given by $\hat{m}^i \equiv -\hat{t}^i \cdot p^i_{\bar{p},\bar{t}^i,\bar{M}^i}\left(\hat{t}^i\right) + \theta^i \cdot \pi$, which is the maximal possible consumption of numeraire available to i. $\hat{t}^i$ is located between the average of the equilibrium trade $\bar{t}$ and the anchor trade $-\left(\bar{M}^i\right)^{-1}\bar{p}$, as in Figure 1a.

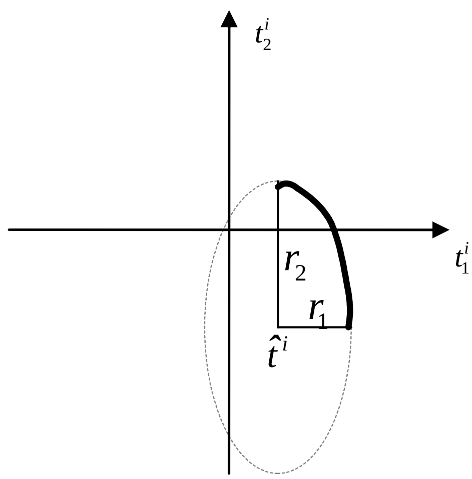

Figure 2. Elliptic budget set

In the non-competitive setting, the vector normal to the surface of a budget set determining the slope of a budget set is given by

$$\begin{pmatrix} MR\left(t^i\right) \\ 1 \end{pmatrix} \tag{15}$$

where the marginal revenue is defined as in (7). At the equilibrium trade $\left(\bar{t}^i, \bar{m}^i\right)$, the normal vector becomes

$$\begin{pmatrix} \bar{p} + \bar{M}^n \bar{t}^n \\ 1 \end{pmatrix}. \tag{16}$$

[10] In the case of more than two non-numeriare goods, it is the extension of an elliptic paraboloid to a higher dimensional space.

Note that at $\bar{t}^i$, the normal vector is different from $(\bar{p}, 1)$, the actual price at which goods are traded. Intuitively, a marginal change of a trade, Δt^i, raises the total spending directly by $\Delta t^i \cdot \bar{p}$. In addition, Δt^i changes prices by $\Delta p = \bar{M}^i \Delta t^i$, and therefore it increases the marginal spending on all goods that are traded by $\Delta p \cdot \bar{t}^i$. The constant total spending implies

$$\begin{aligned} 0 &= \Delta t^i \cdot \bar{p} + \Delta p \cdot t^i + \Delta m^i = \qquad (17) \\ &= \Delta t^i \cdot \bar{p} + \Delta t^i \cdot \bar{M} \bar{t}^i + \Delta m^i = \\ &= \begin{pmatrix} \Delta t^i \\ \Delta m^i \end{pmatrix} \cdot \begin{pmatrix} \bar{p} + \bar{M}^i \bar{t}^i \\ 1 \end{pmatrix}. \qquad (18) \end{aligned}$$

At the autarky trade $\bar{t}^i = 0$, the slope is given by

$$\begin{pmatrix} \bar{p} - \bar{M}^i \bar{t}^i \\ 1 \end{pmatrix}. \qquad (19)$$

In this case, the slope is different from $(\bar{p}, 1)$ because at the autarky trade, the prices are exactly equal to $\bar{p} - \bar{M}^i \bar{t}^i$. Finally, when the price impact is equal to zero, $\bar{M}^i = 0$, as in the Walrasian framework, the vector normal to the budget set coincides with the vector of prices for all trades.

Apart from its shape, the other feature of a budget set that distinguishes it from a Walrasian budget set is that it depends endogenously on the choice of trade. This point is made clear in the next section.

4.2 Stable Consumer's Choice (of Trade)

In this section, I define a *stable* non-numeraire trade of consumer i, $\bar{t}^i$, given observed values of $\bar{M}^i$ and $\bar{p}$. The notion of stability replaces the optimality condition in the Walrasian model. Intuitively, $\bar{t}^i$ is stable at $\bar{p}$ and $\bar{M}^i$, if $\bar{t}^i$, together with the implied consumption of numeraire, $\bar{m}^i = -\bar{t}^i \cdot \bar{p}$, maximizes utility among all trades that are affordable, given the inverse demand function $p^i_{\bar{p}, \bar{t}^i, \bar{M}^i}(\cdot)$.

Definition 1 *The consumer's trade $\bar{t}^i$ is stable at $\bar{p}$ and $\bar{M}^i$ if*

$$U(\bar{t}^i + e^i, \bar{m}^i) \geq U(t^i + e^i, m^i), \qquad (20)$$

$$\textit{for all } \left(t^i, m^i\right) \in B(\bar{p}, \bar{t}^i, \bar{M}^i).$$

It is theoretically possible that two trades exist, $\bar{t}^i$ and $\bar{t}'^i$ stable at $\bar{p}$, $\bar{M}^i$, and one of them is strictly better than the other. This might occur because the budget set $B(\bar{p}, \bar{t}^i, \bar{M}^i)$ is endogenous with respect to a stable trade $\bar{t}^i$, and $\bar{t}'^i$ does not necessarily have to belong to that set. In other words, the stable trade is not simply an optimal response to observed $\bar{p}$ and $\bar{M}^i$. Rather it is an optimal response to the observed triple $(\bar{p}, \bar{t}^i, \bar{M}^i)$, as *all three* elements are needed to determine affordable alternatives completely.

More formally, the stable trade can be interpreted as a fixed point of a procedure, mapping the set of non-numeraire trades into itself, $t'^i \to t''^i$, in the following way: Given $\bar{p}$ and $\bar{M}^i$, and hence $B(\bar{p}, t'^i, \bar{M}^i)$, find t''^i that maximizes the utility on that set (see Figure 3.a). Since t''^i defines a new budget set (see Figure 3.b), t''^i does not have to be stable, as there might be some better choice in $B(\bar{p}, t''^i, \bar{M}^i)$. In fact, t''^i is stable at $\bar{p}$ and $\bar{M}^i$, if and only if it is a fixed point of the mapping from $\bar{t}^i$ to t'^i. Because of the dependence of a budget set on an optimal choice of $\bar{t}^i$, neither existence nor uniqueness of $\bar{t}^i$ is guaranteed by the standard arguments that use the Weierstrass theorem and strict concavity.

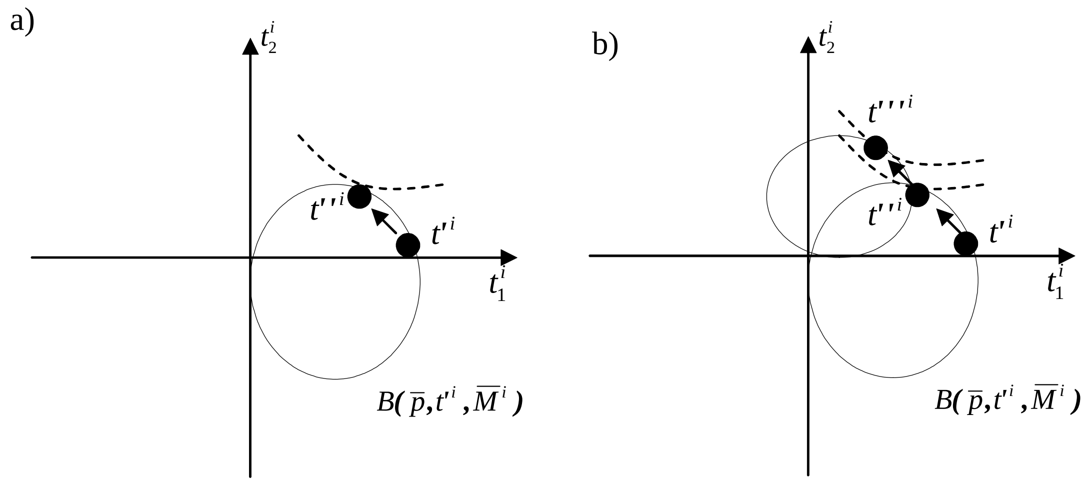

Figure 3. Optimal choice on (a) $B\left(\bar{p}, \bar{t}'^i, \bar{M}^i\right)$ and on (b) $B\left(\bar{p}, \bar{t}''^i, \bar{M}^i\right)$.

The next proposition establishes the existence and the uniqueness of a stable trade $\bar{t}^i$ in a smooth and separable economy.

Proposition 1 *In a smooth and separable economy, for any positive definite diagonal $\bar{M}^i$ and $\bar{p} \geq 0$, there exists a unique $\bar{t}^i$ that is stable at $\bar{p}$ and $\bar{M}^i$.*

The necessary condition for interior $\bar{t}^i$ to be optimal on $B(\bar{p}, \bar{t}, \bar{M}^i)$ is the tangency of a budget set with an upper contour set of the utility function at $\left(\bar{t}^i, \bar{m}^i\right)$. Since the marginal utility of numeraire and its price are always equal to one, the tangency condition reduces to

$$Du^i\left(\bar{t}^i + e^i\right) = MR\left(\bar{t}^i\right) = \bar{p} + \bar{M}^i\bar{t}^i. \tag{21}$$

Proposition 1 assures that the system (21) has a unique solution, and hence there exists a stable trade function $t^h\left(\bar{p}, \bar{M}\right)$ mapping vector of non-negative prices and strictly positive definite, diagonal matrices into stable trades. Corollary 1 characterizes the properties of this function.

Corollary 1 *In a smooth and separable economy, a stable trade function $\bar{t}^i\left(\bar{p}, \bar{M}^i\right)$ is continuous for all $\bar{p}$. This function is also differentiable for all $\bar{p}$ and $\bar{M}^i$, such that the trade is interior, that is $\bar{t}^i\left(\bar{p}, \bar{M}^i\right) >> -e^i$.*

Equation (21) implicitly defines a stable trade function for any interior trade. Its derivative with respect to price is equal to

$$D_p t^i\left(\bar{p}, \bar{M}^i\right) = -\left(\bar{M}^i - D^2 u^i\right)^{-1}. \tag{22}$$

The response of a non-numeraire stable demand to price can be decomposed into two effects: a substitution effect associated with the inverse of the Hessian of a utility function, and a strategic effect, a price impact of i on prices, $\bar{M}^i$.

Gossen's Law asserts that when the choice of the Walrasian consumer is optimal, the marginal rate of substitution must coincide with the price ratio. The tangency condition from (21) allows me to extend this result to a non-competitive setting. The *Generalized Gossen's Law* says that the marginal rate of substitution is equal to a ratio of marginal revenues:

$$MRS_{l,l'}\left(\bar{t}^i\right) = \frac{MR_l\left(\bar{t}^i\right)}{MR_{l'}\left(\bar{t}^i\right)}. \tag{23}$$

When traders do not have price impacts, and hence $\bar{M}^i = 0$, condition (23) collapses to standard Gossen's Law.

Another implication of condition (21) is that consumers with strictly monotone preferences chose demands for which marginal revenue is strictly positive (and equal to marginal utility). This implies that the relevant part of the budget set consists only of trades satisfying $t^i > \hat{t}^i$. In Figure 1, this part of the budget set is marked by a bold line. Consequently, consumers operate solely on the elastic part of the faced demand.

5 Firm's Choice

Modeling non-competitive firms in a general equilibrium framework is associated with two complications. The equilibrium outcome may vary depending on in which goods firms transfer profits to their shareholders. In addition, individual owners typically do not agree on an optimal production plan[11] for the firm. I discuss these complications in detail in the following paragraphs.

In a Walrasian equilibrium, price taking behavior assures that marginal production cost is equal to the marginal utilities of the consumers. When traders have market power, however, in equilibrium there is a wedge between the two values. If consumers are also shareholders, a gap creates natural incentives to transfer non-numeraire goods from a firm directly to its owners, and to bypass the markets. For example, shareholders of General Motors would benefit from receiving their dividends in cars rather than in dollars[12].

In addition, even if the dividends are paid solely in terms of numeraire, there does not exist one objective function for the firm that all shareholders would agree on. In particular the level of production maximizing profit does not coincide with the optimal production plan from the perspective of each individual shareholder. This is because apart from receiving dividends, each consumer is tempted to use the firm as a leverage to profitably affect market prices. In the General Motors example, shareholders benefit from increasing the supply of cars above the profit maximizing level if they buy cars. This is because at the production

[11] For example, this problem was recognized by Gabszewicz and Vial [1972], who studied monopolistic interactions in the Cournot-Chamberlain-Walras framework.

[12] Paying dividends in terms of non-numeraire goods (at a marginal cost) is to some extent similar to a monopoly price discrimination. The owners, apart from buying goods on the market, obtain them directly from the firm at the discounted price.

maximizing profit, increasing the supply only has a second order negative effect on profit, while the benefit from a lower price of a car is of the first order magnitude.

A complete characterization of a firm must specify which goods can be transferred to its owners as dividends and what the objective of a firm is. In the case of a *sole proprietorship,* the firm should be expected to maximize a utility of the owner and there should be no restrictions on transfers of non-numeraire commodities. The goal of (for profit) *corporations* (independent legal entities) is to make a profit. In addition, corporations do not transfer any other commodities to their owners but money.

5.1 Sole Proprietorship

A firm owned by one consumer can be modeled indirectly by modifying the preferences of an owner. Suppose the utility of a consumer-owner is $u^i(\cdot)$. The total amount of non-numeraire at the consumer's disposal is equal to

$$\bar{e}^i = e^i + y^i, \tag{24}$$

where e^i is the initial endowment and y^i is a production level. For any value $\bar{t}^i$, trader's utility is given by $u^i\left(\bar{t}^i + e^i + y^i\right)$. Since non-numeraire transfers between the owner and the firm are feasible, for any fixed $\bar{t}^i$ the optimal production plan assures the equality of marginal utility and marginal cost of production[13]. Otherwise, the owner would benefit from adjusting the production to match the two values, without affecting $\bar{t}^i$. The equality condition

$$Du^i\left(\bar{t}^i + e^i + y^i\right) = Df^i\left(y^i\right) \tag{25}$$

defines a smooth policy function $y\left(\bar{t}^i\right)$, which is an optimal level of production for arbitrary value of trade. The optimal production is negatively correlated with trade $\bar{t}^i$. This is because higher supply on the market (and hence lower t^i) is partially offset by an increased level of production. The value function defined as

$$\hat{u}^i\left(\bar{t}^i\right) = u\left(\bar{t}^i + e^i + y^i\left(\bar{t}^i\right)\right) \tag{26}$$

is increasing concave and smooth. If the assumptions A1-A9 are satisfied, the value function $\hat{u}^i\left(\bar{t}^i\right)$ is also additively separable. This shows that firms owned by individual consumers-owners can be modeled within the framework in a pure exchange economy, with consumers-owners' utility functions $u^i(\cdot)$ replaced by the value functions $\hat{u}^i\left(\bar{t}^i\right)$.

5.2 Corporations

In this paper I focus on incorporated firms. Such firms maximize profits in terms of numeraire and pay dividends exclusively in this good (money). Profit of incorporated firm j is given by

$$\pi^j_{\bar{p},\bar{t}^j,\bar{M}^j}\left(t^j, m^j\right) = -\left(p^j_{\bar{p},\bar{t}^j,\bar{M}^j}\left(t^j\right) t^j + m^j\right). \tag{27}$$

By convention, m^j is money demanded by firm j and trade t^j is a negative of the supply $t^j \equiv -y^j$, and hence the profit function has a minus sign. With the equilibrium values of

[13] The necessity of equality of the marginal utility and marginal cost for any value of trade is analogous to the cost minimization for any value of production.

$\bar{p}$, $\bar{t}^j$ and $\bar{M}^j$, equation (27) defines a map of iso-profit curves. In Example 2, I discuss the properties of such a map.

Example 2 *Consider the following equilibrium price, trade and price impact:*

$$\bar{p} = 1,\ \bar{t}^j = -1, \bar{m}^j = 0.5,\ \bar{M} = 1. \tag{28}$$

In equilibrium, a firm sells one unit of non-numeraire for one dollar, and hence its profit is $\bar{\pi}^j = 0.5$*. The inverse demand faced by a firm is given by*

$$p^j_{\bar{p},\bar{t}^j,\bar{M}^j}\left(t^j_l\right) = 2 + t^j_l = 2 - y^j_l, \tag{29}$$

and the profit function is given by

$$\pi^j_{\bar{p},\bar{t}^j,\bar{M}^j}\left(t^h, m^j\right) = -\left(2 + t^j_l\right) t^j_l - m^j. \tag{30}$$

The iso-profit curves are depicted in Figure 4 (in inverted space $\left(-m^j, -t^j\right)$*). For any value of profit* π^j*, equation* (30) *implicitly defines a corresponding iso-profit curve. In the Walrasian framework it is a straight line. With non-zero price impact, however, it is a parabola. The intuition is analogous to that behind the parabolic shape of a consumers budget set. The increased supply of a non-numeraire commodity adversely affects its price and therefore the marginal revenue is decreasing. At the trade* $\hat{t}^j_l = -1$*, marginal revenue becomes zero and the further increase of the supply reduces total revenue. The zero profit parabola passes through the inaction point* $(0, 0)$*, and separates all profitable trades (all trades to the right of it) from not-profitable ones (all trades to the left). Finally, equilibrium profit is determined by the intersection point of the equilibrium iso-profit parabola and the numeraire axes.*

The general properties of the iso-profit map are as follows. Similar to a consumer's budget constraint, profit function (27) is quadratic in t^j and linear in numeraire m^j, and therefore iso-profit curves are paraboloids. For any level of profit, the iso-profit surface bends backwards at the zero marginal revenue trade $\hat{t}^j$. The iso-profit paraboloid associated with no profit is always passing thorough the inaction point $(0, 0)$, and trades in the interior of this paraboloid are associated with a positive profit. The distance along the numeraire axes between $\bar{\pi}^j = 0$ manifold and any other iso-profit paraboloid determines the value of profit of the curve. The vector normal to any iso-profit curve is equal to a negative of a marginal revenue

$$-\begin{pmatrix} MR\left(t^j\right) \\ 1 \end{pmatrix}. \tag{31}$$

The iso-profit map depends on the equilibrium choice of trade. Consequently, as in the case of consumers, the Walrasian optimality condition must be replaced with the notion of stability.

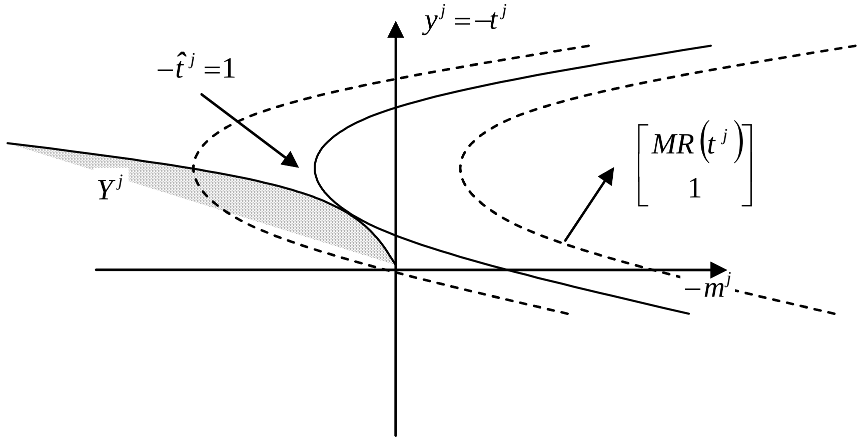

Figure 4. Iso–profit map and the firm's stable trade.

5.3 Stable Firms's Choice (of Trade)

A stable trade of a firm is defined in a similar way to the stable trade of a consumer. Trade $\bar{t}^j$ is stable at observed values of $\bar{M}^j$ and $\bar{p}$, if $\bar{t}^j$ together with the required level of numeraire input to produce it

$$\bar{m}^j \equiv f^j\left(-\bar{t}^j\right) \tag{32}$$

maximizes the profit among all trades technologically feasible, given the inverse demand function $p^j_{\bar{p},\bar{t}^j,\bar{M}^j}(\cdot)$.

Definition 2 *The firm's trade $\bar{t}^j$ is stable at $\bar{p}$ and $\bar{M}^j$, if*

$$\pi^j_{\bar{p},\bar{t}^j,\bar{M}^j}\left(\bar{t}^j, \bar{m}^j\right) \geq \pi^j_{\bar{p},\bar{t}^j,\bar{M}^j}\left(t^j, m^j\right), \tag{33}$$

for all $\left(t^j, m^j\right)$ such that

$$m^j \geq f^j\left(-t^j\right) \text{ and } t^j \leq 0. \tag{34}$$

The firm's objective function (27) is not fixed, as the demand function depends on a stable trade $\bar{t}^j$. The endogeneity of the objective function implies that $\bar{t}^j$ is defined as a fixed point rather than a simple maximand, and hence I need to give the results on its existence and uniqueness. The next proposition establishes the existence of a stable trade function $\bar{t}^j(\cdot)$ in a smooth and separable economy.

Proposition 2 *In a smooth and separable economy, for any positive definite diagonal $\bar{M}^j$ and $\bar{p} \geq 0$, there exists a unique trade function $\bar{t}^j\left(\bar{p}, \bar{M}^j\right) \leq 0$ that is stable. The function $\bar{t}^j\left(\bar{p}, \bar{M}^j\right)$ is differentiable for all $\bar{p}$ and $\bar{M}^j$ such that $\bar{t}^j\left(\bar{p}, \bar{M}^j\right) << 0$.*

At the stable trade a production set must be tangent to iso-profit curve and hence their normal vectors must be colinear. Formally the stable demand function is implicitly defined by the condition that the marginal revenue is equal to the marginal cost

$$Df^j\left(-\bar{t}^j\right) = MR\left(\bar{t}^j\right) = \bar{p} + \bar{M}^j\bar{t}^j, \tag{35}$$

and therefore the derivative of a stable demand function with respect to price is given by

$$D_p\bar{t}^j\left(\bar{p}, \bar{M}^j\right) = -\left(\bar{M}^j + D^2 f^j\right)^{-1}. \tag{36}$$

6 Consistency of $\bar{M}$

Suppose $(\bar{p}, \bar{t}, \bar{M})$ is such that all markets clear and $\bar{t}^n$ is stable at $\bar{p}$ and $\bar{M}^n$, for any $n = i$, j. In this section, I address the following question: What is a reasonable system of impact matrices $\bar{M}$, at $(\bar{p}, \bar{t})$? It is natural to assume that the market power conjectured by each n should coincide with the true response of market prices to off-equilibrium trade deviation of n. However, as Shapley and Shubik [1977] pointed out, the off-equilibrium behavior of the economy is not defined in competitive equilibrium theory. Here I propose the mechanism based on the premise that the markets clear both in and out of equilibrium. In particular, after any unilateral off-equilibrium deviation by trader n, prices adjust in order to encourage other traders to absorb the deviation and keep demand equal to supply. Such a price response to the deviation is considered here to be the true price impact of n. The remaining traders respond optimally to prices both in and out of equilibrium. I will now make precise the idea of disequilibrium market clearing.

The disequilibrium demand deviation by n is given by

$$\Delta t^n = t^n - \bar{t}^n. \tag{37}$$

If $\bar{t}^n$ is interior, then $t^i \geq -e^i$, or $t^j \leq 0$ is not binding for any t^n sufficiently close to $\bar{t}^n$, and trader n can consider a deviation in any possible direction.

It is convenient to assume that a fraction of $\rho^n \in [0,1]$ of the disequilibrium trade, Δt^n, is absorbed outside of the economy $\mathcal{E}$ at the prevailing prices and only $(1-\rho^n)\,\Delta t^n$ is purchased by the remaining traders within $\mathcal{E}$. Exogenous vector $\rho \in [0,1]^N$ specifies a level of external absorption for all traders. It allows us to map most of the exiting (non-)competitive models into the proposed framework by properly choosing respective values of ρ (see Section 10). For example, when $\rho = 1$, each trader may sell the whole Δt^n outside of $\mathcal{E}$ at the given prices, and hence this does not have any price impact. (This is the standard competitive Walrasian model.). When $\rho^n = 0$, for all n, then no good is even partially absorbed outside of the economy and any deviation by n stays in $\mathcal{E}$. It will be shown that none of the theoretical results depend on the introduction of ρ, except when there are only two traders, $N = 2$. Given ρ^n, the off-equilibrium market clearing condition for trader n is

$$(1-\rho^n)\,\Delta t^n + \bar{t}^n + \sum_{n' \neq n} t^{n'}\left(p, \bar{M}^{n'}\right) = 0, \tag{38}$$

where $\bar{t}^{n'}\left(p, \bar{M}^{n'}\right)$ is a stable trade function of trader $n' \in \mathcal{N}\setminus\{n\}$. Since in equilibrium markets clear, pair $p = \bar{p}$, and $\Delta t^n = 0$ satisfies equation (38). When for all $n' \neq n$ matrices

$\bar{M}^{n'}$ are positive definite, equation (38) defines a smooth price impact function $p^n(t^n)$ around $\bar{t}^n$. The Jacobian of $p^n(t^n)$ is given by

$$(39) \qquad D_t p^n|_{\bar{t}^n} = -(1-\rho^n)\left[\sum_{n\neq n'} D_p t^{n'}\left(\bar{p}, \bar{M}^{n'}\right)\right]^{-1} = $$
$$= (1-\rho^n)\left[\sum_{n'\neq n}\left(M^{n'}+V^{n'}\right)^{-1}\right]^{-1},$$

where V^n is a positive definite matrix given by

$$(40) \qquad V^i \equiv -D^2u^i \text{ and } V^j \equiv D^2 f^j,$$

specifying the convexity of preferences of a consumer or convexity of firm's cost function.

The key idea is that in equilibrium the inverse demand conjectured by each trader n coincides with the true price impact $p^n(t^n)$, up to a level approximation of order one. This requirement is called *$\rho-$consistency of $\bar{M}$ with disequilibrium market clearing.*

Definition 3 *Let $(\bar{p}, \bar{t}, \bar{M})$ be such that markets clear and $\bar{t}$ is stable at $\bar{p}, \bar{M}$. A system of impact matrices, $\bar{M}$, is said to be $\rho-$consistent with disequilibrium market clearing if for every $n \in \mathcal{N}$, $\bar{M}^n$ is a semi-positive definite matrix satisfying*

$$(41) \qquad \bar{M}^n = D_t p^n\left(\bar{t}^n\right).$$

Why in equation (39) does the price impact depend on the impact matrices of the other traders, $\bar{M}^{n'}$ for $n' \neq n$? With the substantial price impact, the remaining traders are reluctant to depart from their equilibrium demands and to absorb Δt^n, they must be compensated by significant price changes. In other words, prices respond more to the deviations by trader n and price impact is higher. I call such mutual interdependence of price impacts a *mutual reinforcement effect.*

Next I give a result establishing the existence of $\rho-$consistent $\bar{M}$ at some $\bar{t}$ in a smooth and separable economy .

Proposition 3 *In a smooth separable economy, at any interior $(\bar{p}, \bar{t})$ such that all markets clear and $\bar{t}^n$ is stable, there exists a system of $\rho-$consistent matrices $\bar{M}$ if and only if $N > 2$ or $\rho > 0$.*

Whenever there are more than two traders, the consistent system of matrices exists. For economies with two traders, $\bar{M}$ exists only if at least one of the traders has positive external absorption. The canonical examples of economies for which partial external absorption is necessary are standard Edgeworth Box economy ($I = 2$ and $J = 0$), and Robinson Crusoe economy ($I = 1$ and $J = 1$). The intuition behind the negative result in case of the economies with $N = 2$ and $\rho = 0$ is the following: Market power of n is met by the same power of the other trader, n'. If there is no external absorption of trade, this power in turn reinforces the price impact of n by exactly the same amount. The price impacts of both traders explode to infinity, so that the system (41) has no solution.

7 $\rho-$Competitive Equilibrium

Now that all the elements of an equilibrium have been introduced, I am in the position to define it formally, and to state the theoretical results characterizing it. I call this equilibrium a $\rho-$competitive equilibrium.

Definition 4 *A $\rho-$competitive equilibrium in the private ownership sub-economy $\mathcal{E}$ is a triple $(\bar{p}, \bar{t}, \bar{M})$ such that:*
1) for all n, $\bar{t}^n = \bar{t}^n\left(\bar{p}, \bar{M}^n\right)$;
2) all markets clear $\sum_{n\in\mathcal{N}} \bar{t}^n = 0$;
3) $\bar{M}$ is a system of $\rho-$consistent matrices.

I state five theoretical results regarding $\rho-$competitive equilibrium in a smooth and separable economy. (Recall that a smooth and separable economy is one that satisfies assumptions A1-A9.) I refer to $\rho-$competitive equilibrium as a *diagonal* equilibrium, if matrix $\bar{M}^n$ is diagonal for all n. The first two results are technical and are relevant for the computation of an equilibrium. The third result is a version of the first welfare theorem, while the fourth establishes the convergence to a unique Walrasian equilibrium in a large economy. The last result addresses the problem of whether a $\rho-$competitive equilibrium is testable in general, and it can be distinguished empirically from a Walrasian equilibrium.

A good economic model should be able to give the predictions for any economy. The first theorem establishes the existence of such equilibria.

Theorem 1 *In a smooth and separable economy, a diagonal $\rho-$competitive equilibrium exists if and only if $N > 2$ or $\rho > 0$.*

The assumptions of Theorem 1 are relatively weak. Whenever there are more than two traders, or when at least one of the two traders has some outside option of trade, a diagonal equilibrium exists. Only when the trade is restricted to two traders, as in the Edgeworth box economy or in the Robinson Crusoe economy, and the external absorption is zero for the two traders, does the $\rho-$competitive equilibrium breakdown.

A diagonal $\rho-$competitive equilibrium has two important properties. First, the equilibrium price vector is *always* strictly positive, $\bar{p} >> 0$. As shown in Weretka [2005a], this is not necessarily true when the preferences are not separable, even when they are strictly monotone. In addition, the equilibrium is homogenous of degree zero with respect to $\bar{p}$ and $\bar{M}$. This implies that prices and price impact matrices can be arbitrarily normalized without changing the real outcomes. Note, however, that homogeneity does not contradict the findings of Gabszewicz and Vial [1972], or more recently of Dierker and Grodal [1998], that the choice of a numeraire has severe real effects when the competition is not perfect. In their models the real effects result from the normalization of off-equilibrium prices relative to the equilibrium ones, and not normalization of the absolute level of prices in equilibrium.

The second desirable feature of the model is that its predictions are sharp. Technically the set of all equilibria should be small, possibly a singleton. It turns out that diagonal $\rho-$competitive equilibrium can be expressed as a solution to a system of non-linear equations, where the number of equations is exactly equal to the number of endogenous unknowns. If no equation is (locally) redundant, the equilibria are locally unique. Unfortunately, local "no-redundancy" cannot be guaranteed for any arbitrary economy $\mathcal{E}$. The next theorem formalizes the idea that economies for which the equilibria are not locally unique are not robust. By $\mathcal{E}_{\delta',\delta''}$, I denote an economy $\mathcal{E}$ perturbed by adding term $\delta'^i \cdot x^i + x^i \cdot \delta''^i x^i$ to the utility function and $\delta'^j \cdot y^j + y^j \cdot \delta''^j y^j$ to the cost function. Vector $(\delta', \delta'') \in R^{N\times 2L}$ denotes the perturbation for all traders and $\mathcal{P} \in R^{N\times 2L}$ is *an* open set of the perturbations that satisfies: each perturbations preserve monotonicity and convexity of preferences and cost functions on the set of trades possibly observed in equilibrium[14], has non-zero Lebesque measure in $R^{N\times 2L}$ and $0 \in \mathcal{P}$. Note that each element $(\delta', \delta'') \in \mathcal{P}$ defines some economy and hence $\mathcal{P}$ parameterizes a family of perturbed economies around $\mathcal{E}$.

Theorem 2 *There exists a subset $\hat{\mathcal{P}} \subset \mathcal{P}$ with full Lebesque measure of $\mathcal{P}$, such that for any $(\delta', \delta'') \in \hat{\mathcal{P}}$, all equilibria of the economy $\mathcal{E}_{\delta',\delta''}$ are locally unique.*

Theorem 2 can be interpreted in the following way. For any economy $\mathcal{E}$, including the ones for which the equilibria are not determinate, there exists an arbitrarily small perturbation of preferences and technology, such that in the perturbed economy all equilibria are locally unique. One of the implications of this theorem is that in typical economies the number of equilibria is finite. It should be noted that in the case of quadratic utility functions, for any ρ, the equilibrium is *globally* unique.

The third theorem is related to inefficiency of an equilibrium allocation. In a Walrasian framework, the first welfare theorem guarantees that the equilibrium allocation is Pareto efficient. Theorem 3 provides an analogous though opposite result for a diagonal $\rho-$competitive equilibrium.

Theorem 3 *In a smooth and separable economy, in a diagonal $\rho-$competitive equilibrium with $\rho << 1$, the allocation is Pareto efficient if and only if $J = 0$ and the initial allocation is Pareto efficient.*

It follows that when $\rho << 1$, in economies with firms the equilibrium allocations are always Pareto inefficient, and in a pure exchange economy the allocations are inefficient generically in endowments. The allocation inefficiency in the economy with two traders is demonstrated in the Edgeworth Box example in Figure 6 and in the Robinson Crusoe economy in Figure 7.

The next theorem asserts that although an equilibrium allocation is not Pareto efficient, in large economies the inefficiency is negligible, as a diagonal $\rho-$competitive equilibria converge to a singleton—the unique Walrasian equilibrium. To show this I take advantage of a construct of a $k-$replica economy.

Theorem 4 *For any smooth and separable economy, and any $\varepsilon > 0$, there exists $k^\varepsilon \in \{2, 3\ldots\}$ such that all diagonal $\rho-$competitive equilibria in a $k-$replica economy, for any $k \geq k^\varepsilon$, satisfy*

1) $\left\|\bar{M}^n(k^\varepsilon)\right\| \leq \varepsilon$, *for any n;*
2) $\left\|\bar{t}^n(k^\varepsilon) - \bar{t}^{n,Walras}\right\| \leq \varepsilon$, *for any n;*
3) $\left\|\bar{p}(k^\varepsilon) - \bar{p}^{Walras}\right\| \leq \varepsilon$.

[14] This set is defined in the appendix. Since the set of trades is compact, set Δ exists.

The next theorem indirectly addresses the question of testability of $\rho-$competitive equilibrium and the empirical distinction from the Walrasian equilibrium, given information about observables: trades, endowments and prices $(\bar{t}, e, \bar{p})$. As argued before, the Walrasian equilibrium in $\mathcal{E}$ is equivalent to a $1-$competitive equilibrium, where price impact matrices are $\bar{M}^n = 0$ for all n. This is because when $\rho = 1$, each trader $n \in \mathcal{N}$ may sell her goods outside of $\mathcal{E}$ at the prevailing price, so that all traders in $\mathcal{E}$ are price takers. In general, for $\rho < 1$, the relation between Walrasian and $\rho-$competitive equilibrium can be described in the following way.

Theorem 5 *The triple $(\bar{p}, \bar{t}, \bar{M})$ is a diagonal $\rho-$competitive equilibrium in $\mathcal{E} = (u, e, \theta, f)$ only if $(\bar{p}, \bar{t})$ is a Walrasian equilibrium in the economy with modified preferences and technology, $\tilde{\mathcal{E}} = \left(\hat{u}, e, \theta, \hat{f}\right)$, where*

$$\hat{u}^i\left(x^i\right) = u^i\left(x^i\right) - \frac{1}{2}\left(x^i - e^i\right) \cdot \bar{M}^i\left(x^i - e^i\right), \tag{42}$$

for any consumer $i \in \mathcal{I}$, and

$$\hat{f}^j(y^j) = f^j + \frac{1}{2} y^j \cdot \bar{M}^j y^j,$$

for any firm $j \in \mathcal{J}$.

Any $\rho-$competitive equilibrium allocation and price vector in an economy can be rationalized as a Walrasian equilibrium by some other economy. As a result, any observation $(\bar{t}, e, \bar{p})$ that refutes the Walrasian equilibrium also refutes a $\rho-$competitive one. At the same time, $\rho-$competitive equilibrium cannot be empirically distinguished from the competitive one, using only information about $(\bar{t}, e, \bar{p})$.

Note that the preferences rationalizing $\rho-$competitive equilibrium as a Walrasian one exhibit a peculiar property: each consumer has preferences biased toward its own endowment. Therefore when one restricts attention to a class of preferences that are independent from the endowments[15], the non-competitive hypothesis could be tested by verifying the significance of the initial endowment in explaining the non-numeraire consumption. I illustrate this claim with utility functions characterized by constant relative risk aversion:

$$U^i\left(x^i, m^i\right) = \sum_{l \in \mathcal{L}} \alpha_l \frac{\left(x_l^i\right)^{1-\theta}}{1-\theta} + \lambda^i m^i, \tag{43}$$

where λ^i is a normally distributed marginal value of wealth with $\ln \lambda^i \sim N(0, \sigma)$ and α_l is some positive scalar. From the first order conditions, a competitive demand for good l is given by

$$\ln \bar{x}_l^i = \beta_l^0 + \beta_l^1 \ln \bar{p}_l + \varepsilon^i, \tag{44}$$

[15] The independence of preferences from the endowments can be loosely interpreted as an *a priori* assumption, such that knowledge about endowments contains no information about utility functions, and each utility function is "equally likely" for any arbitrary endowment. In particular, independence excludes phenomena like habit formation and addiction.

where $\beta_l^0 = \frac{1}{\theta} \ln \alpha_l$ is a dummy coefficient on good l, $\beta_l^1 = \frac{1}{\theta}$ and ε^i is normally distributed with $N\left(0, \left(\frac{1}{\theta}\right)^2 \sigma\right)$. Consequently, if the market interactions are competitive, the endowments should have no explanatory power in the regression

$$\ln \bar{x}_l^i = \beta_l^0 + \beta_l^1 \ln \bar{p}_l + \beta_l^2 \ln e_l^i + \varepsilon^i, \tag{45}$$

and therefore hypothesis $H_0 : \beta_l^2 = 0$ can be associated with competitive interactions against $H_1 : \beta_l^2 > 0$, suggesting market power of the traders.

8 Model Predictions

In the following section, I examine the effects of a non-competitive trading on the equilibrium allocation, prices and welfare. First, I study a pure exchange economy with quadratic utility functions. The quadratic utility functions are particularly convenient to work with because their second derivatives do not depend on the level of equilibrium consumption, which allows me to calculate a closed form solution of the equilibrium. Some of the properties of the equilibrium, however, are not robust in the case of more general utility functions, therefore I supplement the analyses with numerical simulations for the economy with utility function characterized by constant relative risk aversion. Subsequently, I study the economies with oligopolistic firms.

8.1 Pure Exchange Economy with Quadratic Utilities.

Consider a pure exchange economy with L commodities and I consumers, each characterized by identical quadratic utility function,

$$U^i\left(x^i, m^j\right) = \sum_{l \in \mathcal{L}} u_l\left(x_l^i\right) + m^i = \sum_{l \in \mathcal{L}} \left(\left(x_l^i\right) - \frac{\bar{\gamma}_l}{2}\left(x_l^i\right)^2\right) + m^i. \tag{46}$$

The initial endowment is equal to $e^i >> 0$. The external absorption is identical for all traders and equal to $\rho^i = \bar{\rho}$. Theorem 1 implies that a $\rho-$competitive equilibrium exists if the number of traders is more than two, $I > 2$, or the external absorption is non-zero $\bar{\rho} > 0$, which is assumed.

8.1.1 Equilibrium Allocation

Given identical, strictly concave utility functions in the Walrasian equilibrium, all traders consume the same amounts of non-numeraire goods equal to the endowment per capita in the whole economy

$$\bar{x}^{Walras} = \frac{1}{I} \sum_i e^i = \bar{e}, \tag{47}$$

and the price of good l is equal to

$$\bar{p}_l^{Walras} = 1 - \bar{\gamma}_l \bar{e}_l.$$

Let me now compute $\rho-$competitive equilibrium. With quadratic utilities consistent price impacts are independent from the level of consumption. The consistency conditions on $\bar{M}$, (39) and (41) define the following systems of $I \times L$ equations and the same number of unknowns (off-diagonal entries of matrices are zeros),

$$\bar{M}^i_{l,l} = (1-\bar{\rho}) \left(\sum_{i' \neq i} \left(\bar{M}^{i'}_{l,l} + \bar{\gamma}_l \right)^{-1} \right)^{-1}. \tag{48}$$

In the economy with two traders and one non-numeraire good equation (48) reduces to

$$\bar{M}^i = (1-\bar{\rho}) \left(\bar{M}^{i'} + \bar{\gamma} \right). \tag{49}$$

(49) is a system of two linear equation and two unknowns $\bar{M}^1$ and $\bar{M}^2$. They are depicted in Figure 5 in the Euclidian space $\left(\bar{M}^1, \bar{M}^2\right)$. In the first equation, market power of the first consumer is increasing in the market power of the second one, which in return, by the second equation, reinforces back the price impact of the first one. The effect of mutual reinforcement of price impact was already discussed: the higher the price impact, the less elastic the demand with respect to the price and hence the greater the price change required to make the other consumers absorb the deviation. Provided that $\bar{\rho} > 0$, the slopes of both lines are smaller than one, and hence the two lines intersect so the considered system has a unique solution. When $\bar{\rho} = 0$, the two lines in Figure 5 become parallel and the system (49) has no solution. This is consistent with the negative result of Proposition 3.

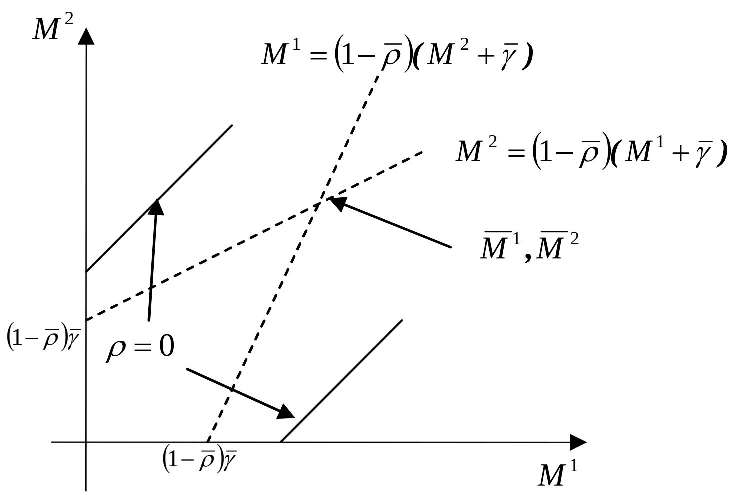

Figure 5. Determination of the price impacts

The solution to system (48) must satisfy[16] $\bar{M}^i = \bar{M}^{i'}$ for any i, i', therefore it is given by

$$\bar{M}^i_{l,l} = \frac{1-\bar{\rho}}{I+\bar{\rho}-2}\bar{\gamma}. \tag{50}$$

Price impact $\bar{M}^i$ is inversely related to external absorption $\bar{\rho}$ and to the number of traders. For $\bar{\rho} = 1$ it is equal to zero, $M^i = 0$, and for $\bar{\rho}$ approaching zero, the price impact is increasing to $1/(I-2)$.

The remaining elements of the equilibrium: the allocation and the price system can be found by solving the system of the first order stability conditions (21) and equilibrium market clearing condition. The equilibrium consumption is given by

$$\bar{x}^i = \frac{I+\bar{\rho}-2}{I-1}\bar{x}^{Walras} + \frac{1-\bar{\rho}}{I-1}e^i. \tag{51}$$

The consumption is a convex combination of the Walrasian consumption and the initial endowment, with weights $(I+\bar{\rho}-2)/(I-1)$ and $(1-\bar{\rho})/(I-1)$, respectively. The allocation becomes more autarkic when the external absorption $\bar{\rho}$ becomes smaller, or when the number of consumers in the economy goes down.

In the economy with identical quadratic preferences, the price vector $\bar{p}$ is the same for all values of $\bar{\rho}$. In particular, it is equal to the Walrasian one $\bar{p}^{Walras}$. As shown in Section 8.1.2, this is a knife-edge case, and this result is not robust with different utility functions.

8.1.2 Price Bias

To make the presentation more transparent, I first discuss the price bias mechanism in an economy with one non-numeraire good and two traders called a buyer and a seller. The buyer has an initial endowment e^b and the seller $e^s > e^b$. Both traders have quadratic utility functions as in (46) with possibly different convexity coefficients γ^i. For any of the consumers $i = b, s$ the consistent price impacts can be found by solving (39) to obtain

$$\bar{M}^i = \frac{(1-\bar{\rho})}{\bar{\rho}(2-\bar{\rho})}\left(\gamma^{i'} + (1-\bar{\rho})\gamma^i\right). \tag{52}$$

The price impact is increasing in the convexities of both consumers γ^i and $\gamma^{i'}$. The greater weight, however, is put on the convexity of the trading partner, and therefore $\gamma^i > \gamma^{i'}$ implies $M^{i'} > M^i$. Solving for a stable trade

$$\bar{t}^i\left(\bar{p}, \bar{M}^i\right) = \frac{1-\gamma^i e^i - \bar{p}}{\gamma^i + \bar{M}^i} = \frac{\gamma^i}{\gamma^i + \bar{M}^i}\frac{1-\gamma^i e^i - \bar{p}}{\gamma^i} = \frac{\gamma^i}{\gamma^i + \bar{M}^i}\bar{t}^i(\bar{p}, 0), \tag{53}$$

where function $\bar{t}^i(\bar{p}, 0)$ corresponds to a Walrasian demand.

[16] To see it, observe that for each i and l, $M^i_{l,l}$ is defined as a harmonic mean of $M^{i'}_{l,l} + \bar{\gamma}_l$ for all $i' \neq i$, multiplied by a scalar $(1-\bar{\rho})/(I-1)$, and as such it is increasing in the corresponding elements. If the consistent price impacts where not equal, it would be possible to find i and i' such that $M^{i'}_{l,l} > M^i_{l,l}$. Consequently, M^i would be a harmonic mean of elements $\left(M^{i'}_{l,l} + \bar{\gamma}\right)$ that are weakly greater than the ones defining $M^{i'}$ and one could conclude that $M^i_{l,l} \geq M^i_{l,l}$, which is a contradiction.

Suppose for the moment that $\gamma^b = \gamma^s$. By equation (52), the price impacts of both traders also coincide, $M^b = M^s$, and the equilibrium market clearing condition, implicitly defining the equilibrium price, can be written as

$$\sum_{i=b,s} \bar{t}^i(\bar{p}, \bar{M}^i) = \frac{\gamma^i}{\gamma^i + \bar{M}^i} \sum_{i=b,s} \bar{t}^i(\bar{p}, 0) = \sum_{i=b,s} \bar{t}^i(\bar{p}, 0) = 0. \tag{54}$$

The unique price that solves the last equality in (54) is the Walrasian one $\bar{p}^{Walras}$ and hence $\bar{p} = \bar{p}^{Walras}$. The intuition behind this result is as follows: the identical convexity for the buyer and seller leads to the same market power of both consumers. For any fixed price, the buyer with price impact reduces its demand in the same proportion $\gamma^i / \left(\gamma^i + M^i\right)$ as the seller decreases its supply, and the market clears and the price remains unchanged.

Now suppose that traders' convexities differ, for example $\gamma^b > \gamma^s$. Then the price impacts satisfy $M^b < M^s$ and the ratios $\gamma^b / \left(\gamma^b + M^b\right) > \gamma^s / \left(\gamma^s + M^s\right)$. This and the stable market clearing condition

$$0 = \sum_{i=b,s} \bar{t}^i(\bar{p}, \bar{M}^i) = \sum_{i=b,s} \frac{\gamma^i}{\gamma^i + M^i} \bar{t}^i(\bar{p}, 0) \tag{55}$$

imply that at the $\rho-$competitive equilibrium price, $\bar{p}$, the Walrasian excess demand is strictly negative

$$\sum_{i=b,s} \bar{t}^i(\bar{p}, 0) < 0. \tag{56}$$

The Walrasian aggregate demand function is strictly decreasing in price, and therefore in order to establish the market clearing in a competitive framework, the Walrasian price must go down. One can conclude that

$$\bar{p} > \bar{p}^{Walras}. \tag{57}$$

By a similar argument, the price bias is negative when the seller's convexity is greater, $\gamma^b < \gamma^s$, that is

$$\bar{p} < \bar{p}^{Walras}. \tag{58}$$

This example illustrates that in a $\rho-$competitive equilibrium, price can be greater, the same or below the competitive one, depending on the values of parameters of the considered economy.

In general, when consumers have identical quasi-linear and concave but not necessarily quadratic utility functions, their Walrasian consumption vectors are the same, $\bar{x}^i = \bar{x}^{Walras}$. In a $\rho-$competitive equilibrium however, the sellers are given incentives to reduce their supply while the buyers cut down their demands. This leads to a disparity between the consumption of the buyers and the sellers, $\bar{x}^b < \bar{x}^{Walras} < \bar{x}^s$. If the second derivative of the utility function is increasing ($u'''(\cdot) > 0$), then the convexity of the utility function of the buyer, $\gamma^b = \left|u''\left(\bar{x}^b\right)\right|$, is higher than the one of the seller's $\gamma^s = |u''(\bar{x}^s)|$. The quadratic example suggests that such endogenous asymmetry in convexity may lead to a positive price bias over the Walrasian price, $\bar{p} > \bar{p}^{Walras}$. To make this observation formal, for any $\rho-$competitive equilibrium $\left(\bar{p}, \bar{t}, \bar{M}\right)$, I partition the set of all consumers into three groups: buyers, $\mathcal{I}^b = \left\{i \in \mathcal{I} | \bar{t}^i_l > 0\right\}$, sellers, $\mathcal{I}^s = \left\{i \in \mathcal{I} | \bar{t}^i_l < 0\right\}$, and non-active traders, $\mathcal{I}^{na} = \left\{i \in \mathcal{I} | \bar{t}^i_l = 0\right\}$.

Proposition 4 *Consider a smooth and separable pure exchange economy, in which initial allocation of any of the non-numeraire goods is not Pareto efficient, and the utility functions and external absorption is the same for all consumers. Then in any $\rho-$competitive equilibrium $\left(\bar{p},\bar{t},\bar{M}\right)$, on the market l for any $b\in\mathcal{I}^{b}$, $s\in\mathcal{I}^{s}$ and $na\in\mathcal{I}^{na}$*

1) $u_l'''(\cdot)>0$ implies upward price bias $\bar{p}_l>\bar{p}_l^{Walras}$ and $\bar{M}_{l,l}^{s}>\bar{M}_{l,l}^{na}>\bar{M}_{l,l}^{b}$,

2) $u_l'''(\cdot)=0$ implies no price bias $\bar{p}_l=\bar{p}_l^{Walras}$ and $\bar{M}_{l,l}^{s}=\bar{M}_{l,l}^{na}=\bar{M}_{l,l}^{b}$,

3) $u_l'''(\cdot)<0$ implies downward price bias $\bar{p}_l<\bar{p}_l^{Walras}$ and $\bar{M}_{l,l}^{s}<\bar{M}_{l,l}^{na}<\bar{M}_{l,l}^{b}$.

There are two effects generating the price bias. First, with $u_l'''(\cdot)>0$, the sellers tend to reduce their supply more than the buyers for any level of price impact $M_{l,l}^{i}$, and hence price must go up to clear the market. This is a direct effect. Second, the endogenous asymmetry in convexity implies that the sellers' price impacts exceed the ones of the buyers, further increasing the price.

The utility functions characterized by constant relative risk aversion (CRRA) satisfy condition $u_l'''(\cdot)>0$ and therefore in a pure exchange economy with such utility functions one should expect upward price bias and high relative market power of the sellers.

8.1.3 Welfare

Now I focus on the effects of non-competitive interactions on welfare. In an economy with quasi-linear preferences, the utility is transferrable and one can measure "welfare" in terms of a numeraire.

The aggregate gains to trade on market l are defined as welfare attained at the unique Pareto optimal allocation, less the utility in the autarky. In the considered economy, the aggregate gains to trade are equal to

$$\Delta W_l=\sum_i u_l^i\left(\bar{e}\right)-u_l^i\left(e^i\right)=\frac{1}{2}I\bar{\gamma}_l Var\left(e_l^i\right). \tag{59}$$

The aggregate gains to trade do not depend on the distribution of numeraire m^i. They are positively correlated with the heterogeneity of the initial endowments measured by variance $Var\left(e_l^i\right)$, the convexity of the utility function $\bar{\gamma}_l$ and the number of traders. If the endowments are homogenous, the initial allocation is Pareto efficient and no trade can improve welfare. This is the case when $Var\left(e_l^i\right)=0$ and hence $\Delta W_l=0$. Similarly, when $\bar{\gamma}_l$ is close to zero, marginal utility is virtually constant for all traders and thus the distribution of goods has minimal impact on aggregate welfare.

Theorem 3 implies that if the initial allocation is Pareto inefficient, then a $\rho-$competitive allocation is not Pareto efficient either. This implies that some part of the aggregate gains to trade is forgone. The dead weight loss on market l, denoted by DWL_l, gives the fraction of the aggregate gains to trade ΔW_l lost due to not perfectly competitive trading,

$$DWL_l\equiv\frac{\sum_i u_l\left(\bar{e}_l\right)-u_l^i\left(\bar{x}_l^i\right)}{\Delta W_l}=\left(\frac{1-\bar{\rho}}{I-1}\right)^2. \tag{60}$$

Equation (60) shows that the more traders in the economy I, the more competitive interactions are and hence DWL_l is smaller. Since the inverse of DWL_l is quadratic in I, and ΔW_l

is linear, in absolute terms the welfare loss $DWL_l \cdot \Delta W_l$ also disappears with I approaching infinity, as long as $Var\left(e_l^i\right)$ stays bounded. The external absorption also lessens the effects of non-competitive trading.

Now I discuss how the welfare loss is distributed among different traders in market l. By $\bar{m}_l^i = -\bar{p}_l \bar{t}_l^i$, I denote the amount of numeraire spent on good l. The *individual welfare loss* of consumer i on market l, IWL_l^i is a difference between the utility in a unique Walrasian equilibrium and in a $\rho-$competitive equilibrium, as a fraction of the individual gains to trade in the Walrasian equilibrium,

$$IWL_l^i = \frac{u_l\left(\bar{t}_l^{Walras}\right) + \bar{m}_l^{i,Walras} - u_l\left(\bar{t}^i\right) - \bar{m}_l^i}{u_l\left(\bar{t}_l^{Walras}\right) + \bar{m}_l^{i,Walras} - u_l\left(e_l^i\right)}. \tag{61}$$

The denominator of IWL_l^i is always non-negative, therefore $IWL_l^i > 0$ indicates that the trader is better off in a Walrasian than in a $\rho-$competitive equilibrium. The negative value of IWL_l^i implies that the consumer i gains extra utility at the expense of other traders, taking advantage of its market power. For the identical quadratic utility functions, the individual welfare loss is given by

$$IWL_l^i = \left(\frac{1-\bar{\rho}}{I-1}\right)^2. \tag{62}$$

Each consumer is losing exactly the same fraction of its individual Walrasian gains to trade, regardless of whether she is a buyer, a seller or a non-active trader. In addition, IWL_l^i coincides with the aggregate dead weight loss. This shows that the welfare loss is distributed uniformly across all traders and no trader gains utility at the expense of other consumers. As will be shown in the next sections, this result also relies on the assumption of identical quadratic utility functions.

8.1.4 Convergence to a Walrasian Equilibrium

Is the Walrasian equilibrium a good approximation of market interactions in $\mathcal{E}$, provided that $\mathcal{E}$ is large? Theorem 4 gives a convergence result for a $k-$replica of a smooth and separable economy. In this section, I illustrate this result geometrically in a well-known Edgeworth box example. I consider a $k-$replica of the economy with *two* non-numeraire goods $l = 1, 2$ and two types of consumers $i = 1, 2$. The consumers of the first type are initially endowed with one unit of good one and zero of good two, $e^1 = (1, 0)$. The endowment of the second type is $e^2 = (0, 1)$. Both types have preferences as in (46). Note that for any replica with $k \geq 2$, the economy is composed of at least four consumers, $I \geq 4$, and hence the equilibrium exists[17] for any level of external absorption, including $\rho = 0$. Observe that this economy is a special case of the economy considered in the previous section.

In the Walrasian equilibrium, the consumption of each trader is equal to $\bar{x}^{i,Walras} = (0.5, 0.5)$, and is independent from k. In $0-$competitive equilibrium consumption it is given by

$$\bar{x}^i(k) = \frac{2k-3}{2k-2}\bar{x}^{i,Walras} + \left(\frac{1}{2k-2}\right)e^i. \tag{63}$$

[17] In fact, one should also assume strictly positive initial endowments for all traders. This assumption is superfluous in the case of quadratic utility functions.

For $k = 2$, the $\rho-$competitive equilibrium is depicted in Figure 6, panel a. The stable consumption of trader 1 is given by $\bar{x}^1 = (0.75, 0.25)$. At the equilibrium consumption point the budget set (shaded area) is tangent to the indifference curve. For trader 2 the consumption vector is symmetric. The indifference curves of traders are not tangent to each other, which demonstrates Pareto inefficiency of the equilibrium allocation. In panel b) it is shown how equilibrium allocation converges to a Walrasian one. As k approaches infinity, the weight put on the Walrasian consumption is closer to one and hence the equilibrium becomes competitive.

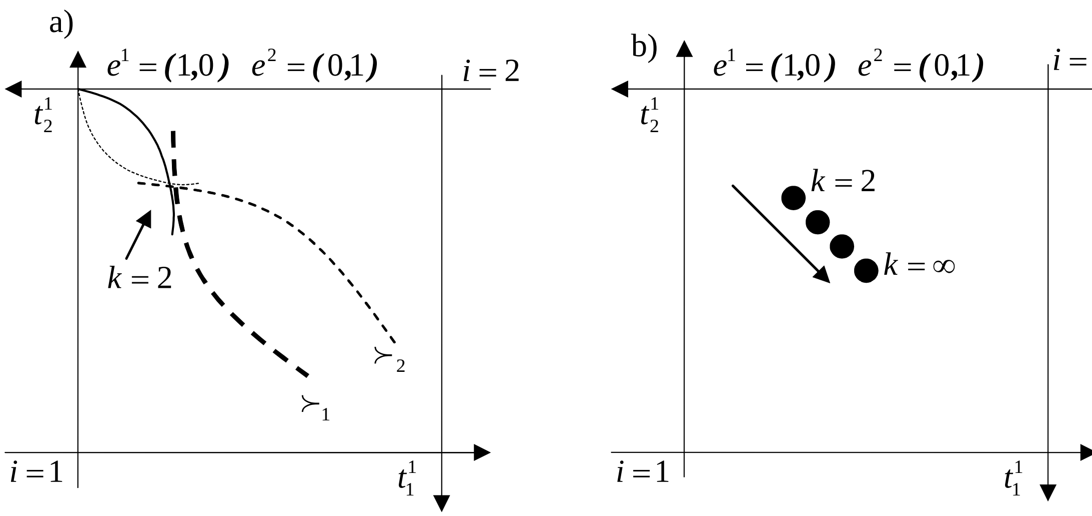

Figure 6. Convergence to the Walrasian equilibrium in the Edgeworth box economy.

8.2 Bilateral Monopoly and Constant Relative Risk Aversion

In this section I examine how non-constant convexity affects the results from the previous section. In particular, I study the economy with only one good and two consumers—a buyer and a seller. The consumers have an identical utility function characterized by a constant relative risk aversion,

$$U\left(x^i, m^i\right) = \frac{\left(x^i\right)^{1-\theta}}{1-\theta} + m^i. \tag{64}$$

(For $\theta = 1$, I assume a logarithmic utility function.)

The utility function (64) does not allow me to calculate equilibrium analytically, and therefore I give numerical results. With such a utility function, a $\rho-$competitive equilibrium has a convenient property that some of the variables characterizing it do not depend on the scale of the endowments. Formally,

Lemma 1 *Let $\mathcal{E} = (u, e)$ be a pure exchange economy with consumers characterized by constant relative risk aversion, with identical coefficient θ for all i. Then $\left(\bar{p}, \bar{t}, \bar{M}\right)$ is a diagonal $\rho-$competitive equilibrium in $\mathcal{E}$ iff $\left(\lambda^{-\theta}\bar{p}, \lambda\bar{t}, \lambda^{-(1+\theta)}\bar{M}\right)$ is a diagonal $\rho-$competitive equilibrium in the scaled economy $\mathcal{E}^\lambda \equiv (u, \lambda e)$ for arbitrary scalar $\lambda > 0$.*

Consequently, for any fixed ρ and θ, the relative allocation of non-numeraire good, x^b/x^s, the relative market power, M^b/M^s, and the price bias, $\Delta\bar{p}[\%]$, defined as

$$\Delta\bar{p}[\%] = \left(\frac{\bar{p}}{\bar{p}^{Walras}} - 1\right) \times 100\%,$$

are function of a relative, e^b/e^s, rather than the absolute level of initial endowment.

In the following sections I discuss the effects of the asymmetry in endowments, relative risk aversion, and external absorption ρ on the equilibrium allocation, price and welfare.

8.2.1 Endowments

When $e^b/e^s = 1$, the buyer and the seller is endowed with exactly the same amount of non-numeraire and hence the traders do not benefit from trading. On the other extreme, when the endowment of the buyer is equal to zero and hence in autarky, she has minus infinite utility, thus the gains to trade are maximal.

I report the equilibrium values of relative consumption, price impacts and price bias, as a function of relative endowment. The external absorption is set to be equal to $\rho^b = \rho^s = 0.5$ and the relative risk aversion is $\theta = 1$ so that the utility is logarithmic.

Table 1a. Effects of e^b/e^s: equilibrium.

e^b/e^s	0	0.2	0.4	0.6	0.8	1
$\bar{x}^b/\bar{x}^s$	12.0%	44.7%	63.2%	77.4%	89.4%	100%
$\bar{M}^b/\bar{M}^s$	50.9%	63.6%	75.0%	84.6%	92.8%	100%.
$\Delta\bar{p}[\%]$	235.0%	27.0%	8.7%	2.7%	0.5%	0.0%

Notes: The first row reports the value of a relative buyers endowment e^b/e^s for which the simulation was conducted. x^b/x^s is a consumption of the buyer relative to the consumption of the seller; $\bar{M}^b/\bar{M}^s$ is a buyers relative price impact; $\Delta p[\%]$ is the percentage increase in the equilibrium price over the Walrasian one.

In the Walrasian equilibrium, traders consume the same amounts of non-numeraire and therefore $\bar{x}^b/\bar{x}^s$ is equal to one for arbitrary value of e^b/e^s. In a $\rho-$competitive equilibrium, the buyer consumes less than the seller, provided that $e^b/e^s < 1$. This is because both traders have incentives to strategically reduce their trades to affect prices. The ratio $\bar{x}^b/\bar{x}^s$ is positively correlated with the relative endowment e^b/e^s.

The simulations are consistent with the result from Proposition 4. The second derivative of a logarithmic utility function is increasing

$$\frac{\partial^3 \ln\left(x^i\right)}{\partial\left(x^i\right)^3} = 2\left(x^i\right)^{-3} > 0, \tag{65}$$

and hence one observes an upward price bias.

The dead weight loss DWL, and individual welfare losses IWL^b and IWL^s do not depend on the scale of the endowment, and they are functions of a relative endowment e^b/e^s, too. The effects on welfare are given in Table 1b.

Table 1b. Effects of e^b/e^s: welfare.						
e^b/e^s[18]	0	0.2	0.4	0.6	0.8	1
DWL	37.9%	28.1%	25.6%	25.2%	25.0%	25.0%
IWL^b	42.5%	40.8%	36.4%	32.3%	28.5%	25%.
IWL^s	-70.0%	-11.8%	6.0%	15.3%	21.0%	25.0%

Notes: The first row reports the value of a relative buyers endowment e^b/e^s for which the simulation was conducted. DWL is an aggregate loss of welfare associated with non-competitive trade as a fraction of total gains to trade in the economy; IWL^b and IWL^s are buyer's and seller's individual losses/gains of welfare due to a non-competitive trade as a fraction of the individual gains to trade in a competitive equilibrium.

It can be seen that the welfare loss, as a fraction of total gains to trade, is relatively stable. It varies from -33% for the lowest relative endowment to -25% in the case when there are no gains to trade at all. When e^b/e^s approaches 1, the aggregate gains to trade are minimal and therefore in absolute terms the loss of utility is also close to zero. The last column suggests that when the endowments are nearly symmetric, the welfare loss is uniformly spread between the buyer and the seller: each of them is losing 25% of their potential gains. When endowments differ substantially, the seller is able to pass most of the welfare cost onto the buyer. For $e^b/e^s \leq 0.3$, the seller's benefit from being a monopoly is sufficient to compensate for the welfare loss associated with the reduced volume of trade. In this case, the buyer not only "pays" the cost of a dead weight loss but also a monopoly rent to the seller.

8.2.2 Relative Risk Aversion

Now I examine the effects of a different level of relative risk aversion. Table 2a reports the numerical simulations of a 0.5−competitive equilibrium ($\rho^i = 0.5$), with the relative endowment equal to $e^b/e^s = 40\%$, for the following values of relative risk aversion $\theta = 0.5$, 1, 2 and 3.

Table 2a. Effects of θ: equilibrium.				
θ	0.5	1	2	3
$\bar{x}^b/\bar{x}^s$	63.8%	63.2%	61.6%	59.5%
$\bar{M}^b/\bar{M}^s$	80.4%	75.0%	65%	58.9%
$\Delta\bar{p}[\%]$	3.1%	8.7%	29.6%	69.7%

Notes: The first row reports the value of a relative risk aversion θ for which the simulation was conducted. x^b/x^s is a consumption of the buyer relative to the consumption of the seller; $\bar{M}^b/\bar{M}^s$ is a buyers relative price impact; $\Delta p[\%]$ is the percentage increase in the equilibrium price over the Walrasian one.

The relative consumption $\bar{x}^b/\bar{x}^s$ is decreasing in θ. This seems paradoxical as the more concave the utility functions, the greater the gains to trade. Note, however, that market power is increasing in the convexity of the utility functions $|u''(\cdot)|$, which in turn negatively

[18] In the first and last column I give the limits of the statistics, as e^b/e^s is approaching 0 and 1 respectively, as the statistics are not defined for the two values of relative endowment.

affects the trade. The relative risk aversion also magnifies the asymmetry of market power. For $\theta = 3$, the buyer only has 58% of the sellers price impact. This in turn positively affects the markup over a competitive price. The effects of relative risk aversion on welfare are given in Table 2b.

Table 2b. Effects of θ: welfare.

θ	0.5	1	2	3
DWL	25.4%	25.6%	26.4%	27.9%
IWL^b	34.0%	36.4%	40.6%	44.1%
IWL^s	11.9%	6.0%	-8.7%	-28.3%

Notes: The first row reports the value of a relative risk aversion θ for which the simulation was conducted. DWL is an aggregate loss of welfare associated with non-competitive trade as a fraction of total gains to trade in the economy; $\mathrm{IWL^b}$ and $\mathrm{IWL^S}$ are buyer's and seller's individual losses/gains of welfare due to a non-competitive trade as a fraction of the individual gains to trade in a competitive equilibrium.

The dead weight loss is only modestly increasing in relative risk aversion, but the distribution of the welfare loss is very sensitive to changes in θ. For relatively small values of θ, both traders are worse-off in a $\rho-$competitive equilibrium. When θ is high, the effect of endogenous convexity implies that the majority of the aggregate welfare loss comes from the reduction of utility of the buyer. For θ greater than one, the seller extracts the utility using its superior market power.

8.2.3 Symmetric External Absorption

In this section, I investigate the consequences of the symmetric external absorption $\rho^b = \rho^s = \bar{\rho}$ and in the following section I discuss the effects of its asymmetry. In both sections, I assume a logarithmic utility function and the value of a relative endowment equal to $e^b/e^s = 0.40\%$.

Table 3a gives the results for different values of $\bar{\rho}$.

Table 3a. Effects of $\bar{\rho}$: equilibrium.

$\bar{\rho}$	0.1	0.3	0.5	0.7	0.9	1
$\bar{x}^b/\bar{x}^s$	43.2%	51.8%	63.2%	76.7%	91.7%	100%
$\bar{M}^b/\bar{M}^s$	93.0%	81.5%	75.0%	75.5%	86.9%	n.d.
$\Delta\bar{p}[\%]$	19.4%	14.8%	8.7%	3.6%	0.5%	0.0%

Notes: The first row reports the value of the symmetric external absorption $\bar{\rho}$ for which the simulation was conducted. $\mathrm{x^b/x^S}$ is a consumption of the buyer relative to the consumption of the seller; $\mathrm{\bar{M}^b/\bar{M}^S}$ is a buyers relative price impact; Δp[%] is the percentage increase in the equilibrium price over the Walrasian one.

The relative consumption is increasing in $\bar{\rho}$ to a competitive level $x^b/x^s = 100\%$, for $\bar{\rho} = 1$. Surprisingly, the relative market power is decreasing in $\bar{\rho}$. This shows that for low values of $\bar{\rho}$, the mutual reinforcement effect of price impacts dominates the endogenous asymmetry of the convexity (see Proposition 4). Price bias is also negatively related to $\bar{\rho}$.

The dead weight loss is decreasing in $\bar{\rho}$. Interestingly, the seller is better off in a $\rho-$competitive equilibrium provided that the external absorption is not too small. Intuitively this is because for high values of $\bar{\rho}$, the relative market power of the seller is high, and hence she benefits from a higher price, while the trade volume is still relatively large. When $\bar{\rho}$ is small, the sellers utility gains associated with higher price are not sufficient to compensate the welfare loss due to a reduced volume of trade.

Table 3b. Effects of $\bar{\rho}$: welfare.

$\bar{\rho}$	0.1	0.3	0.5	0.7	0.9	1
DWL	84.3%	52.3%	25.6%	8.6%	0.9%	0.0%
IWL^b	87.3%	61.7%	36.4%	15.5%	2.2%	0.0%
IWL^s	78.7%	35.3%	6.0%	-3.9%	-1.37%	0.0%

Notes: The first row reports the value of the symmetric external absorption $\bar{\rho}$ for which the simulation was conducted. DWL is an aggregate loss of welfare associated with a non-competitive trade as a fraction of total gains to trade in the economy; IWL^{b} and IWL^{s} are buyer's and seller's individual losses/gains of welfare due to a non-competitive trade as a fraction of the individual gains to trade in a competitive equilibrium.

8.2.4 Asymmetric Bilateral Monopoly

In this section, I examine the effects of the asymmetry in external absorption. In Tables 4a and 4b, the column associated with $\rho^b = 0$ and $\rho^s = 1$ gives results for the economy with a monopolistic seller and a competitive buyer (pure monopoly). In last column, with $\rho^b = 1$ and $\rho^s = 0$, the buyer has full market power while the seller is competitive (pure monopsony). The intermediate columns represent the case of bilateral monopoly with different levels of asymmetry.

The relative consumption of the buyer $\bar{x}^b/\bar{x}^s$ is "U" shaped. It is large for extreme monopolies and smaller for a symmetric bilateral monopoly. This is because the mutual reinforcement of price impacts is not present in a pure monopoly or monopsony and therefore the market power is relatively low in general. In addition, the incentives to reduce the trade are partially offset by the favorable price change for the two extremes. The relative consumption is greater in the case of a pure monopsony than in a monopoly. With logarithmic utility function, the buyer's gains to trade are substantially higher than for the seller, and the buyer is not willing to sacrifice such benefits as much as the seller does in order to decrease the price. Not surprisingly, the price bias is negative for the pure monopsony, and positive for the pure monopoly.

Table 4a. Effects of asymmetric ρ: equilibrium.

ρ^s	1	0.8	0.6	0.5	0.4	0.2	0
ρ^b	0	0.2	0.4	0.5	0.6	0.8	1
$\bar{x}^b/\bar{x}^s$	78.4%	69.9%	64.6%	63.2%	62.8%	64.33%	69.6%
$\bar{M}^b/\bar{M}^s$	∞%	197.8%	98.3%	75.0%	57.2%	28.3%	0.0%
$\Delta\bar{p}\,[\%]$	-10.8%	-2.8%	5.1%	8.7%	12.1%	17.7%	21.8%

Notes: The first two rows report the values of external absorption for the buyers and the seller for which the simulation was conducted. $\bar{x}^b/\bar{x}^s$ is a consumption of the buyer relative to the consumption of the seller; $\bar{M}^b/\bar{M}^s$ is a buyer's relative price impact; $\Delta p\,[\%]$ is the percentage increase in the equilibrium price over the Walrasian one.

The pattern of aggregate welfare loss is a mirror image of the relative consumption and therefore DWL has an inverted "U" shape. On the individual level, the buyer is better-off as a pure monopsony and the seller as a pure monopoly. In general, the seller is willing to use market power more aggressively than the buyer as her welfare loss associated with reduced volume of trade is smaller than the one of the buyer.

Table 4b. Effects of asymmetric ρ: welfare.

ρ^s	1	0.8	0.6	0.5	0.4	0.2	0
ρ^b	0	0.2	0.4	0.5	0.6	0.8	1
DWL	7.2%	15.2%	23.4%	25.6%	26.5%	23.8%	16.0%
IWL^b	-19.2%	8.3%	29.0%	36.4%	42.1%	49.7%	55.5%
IWL^s	55.5%	29.3%	13.5%	6.0%	-1.9%	-23.6%	-55.7%

Notes: The first two rows report the values of external absorption for the buyers and the seller for which the simulation was conducted. DWL is an aggregate loss of welfare associated with non-competitive trade as a fraction of total gains to trade in the economy; IWL^b and IWL^s are buyer's and seller's individual losses/gains of welfare, due to a non-competitive trade as a fraction of the individual gains to trade in a competitive equilibrium.

8.3 Equilibrium with Producers: Robinson Crusoe Inc.

So far I have focused on the non-competitive interactions in a pure exchange economy. Now to the economy from Section 8.1, I introduce production firms and demonstrate in the simple example how the ownership structure affects the equilibrium outcome.

I first study an economy with J incorporated firms, with technology characterized by a quadratic cost function for each good $l \in \mathcal{L}$

$$f_l^j = \frac{\bar{\gamma}_l}{2}\left(y_l^j\right)^2. \tag{66}$$

The quadratic preferences of the consumers are as in equation (46), and the convexities of the utility function and the cost function, $\bar{\gamma}_l$, coincide. The consumers' non-numeraire endowments are equal to zero and the shares are distributed according to some vector $\theta \in \mathbb{R}^{I\times J}$. Throughout this section I assume the same level of external absorption for all consumers and firms, $\rho_l^i = \rho_l^j = \bar{\rho}$.

As in the pure exchange economy from Section 8.1, $\rho-$competitive equilibrium has a closed form solution. The equilibrium prices do not depend on the absolute number of the traders but on the composition of the traders in the economy

$$\bar{p}_l = \bar{p}_l^{Walras} = \frac{I}{I+J}. \tag{67}$$

Moreover, prices are independent from the level of external absorption. The price impacts are identical for all firms and consumers and are given by

$$\bar{M}_{l,l}^i = \bar{M}_{l,l}^j = \frac{1-\bar{\rho}_l}{I+J+\bar{\rho}_l-2}\bar{\gamma}_l. \tag{68}$$

They are increasing in $\bar{\gamma}_l$, and decreasing in $\bar{\rho}$ and absolute number of traders $N = I + J$. Finally, the non-numeraire allocation is given by

$$\bar{x}_l^i = \bar{t}_l^i = \frac{I+J+\bar{\rho}_l-2}{I+J-1}\bar{x}_l^{i,Walras} \text{ where } \bar{x}_l^{i,Walras} = \frac{1}{\bar{\gamma}_l}\frac{J}{I+J}, \tag{69}$$

and the individual stable supply is

$$\bar{y}_l^j = -\bar{t}_l^j = \frac{I+J+\bar{\rho}_l-2}{I+J-1}\bar{y}_l^{j,Walras} \text{ where } \bar{y}_l^{j,Walras} = \frac{1}{\bar{\gamma}_l}\frac{I}{I+J}. \tag{70}$$

Consider the simplest version of the considered economy, with one non-numeraire good, one consumer, and one firm $I = J = 1$, and also assume $\bar{\gamma} = 1$. The equilibrium in such an economy is depicted in Figure 7a. For $\bar{\rho} = 1$, it coincides with a Walrasian equilibrium and is given by

$$\bar{x}^{i,Walras} = \bar{y}^{j,Walras} = \frac{1}{2},\ \bar{p}^{Walras} = \frac{1}{2} \text{ and } \bar{\pi}^{Walras} = \frac{1}{8}. \tag{71}$$

The equilibrium allocation is Pareto efficient, therefore it can be determined as a tangency point of the indifference curve and the production set. The consumer's budget set at equilibrium prices consists of all $\left(t^i, m^i\right)$ below the iso–profit line $\bar{\pi}^{Walras}$. When Robinson is a sole proprietor of the firm, then he would chose the Pareto optimal (Walrasian) allocation, regardless of the value of $\bar{\rho}$.

Now suppose that the firm is incorporated and therefore it maximizes profit. With external absorption strictly smaller than one, for example $\bar{\rho} = 0.5$, a unique $\rho-$competitive equilibrium is given by

$$\bar{x}^i = \bar{y}^j = \frac{1}{4},\ \bar{p} = \frac{1}{2},\ \bar{M}^i = \bar{M}^j = 1 \text{ and } \bar{\pi} = \frac{3}{32} < \bar{\pi}^{Walras}. \tag{72}$$

The stable choice of the firms is depicted in panel b). The equilibrium iso–profit parabola is tangent to a production set at $\left(\bar{y}^j, \bar{m}^j\right)$, and it cuts the numeraire axes at the equilibrium profit $\bar{\pi} = \frac{3}{32}$. The stable choice of the consumer is shown in panel c). The budget set corresponds to the interior of the parabola that crosses the numeraire axes at $\bar{\pi} = \frac{3}{32}$ and also passes through the equilibrium allocation $\left(-\frac{1}{8}, \frac{1}{4}\right)$. By definition of stability, the indifference curve is tangent at $\left(-\frac{1}{8}, \frac{1}{4}\right)$ to the budget set. Note, however, that the indifference curve is *not tangent* to the production set, and hence the equilibrium allocation is Pareto inefficient.

It should be noted that in the case of a sole proprietorship, Pareto efficiency results from the fact that Robinson is the *only* trader in the economy. The sole proprietorship is not necessarily more efficient than the incorporated ownership.

Any allocation on the frontier of the production set located between the Walrasian allocation $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and the autarky $(0, 0)$ can be a part of an equilibrium for some value of $\bar{\rho}$ (this is shown in panel d). In addition, any $0-$competitive allocation in the k- replica economy is also on the frontier, between allocation $\left(-\frac{1}{6}, \frac{1}{3}\right)$ corresponding to $k = 2$, and the Walrasian allocation $\left(-\frac{1}{4}, \frac{1}{2}\right)$ for k approaching infinity.

With asymmetric external absorption, the geometric representation of an equilibrium differs from the one depicted in Figure 7 as follows: For a pure monopoly ($\rho^j = 0$, and $\rho^i = 1$), the budget set of a consumer in panel c) becomes linear and in the case of a pure monopsony, ($\rho^j = 1$ and $\rho^i = 0$) the iso–profit curves in panel b) are linear.

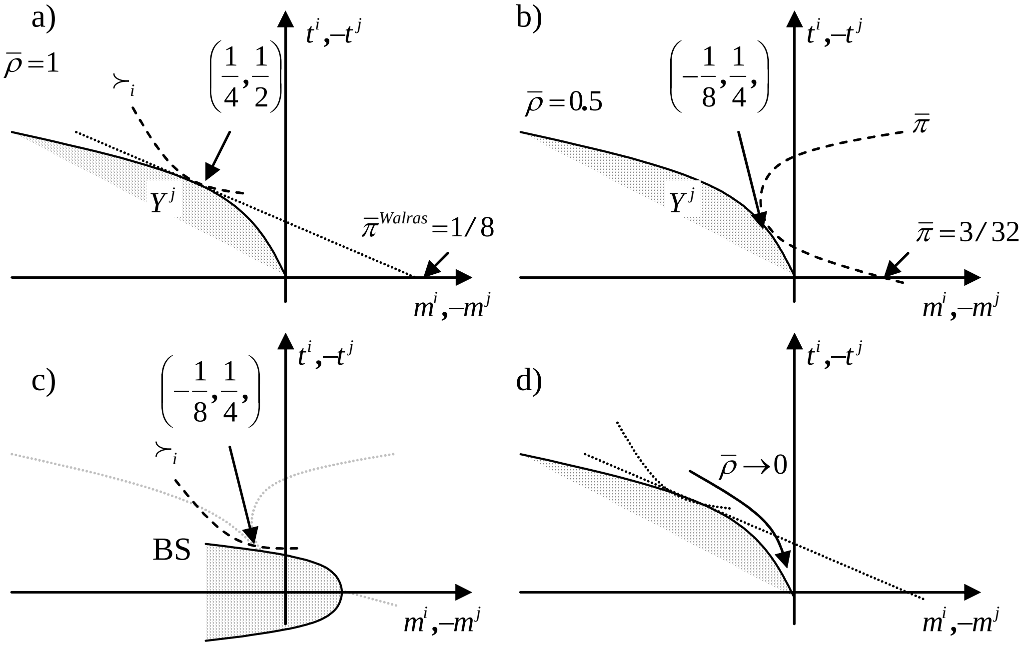

Figure 7. Equilibrium in the economy with a firm.

8.4 Increasing Returns to Scale and a Natural Monopoly.

Sometimes establishing a firm is associated with large fixed costs while the production is characterized by a constant marginal cost. In such case the average cost is monotonically decreasing, and the technology exhibits increasing returns to scale. I call such a firm a natural monopoly. The behavior of a natural monopoly cannot be studied within the Walrasian framework. This is because the firm's competitive supply is either infinite or zero, depending on whether the price is above the marginal cost or not. In this section, I demonstrate how the natural monopoly can be incorporated in a general equilibrium framework when traders have market power.

I consider an economy with one non-numeraire good, one firm characterized by the following cost function

$$f^j\left(y^j\right) = \left\{ \begin{array}{c} F + cy^j \text{ if } y^j > 0 \\ 0 \text{ otherwise} \end{array} \right\}, \tag{73}$$

and I consumers with a quadratic utility function given by (46). The initial endowments are normalized to zero. In equation (73), F corresponds to a fixed cost and c is a marginal cost. The production set of the monopoly is depicted in Figure 8a. The cost function in (73) does not satisfy assumptions A7 and A9 and hence the considered economy is not smooth and separable. Consequently, the theorems from Section 7 do not apply here.

When F is already a sunk cost, the production set does contain inaction point $(0, 0)$ and hence is convex. In such case, a stable supply function can be determined from the first order stability condition (35) . The equality between marginal cost and marginal revenue implies

$$\bar{y}^j \equiv -\bar{t}^j = \frac{1}{\bar{M}^j}(\bar{p} - c), \tag{74}$$

for any $\bar{p} \geq c$ and 0 otherwise. In Figure 8, the stable supply is geometrically determined as a tangency point of the iso–profit parabola and the production set.

If F is a fixed rather than a sunk cost, the inaction point is a feasible option for the firm, and hence the production plan $\bar{y}^j$ from (74) may fail to be stable. This would be the case when $\bar{y}^j$ is associated with a negative profit, and hence the natural monopoly prefers the inaction point. In Figure 8, the zero profit parabola passing through trade (0.0) is represented by a dashed line. If $\bar{y}^j$ is in the interior of this parabola, it dominates the inaction point in terms of profit and hence it is stable on the whole production set (see panel a). In panel b), $\bar{y}^j$ is not profitable and therefore it is not stable.

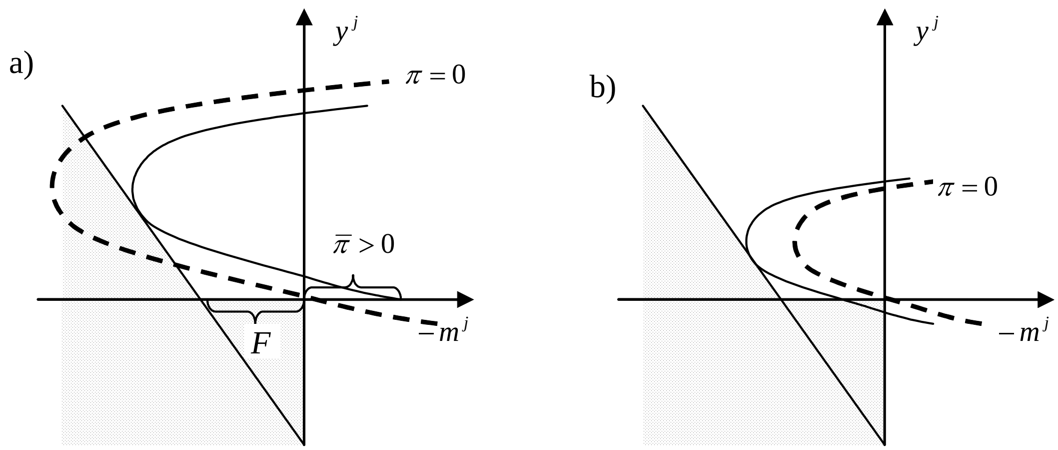

Figure 8. The existence of a stable supply.

The individual inverse supply function is depicted in Figure 9 in space $(\bar{y}^j, \bar{p})$. The tangency condition (74) defines a straight line originating in point $(0, c)$ and with the slope equal to $\bar{M}^j$.

The firm's profit is non-negative if and only if the observed price is greater or equal to the average cost

$$\bar{p} \geq AC \equiv c + F/\bar{y}^j, \tag{75}$$

and therefore all combinations $(\bar{y}^j, \bar{p})$ below the average cost curve are associated with a negative profit. Consequently, the stable supply function is a half line to the right of the AC

curve.

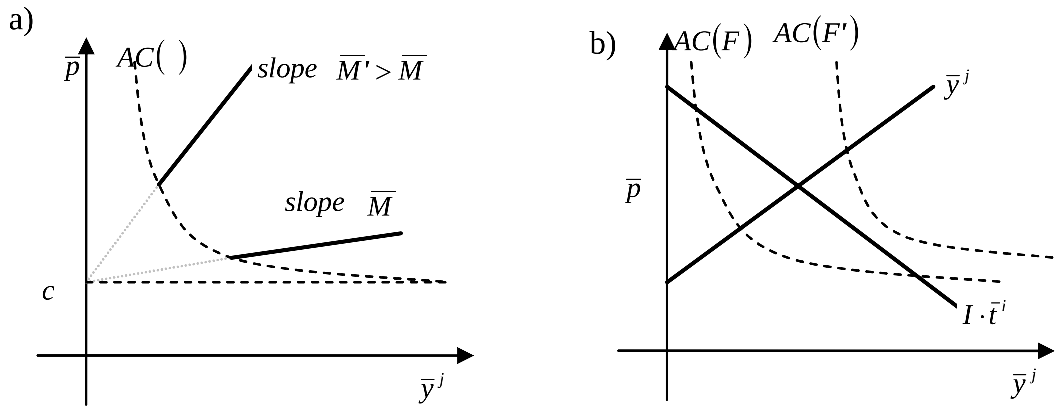

Figure 9. Stable supply function of a natural monopoly and equilibrium.

The aggregate stable demand of the consumers is as in the previous section. In Figure 9b, a natural candidate for a $\rho-$competitive equilibrium is the trade and the price for which the stable supply and demand intersect. Solving the model analytically reveals that at this point, the price is equal to

$$\bar{p} = \frac{(1+c)}{2}. \tag{76}$$

The price does not depend on the number of consumers I. This is because for higher I, the larger aggregate demand is accompanied with increased supply due to the reduced price impact of the monopoly and the prices may stay the same.

With the possibility of inaction, the intersection point determines $0-$competitive equilibrium only if the average cost does not exceed price (76). This happens in Figure 9b for a fixed cost equal to F. When, however, $F' > F$, at the price $\bar{p}$ the firm incurs a loss, therefore it shuts down the production. In such case, an equilibrium with an active firm fails to exist.

It can be shown that for any fixed value of $F \geq 0$, there exists a critical mass of consumers, $\bar{I}$, such that in the economy with $I \geq \bar{I}$ consumers, the equilibrium with an active monopoly exists, and with $I < \bar{I}$ it does not. Similarly, for any number of consumers, $I \geq 2$, there exists a threshold value of $\bar{F}$, for which the equilibrium with a monopoly exists if and only if $F \leq \bar{F}$. In Figure 9, threshold $\bar{F}$ can be determined as the fixed cost for which the AC locus passes through the intersecting point of the demand and supply.

8.5 Endogenous Market Structure

In the previous sections, the number of operating firms was exogenously fixed. Here I will assume that in the considered economy, there is an infinite number of entrepreneurs who have access to the most efficient, commonly known technology, and they may potentially use it to form a production firm. Formation of a firm is motivated by a non-numeraire profit. The technology to produce non-numeraire commodities is given by some cost function f. An economy consists of two elements: A pure exchange economy satisfying A1–A5 and a technology f,

$$\mathcal{E}^{LR} = \left(\left(u^i, e^i\right)_{i \in I}, f \right). \tag{77}$$

In a Marshallian long run equilibrium, the assumptions of free entry and price taking behavior imply that price is equal to a minimal average cost and each firm produces at the most efficient scale. The number of active firms is determined as a ratio of the aggregate demand divided by the individual production. This approach has the following drawbacks: The equilibrium is well defined only for the technology characterized by a "U" shaped average cost, and hence the model does not give any predictions for the industries with increasing returns to scale. In addition, the model assumes price taking behavior by all market participants even if the predicted number of firms is small. Finally, the assumption of "profit taking" by potential entrants rules out the possibility of strictly positive profit in equilibrium. Here I redefine the notion of long run equilibrium to free it from such drawbacks.

By $\mathcal{E}^J = \left(u, e, \theta^J, f \times J\right)$, I denote an exchange economy with J identical firms using technology f, and θ^J is an arbitrary distribution of shares of the firms among the consumers. The $\rho-$competitive equilibrium in $\mathcal{E}^J$ is symmetric if $\bar{t}^j$ is the same for all $j \in \mathcal{J}$ and firm j is active if $\bar{t}^j \neq 0$. The long run equilibrium is defined iteratively in the following way:

Definition 5 *A long run equilibrium in economy $\mathcal{E}^{LR}$ is a vector $(\bar{p}, \bar{t}, \bar{M}, \bar{J})$ such that*
1) $\left(\bar{p}, \bar{t}, \bar{M}\right)$ is a symmetric $0-$ competitive equilibrium in $\mathcal{E}^{\bar{J}}$ with $\bar{J}$ active firms.
2) No profitable or profit reducing entry, that is for any $J > \bar{J}$:
a) if $\bar{\pi}^{\bar{J}} > 0$, there does not exist an $0-$ competitive equilibrium in economy $\mathcal{E}^J$,
b) if $\bar{\pi}^{\bar{J}} = 0$ there does not exist a symmetric $0-$ competitive equilibrium in economy $\mathcal{E}^J$ with $\bar{\pi}^J > 0$.

The first condition assures that none of the active firms have incentives to change the traded quantities and all markets clear. The second condition prevents the non-active firms from entering the industry. I assume that a new firm is formed if the current observed profit is strictly positive and the industry may accommodate additional firms. A firm will also be formed even if the observed profit is zero, provided that after the entry in a new equilibrium the profit is positive. Observe that there are no incentives to entry when the current observed profit and the profit after entry is zero.

By Theorem 1, there exists a $0-$competitive equilibrium in a pure exchange economy with $J = 0$, provided $I > 2$. Such equilibrium is not a long run equilibrium only if there exists a $0-$competitive equilibrium in the economy $\mathcal{E}^J$ for some $J > 0$, with firms making a positive profit. This in turn is not a long run equilibrium only if there is an equilibrium in the economy with $J' > J$ firms. This shows that the long run equilibrium exists or the economy can accommodate an arbitrarily large number of firms, all making a strictly positive profit. With many firms the price impact is negligible, therefore the economies without long run equilibrium are called *competitive.*

In the following sections I verify what market structures are compatible with different types of technology.

8.5.1 Natural Monopoly

The active or inactive monopoly studied in Section 8.4 is the only market structure compatible with the cost function given by (73). This is because with two or more operating firms, constant marginal cost makes positive price impacts inconsistent with off-equilibrium market clearing condition[19].

Lemma 2 *Suppose the second derivative of the cost function is equal to zero, $f''(\cdot) = 0$. In a symmetric $0-$competitive equilibrium, the price impacts $\bar{M}^n > 0$ are consistent with off-equilibrium market clearing condition if and only if $J = 1$.*

Intuitively, with the price impact of firm j equal to $M^j > 0$, and without extra convexity "added" by the Hessian of the cost function $D^2 f^j = 0$, any unilateral deviation by firm j' is followed by the response of firm j sufficient to make the price impact of j' only a fraction of $\bar{M}^j$. Because firms are symmetric, this is possible only if $\bar{M}^j = \bar{M}^{j'} = 0$.

Strictly positive price impacts $\bar{M}^j > 0$ are necessary for the existence of a stable supply function given the cost function (73). Hence there does not exist a $\rho-$competitive equilibrium with $J \geq 2$, and the monopoly is the richest market structure compatible with such a cost function. Intuitively, new firms are not willing to challenge the natural monopoly as their entry triggers perfectly competitive interactions resulting in strictly negative profits for all firms.

8.5.2 Perfectly Competitive Oligopoly

For the cost function

$$f^j\left(y^j\right) = cy^j, \tag{78}$$

associated with constant returns to scale, the model predicts oligopolistic market structure, including a duopoly. The equilibrium price is equal to a marginal cost, $p = c$, and the profit equal to zero, $\bar{\pi}^j = 0$. This is because with constant marginal cost and $J \geq 2$, Lemma 2 implies that the positive price impact $\bar{M}^j > 0$ is inconsistent with $\rho-$competitive equilibrium. For $\bar{M}^j = 0$, the stable supply of the oligopolistic firm is well defined, and it coincides with the Walrasian supply

$$\bar{y}^j \equiv -\bar{t}^j = \left\{ \begin{array}{c} 0 \text{ if } p < c \\ 0 \leq \bar{y}^j \leq \infty \text{ for } p = c \\ \infty \text{ otherwise} \end{array} \right\}. \tag{79}$$

With such supply function, any unilateral deviation of a trader is absorbed by other active oligopolists without any effect on price. Consequently, $\bar{M}^n = 0$ for all traders $n = i, j$ is consistent with off-equilibrium market clearing. The number of firms and the individual level of production is not determined by the model. In fact, any $\bar{J} \geq 2$ and $\bar{t}^j$ satisfying

$$\bar{t}^j = \frac{\sum_{i \in I} \bar{t}^i (c, 0)}{\bar{J}} > 0 \tag{80}$$

is consistent with equilibrium. Observe that such equilibrium is Pareto efficient.

A monopoly with constant returns to scale will not be observed in equilibrium. The monopolistic firm is trading with consumers, and hence it faces a downward sloping demand,

[19] This result is consistent with the findings of Bresnahan [1981], in Example 6 for a duopoly and Perry [1983] Section 3 for oligopoly.

and hence has strictly positive profit. This in turn encourages other firms to enter the industry. By a similar argument, the economy with $\bar{J} = 0$ is not a long run equilibrium because with $\bar{J} = 1$ the firm makes a strictly positive profit, as long as $\sum_{i \in I} \bar{t}^i(c, 0) > 0$, and hence there are incentives to form a monopoly.

8.5.3 Imperfectly Competitive Oligopoly

In the following two sections, I focus on strictly convex cost functions. First I consider a cost function with a positive fixed cost that is given by

$$f\left(\bar{y}^j\right) = \left\{ \begin{array}{c} F + c\left(\bar{y}^j\right) \text{ if } \bar{y}^j > 0 \\ 0 \text{ otherwise} \end{array} \right\}. \tag{81}$$

$c\left(y^j\right)$ is strictly increasing, strictly convex and marginal cost satisfies $c'(0) = 0$.

The average cost is either monotonically decreasing and hence the technology exhibits increasing returns to scale[20], or the average cost has a "U" shape and has a well defined minimal efficient scale. In the long run, such technology induces a monopolistic or oligopolistic market structure in which active firms have positive price impacts and typically positive profits.

Market structure with "U" shaped average cost was extensively studied within the competitive framework. For example, with quadratic cost function

$$f\left(y^j\right) = \left\{ \begin{array}{c} F + \frac{\bar{\gamma}}{2}\left(y^j\right)^2 \text{ if } y^j > 0 \\ 0 \text{ otherwise} \end{array} \right\}, \tag{82}$$

the competitive long run equilibrium predicts that each firm produces at the minimal efficient scale, and price is equal to a minimal average cost,

$$\bar{y}^j = \sqrt{\frac{2F}{\bar{\gamma}}},\ p = \sqrt{2F\bar{\gamma}}. \tag{83}$$

The number of consumers per each firm can be found as a ratio of the level of production of a typical firm and the demand of each consumer. With I consumers characterized by a quadratic utility functions

$$U\left(x^i, m^j\right) = \alpha x^i - \frac{\bar{\gamma}}{2}\left(x^i\right)^2 + m^i, \tag{84}$$

this ratio is equal to

$$\frac{I}{J} = \frac{\sqrt{2F\bar{\gamma}}}{\alpha^i - \sqrt{2F\bar{\gamma}}}. \tag{85}$$

The ratio is independent from the absolute scale of the economy.

[20] The example of the cost function that is strictly increasing, strictly convex and with monotonically decreasing average cost is

$$f\left(y^j\right) = F + 1 + y^j - \frac{y^j}{y^j + 1}.$$

As argued before, the concept of competitive long run equilibrium is not coherent unless the economy is large. With the relatively small number of active firms, the price and profit taking behavior is not justified. In addition, with only a few consumers, one should also expect some monopsony power on the buyer's side.

Using the framework from this paper, I find the magnitude of the small scale biases in the predictions of the competitive long run equilibrium in the economy with consumers' utility function given by (84) and technology (82) . Table 5 summarizes the results for the following values of parameters $F = 10$, $\bar{\gamma} = 10$ and $\alpha = 15$.

Table 5. Small scale correction.

J	1	2	3	10	100	Walras
I	18-33	34-49	50-66	165-181	1649-1665	1-∞
I/J	18-33	17-24.5	16.66-22.0	16.5-18.1	16.48-16.65	16.48
$\bar{y}$	1.34-1.41	1.38-1.41	1.38-1.41	1.406-1.413	1.413-14.14	1.414
$\bar{y}$	14.21-14.56	14.17-14.41	14.15-14.35	14.134-14.215	14.142-14.15	14.142
$\bar{\pi}$	0.07-0.59	0.026-0.380	0.008-0.290	0.00-0.103	0.00-0.011	0

Notes: The first row, J reports the number of firms in the industry for which the simulation was conducted. I is the number of consumers compatible with J firms; I/J is the number of consumers per firm. y gives the range of production levels; p denotes a price and π is profit.

The first row denoted by J, defines the market structure in question. For example, column $J = 1$ represents a monopoly, $J = 2$, is a duopoly and for $J > 2$ it is an oligopoly with J firms. In the column "Walras" I report the predictions of the competitive long run equilibrium. Row I gives the the numbers of consumers supporting considered market structure, and row I/J informs about the number of consumers per each active firm. The last three rows give the level of firms production, equilibrium price and profit.

It can be seen that the profits of the oligopolistic firms are positive and the price is above the minimal average cost. In the small economy, the number of consumers per active firm is significantly above the value predicted by the Walrasian equilibrium. For example, the monopoly serves from 18 to 33 consumers and not the predicted 16.48. One of the reasons for this is that indivisibility of a fixed cost prevents the other firms from entering the industry, even if the incumbent has a relatively high profit. In addition, in a small economy, consumers have some market power and hence the aggregate demand is below the competitive one. On the other hand, the simulations suggest that the Walrasian long run equilibrium is a good approximation when the number of operating firms is large. In the economy with more than one hundred active firms, the gap in the predictions of the two frameworks becomes negligible.

8.5.4 Competitive Economy

When in the cost function given by (81) the value of fixed cost is zero, $F = 0$, the long run equilibrium fails to exist. For any value of J, Theorem 1 asserts that one can find a $\rho-$competitive one. It is easy to show that given identical cost function, all firms produce the same amount of output and hence the equilibrium is symmetric, and with increasing marginal cost all firms make a positive profit. Hence for any J there are always incentives to form new firms. By definition, the economy without the long run equilibrium is competitive.

9 Determinants of Market Power and Policy Implications

The key to understanding the phenomenon of market power is a source of trader's price impact defined as her ability to affect prices by marginally changing her trade. The presented framework gives some insights into the nature of determinants of price impacts, a phenomenon long ignored in the economic literature.

9.1 Determinants of Price Impacts

The essential ingredient of a price impact of each trader is a convexity of utility (negative of the second derivative) or cost functions of the trading partners. The steeper the marginal utility or cost of the other traders, the greater the price change needed to make others willing to absorb the off-equilibrium deviation. Therefore the trader with the highest convexity has the lowest price impacts, as she trades with a group of traders characterized by the lowest convexity. More generally, the traders' ranking with respect to the strength of price impact is a reversed order of their convexities.

Some economies endogenously assign higher market power to sellers than to buyers. For example, this is the case in a pure exchange economy with identical utility functions with a positive third derivative. In equilibrium, the convexities of buyers are naturally above the sellers' and hence sellers have more substantial price impacts.

Price impacts are subject to a mutual reinforcement effect. They "add" to the convexities of utility or cost functions, making the traders more reluctant to absorb the unilateral deviations of other traders. This positively affects price impacts of the remaining traders. The mutual reinforcement effect amplifies the overall level of price impact in the economy.

It does not matter for the price impact whether the trader is a firm maximizing profit or a consumer. In equilibrium, consumers and firms with the same convexity have identical price impacts.

Surprisingly, price impact *does not* depend on the trader's share in the total volume of trade. Even if one of the traders is a sole supplier of a commodity on the market, she has the same price impact as others, provided that the convexities are the same.

Price impacts are *always* decreasing in the absolute number of traders. Note that this is not necessarily the case in the model of Cournot. With a linear demand function, the Cournot model predicts/assumes that the price impact of an oligopolist is the same, regardless of the number of firms operating in the industry and it is equal to the slope of a demand of the industry. This is because in the Nash equilibrium, strategic traders do not react to unilateral off-equilibrium deviations, and the price impacts are solely determined by the reaction of competitive consumers.

In the economic literature, market power is associated with large traders. This is because the same price impact affects traders' decisions in a different way, depending on her volume of trade. For example, with the price impact and the supply equal to one, the producer, by selling additional unit of a good, is losing only one dollar on the previous sales due to a reduced price. When the original volume of trade is equal to one hundred units, the loss of profit amounts to one hundred dollars. Consequently, large traders are affected more severely than the smaller ones, and hence their non-competitive behavior is more pronounced. In other words, what matters is the elasticity of the demand function, and price impact is only one of its components.

Finally, I would like to stress the role of a harmonic mean in the framework. The consistency condition on price impacts implies that a trader's price impact is equal to a harmonic mean of price impacts and convexities of utility or cost functions of the remaining traders,

$$M^n = \frac{1}{N-1}\mathcal{H}\left(M^n + V^n | n' \neq n\right), \tag{86}$$

and multiplied by a discount factor $1/(N-1)$. Harmonic mean arises naturally in the consistency condition as it puts higher weight on the traders with smaller convexities. Observe that such traders absorb most of the unilateral deviation and hence are key in price impact determination. The traders with almost linear utility or cost functions are willing to absorb off-equilibrium deviation without any price reward, making price impacts of other traders negligible. This corresponds to the property of the harmonic mean that whenever one of its elements converges to zero, the value of the mean also becomes zero.

9.2 Mergers and Antitrust Policy

The analytical framework used by the Department of Justice and the Federal Trade Commission to determine whether mergers are likely to lessen the competition of the industry is primarily based on the Herfindahl-Hirschman Index (HHI). HHI measures post-merger market concentration by summing the squares of the market shares of all market participants.

$$HHI = \sum_{j \in J} \left(s^j\right)^2, \tag{87}$$

where s^j is a share in the sales on the market by firm j. Markets with HHI above 1800 are considered highly concentrated, and mergers increasing HHI by more than 100 points are assumed to be likely to "create or enhance market power or facilitate its exercise."

The model presented in this paper suggests that HHI does not contain sufficient information for the evaluation of the *unilateral*[21] effects of mergers on competitiveness of the market. This is because, other things being equal, oligopolistic market power critically depends on the convexity of the cost functions of the other firms. In the industries characterized by large fixed costs and approximately constant marginal costs, the interactions should be competitive even if the number of firms is small. Consider a software industry with three firms of equal size, and suppose that two of them are considering a merger to save on fixed costs. The post merger Herfindahl-Hirschman index is equal to

$$HHI = 5445, \tag{88}$$

so the market is considered as highly concentrated. In addition, the merger increases HHI by 2178, and therefore the merger is likely to be blocked. If, however, the marginal cost is constant, such a merger does not enhance market power at all! In addition, with increasing returns to scale such a merger is Pareto improving, as it reduces the fixed cost expenses.

[21] The model by assumption rules out the merger effects on lessening the competition through a coordinated interaction.

10 Relation to Other Non-competitive Models

When external absorption is equal to one for all traders, the framework becomes the Walrasian one. In this section, I argue that several other non-competitive models of market interactions are also embedded in the proposed framework.

10.1 Unilateral Monopoly, Dixit-Stiglitz Monopolistic Competition

The models of unilateral monopoly assume that only one trader in each market has market power. An example of such models include a standard monopoly or monopsony. Such models have been extensively studied in economic literature.

Technically, the models of a unilateral monopoly can be obtained within the proposed framework by setting the value of external absorption of one trader to zero, and for the remaining traders to one. For example, in the economy with one consumer and one firm, setting $\rho^i = 1$ for the consumer and $\rho^j = 0$ for the producer transforms the framework into a standard model of monopoly. When the values are reverted, i.e., $\rho^i = 0$ and $\rho^j = 1$, it becomes a model of monopsony. The proper values of external absorption for a buyer and a seller in the Edgeworth box economy define a model of monopoly or monopsony.

The most prominent example of a unilateral monopoly is a model of monopolistic competition proposed by Dixit and Stiglitz [1977]. Dixit and Stiglitz consider an economy with one competitive consumer and J firms, each producing one non-numeraire good. Their representative consumer has preferences given by a utility function

$$U\left(m, \left(\sum_l (x_l)^{\frac{s-1}{s}}\right)^{\frac{s}{s-1}}\right). \tag{89}$$

$U(\cdot,\cdot)$ is a homogenous, strictly quasi-concave and increasing function and $s \in (1,\infty)$. Such preferences are more general than the ones assumed in this paper, as they allow for different values of marginal rate of substitution between money and "utility" from non-numeraire goods. I will argue, however, that in deriving their equilibrium, Dixit and Stiglitz implicitly assumed a constant marginal rate of substitution on the individual level (the marginal rate of substitution is determined on the aggregate level).

If one defines a modified utility function as $\tilde{U}(m,u) \equiv U\left(m, u^{\frac{s-1}{s}}\right)$, the utility function

$$\tilde{U}\left(m, \sum_l (x_l)^{\frac{s-1}{s}}\right) \tag{90}$$

represents the same preferences as the one in (89), and its non-numeraire argument is additively separable. The aggregate price index $\bar{p}$ and an index of aggregate non-numeraire consumption $\bar{x}$ in the economy is defined as

$$\bar{p} = \left(\sum_l (\bar{p}_l)^{1-s}\right)^{\frac{1}{1-s}} \text{ and } \bar{x} = \left(\sum_l (x_l)^{\frac{s-1}{s}}\right)^{\frac{s}{s-1}}. \tag{91}$$

The maximization of the utility function on a budget set,

$$m + \sum_l p_l x_l \le I, \tag{92}$$

yields the following optimal non-numeraire demand of each l:

$$x_l = \left(\frac{1}{p_l}\right)^s \bar{x}\bar{p}^s. \tag{93}$$

(93) and the budget constraint in (92) determine the consumption of numeraire. In general it is very hard to find a closed form solution for the system of $L+1$ non-linear equations and $L+1$ unknowns. Dixit and Stiglitz argue, however, that when the number of non-numeraire goods is large, the change of an individual price should have a negligible effect on the aggregate price level $\bar{p}$ and also on the aggregate consumption $\bar{x}$. They conclude that the elasticity of the demand for individual good should be approximately equal to s.

It is immediate to show that in optimum, the marginal rate of substitution between a numeraire and non-numeraire utility is equal to

$$\frac{\tilde{U}_m(\cdot)}{\tilde{U}_x(\cdot)} = \frac{s-1}{s}\left(\bar{x}\bar{p}^s\right)^{-\frac{1}{s}}. \tag{94}$$

Therefore, fixing the value of $\bar{x}\bar{p}^s$ is equivalent to making $\tilde{U}_m/\tilde{U}_x$ constant, too. In other words, Dixit and Stiglitz assume that from the perspective of an individual firm, the representative consumer behaves as if she had a quasi-linear utility function, additively separable with respect to non-numeraire goods

$$\tilde{U}\left(m, \sum_l (x_l)^{\frac{s-1}{s}}\right) = \left(\tilde{U}_m/\tilde{U}_x\right) m + \sum_l (x_l)^{\frac{s-1}{s}}.$$

Such a utility functional satisfies the assumptions A1-A5 in this paper.

To close the model, Dixit and Stiglitz endogenize $\tilde{U}_m/\tilde{U}_x$ on the aggregate level. The unique equilibrium value of $\tilde{U}_m/\tilde{U}_x$, and the production level $\bar{x}_l$ is determined[22] by the condition that the consumers' demand is sufficiently large to insure that the firm breaks even, given the elasticity of the demand equal to s and fixed cost equal to F. The number of goods produced in the economy, L, is then determined to make the equilibrium marginal rate of substitution $\tilde{U}_m/\tilde{U}_x$ to be a true one. Formally, L is implicitly defined by the condition

$$\frac{\tilde{U}_m\left(m, Lx_l^\gamma\right)}{\tilde{U}_x\left(m, Lx_l^\gamma\right)} = \frac{\tilde{U}_m}{\tilde{U}_x}. \tag{95}$$

It is straightforward to generalize the framework in this paper to allow for the preferences as in the monopolistic competition model of Dixit and Stiglitz. The derivation of an equilibrium requires some approximations, and therefore such a generalization come at the cost of not perfect internal consistency. In this paper, I focus on the interdependencies of market power among traders rather than income effects in non-competitive trading. Therefore, I leave such an extension for future research.

[22] In the version of monopolistic competition with a fixed number of operating firms, the marginal rate of substitution is determined as a solution to the system of two equations and two unknowns (93), with MRS substituted from (94), giving $x_l\left(\tilde{U}_m/\tilde{U}_x\right)$ and $\tilde{U}_m(m, x_l)/\tilde{U}_x(m, x_l)$.

10.2 Bilateral/Multilateral Monopoly, Game-theoretic Approach.

Modelling a bilateral monopoly is more complicated than the unilateral one. The complication results from the fact that the price impact of the oligopolist depends on the behavior and hence the price impact of the other oligopolist and *vice versa*. Such mutual interdependence of price impacts is explicitly studied in this paper. The game theoretic models solve the problem in an alternative way. They cut the link between the price impacts of different traders by imposing the condition that only price takers react to unilateral deviation of strategic players. For example, this is the case in the Cournot model.

In the Cournot model, each firm is maximizing a profit, given a demand of the industry

$$p\left(y^j + \sum \bar{y}^j\right). \tag{96}$$

If the demand is determined by a market clearing condition, as in the monopolistic competition studied by Gabszewicz and Vial [1972], function $p()$ is implicitly determined by the market clearing condition

$$\sum \bar{t}^{i,Walras}(p) = y^j + \sum_{j\prime \neq j} \bar{y}^{j\prime}, \tag{97}$$

which, given the inelastic supply function, $y^j = -t^j$, can be written as

$$t^j + \sum_{j\prime \neq j} \bar{t}^{j\prime} + \sum \bar{t}^i(p, 0) = 0. \tag{98}$$

This shows that the allocation in a Cournot-Nash equilibrium is numerically equivalent to a $\rho-$competitive equilibrium, in which external absorption of the consumers is set to one and the firms' off-equilibrium market clearing condition is modified so that the firms do not react to unilateral deviations (98).

The Bertrand model can also be embedded in the framework with modified off-equilibrium price. The Bertrand firms, however, instead of optimally responding to changes in the demands, are assumed to supply *any* demanded quantity of the good at the equilibrium price. In other words, in Nash equilibrium their trade correspondence is "flat" and defined as

$$\bar{t}^{j'}\left(p, \bar{M}\right) = \left\{0 \geq -t^j \geq \infty \text{ for } p = \bar{p}\right\}. \tag{99}$$

With such supply correspondences with "zero" slopes, the off-equilibrium market clearing condition predicts zero price impacts of all traders—any unilateral deviation is absorbed by the other traders without any price change. The similar reasoning generalizes to the Grossman's model of Nash in Supply Function from Grossman [1981].

Another well-known model of a bilateral monopoly, the Shapley and Shubik strategic market game, cannot be mapped into the proposed framework. The reason is that in such a game, *none* of the traders responds optimally to changing prices after a unilateral deviation.

The framework is most closely related to the models studied by Bresnahan [1981], Perry [1982] or Hart [1985]. The key difference is that here traders here have conjectures about the deviations of prices as a function of off-equilibrium *deviations* rather that the absolute level of trade, and they make consistent conjectures not about a best response of the other player, but about the equilibrium in an economic system with many heterogenous traders.

11 Perfect Conjectural Equilibrium

A $\rho-$competitive equilibrium from this paper is an example of a Perfect Conjectural Equilibrium, a solution concept formulated in Weretka [2005c]. to moddel strategic interactions in complex systems in which players have a possibily to revise their strategies and the only information about their environment is limited to the observations of variables relevant in their payofs that comes from their experimentation. Each player forms believes of how she affects the behavior of other players $\Delta s_{-i}(\Delta s_i)$ and take into account when reacting in and out of equilibirum.

12 Non-competitive Second Welfare Theorem

Theorem 3 asserts that the equilibrium allocation is typically Pareto inefficient. It is natural to ask whether it is possible to design a tax system that corrects the incentives, so that Pareto efficient allocation is implementable as a $\rho-$competitive equilibrium.

Consider a smooth tax rule, $T = \left(T^1, T^2 \ldots, T^I, \ldots, T^N\right)$, defined as a vector of N smooth functions, each specifying a tax obligation of trader n in terms of numeraire for any possible value of trade t^n. I restrict attention to separable tax rules, that is the rules for which Hessians D^2T^n are diagonal .

By t_{PE} I denote a unique Pareto efficient trade[23]. In a smooth and separable economy, t_{PE} must satisfy

$$Du^i|_{t^i_{PE}} = Df^j|_{t^j_{PE}} \tag{100}$$

for any $i, j \in \mathcal{N}$. The condition (100) and market clearing is sufficient for Pareto optimality of t_{PE}.

With a smooth tax rule $T(\cdot)$, the necessary condition for stability of consumer's trade is

$$Du^i\left(t^i + e^i\right) = \bar{p} + \Psi^i, \tag{101}$$

where Ψ^i is a distortion associated with a non-competitive trading and taxation. For any producer j, in the equation (101) marginal utility is replaced with the marginal cost. The distortion can be written as

$$\Psi^n = \bar{M}^n_T t^n + DT^n. \tag{102}$$

Since in equilibrium markets clear, to implement the Pareto efficient allocation, it is sufficient to make Ψ^n equal across all traders n, at t_{PE}. I focus on tax rules for which the distortion is null, $\Psi^n = 0$.

A tax rule affects Ψ^n in two ways. Its gradient enters equation (102) directly, changing the first order incentives. In addition, taxes determine the price impacts of all traders in equilibrium, $\bar{M}_T$ (hence $\bar{M}$ has subscript T). Does there exist a smooth tax rule for which two effects offset and hence $\Psi^n = 0$? To make the argument more transparent, I introduce the following notation: $T'^n \equiv DT^n\left(t^n_{PE}\right)$ is a gradient of a tax rule at the Pareto efficient

[23] In a smooth and separable economy, there is a unique allocation x satisfying this condition. Therefore, this will also be an allocation in a unique Walrasian Equilibrium.

allocation and $T''^i \equiv D^2 T^i \left(t^i_{PE}\right)$ is the Hessian. T', and T'' give T'^n and T''^n for all traders. The next proposition establishes the existence of a tax rule implementing the Pareto efficient allocation in a $\rho-$competitive equilibrium:

Proposition 5 *Consider T'' such that $-D^2u^i\,() >> T''^i$ for any consumer $i \in \mathcal{I}$ and $D^2 f^j\,() >> T''^j$ for any producer $j \in \mathcal{J}$. Then there exists a unique T', such that $\Psi^n = 0$ for all n.*

One of the implications of Proposition 5 is that for any T'' one can find a gradient T' that makes the distortion equal to zero. Consequently, the proposition implies the existence of a whole class of tax rules T implementing a Pareto efficient allocation in $\rho-$competitive equilibrium. In Proposition 5 the restrictions on T'' guarantee that the maximization problem of each trader is convex and hence first order stability conditions are sufficient. It should be stressed that it is sufficient for the result that the taxes are paid by at least two of the traders in the economy and not all traders.

The next result demonstrates that the distortion related to non-competitive trading can be made arbitrarily small by choosing a tax rule with Hessians sufficiently close to Hessians of corresponding utility or cost functions.

Proposition 6 *Let $\{T_k\}_{k\in\{1,2,\ldots\infty\}}$ be a sequence of tax rules that satisfy the assumptions of Proposition 5 and for each consumer are such that*

$$\lim_{k\to\infty} T''^i_k = D^2u^i\left(t^i_{PE} + e^i\right), \tag{103}$$

and for a firm

$$\lim_{k\to\infty} T''^j_k = -D^2 f^j\left(-t_{PE}\right), \tag{104}$$

then for each trader

$$\lim_{k\to\infty}\left(M^n_{T_k}\right)t^n = 0. \tag{105}$$

Proposition 6 shows that it is possible to implement Pareto efficient allocation almost without affecting the first-order incentives by *de-convexifying* the optimization problem of each trader around Pareto efficient trade t^n_{PE}. The example of the sequence of tax rules from 6 is

$$\begin{aligned} T^{i,k}\left(t^i\right) &= \left(\frac{k-1}{k}\right)\left(t^i - t^i_{PE}\right)^T\left(D^2u^i\left(t^i_{PE}+e^i\right)\right)\left(t^i - t^i_{PE}\right), \\ T^{j,k}\left(t^j\right) &= -\left(\frac{k-1}{k}\right)\left(t^j - t^j_{PE}\right)^T\left(D^2 f^j\left(t^j_{PE}\right)\right)\left(t^j - t^j_{PE}\right). \end{aligned} \tag{106}$$

Note that such a sequence satisfies the restriction on Hessians from Proposition 6. Since $D^2u^i\left(t^i_{PE}+e^i\right)$ is negative definite and $D^2 f^j\left(-t^j_{PE}\right)$ is positive definite, the maximal tax obligation is associated with trade t^n_{PE}, and it is equal to zero. For all other trades, $t^n \neq t^n_{PE}$, the value of the tax is negative. In other words, trader n is paid to deviate from the Pareto efficient allocation. Why is t_{PE} implemented in a $\rho-$competitive equilibrium by a tax rule that gives second-order incentives to deviate from it? Intuitively, the tax rule T^k makes the marginal utility or cost nearly constant around $t_{PE}^{n'}$ for all other traders n'. Therefore for any unilateral deviation by n, the remaining traders are willing to take the deviation out of the market without any change in price and hence market power of n is zero. Since this is true for all traders, the distortion associated with market power disappears and the equilibrium becomes competitive.

13 Conclusions

In this paper, I propose a framework for studying the outcomes of market interactions and the determinants of market power in an economy with a small number of traders or with heterogenous traders. I study economies with consumers characterized by separable utility function and firms, using a numeraire as the only input. Though these separability assumption are strong, the model has important applications. For example, when modeling trade under uncertainty, the traders are often assumed to maximize expected utility—a utility function that is separable with respect to consumption in different states of the world. This model also makes it possible to study phenomena associated with non-competitive trading of firms that do not depend on cross-market effects.

The assumption of separability is relaxed in Weretka[2005a]. There I study phenomena associated with strategic pricing across markets. In particular, I investigate cross-subsidization of markets and substitution of inputs traded in noncompetitive markets by competitive inputs. I also discuss the problem of existence of equilibrium, determinacy and testability in a non-separable economy.

A vast empirical literature suggests that the trades of large institutional investors do impact asset prices and hence returns, and that investors take this impact into account when trading. Yet the existing models of equilibrium asset pricing assume that all investors are "return takers." In Weretka [2005b], I use the abstract framework of non-competitive trading to explain the effect of institutional trading on asset prices and allocations. I discuss the effects of non-competitive trading in a model with mean-variance optimizing traders in a strategic version of CAPM (S-CAPM). I also show that market power may account for the equity premium and the risk-free interest rate puzzles when the utility functions are characterized by prudence.

In [2005c], I formulate a concept of *perfect conjectural equilibrium* in an abstract non-cooperative smooth game, possibly with more than two heterogenous players. Intuitively, in such a game any marginal off-equilibrium deviation of player i triggers a game among all remaining players, and the equilibrium outcome of this game is predicted correctly by player i. When playing the game, other players are assumed to take into account their impacts on the strategies of the other players. I study the properties of such equilibria.

References

[1] Arrow, K.J., and Hahn, F.H., (1971), General Competitive Analysis, Holden Day

[2] Bowley, A.L., (1924), The Mathematical Groundwork of Economics, Oxford University Press, Oxford

[3] Boyer, M., and Moreaux, M., (1983a), Consistent versus Non-Consistent Conjectures in Duopoly Theory: Some Examples, *Journal of Industrial Economics*, 32(1), pp. 97-110

[4] Boyer, M., and Moreaux, M., (1983b), Conjectures Rationality and Duopoly Theory, *Intl. Journal of Industrial Organization* 1, pp. 23-41

[5] Bresnahan, T.F., (1981), Duopoly Models with Consistent Conjectures, *American Economic Review*, 71(5), pp. 934-945

[6] Bresnahan, T.F., (1983), Existence of Consistent Conjectures: Reply, *American Economic Review,* 73, pp. 457-458

[7] Chan, K.C., and Lakonishok J., (1993) Institutional Traders and Intra-day Stock Price Behavior *Journal of Financial Economics,* 33, pp. 173-199

[8] Chan, K.C., and Lakonishok, J., (1995) The Behavior of Stock Price Around Institutional Trades *Journal of Finance,* 50, pp. 1147-1174

[9] Dixit, K., and Stiglitz J.E., (1977), Monopolistic Competition and Optimum Product Diversity, *American Economic Review*, 67 (3), pp. 297-308

[10] Frish, R., (1951), Monopoly, Polypoly. The concept of force in the economy, *International Economic Papers*, 1, pp. 23-36

[11] Figueieres, C., Jean-Marie A., Quirou N., Tidball M., (2004), Theory of Conjectural Variation, Series on Mathematical Economics and Game Theory Vol. 2, World Scientific.

[12] Holthousen, R., Leftwich, R., and Mayers, D., (1987) The Effects of Large Block Transations The Speed of Response and Temporary and Permanent Stock-Price Effects, *Journal of Financial Economics* 26, pp. 71-95 ol 2., World Scientific Publishing

[13] Gabszewicz, J.J., and Vial, J.P., (1972), Oligopoly "A la Cournot" in a General Equilibrium Framework, *Journal of Economic Theory,* 4, pp. 381-400

[14] Gantmacher, F.R., (2000), The Theory of Matrices, Providence, RI: AMS Chelsea

[15] Grossman, S., (1981), Nash Equilibrium and the Industrial Organization of Markets with Large Fixed Costs, *Econometrica,* 49, pp.1149-1172

[16] Hahn, F.H., (1978), On Non-Walrasian Equilibria, *Review of Economics Studies*, 45 (1), pp.1-17

[17] Hahn, F.H., (1977), Exercises in Conjectural Equilibria, *Scandinavian Journal of Economics*, 79(2), pp. 210-26

[18] Hart, O., (1979), Monopolistic Competition in a Large Economy with Differentiated Commodities, *Review of Economics Studies,* 46 (1), pp. 1-30

[19] Hart, O., (1982), Reasonable Conjectures, mimeo London School of Economics

[20] Hausman, J.A., Lo J., and McKinlay, C.A., (1992) An Ordered Probit Analysis of Transaction Stock Prices, *Journal of Financial Economics*, 31, pp. 319-397

[21] Holthousen, R., Leftwich R., and Mayers, D., (1987) The Effects of Large Block Transations on Security Prices: A Cross Sectional Analysies, *Journal of Financial Economics* 19, pp. 237-267

[22] Holthousen, R., Leftwich R., and Mayers, D., (1990) Large Block Transations The Speed of Response and Temporary and Permanent Stock-Price Effects, *Journal of Financial Economics* 26, pp. 71-95.

[23] Kamien, M., and Schwartz, N., (1983), Conjectural Variation, *Canadian Journal of Economics*, 16(2), pp.192-211

[24] Keim, D.B., and Madhavan, A., (1996) The Upstairs Markets for Large-Block Transations. Analisis and Meassurement of Price Effects *Review of Financial Studies* 9, pp. 1-39

[25] Klemperer, P., and Meyer, M.A., (1989), Supply Function Equilibria in Oligopoly under Uncertainty, *Econometrica*, 57, pp. 1243-1277

[26] Kraus, A., and Stoll, H., (1972) Price impact on block trading on the New York Stock Exchange, *Journal of Finance* 27, pp. 569-588

[27] Laitner, J., (1980), “Rational” Duopoly Equilibria, *Quarterly Journal of Economics*, 95(4), pp. 641-662

[28] Negishi, T., (1960), Monopolistic Competition and General Equilibrium, *Review of Economics Studies*, 28 (3), pp.196-201

[29] Perry, M., (1982), Oligopoly and Consistent Conjectural Variation, *Bell Journal of Economics*, 13, pp. 197-205

[30] Roberts, J., and Sonnenschein, H., (1977), On the Foundations of the Theory of Monopolistic Competition, *Econometrica,* 45, pp.101-113

[31] Robson, A., (1983), Existence of Consistent Conjectures: Comment *American Economic Review,* 73, pp. 454-456

[32] Shapley, L., and Shubik, M., (1977), Trade Using One Commodity as a Means of Payments, *Journal of Political Economy*, 85(5), pp. 937-968

[33] Triffin, R., (1940), Monopolistic Competition an General Equilibrium Theory, Harvard University Press, Cambridge, Mass.

[34] Weretka, M., (2005a), Perfect Conjectural Equilibrium and Endogenous Market Power: a Non-separable Case, mimeo Yale University

[35] Weretka, M., (2005b), The Institutional Traders and Non-competitive Financial Markets, mimeo Yale University

[36] Weretka, M., (2005c), Perfect Conjectural Equilibria in Games, mimeo Yale University

14 Appendix

Proof. (Proposition 1: Existence and uniqueness of a stable trade for a consumer.)

Consider a trade that maximizes preferences $U^i\left(t^i+e^i,m^i\right)$ on a budget set $B\left(\bar{p},\bar{t}^i,\bar{M}^i\right)$, and is feasible $t^i+e^i\geq 0$. Formally, such $\left(t^i,m^i\right)$ is a solution to the program

(107) $$\max_{t^i,m^i} U\left(t^i+e^i,m^i\right),$$

subject to

$$\left(t^i,m^i\right)\in B\left(\bar{p},\bar{t}^i,\bar{M}^i\right) \text{ and } t^i\geq -e^i.$$

$U\left(\cdot\right)$ is strictly concave in t^i, and constraints are convex, provided that $\bar{M}^i$ is positive definite. In addition, by strict monotonicity of preferences (in money), in optimum budget constraint will be satisfied with equality. With symmetric $\bar{M}^i$, the necessary Khun – Tucker condition of optimality is given by

(108) $$\begin{aligned} Du^i-\bar{p}-\bar{M}^i(2t^i-\bar{t}^i)&\leq 0,\\ p_{\bar{p},\bar{t}^i,\bar{M}^i}\left(t^i\right)\cdot t^i+m^i &=0,\\ p_{\left(\bar{p},\bar{t}^i,\bar{M}^i\right)}\left(t^i\right)&=\bar{p}+\bar{M}^i\left(t^i-\bar{t}^i\right), \end{aligned}$$

where the first inequality holds with equality for all commodities such that $t^i_l>-e^i_l$. By the definition of a stable trade, $\bar{t}^i$ is optimal on a budget set generated by itself and therefore the additional necessary stability condition is

(109) $$t^i=\bar{t}^i.$$

Equation (109) allows me to reduce (108) to

(110) $$\begin{aligned} Du^i-\bar{M}^i\bar{t}^i &\leq \bar{p},\\ \bar{m}^i+\bar{p}\cdot\bar{t} &= 0. \end{aligned}$$

The inequality asserts that at the stable trade, marginal utility can be at most equal to marginal revenue. For any positive definite, diagonal matrix $\bar{M}^i$, equation (110) has a unique solution $\bar{t}^i$. To see it observe that the first group of inequalities in (110) forms a system of L independent inequalities, each just in one unknown. Each diagonal element is $\bar{M}^i_{l,l}>0$, therefore the left hand side is decreasing in $\bar{t}^i_l$. This function attains its maximal value, $p_l^{reservation}$, at $\bar{t}^i_l=-e^i$ and is equal to zero for some finite value of $\bar{t}^i_l$. Therefore, for any p_l

in the interval $[0, p_l^{reservation}]$, there exists a unique positive value of $\bar{t}_l^i$ that solves inequality (110) with equality, and for p_l in $\left(p_l^{reservation}, \infty\right)$, the unique value of $\bar{t}^i$ satisfying (110) is zero. Given $\bar{t}^i$ that solves all inequalities (110), a unique stable $\bar{m}^i$ is determined by the last equality (110). This shows that for any $\bar{p} \geq 0$ and positive definite $\bar{M}^i$ diagonal matrix, there exists a unique $\bar{t}^i$ that is stable. ■

Proof. (**Corollary 1: Stable trade function is differentiable in $\bar{p}$.**).

Proposition 1 asserts that for any positive definite diagonal matrix $\bar{M}^i$ and a price vector $\bar{p}$ such that $\bar{p} \geq 0$, there exists a unique stable trade $\bar{t}^i$. This defines a function $t^i\left(\bar{p}, \bar{M}^i\right)$. In addition, when $\bar{t}^i >> -e^i$, non-numeraire stable demand is implicitly defined by the equality of the marginal utility and the marginal revenue (see (110))

$$Du^i\left(\bar{t}^i + e^i\right) = \bar{p} + \bar{M}^i \bar{t}^i \ . \tag{111}$$

The derivative of the implicit function with respect to $\bar{t}^i$ is given by

$$\bar{M}^i - D^2 u^i\left(\cdot\right). \tag{112}$$

By assumption, $-D^2 u^i\left(\cdot\right)$ and $\bar{M}^i$ are positive definite, therefore matrix (112) has full rank. The implicit function theorem says that in the neighborhood of $\bar{p}$, a stable trade function, $t^{*i}\left(\bar{p}, \bar{M}^i\right)$, solving equation(110) is differentiable. Since this holds for any interior trade, $t^i\left(\bar{p}, \bar{M}^i\right)$ is smooth for all interior trades. The implicit function theorem implies that the derivative of the trade with respect to price

$$D_p^2 t\left(\bar{p}, \bar{M}^i\right) = -\left(\bar{M}^i - D^2 u^i\right)^{-1} \tag{113}$$

is an inverse of the Hessian and the price impact matrix. ■

Proof. (Proposition 2: Existence of a differentiable, stable trade function for a firm.)

Consider a problem of a firm choosing a trade t^j to maximize a profit

$$\max_{t^i, m^j} \pi = -\left(p_{\bar{p}, \bar{t}^j, \bar{M}^j}\left(t^j\right) \cdot t^j + m^j\ \right), \tag{114}$$

and is constrained by the production set determined by:

$$m^j \geq f^j\left(-t^j\right) \text{ and } t^j \leq 0. \tag{115}$$

With positive definite $\bar{M}^j$ price impact matrix, the objective function is strictly concave. With $\bar{M}^j$ symmetric, the necessary conditions for t^j to be optimal on the production set are

$$\begin{gathered} \bar{p} + \bar{M}^j (2t^j - \bar{t}^j) - Df^j \leq 0, \\ m^i = f^j\left(-t^j\right), \\ p_{\left(\bar{p}, \bar{t}^j, \bar{M}^j\right)}\left(t^j\right) = \bar{p} + \bar{M}^j\left(t^j - \bar{t}^j\right), \end{gathered} \tag{116}$$

where the inequalities hold with equality for all interior trades $\bar{t}_l^j < 0$. The stability requires that the trade is optimal with respect to "itself" and therefore

$$t^j = \bar{t}^j. \tag{117}$$

The necessary and sufficient stability condition becomes

$$\bar{p} \leq Df^j\left(-\bar{t}^j\right) - \bar{M}^j\bar{t}^j, \\ \bar{m}^i = f^j\left(-t^j\right), \tag{118}$$

where again the inequality holds with equality for all interior goods such that $\bar{t}^j_l < 0$. Given positive definite diagonal $\bar{M}^j$ for any l, each of the first L inequalities depends only on one variable, $\bar{t}^j_l$. The right hand side is continuous and monotonically decreasing in $\bar{t}^j_l$, from infinity for $\bar{t}^j_l$ approaching minus infinity (convexity), to zero for $\bar{t}^j_l = 0$. There therefore exists a unique non-positive $\bar{t}^j$ satisfying the equation with equality for any price $p_l \geq 0$. This establishes the existence of a function $\bar{t}^j\left(\bar{p}, \bar{M}\right)$. By analogous arguments, as in the case of a consumer, such function is differentiable with the derivative given by

$$D^2_p t\left(\bar{p}, \bar{M}^j\right) = -\left(M^j + D^2 f^j\right)^{-1},$$

which is symmetric to the derivative of the consumer's demand. ■

Proof. (Proposition 3: Existence of $\rho-$consistent matrices.)
First I prove the proposition for $N > 2$ in the following four steps: I show that the existence of $\rho-$consistent $\bar{M}$ can be reformulated as a problem of the existence of a fixed point of some function $H(\cdot)$ that maps the set of vectors of positive semi-definite matrices into itself. In the case of smooth and separable economies, $H(\cdot)$ maps the set of positive semi-definite *diagonal* matrices into itself (Claim 1). Next I construct a non-empty, compact, and convex subset of positive semi-definite diagonal matrices (Claim 2) and show that $H(\cdot)$ maps this set into itself (Claim 3). Since $H(\cdot)$ is continuous, standard argument in Claim 4 establishes the existence of consistent matrices. Finally, in Claim 5, I consider the case economy with only two traders, $N = 2$. In the proof, ρ and Hessians $D^2u^i\left(\bar{t}^i + e^i\right)$, and $D^2 f^j\left(-\bar{t}^j\right)$ are assumed to be fixed for all $n = i, j$.

Claim 1 *($\rho-$consistent $\bar{M}$ is a fixed point of $H(\cdot)$.)*

(Argument for Claim 1): A system $\bar{M}$ is $\rho-$consistent with disequilibrium market clearing at $\bar{t}$, if for any $n \in \mathcal{N}$

$$M^n = D_t p^n\left(\bar{t}, M^{-n}\right). \tag{119}$$

With (39), equation (119) can be written as

$$\bar{M}^n = (1-\rho^n)\left[\sum_{n' \neq n}\left(\bar{M}^{n'} + \bar{V}^{n'}\right)^{-1}\right]^{-1}, \tag{120}$$

where $\bar{V}^i = -D^2u^i\left(e^i + \bar{t}^i\right)$ is a negative of a Hessian of a utility function and $\bar{V}^j = D^2 f^j\left(-\bar{t}^j\right)$ is a Hessian of a cost function. Note that $\bar{V}^n$ is (strictly) positive definite for all $n \in \mathcal{N}$. Let $\bar{V}$ be a specification of all $\bar{V}^n$,

$$\bar{V} = \left(\bar{V}^1, \ldots, \bar{V}^I, \bar{V}^{I+1}, \ldots, \bar{V}^N\right). \tag{121}$$

The consistency condition on $\bar{M}$ (120) is mathematically equivalent to $\bar{M}$ being a fixed point of a function $H\left(M,\bar{V}\right)$

$$H\left(\cdot\right)=\left(H^{1}\left(\cdot\right),\ldots,H^{I}\left(\cdot\right),H^{I+1}\left(\cdot\right)...,H^{N}\left(\cdot\right)\right), \tag{122}$$

where

$$H^{n}\left(M,\bar{V}\right)=\left(1-\rho^{n}\right)\left[\sum_{n'\neq n}\left(M^{n'}+\bar{V}^{n'}\right)^{-1}\right]^{-1}, \tag{123}$$

and $\bar{V}$ is fixed.

Claim 2 *(Construction of a non-empty convex and compact set)*

(Argument for Claim 2): Let $M=\left\{M^{1},\ldots,M^{I},M^{I+1},\ldots,M^{N}\right\}$, where M^{n} is a $L\times L$, positive definite, diagonal matrix, and $\mathcal{M}$ is a set of all such M. Define two functions on $\mathcal{M}$, mapping $\mathcal{M}\rightarrow\mathbb{R}_{+}$, the lower and upper "norm", $\overline{\|M\|}$ and $\underline{\|M\|}$ in the following way:

$$\overline{\|M\|}=\max_{n\in\mathcal{N}}\overline{\|M^{n}\|^{*}}, \text{ where } \overline{\|M^{n}\|^{*}}=\max_{l\in\mathcal{L}}\left(M_{l,l}^{n}\right), \tag{124}$$

and

$$\underline{\|M\|}=\min_{n\in\mathcal{N}}\underline{\|M^{n}\|^{*}}, \text{ where } \underline{\|M^{n}\|^{*}}=\min_{l\in\mathcal{L}}\left(M_{l,l}^{n}\right). \tag{125}$$

The two functions (and also their "matrix" equivalents $\overline{\|\cdot\|^{*}}$ and $\underline{\|\cdot\|^{*}}$) have the following properties, similar to the properties of a norm in a vector space:

a) $\overline{\|M\|}=0$ if and only if $M=0$,

b) scalar linearity: for all $\alpha\geq0$

$$\overline{\|\alpha M\|}=\alpha\overline{\|M\|}, \tag{126}$$

and

$$\underline{\|\alpha M\|}=\alpha\underline{\|M\|}, \tag{127}$$

c) triangle inequalities:

$$\overline{\|M_{1}+M_{2}\|}\leq\overline{\|M_{1}\|}+\overline{\|M_{2}\|}, \tag{128}$$

and

$$\underline{\|M_{1}+M_{2}\|}\geq\underline{\|M_{1}\|}+\underline{\|M_{2}\|}, \tag{129}$$

d) functions $\overline{\|M\|}$ and $\underline{\|M\|}$ are continuous with respect to elements of M.

Next, for any two non-negative scalars $\underline{\lambda}$ and $\bar{\lambda}$, satisfying $0\leq\underline{\lambda}\leq\bar{\lambda}<\infty$, define $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}\subset\mathcal{M}$

$$\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}=\left\{M\in\mathcal{M}\,|\,\overline{\|M\|}\leq\bar{\lambda}\text{ and }\underline{\|M\|}\geq\underline{\lambda}\right\}. \tag{130}$$

Set $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ is non-empty, convex and compact in $\mathbb{R}^{N\times L\times L}$. For non-emptiness, observe that diagonal M with diagonal entries $\underline{\lambda}$ are always members of $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ for any $\underline{\lambda} \leq \bar{\lambda}$. For convexity, suppose $M_1,\ M_2 \in \mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$. Then $\alpha M_1 + (1-\alpha) M_2$ consists of matrices $\alpha M_1^n + (1-\alpha) M_2^n$ for each $n \in \mathcal{N}$ that are positive semi-definite and diagonal. In addition, by the triangle inequality and the linearity of norms

$$\overline{\|\alpha M_1 + (1-\alpha) M_2\|} \leq \alpha \overline{\|M_1\|} + (1-\alpha) \overline{\|M_2\|} \leq \bar{\lambda}, \tag{131}$$

and

$$\underline{\|\alpha M_1 + (1-\alpha) M_2\|} \geq \alpha \underline{\|M_1\|} + (1-\alpha) \underline{\|M_2\|} \geq \underline{\lambda}. \tag{132}$$

This shows that $(\alpha M_1 + (1-\alpha) M_2) \in \mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$, and hence $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ is convex. Now I will argue that $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ is compact in $\mathbb{R}^{\ N\times L\times L}$. It is sufficient to show that $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ is closed and bounded in $R^{N\times L\times L}$. For closedness, let $\{M_k\}_{k=1,2\ldots,\infty}$ be a sequence in $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$, converging to some M. For any k, n^{th} matrix M_k^n is diagonal, so that M^n must also be diagonal. Since $\overline{\|\cdot\|}$ and $\underline{\|\cdot\|}$ are continuous and for any k, $\overline{\|M_k\|} \leq \bar{\lambda}$ and $\underline{\|M_k\|} \geq \underline{\lambda}$, therefore

$$\lim_{k\to\infty} \overline{\|M_k\|} = \overline{\left\| \lim_{k\to\infty} M_k \right\|} = \overline{\|M\|} \leq \bar{\lambda}, \tag{133}$$

and

$$\lim_{k\to\infty} \underline{\|M_k\|} = \underline{\left\| \lim_{k\to\infty} M_k \right\|} = \underline{\|M\|} \geq \underline{\lambda}. \tag{134}$$

This implies that $M \in \mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ and hence $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ is closed. The boundedness of $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ follows from the fact that all off-diagonal entries are 0 and the diagonal ones are bracketed between $\underline{\lambda} \geq M_{l,l} \geq \bar{\lambda}$. Closedness and boundedness of $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$ is sufficient for the compactness of $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$.

Let $\mathcal{V}$ and $\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$ be defined analogously to $\mathcal{M}$ and $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$, respectively. By identical argument, as in case of $\mathcal{M}_{\underline{\lambda}}^{\bar{\lambda}}$, for any $\underline{\gamma}$ and $\bar{\gamma}$ satisfying $0 < \underline{\gamma} \leq \bar{\gamma} < \infty$, set $\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$ is non-empty convex and compact.

Claim 3 *(Suppose $N > 2$. For any $\underline{\gamma}$ and $\bar{\gamma}$ such that $0 < \underline{\gamma} \leq \bar{\gamma} < \infty$, there exists $\bar{\lambda} < \infty$ such that function $H(M,V)$ maps $\mathcal{M}_0^{\bar{\lambda}} \times \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}} \to \mathcal{M}_0^{\bar{\lambda}}$ in a continuous way.)*

(Argument for Claim 3): Define an auxilirary function $A : \mathcal{M} \times \mathcal{V} \to \mathcal{M}$ in the following way:

$$A(\cdot) = \left(A^1(\cdot), \ldots, A^I(\cdot), A^{I+1}(\cdot) \ldots, A^N(\cdot)\right), \tag{135}$$

where each

$$A^n\left(M, \bar{V}\right) = \frac{1}{N-1}\left(\frac{1}{N-1}\sum_{n'\neq n}\left(M^{n'} + \bar{V}^{n'}\right)\right). \tag{136}$$

The n^{th} component of $A(\cdot)$, $A^n(\cdot)$ is an arithmetic mean of elements of $M^{n'} + V^{n'}$ for all $n' \in \mathcal{N} \setminus \{n\}$, and "discounted" by factor $\frac{1}{N-1} < 1$. Mapping $A(\cdot)$ preserves positive definiteness and diagonality of the matrices. Let $\bar{\lambda}$

$$\bar{\lambda} = \frac{1}{N-2}\bar{\gamma}. \tag{137}$$

Since $N > 2$, therefore $0 < \bar{\lambda} < \infty$. Observe that by the triangle inequality and the scalar linearity of $\overline{\|\cdot\|^*}$, for each n

$$\begin{aligned} \overline{\|A^n(M,V)\|^*} &= \frac{1}{N-1}\left(\frac{1}{N-1}\sum_{n\neq n'}\left(M^{n'} + \bar{V}^{n'}\right)\right) \leq \\ &\leq \frac{1}{N-1}\left(\frac{1}{N-1}\sum_{n\neq n'}\left(\overline{\|M^{n'}\|^*} + \overline{\|V^{i'}\|^*}\right)\right). \end{aligned} \tag{138}$$

For any $(M,V) \in \mathcal{M}_0^{\bar{\lambda}} \times \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$ inequality (138) implies that

$$\begin{aligned} \overline{\|A^n(M,V)\|^*} &\leq \frac{1}{N-1}\left(\bar{\lambda} + \bar{\gamma}\right) = \\ &= \frac{1}{N-1}\bar{\lambda} + \frac{N-2}{N-1}\bar{\lambda} = \bar{\lambda}, \end{aligned} \tag{139}$$

where the first equality holds by (137). Since inequality (139) holds for any n,

$$\overline{\|A(M,V)\|} \leq \bar{\lambda}. \tag{140}$$

Consider function $H^n(M,V)$

$$H^n(M,V) = (1-\rho^n)\left[\sum_{n'\neq n}\left(M^{n'} + V^{n'}\right)^{-1}\right]^{-1}. \tag{141}$$

If $V \in \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$ with $\underline{\gamma} > 0$, matrix $M^n + V^n$ is strictly positive definite for all n, hence it is invertible, so $H(M,V)$ is a well defined, continuous function on $\mathcal{M} \times \mathcal{V}$. In addition, $H(\cdot)$ preserves positive definiteness and diagonality of matrices, so that $H(\cdot) : \mathcal{M} \times \mathcal{V} \to \mathcal{M}$. By the arithmetic – harmonic mean[24] inequality and the fact that $(1-\rho^n) \leq 1$ for any n,

$$H^n(M,V) \leq A^n(M,V), \tag{142}$$

(element by element) so that

$$\overline{\|H^n(M,V)\|^*} \leq \overline{\|A^n(M,V)\|^*} \tag{143}$$

[24] Each function $H^n(M,V)$ can be written as $H^n(M,V) = \frac{(1-\rho^n)}{N-1}\mathcal{H}\left(\bar{M}^{n'} + \bar{V}^{n'} | n' \neq n\right)$

where $\mathcal{H}()$ is a Harmonic Mean of $N-1$ positive definite matrices $\left(\bar{M}^{n'} + \bar{V}^{n'}\right)$ for all traders $n' \neq n$, while function $A^n(M,V) = \frac{1}{N-1}\mathcal{A}\left(\bar{M}^{n'} + \bar{V}^{n'} | n' \neq n\right)$ where $\mathcal{A}()$ is the arithmetic mean.

and hence by inequality (140), for every $(M,V) \in \mathcal{M}_0^{\bar{\lambda}} \times \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$

$$\overline{\|H(M,V)\|} \leq \overline{\|A(M,V)\|} \leq \bar{\lambda}. \tag{144}$$

In addition, since $H(M,V)$ preserves positive definiteness of matrices, $\underline{\|H(M,V)\|} \geq 0$. This proves that $H(\cdot)$ maps in a continuous way $\mathcal{M}_0^{\bar{\lambda}} \times \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}} \to \mathcal{M}_0^{\bar{\lambda}}$.

Claim 4 *(Existence for $N > 2$.)*

(Argument for Claim 4): For $\bar{V}$ defined as in (121), find $\underline{\gamma} = \underline{\|\bar{V}\|} > 0$ and $\bar{\gamma} = \overline{\|\bar{V}\|} < \infty$ and let $\bar{\lambda}$ be as in (137). Since $H : \mathcal{M}_0^{\bar{\lambda}} \times \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}} \to \mathcal{M}_0^{\bar{\lambda}}$, and $\bar{V} \in \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$, the restriction $H(\cdot, \bar{V})$ maps $\mathcal{M}_0^{\bar{\lambda}} \to \mathcal{M}_0^{\bar{\lambda}}$ and is continuous, and $\mathcal{M}_0^{\bar{\lambda}}$ is a non-empty convex and compact set. Therefore, by the Brouwer fixed point theorem, there exists $\bar{M}$ such that

$$H(\bar{M}, \bar{V}) = \bar{M}. \tag{145}$$

This implies that $\bar{M}$ is $\rho-$consistent with disequilibrium market clearing at $\bar{t}^n$. In addition, since $\bar{V}$ is *strictly* positive definite, $H(\cdot, \bar{V})$ maps into a subset of strictly positive definite matrices. Hence $\bar{M} = H(\bar{M}, \bar{V})$ consists of strictly positive diagonal matrices.

Claim 5 *((Non-) existence for $N = 2$.)*

(Argument for Claim 5): In the case of $N = 2$, the necessary condition for $\rho-$consistency becomes (See equation (120))

$$M^1 = (1-\rho^1)(M^2 - D^2u^2), \tag{146}$$

$$M^2 = (1-\rho^2)(M^1 - D^2u^1). \tag{147}$$

If $\rho^1 = \rho^2 = 0$, equations (146) and (147) imply that

$$D^2u^1 = -D^2u^2. \tag{148}$$

But this is impossible since the Hessians of both utility functions are negative definite. When for at least one of the two traders external absorption is positive, $\rho^n > 0$, the system (146) and (147) can be solved to obtain

$$M^1 = -\frac{(1-\rho^1)}{1-(1-\rho^1)(1-\rho^2)}\left((1-\rho^2) D^2u^1. + D^2u^2\right), \tag{149}$$

$$M^2 = -\frac{(1-\rho^2)}{1-(1-\rho^1)(1-\rho^2)}\left((1-\rho^1) D^2u^2. + D^2u^1\right), \tag{150}$$

The matrices M^1 and M^2 are consistent. ■

Proof. (Theorem 1: Existence of a diagonal $\rho-$competitive equilibrium.)

One of the complications of proving the existence of equilibrium in a non-competitive setting is that the second derivatives are defined only on R^L_{++} (an open set) and may not be uniformly bounded. In such case, the set of Hessians of utility functions for different values of trade may not be bounded (and hence not compact either). To give an example, for logarithmic utility function $\ln(x)$, the second derivative is equal to $-\frac{1}{x^2}$. It is not possible to find an upper bound on the absolute value of the derivative, as $lim_{x^i \to 0} \left|1/x^2\right| = \infty$, even if the set of trades is compact.

It can be shown, however, that given strictly positive endowments, the consumption near zero will never be observed in equilibrium. Therefore to overcome the problem of unboundedness of the Hessians, I adopt the following strategy. I truncate the space of feasible trades so that the consumption is bounded away from zero for all consumers. Then I define a non-empty, convex, and compact set of trades, prices and price impact matrices, and a continuous function mapping this set into itself. I establish the existence of a fixed point of this map (Claim 7). Next I show that the constraints in the truncation are not binding so that optimality extends to a whole R^L_+ commodity space. (Claim 8). Finally I show that the fixed point of the map defines an equilibrium in $\mathcal{E}$ (Claim 9).

Claim 6 *(Truncated space of trades, $\mathcal{T}$.)*

(Argument for Claim 6): For each trader I replace the original commodity space R^L_+ with a compact box. In order to do so, for any non-numeraire good l, I find the maximal value possibly observed as a price in a Walrasian equilibrium in $\mathcal{E}$

$$\bar{u}_l \equiv \max_{i \in \mathcal{I}} \frac{\partial u^i_l\left(e^i_l\right)}{\partial x^i_l}. \tag{151}$$

Then for any consumer i I find a lower bound of the consumption of good l, $x^{i,\min}$

$$\bar{u}_l = \frac{\partial u^i_l\left(x^{i,\min}_l\right)}{\partial x^i_l}. \tag{152}$$

By the interiority assumption, A5, and convexity of the utility function, $x^{i,\min}_l$, is well defined and $0 < x^{i,\min}_l \leq e^i_l$. By $x^{i,\min}$, I denote a vector of lower bounds for all goods, $x^{i,\min} \equiv \left\{x^{i,\min}_1, \ldots, x^{i,\min}_L\right\}$. For any firm j I define an upper bound on the production of good l, $y^{j,\max}_l$, as follows:

$$\frac{\partial f^j_l\left(y^{j,\max}_l\right)}{\partial y^i_l} = \bar{u}_l. \tag{153}$$

By assumption of monotonicity, A8, and convexity, A9, strictly positive $y^{j,\max}_l$ exists and is unique. Then $y^{j,\max} = \left\{y^{j,\max}_1, \ldots, y^{j,\max}_L\right\}$ is a vector of upper bounds on production for all goods. A truncated space of trades, denoted by $\mathcal{T}$, is defined as a set of all trades such that markets clear and individual trades are within the defined bounds, namely

$$\mathcal{T} \equiv \left\{ t \in \mathbb{R}^{N \times L} | \sum_{n \in \mathcal{N}} t^n = 0, \text{ and } t^i \geq -e^i + \frac{1}{2} x^{i,\min}, \text{ and } 0 \geq t^j \geq -y^{j,\max} \right\}. \tag{154}$$

The trade space $\mathcal{T}$ is non-empty (for example, autarky is a member of $\mathcal{T}$), convex (union of three convex sets) and compact (closed – it is a preimage of a closed set by a continuous function, and bounded—all trades are bounded from below, and they all sum up to zero hence they are also bounded from above).

Claim 7 *(Construction of a compact set and a continuous map, the existence of a fixed point.)*

(Argument for Claim 7): First I define function $V : \mathcal{T} \to \mathcal{V}$, mapping trades into a space of vectors of positive definite matrices as

$$V(t) = \left(V^1\left(t^1\right), \ldots, V^I\left(t^I\right), V^{I+1}\left(t^{I+1}\right), \ldots, V^N\left(t^N\right)\right) \tag{155}$$

where

$$V^i\left(t^i\right) = -D^2u^i\left(e^i + t^i\right) \text{ and } V^j\left(t^j\right) = D^2 f^j\left(-t^j\right). \tag{156}$$

Note that V is not well defined for trades in which the firm's supply of some good is zero. This is because the Hessians of cost functions are not defined on the boundary of R^L_+. Therefore for any t^j such that $t^j_l = 0$ for some l, $V^j\left(t^j\right)$ is defined as a limit of the Hessian as t^j_l approaches zero

$$V^j\left(t^j\right) = \lim_{t^j_l \to 0} D^2 f^j\left(-t^j\right). \tag{157}$$

By assumption A9, such limit is well defined, is finite and strictly greater than zero for any diagonal entry of $V^j\left(t^j\right)$. By interiority of the initial endowments, such problem does not arise in case of consumers, as consumption induced by any trade in $\mathcal{T}$ is bounded away from zero. Utility and cost functions are twice continuously differentiable on $\mathbb{R}_{++}$, therefore $V(t)$ defined by (**??**) and (157) is continuous on $\mathcal{T}$. Let

$$\bar{\gamma} = \sup_{t \in \mathcal{T}} \overline{\|V(t)\|} \text{ and } \underline{\gamma} = \inf_{t \in \mathcal{T}} \underline{\|V(t)\|}. \tag{158}$$

Because $\overline{\|\cdot\|}$, $\underline{\|\cdot\|}$, and $V(t)$ are continuous on $\mathcal{T}$ and $\mathcal{T}$ is compact, sup and inf are attained on $\mathcal{T}$, therefore $0 < \underline{\gamma} \leq \bar{\gamma} < \infty$. For such two scalars define a set of vectors of positive definite matrices, $\mathcal{V}^{\bar{\gamma}}_{\underline{\gamma}}$, as in Claim 2 in Proposition 3. As argued before, $\mathcal{V}^{\bar{\gamma}}_{\underline{\gamma}}$ is non-empty, convex, and compact and function $V : \mathcal{T} \to \mathcal{V}^{\bar{\gamma}}_{\underline{\gamma}}$ is continuous. Finally, define $\bar{\lambda}$ and $\mathcal{M}^{\bar{\lambda}}_0$ as in Claim 3 in Proposition 3 and allocation function $T : \mathcal{M}^{\bar{\lambda}}_0 \to \mathcal{T}$, by the equation

$$T(M) \equiv \arg\max_{t \in \mathcal{T}} \left[\sum_{i \in \mathcal{I}} \left(u^i\left(t^i + e^i\right) - \frac{1}{2} t^i \cdot M^i t^i\right) - \sum_{j \in \mathcal{J}} \left(f^j\left(-t^j\right) + \frac{1}{2} t^j \cdot M^i t^j\right)\right]. \tag{159}$$

The objective function in definition (159) is continuous, strictly concave, and $\mathcal{T}$ is compact, therefore allocation function $T(M)$ is well defined. In addition, $T(M)$ is continuous by a maximum principle. Finally define a map $\mathcal{F}$

$$\mathcal{F} : \mathcal{T} \times \mathcal{V}^{\bar{\gamma}}_{\underline{\gamma}} \times \mathcal{M}^{\bar{\lambda}}_0 \to \mathcal{T} \times \mathcal{V}^{\bar{\gamma}}_{\underline{\gamma}} \times \mathcal{M}^{\bar{\lambda}}_0, \tag{160}$$

as

$$\mathcal{F} = \{T(\cdot), V(\cdot), H(\cdot)\}.$$

In was shown in Claim 3 of Proposition 3 that $H(\cdot)$ maps in a continuous way $\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}} \times \mathcal{M}_0^{\bar{\lambda}}$ into $\mathcal{M}_0^{\bar{\lambda}}$. Since each component of $\mathcal{F}$ is continuous, $\mathcal{F}$ is continuous on a non-empty, convex and compact set $\mathcal{T} \times \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}} \times \mathcal{M}_0^{\bar{\lambda}}$. By the Brouwer fixed point theorem, there exists a fixed point (t^*, V^*, M^*) such that

$$\mathcal{F}(t^*, V^*, M^*) = (t^*, V^*, M^*). \tag{161}$$

(For the case with $N = 2$ and $\rho > 0$, the proof is modified in a following way: By $\mathrm{Im}\left(\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}\right) \subset \mathcal{M}$, denote the image of set $\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$ by the function (149) and then let $\bar{\lambda} \equiv \sup_{M \in \mathrm{Im}\left(\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}\right)} \overline{\|M\|}$. Since $\mathrm{Im}\left(\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}\right)$ is compact, $\bar{\lambda}$ is positive and finite. Then in the proof above use this $\bar{\lambda}$ and replace map $H(\cdot)$ with functions (149).)

Claim 8 *(t^* is in the interior of $\mathcal{T}$)*

(Argument for Claim 8.): I show that all inequality constraints in the definition of $\mathcal{T}$, (154) are satisfied with strict inequality at t^* and hence are not binding.

a) $t_l^j \leq 0$ is not binding.

Partial derivative of the objective function (159) with respect to t_l^j is positive for all $t_l^j < 0$ and equal to zero for $t_l^j = 0$. If for some j, $t_l^{j*} = 0$, then $t_l^{j'*} = 0$ must hold for all firms $j' \in \mathcal{J}$. Otherwise, one could benefit by reducing the supply of j' and offset this change by increased supply by firm j. With all firms producing zero, there must exist at least one consumer with non-positive trade t^{i*} (as all trades sum up to zero) and partial derivative of (159) with respect to t_l^i evaluated at t_l^{i*},

$$\frac{\partial u_l^i\left(t^{i*} + e^i\right)}{\partial x_l^i} - M_{ll}^{i*} t_l^{i*} > 0 \tag{162}$$

is strictly positive as $M_{l,l}^{*i} \geq 0$. Consequently marginally reducing t_l^{j*}(hence increasing supply of j) and at the same time increasing t_l^{i*} by the same amount increases the value of the objective function and does not violate the constraints defining $\mathcal{T}$. This contradicts the optimality of t^* on $\mathcal{T}$.

b) $t^j \geq -y^{j,\max}$ is not binding.

Suppose for some j and l, $t_l^{j*} = -y_l^{j,\max} < 0$. By definition of $y_l^{j,\max}$, the partial derivative of the objective function (159) with respect to t_l^j, evaluated at t_l^{j*} is (153)

$$\frac{\partial f_l^i\left(-t_l^{i*}\right)}{\partial y_l^i} - M_{ll}^{i*} t_l^{i*} > \bar{u}_l = \max_{i \in \mathcal{I}} \frac{\partial u_l^i\left(e_l^i\right)}{\partial x_l^i}. \tag{163}$$

In addition, by the fact that all trades sum up to zero, there must be at least one consumer i with strictly positive trade $t^i > 0$. For such consumers partial derivative

$$\frac{\partial u_l^i\left(t_l^{i*} + e_l^i\right)}{\partial x_l^i} - M_{ll}^{i*} t_l^{i*} < \frac{\partial u_l^i\left(e_l^i\right)}{\partial x_l^i} \leq \bar{u}_l. \tag{164}$$

Equations (163) and (164) imply that increasing marginally t_l^j (cutting the supply) and reducing the demand t_l^i by the same amount increases value of (159) and such change does not violate the constraints. This contradicts optimality of t^*.

c) $t^i \geq -e^i + \frac{1}{2}x^{i,\min}$ is not binding.

Suppose for some i, $t_l^{i*} = -e_l^i + \frac{1}{2}x_l^{i,\max}$. By construction $x_l^{i,\max} < e_l^i$, therefore $t_l^{i*} < 0$. In addition,

$$\frac{\partial u_l^i\left(t_l^{i*}+e_l^i\right)}{\partial x_l^i} - M_{l,l}^{i*}t_l^{i*} = \frac{\partial u_l^i\left(\frac{1}{2}x_l^{i,\max}\right)}{\partial x_l^i} - M_{l,l}^{i*}t_l^{i*} > \bar{u}_l. \tag{165}$$

Since the trades sum up to zero, there must exist some other consumer i' with $t_l^{i*} > 0$ and for such consumer

$$\frac{\partial u_l^{i'}\left(t_l^{i'*}+e_l^{i'}\right)}{\partial x_l^{i'}} - M_{l,l}^{i'*}t_l^{i'*} < \frac{\partial u_l^{i'}\left(e_l^{i'}\right)}{\partial x_l^{i'}} \leq \bar{u}_l. \tag{166}$$

Equations (165) and (166) imply that increasing the demand of i and offsetting this change by the reduction of trade i' does not violate the constraints and increases the value of (159). This contradicts optimality of t^*on $\mathcal{T}$.

Claim 9 *(Fixed point (t^*, V^*, M^*) implies equilibrium (p^*, t^*, M^*) in $\mathcal{E}$.)*

(Argument for Claim 9): The objective function is strictly concave, and therefore Claim 8 implies that t^* maximizes (159) on the set of trades defined by L constraints $\sum_{n\in\mathcal{N}} t^n = 0$. Therefore, there must exist a vector of Lagrangian multipliers, $p^* \in R^L$ such that t^* solves unconstraint maximization problem, with objective function (159) augmented by the additional term (constraints)

$$-p^* \cdot \sum_{n\in\mathcal{N}} t^n. \tag{167}$$

The first order optimality conditions for this program are: for any consumer i

$$Du^i\left(t^{i*}+e^i\right) - M^{i*}t^{i*} = p^*, \tag{168}$$

and for any firm j

$$Df^j\left(-t^{j^*}\right) - M^{i*}t^{j*} = p^*. \tag{169}$$

For each i, let $m^{i*} = -p^* \cdot t^{i*}$ and for each firm $m^{j*} = f^j\left(-\bar{t}^{j*}\right)$. Such definitions and two conditions (168) and (169) are necessary and sufficient for t^{n*} to be stable at p^* and M^{n*}. (See (110) and (118).) Since $t^* \in \mathcal{T}$, therefore $\sum_{n\in\mathcal{N}} t^{*n} = 0$, all markets clear. In addition, by interiority of trades $V^{i*} = -D^2u^i\left(t^{i*}+e^i\right)$ and $V^{j*} = D^2f^i\left(-t^{j*}\right)$ therefore $M^* = H\left(M^*, V^*\right)$, at t^*, so that M^* is $\rho-$consistent with the disequilibrium market clearing condition. This proves the existence of an equilibrium (p^*, t^*, M^*).

In such equilibrium, $p^* >> 0$. To see it, observe that if there exists some firm such that $t_l^{j*} < 0$ then $\bar{p}_l = \frac{\partial f^j\left(-t^{j*}\right)}{\partial t^j} - M^{i*}t^{j*} > 0$. Otherwise, there must exist at least one consumer with $t_l^{j*} \leq 0$ and hence $\bar{p}_l = \frac{\partial u^i\left(t^{i*}+e^i\right)}{\partial t^{i*}} - M^{i*}t^{j*} > 0$. ■

Proof. (Theorem 2: Local determinacy of an equilibrium.)

For the economy $\mathcal{E} = (u, e, \theta, f)$, define an economy $\mathcal{E}_{\delta',\delta''} = \left(\hat{u}, e, \theta, \hat{f}\right)$ as an economy with preferences and technology perturbed by vector $\left(\delta', \delta''\right) = \left(\left(\delta'^1, \delta''^1\right), \ldots, \left(\delta'^N, \delta''^N\right)\right)$ in the following way:

$$\hat{u}^i\left(x^i\right) = u\left(x^i\right) + \delta'^i \cdot x^i + x^i \cdot \delta''^i x^i, \tag{170}$$

and for any firm

$$\hat{f}^j\left(y^j\right) = f\left(y^j\right) + \delta'^j \cdot y^j + y^j \cdot \delta''^j y^j.$$

For any trader n, vector $\delta'^n \in R^L$ specifies a perturbation of the first derivative of a utility or cost function, while $\delta''^n \in R^{L\times L}$ is $L \times L$ diagonal matrix that perturbs the second derivative. Observe that δ''^n is an L dimensional object. Consider *any* open connected set of perturbations $\mathcal{P} \in R^{N\times 2L}$ such that 1) $0 \in \mathcal{P}$; and 2) the perturbation preserves the monotonicity and concavity (convexity) of the utility functions (cost functions) on the compact set of trades, $\mathcal{T}$, defined as in (154); and 3) has non-zero Lebesgue measure. Such $\mathcal{P}$ exists as $\mathcal{T}$ is compact so that the perturbation for any $t \in \mathcal{T}$ can be made arbitrarily small. Note that each element of $\mathcal{P}$ defines some economy and hence $\mathcal{P}$ parameterizes a family of perturbed economies. The original economy is represented in $\mathcal{P}$ by vector 0.

For any $\mathcal{E}_{\delta',\delta''}$, the set of all diagonal $\rho-$competitive equilibria is defined as a critical point of the system of equations

$$\Psi\left(\bar{p}, \bar{t}, \bar{M}\right) = 0, \tag{171}$$

with

$$\Psi\left(\bar{p}, \bar{t}, \bar{M}\right) = \begin{pmatrix} D\hat{u}^1\left(\bar{t}^1\right) - \bar{M}^i\bar{t}^1 - \bar{p} \\ \ldots \\ D\hat{u}^I\left(\bar{t}^I\right) - \bar{M}^I\bar{t}^I - \bar{p} \\ D\hat{f}^{I+1}\left(-\bar{t}^{I+1}\right) - \bar{M}^{I+1}\bar{t}^{I+1} - \bar{p} \\ \ldots \\ D\hat{f}^N\left(-\bar{t}^N\right) - \bar{M}^N\bar{t}^N - \bar{p} \\ \bar{M}^1 - \left(1-\rho^1\right)\left(\sum_{n\neq 1}\left(\bar{M}^n + \hat{V}^n\left(\bar{t}^n\right)\right)^{-1}\right)^{-1} \\ \ldots \\ \bar{M}^N - \left(1-\rho^N\right)\left(\sum_{n\neq N}\left(\bar{M}^n + \hat{V}^n\left(\bar{t}^n\right)\right)^{-1}\right)^{-1} \\ \sum_{n\in\mathcal{N}}\bar{t}^n \end{pmatrix} \tag{172}$$

and where $\hat{V}^i\left(\bar{t}^i\right) = -D^2\hat{u}^i\left(\bar{t}^i + e^i\right)$ and $\hat{V}^j\left(\bar{t}^j\right) = D^2\hat{f}^j\left(-\bar{t}^i\right)$. Equation (172) is a system of $L\times N + L\times N + L$ equations and the same number of unknowns. In the economy $\mathcal{E}_{\delta',\delta''}$, all

$\rho-$competitive equilibria are locally unique if $\Psi\left(\bar{p}, \bar{t}, \bar{M}\right)$ is transverse to 0. Unfortunately, the transversality condition is not satisfied for arbitrary $\mathcal{E}_{\delta', \delta''}$. Next, I show that transversality holds for "almost all" economies in $\mathcal{P}$, so that local indeterminacy may occur only as a degenerate case. I first extend $\Psi(p, t, M)$ to a domain that includes the perturbation parameters δ', δ'' and I show that extended $\Psi(\cdot)$ is transverse to zero. Then, using the transversality theorem, I establish a generic determinacy.

Claim 10 *($\Psi\left(\bar{p}, \bar{t}, \bar{M}, \delta', \delta''\right)$ is transverse to* 0*.)*

(Argument for Claim 10): I prove the claim using the method of perturbation. Any of $N \times L$ equations defined by the trader's first order condition (the first of the three main groups of equations) can be perturbed by varying $\delta_l'^n$, without affecting any other equation in the system. To see it, observe that

$$\frac{\partial \hat{u}_l^i\left(\bar{t}_l^i\right)}{\partial \bar{t}_l^i} = \frac{\partial u^i\left(\bar{t}_l^i\right)}{\partial t_l^i} + \delta_l'^i + \delta_l''^i \bar{t}^i \tag{173}$$

and similarly for a firm

$$\frac{\partial \hat{f}_l^j\left(-\bar{t}_l^j\right)}{\partial t_l^i} = \frac{\partial \hat{f}_l^i\left(-\bar{t}_l^i\right)}{\partial t_l^i} - \delta_l'^j + \delta_l''^j t^j.$$

The equations defining the consistency of price impacts (second group), for example, associated with the price impact of trader n on market l, we may be perturbed by changing the diagonal entry $\bar{M}_{l,l}^i$ to $\tilde{M}_{l,l}^n$ and offsetting this change by the corresponding adjustment of $\delta_l''^n$ so that the sum of the two matrices stays constant

$$\bar{M}_{l,l}^n + V_{l,l}^n\left(t^n\right) = \tilde{M}_{l,l}^n + V_{l,l}^n\left(t^n\right) + \delta_{l,l}''^n = \text{constant} \tag{174}$$

and also by changing $\delta_l'^n$ to offset the effects of the change in $\delta_{l,l}''^n$ on the first group equations. Finally, the market clearing condition for each l can be perturbed by varying $\bar{t}_l^i$ and compensating this change by adjusting $\left(\delta_l'^i, \delta_l''^i\right)$ to keep the values of equations in the first two groups unchanged. This is sufficient for $\Psi\left(\bar{p}, \bar{t}, \bar{M}, \delta', \delta''\right)$ to be transverse to 0. Applying transversality theorem, one can establish that there exists a dense subset $\hat{\mathcal{P}}$ of perturbations in $\mathcal{P}$ with full Lebesgue measure, such that for $\left(\delta', \delta''\right) \in \hat{\mathcal{P}}$ from this set, function $\Psi_{\left(\delta', \delta''\right)}\left(\bar{p}, \bar{t}, \bar{M}\right) = \Psi\left(\bar{p}, \bar{t}, \bar{M}, \delta', \delta''\right)$ is transverse to zero. Consequently, in any economy $\mathcal{E}_{\delta', \delta''}$, such that $\left(\delta', \delta''\right) \in \hat{\mathcal{P}}$, all equilibria are locally unique. ■

Proof. (Theorem 3: Non-autarkic inefficiency.)

Let $\left(\bar{p}, \bar{t}, \bar{M}\right)$ be a diagonal $\rho-$competitive equilibrium and suppose that $\rho < 0$. For any $i \in \mathcal{I}$, the necessary and sufficient condition of stability of $\bar{t}^i$ at $\bar{p}$ and M^i is

$$Du^i\left(\bar{t}^i + e^i\right) = \bar{p} + \bar{M}^i \bar{t}^i \tag{175}$$

and similarly for firms

$$Df^j\left(-\bar{t}^j\right) = \bar{p} + \bar{M}^j \bar{t}^j. \tag{176}$$

The necessary and sufficient condition for Pareto efficiency of the equilibrium allocation is the equality of marginal utilities and marginal costs for any $n \in \mathcal{N}$

$$Du^i\left(\bar{t}^i + e^i\right) = Df^j\left(-\bar{t}^j\right). \tag{177}$$

First I show that any allocation in a $\rho-$competitive equilibrium is Pareto efficient if and only if the equilibrium allocation is autarkic, that is for any n, $\bar{t}^n = 0$.

Claim 11 *($\rho-$competitive allocation is Pareto efficient if and only if in equilibrium $\bar{t}^n = 0$ for all $n \in \mathcal{N}$)*

(Argument for Claim 11): For "if" part, observe that if for all $n \in \mathcal{N}$, $\bar{t}^n = 0$, then equations (175) and (176) imply Pareto efficiency condition (177). For "only if" direction, suppose that non-autarkic equilibrium allocation is Pareto efficient. This, by (177), implies for all n

$$\bar{M}^n \bar{t}^n = \bar{M}^{n'} t^{n'}. \tag{178}$$

Suppose the allocation is non-autarkic, and hence for at least one $n \in \mathcal{N}$ and $l \in \mathcal{L}$ the trade in non-zero $\bar{t}^i_l \neq 0$. Without loss of generality assume $\bar{t}^n_l > 0$. By the market clearing condition, there must exist trader $n' \neq n$ such that $\bar{t}^{n'}_l < 0$. By diagonality of $\bar{M}^n$, the l^{th} entry of (178) is given by

$$\bar{M}^n_{l,l} \bar{t}^n = \bar{M}^{i'}_{l,l} \bar{t}^{i'}_l. \tag{179}$$

Since $\bar{M}^n$ is positive definite, its diagonal entries satisfy $\bar{M}^n_{l,l}, \bar{M}^{n'}_{l,l} > 0$. But then equation (179) is impossible, as both sides have different signs.

To complete the proof, I show that $\rho-$competitive equilibrium allocation is autarkic if and only if the initial allocation is Pareto efficient.

Claim 12 *(In a $\rho-$competitive equilibrium, the allocation is autarkic if and only if the initial allocation is Pareto efficient and $J = 0$.)*

(Argument for Claim 12): For "only if" part, suppose $J > 0$. Then by (175) for each firm $\bar{t}^j = 0$ trade is stable if and only if $\bar{p} = 0$. But at such price stable trades of consumers are strictly positive and hence markets do not clear, a contradiction. Now suppose $J = 0$ and that in $\rho-$competitive equilibrium $\bar{t}^i = 0$ for all $i \in \mathcal{I}$. Then equation (175) implies that all gradients coincide with price vector and hence

$$Du^i\left(e^i\right) = Du^{i'}\left(e^{i'}\right). \tag{180}$$

But this is sufficient for Pareto efficiency of the initial allocation.

For "if" direction, let $J = 0$ and the initial allocation is Pareto efficient, and therefore for any i and i'

$$Du^i\left(e^i\right) = Du^{i'}\left(e^{i'}\right), \tag{181}$$

and suppose that in a $\rho-$competitive equilibrium the allocation is not autarkic, hence

$$\bar{t}_l^i \neq 0 \tag{182}$$

for some i and l. Without loss of generality, assume that $\bar{t}_l^i > 0$. By the market clearing condition, there exists $i' \neq i$ such that $\bar{t}_l^{i'} < 0$. By strict concavity of utility functions equation (181) implies that

$$\frac{\partial u_l^i\left(\bar{t}_l^i + e_l^i\right)}{\partial x^{i_l}} < \frac{\partial u_l^{i'}\left(\bar{t}_l^{i'} + e_l^{i'}\right)}{\partial x^{i'}}. \tag{183}$$

Equation (175) implies

$$\frac{\partial u_l^i\left(\bar{t}_l^i + e_l^i\right)}{\partial x^{i_l}} - \bar{M}_{l,l}^i t_l^i = \frac{\partial u_l^{i'}\left(\bar{t}_l^{i'} + e_l^{i'}\right)}{\partial x^{i'}} - M_{l,l}^{i'} t_l^{i'}. \tag{184}$$

From the two equations if follows that:

$$\frac{t_l^{i'}}{t_l^i} > \frac{\bar{M}_{l,l}^i}{M_{l,l}^{i'}} > 0. \tag{185}$$

But inequality (185) is impossible, since by assumption $t_l^{i'}/t_l^i < 0$. The two claims show that Pareto efficiency of initial allocation and $J = 0$ is equivalent to autarky of a $\rho-$competitive equilibrium, which in turn is equivalent to Pareto efficiency of the equilibrium allocation. ■

Proof. (Theorem 4: Convergence to a Walrasian Equilibrium.)
By the similar arguments as in Claim 8, the only trades consistent with $\rho-$competitive equilibrium are in set $\mathcal{T} \subset R_+^{N\times L}$ (for its definition see 154). For such $\mathcal{T}$, the set of all possibly observed Hessians of utility functions and cost functions is a subset of a compact set $\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$, where $\mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$ and the scalars $\bar{\gamma}$, $\underline{\gamma}$ are defined as in Claim 7 of Theorem 1. In the following arguments, I consider a $k-$replica of the economy, and therefore I assume that there are k traders of each type n.

Claim 13 *(In any $\rho-$competitive equilibrium, consistent matrices $\bar{M}$ satisfy $\overline{\|M\|} \leq \frac{1}{N\times k-2}\bar{\gamma}$.)*

(Argument for Claim 13): For any fixed $\bar{V}$, $\bar{M}$ is $\rho-$consistent if and only if it is a fixed point of map $H : \mathcal{M} \to \mathcal{M}$ where the n^{th} element of $H(\cdot)$, $H^n(\cdot)$, is given by

$$H^n\left(\bar{M}, \bar{V}\right) = (1-\rho^n)\left[k\sum_{n'\neq n}\left(\bar{M}^{n'} + \bar{V}^{n'}\right)^{-1} + (k-1)\left(\bar{M}^n + \bar{V}^n\right)^{-1}\right]^{-1}. \tag{186}$$

Observe that each type trades with k trades of each other type (there are $N-1$, other types) and $k-1$ traders of its own type. For each type n define a map
(187)

$$A^n(M,V) = \frac{1}{N\times k-1}\left(\frac{1}{N\times k-1}\left(k\sum_{n'\neq n}\left(\bar{M}^{n'} + \bar{V}^{n'}\right) + (k-1)\left(\bar{M}^{n'} + \bar{V}^{n'}\right)\right)\right).$$

Given $\overline{\|V\|} \leq \bar{\gamma}$ (as $V \in \mathcal{V}_{\underline{\gamma}}^{\bar{\gamma}}$), for any $\bar{M}$ with $\overline{\|\bar{M}\|} = \lambda$, equation 7 implies following inequality:

$$\overline{\left\|A^n\left(\bar{M},\bar{V}\right)\right\|^*} \leq \frac{1}{N \times k - 1}\left(\lambda + \bar{\gamma}\right). \tag{188}$$

Suppose that $\lambda > \frac{1}{N \times k - 2}\bar{\gamma}$, then

$$\overline{\|A^n\left(M,V\right)\|^*} < \lambda. \tag{189}$$

By arithmetic—harmonic mean inequality and the fact that $(1-\rho^n) \leq 1$

$$H^n\left(M,V\right) \leq A^n\left(M,V\right), \tag{190}$$

and therefore

$$\overline{\|H^n\left(M,V\right)\|^*} < \lambda, \tag{191}$$

so

$$\overline{\|H\left(M,V\right)\|} < \overline{\|M\|}, \tag{192}$$

which implies that $H\left(\bar{M},\bar{V}\right) \neq \bar{M}$, and therefore $\bar{M}$ is not a fixed point of $H\left(\bar{M},\bar{V}\right)$ and therefore $\bar{M}$ is not consistent. Consequently, $\bar{M}$ is consistent only if $\overline{\|M\|} \leq \frac{1}{N \times k - 2}\bar{\gamma}$.

Claim 14 *(Convergence of $\rho-$competitive equilibrium to a Walrasian one.)*

(Argument for Claim 14): For any $\varepsilon > 0$ define $k\left(\varepsilon\right) \equiv \left(\frac{\bar{\gamma}L}{\varepsilon} + 2\right)\frac{1}{N}$ (ignore the integer problem). Given the result from the previous claim, for such $k\left(\varepsilon\right)$ the vector of consistent price impacts $\bar{M}$ satisfies

$$\overline{\left\|\bar{M}\right\|} \leq \frac{1}{N \times k - 2}\bar{\gamma} = \frac{\varepsilon}{L}. \tag{193}$$

For any trader n, a diagonal element of $\bar{M}^n$ satisfies

$$\bar{M}^n_{l,l}L \leq \varepsilon, \tag{194}$$

and hence the Euclidian norm $\left\|\bar{M}^n\right\| \leq \varepsilon$. In addition, for any $\bar{M}$ there exists a unique $\bar{t}$ and $\bar{p}$ such that $\left(\bar{p},\bar{t},\bar{M}\right)$ is $\rho-$competitive equilibrium. To see it, observe that trade stability for each trader requires for all i and j that

$$Du^i\left(\bar{t}^i + e^i\right) - \bar{M}^i\bar{t}^i = Df^j\left(-\bar{t}^i\right) - \bar{M}^i\bar{t}^j, \tag{195}$$

and since each of the L equations is deceasing in $\bar{t}^n_l$ and

$$\sum_{n\in\mathcal{N}}\bar{t}^n = 0, \tag{196}$$

there is a unique $\bar{t}$ that solves conditions (195) and (196) for some fixed value of $\bar{M}$. The equilibrium price vector is given by $\bar{p} = Du^i\left(\bar{t}^i + e^i\right) - \bar{M}^i\bar{t}^i$ for any i. This shows that there exists a function $\bar{t}\left(\bar{M}\right)$ and $\bar{p}\left(\bar{M}\right)$, mapping vector of matrices into equilibrium trades and prices. Because (195) and (196) are continuous, by the standard argument $\bar{t}\left(\bar{M}\right)$ and $\bar{p}\left(\bar{M}\right)$ are also continuous in $\bar{M}$. Therefore, for any ε there exists ε'such that for any $\left\|\bar{M}\right\| \leq \varepsilon'$, one obtains $\left\|\bar{t}^n\left(\bar{M}\right) - \bar{t}^n(0)\right\| \leq \varepsilon$ and $\left\|\bar{p}\left(\bar{M}\right) - \bar{p}(0)\right\| \leq \varepsilon$. But $\bar{t}^n(0)$ and $\bar{p}(0)$ correspond to the Walrasian trade and price.

To conclude the proof, for any ε define $k(\varepsilon) = \max\left(\left(\frac{\bar{\gamma}L}{\varepsilon} + 2\right)\frac{1}{N}, \left(\frac{\bar{\gamma}L}{\varepsilon'} + 2\right)\frac{1}{N}\right)$. For any $k \geq k(\varepsilon)$, in $k-$replica economy

1) $\left\|\bar{M}^n\right\| \leq \varepsilon$, for any n;
2) $\left\|\bar{t}^n - \bar{t}^{n,Walras}\right\| \leq \varepsilon$, for any n;
3) $\left\|\bar{p} - \bar{p}^{Walras}\right\| \leq \varepsilon$. ■

Proof. (Theorem 5: $\rho-$competitive equilibrium rationalized as a Walrasian equilibrium.)

Let $\left(\bar{p}, \bar{t}, \bar{M}\right)$ be a diagonal $\rho-$ competitive equilibrium for some ρ. For any i, stable $\bar{t}^i$ satisfies

$$Du^i\left(\bar{t}^i + e^i\right) - \bar{M}\bar{t}^i = \bar{p}. \tag{197}$$

But this condition, together with equation $\bar{m}^i = -\bar{p}\cdot\bar{t}^i$ is necessary and sufficient for $\left(\bar{t}^i, \bar{m}^i\right)$ to be optimal on the Walrasian budget set

$$\bar{p}t^i + m^i \leq 0, \tag{198}$$

given the utility function

$$\hat{u}^i\left(x^i\right) = u^i\left(x^i\right) + \frac{1}{2}\left(x^i - e^i\right)\cdot\bar{M}^i\left(x^i - e^i\right) + m^i. \tag{199}$$

Similarly, for a firm stability implies

$$Df^j\left(-\bar{t}^j\right) - \bar{M}\bar{t}^j = \bar{p}, \tag{200}$$

which is also necessary and sufficient for the Walrasian firm to maximize profit, given the cost function

$$\hat{f}^j\left(y^j\right) = f^j\left(y^j\right) - \frac{1}{2}y^j\cdot\bar{M}^iy^j.$$

Since in addition in $\rho-$competitive equilibrium all markets clear, $\sum_{i\in I}\bar{t}^i = 0$, tuple $(\bar{p}, \bar{t})$ is a Walrasian equilibrium in the economy with modified preferences and technology. ■

Proof. (Proposition 4: Price bias with monotone u'''.)

In order to make the proof more transparent, I consider an economy with $u''' > 0$ and with only one non-numeraire good. The arguments for $u''' < 0$ and $u''' = 0$ are straightforward modifications of the presented proof. In addition, in a smooth and separable economy, markets are separated (quasi-linear utility function and separable utility), and therefore the proof extends directly to the case with $L > 1$.

The proof consists of the following steps: First, I prove two useful facts needed in the main part of the proof: In Claim 15, I show that the convexities of buyers' utility functions at the stable consumption are always greater then the ones of the sellers, for any arbitrary level of price impact but evaluated at the equilibrium price. Then in Claim 16, I argue that in considered $\rho-$competitive equilibrium, price impacts of sellers exceed the ones of the buyers. These two results will be used in the proof of the main result in Claim 17, the positive price bias.

The idea behind the proof is as follows: Since in $\rho-$competitive equilibrium markets clear, the sum of stable trade functions evaluated at the equilibrium $\bar{p}$, $\bar{M}$ is zero. The individual trades are monotone in $\bar{M}^i$, therefore by the result of Claim 16, replacing the $\bar{M}^i$ with some common, intermediate value of $\tilde{M}$ reduces the value of the excess aggregate demand and hence makes it strictly negative. Then using the result from Claim 15, I show that aggregate stable demand is increasing in the common value $\tilde{M}$, and hence replacing it with zero further decreases the aggregate demand, making it even more negative. But stable excess demand at $\tilde{M} = 0$ for all traders corresponds to a Walrasian excess demand. This shows that at the price $\bar{p}$, there is an excess supply on a market when traders are competitive. Therefore, in the Walrasian equilibrium, the price must go down to reestablish market clearing and hence, $\bar{p} > \bar{p}^{Walras}$.

For any equilibrium$\left(\bar{p}, \bar{t}, \bar{M}\right)$, the set of all consumers is partitioned into thee groups: buyers, $\mathcal{I}^b = \left\{i \in \mathcal{I} | \bar{t}^i > 0\right\}$, sellers, $\mathcal{I}^s = \left\{i \in \mathcal{I} | \bar{t}^i < 0\right\}$ and traders that are not active, $\mathcal{I}^{na} = \left\{i \in \mathcal{I} | \bar{t}^i = 0\right\}$. Observe that since the initial allocation is not Pareto efficient, the sets $\mathcal{I}^s$ and $\mathcal{I}^b$ are non-empty. For any $p \geq 0$ and scalar $M^i \geq 0$, I define a stable convexity of the trader i, $\gamma^i\left(p, M^i\right)$ as an absolute value of the second derivative, evaluated at the stable trade $\bar{t}^i\left(p, M^i\right)$

$$\gamma^i\left(p, M^i\right) \equiv \left|\frac{\partial^2 u\left(\bar{t}^i\left(p, M^i\right) + e^i\right)}{\partial\left(x^i\right)^2}\right|. \tag{201}$$

Claim 15 asserts that at the equilibrium price, $\bar{p}$, and positive but otherwise arbitrary price impacts M^b, M^{na}, M^s, buyers always have higher stable convexity, provided $u''' > 0$.

Claim 15 *(For any $b \in \mathcal{I}^b$, $s \in \mathcal{I}^s$, and $na \in \mathcal{I}^{na}$, equilibrium price $\bar{p}$, and arbitrary positive M^b, M^{na}, M^s stable convexities satisfy*

$$\gamma^b\left(\bar{p}, M^b\right) > \gamma^{na}\left(\bar{p}, M^{na}\right) > \gamma^s\left(\bar{p}, M^s\right) > 0, \tag{202}$$

provided that $u''' > 0$.)

(Argument for Claim 15): Stable demand $\bar{t}^i\left(\bar{p}, M^i\right)$ is implicitly defined by the equality of the marginal utility and marginal revenue

$$\frac{\partial u^i\left(\bar{t}^i + e^i\right)}{\partial x^i} = \bar{p} + M^i \bar{t}^i. \tag{203}$$

The derivative of $\bar{t}^i\left(\bar{p}, M^i\right)$ with respect to M^i, implicitly defined by (203) is given by

$$\frac{\partial \bar{t}^i\left(\bar{p}, M^i\right)}{\partial M^i} = -\frac{\bar{t}^i\left(\bar{p}, M^i\right)}{M^i + \gamma^i\left(\bar{p}, M^i\right)}. \tag{204}$$

Since the denominator is positive, the trade is strictly increasing in M^i for the buyers, decreasing for the sellers and is constant for non-active traders. In addition, since $\partial u^b\left(e^i\right)/\partial x^b > \bar{p}$, therefore for any M^b, stable trade is positive. Consequently, for any strictly positive price impact $M^b > 0$, the stable buyers demand is bracketed by a Walrasian and zero trade

$$\bar{t}^b\left(\bar{p}, 0\right) < \bar{t}^b\left(\bar{p}, M^b\right) < 0. \tag{205}$$

Similarly, for each seller the trade is bracketed by

$$\bar{t}^s\left(\bar{p}, 0\right) > \bar{t}^s\left(\bar{p}, M^s\right) > 0. \tag{206}$$

The Walrasian consumption of all traders at price $\bar{p}$ is the same, as utilities are identical,

$$\bar{t}^i\left(\bar{p}, 0\right) + e^i = \bar{t}^{i'}\left(\bar{p}, 0\right) + e^{i'}, \tag{207}$$

therefore conditions (205) and (206) imply for any strictly positive price impacts M^b, M^{na}, M^s stable consumption is

$$\bar{t}^b\left(\bar{p}, M^b\right) + e^b < \bar{t}^{na}\left(\bar{p}, M^{na}\right) + e^{na} < \bar{t}^s\left(\bar{p}, M^s\right) + e^s. \tag{208}$$

If the second derivative of the utility function is increasing in consumption, ($u''' > 0$), the relation between stable consumptions (208) induces the following ranking of convexities:

$$\gamma^b\left(\bar{p}, M^b\right) > \gamma^{na}\left(\bar{p}, M^{na}\right) > \gamma^s\left(\bar{p}, M^s\right). \tag{209}$$

Claim 16 *($u''' > 0$ implies that for any $b \in \mathcal{I}^b$, $s \in \mathcal{I}^s$, and $na \in \mathcal{I}^{na}$, the equilibrium price impacts satisfy*

$$\bar{M}^s > \bar{M}^{na} > \bar{M}^b.) \tag{210}$$

(Argument for Claim 16): The consistency condition on $\bar{M}^i$ for any i can be written as

$$\bar{M}^i = \frac{(1-\bar{\rho})}{I-1}\mathcal{H}\left(M^{i'} + \gamma^{i'}\left(\bar{p}, \bar{M}^{i'}\right)|i' \neq i\right) \tag{211}$$

where $\mathcal{H}(x^{i'}||i' \neq i)$ denotes a harmonic mean of elements $x^{i'}$ for all consumers but i. I will argue that I conditions (211) imply that the equilibrium ranking of $\bar{M}^i$ is just a reverted ranking of the convexities. Suppose for two traders i and i', $\gamma^i\left(\bar{p}, \bar{M}^i\right) > \gamma^{i'}(\bar{p}, \bar{M}^{i'})$ and $\bar{M}^i \geq \bar{M}^{i'}$. Condition (211) implies that $\bar{M}^i$ is a harmonic mean of $I-2$ elements that also define $\bar{M}^{i'}$ and one element that is strictly smaller as by assumption $\bar{M}^{i'} + \gamma^{i'}(\bar{p}, \bar{M}^{i'}) < \bar{M}^i + \gamma^i\left(\bar{p}, \bar{M}^i\right)$ and hence $\bar{M}^{i'} > \bar{M}^i$, which is a contradiction. Consequently $\gamma^i\left(\bar{p}, \bar{M}^i\right) > \gamma^{i'}(\bar{p}, \bar{M}^{i'})$ implies that $\bar{M}^i < \bar{M}^{i'}$ for any $i, i' \in \mathcal{I}$. This, together with Claim 15 is sufficient for the result of Claim 16.

Claim 17 *($u''' > 0$ implies $\bar{p}^{Walras} > \bar{p}$)*

(Argument for Claim 17): Let $\tilde{M}$ be some scalar satisfying

$$\bar{M}^s > \tilde{M} > \bar{M}^b, \tag{212}$$

for any buyer $b \in \mathcal{I}^b$, and seller $s \in \mathcal{I}^s$. By Claim 16, such scalar exists. In a $\rho-$competitive equilibrium the excess stable demand is equal to zero

$$\sum_i \bar{t}^i\left(\bar{p}, \bar{M}^i\right) = 0. \tag{213}$$

Now in equality (213), I replace equilibrium price impacts, $\bar{M}^i$, with the intermediate level $\tilde{M}$. Strict monotonicity of $\bar{t}^i\left(\bar{p}, \bar{M}^i\right)$ in M^i shown in (204) and ranking of equilibrium price impacts (212) imply that aggregate excess demand evaluated at $\tilde{M}$ is negative

$$Z\left(\bar{p}, \tilde{M}\right) \equiv \sum_i \bar{t}^i\left(\bar{p}, \tilde{M}\right) < 0. \tag{214}$$

In addition, for arbitrary but common level of M, partial derivative of $Z(\cdot)$ with respect to M is given by

$$\frac{\partial Z(\bar{p}, M)}{\partial M} = -\sum_i \frac{\bar{t}^i(\bar{p}, M)}{M + \gamma^i(\bar{p}, M)}. \tag{215}$$

By $\tilde{\gamma} > 0$, I denote a convexity of the utility function evaluated at x^i satisfying $\bar{p} = u'\left(x^{i,Walras}\right)$. Given u''' and the result of Claim 15, $\gamma^s(\bar{p}, M) < \tilde{\gamma} < \gamma^b(\bar{p}, M)$ for any value of $M > 0$. Then replacing the convexities with a common value $\tilde{\gamma}$ in the the derivative of $Z(\bar{p}, M)$ gives a lower bound of this derivative

$$\frac{\partial Z(\bar{p}, M)}{\partial M} > -\frac{\sum_i \bar{t}^i(\bar{p}, M)}{M + \tilde{\gamma}} = -\frac{Z(\bar{p}, M)}{M + \tilde{\gamma}}. \tag{216}$$

The inequality results from the fact that for the buyers (positive trades) replacing $\gamma^i(\bar{p}, M)$ with $\tilde{\gamma}$ decreases the denominator, while for sellers, (negative trade) the denominator goes up.

Inequality (216) defines a lower bound on the slope of $Z(\bar{p}, M)$. Observe that $Z(\bar{p}, M)$ is upward slopping in M as long as $Z(\bar{p}, M) < 0$ and $M \geq 0$. Since excess demand evaluated at the intermediate level $\tilde{M}$ is negative, $Z\left(\bar{p}, \tilde{M}\right) < 0$, therefore $Z(\bar{p}, M)$ must be increasing in M for all M from the interval $(0, \tilde{M}]$. To see it, consider a subset of all M from $0 \leq M \leq \tilde{M}$, for which the excess demand is greater than zero $Z(\bar{p}, M) \geq 0$. This subset is a preimage of a closed set by a continuous function and hence is closed. It is also bounded and hence it is compact and therefore it must contain some maximal value $\hat{M}$. By construction, on the interval $\hat{M} < M \leq \tilde{M}$, $Z(\bar{p}, M) < 0$ and hence its derivative is strictly positive. But this implies that for any such M, $Z(\bar{p}, M) < Z\left(\bar{p}, \tilde{M}\right) < 0$ and hence by continuity of $Z(\bar{p}, M)$ at $\hat{M}$, $Z\left(\bar{p}, \hat{M}\right) \leq Z\left(\bar{p}, \tilde{M}\right) < 0$. But this contradicts the assumption that $Z\left(\bar{p}, \hat{M}\right) \geq 0$. Consequently, for all $M \in (0, \tilde{M}]$, excess demand $Z(\bar{p}, M)$ is increasing in M and hence

$$Z(\bar{p}, 0) < Z\left(\bar{p}, \tilde{M}\right) < 0. \tag{217}$$

But this shows that the Walrasian excess demand $Z(\bar{p},0)$ is negative, when evaluated at the price $\bar{p}$.

$$Z(\bar{p},0)=\sum_i \bar{t}^i(\bar{p},0)<0. \tag{218}$$

Since $Z(\bar{p},0)$ is strictly decreasing in p,

$$\frac{\partial Z(p,0)}{\partial p}=-\sum_i \frac{1}{\gamma^i(\bar{p},0)}<0, \tag{219}$$

and in the unique Walrasian equilibrium price, $\bar{p}^{Walras}$, solves

$$Z\left(\bar{p}^{Walras},0\right)=0, \tag{220}$$

therefore $\bar{p}^{Walras}$ must satisfy

$$\bar{p}^{Walras}>\bar{p}. \tag{221}$$

The proofs for $u'''<0$ and $u'''=0$ are composed of symmetric arguments but with reversed signs. ■

Proof. (Lemma 1: Scale independence for CRRA.)
For "if" part set $\lambda=1$. For "only if" direction, observe that $u'(x)=x^{-\theta}$ and $u^{i\prime\prime}(x)=-\theta x^{-(\theta+1)}$. Suppose $\left(\bar{p},\bar{t},\bar{M}\right)$ is a $\rho-$competitive equilibrium in $\mathcal{E}$ and hence it satisfies the following conditions: The stability of trade for each i

$$\left(\bar{t}^i+e^i\right)^{-\theta}=\bar{p}+\bar{M}^i\bar{t}^i, \tag{222}$$

$\rho-$consistency of price impact for each i,

$$M^i=\left(1-\rho^i\right)\cdot\left(\sum_{i'\neq i}\left(M^{i'}-\theta\left(\bar{t}^{i'}+e^i\right)^{-(\theta+1)}\right)^{-1}\right)^{-1}, \tag{223}$$

and market clearing condition

$$\sum_{i\in\mathcal{I}}\bar{t}^i=0. \tag{224}$$

For any value of $\lambda>0$, these conditions imply the necessary and sufficient conditions defining equilibrium in $\mathcal{E}^\lambda$. To see it, multiply both sides of equation (222) by $\lambda^{-\theta}$ to obtain

$$\left(\lambda\bar{t}^i+\lambda e^i\right)^{-\theta}=\lambda^{-\theta}\bar{p}+\lambda^{-(1+\theta)}\bar{M}^i\lambda t^i, \tag{225}$$

the equation (223) by $\lambda^{-(1+\theta)}$

$$\lambda^{-(1+\theta)}M^i=\left(1-\rho^i\right)\cdot\left(\sum_{i'\neq i}\left(\lambda^{-(1+\theta)}M^{i'}-\theta\left(\lambda\bar{t}^{i'}+\lambda e^i\right)^{-(\theta+1)}\right)^{-1}\right)^{-1}, \tag{226}$$

and finally market clearing condition by λ

$$\sum_i \lambda \bar{t}^i = 0. \tag{227}$$

Conditions (225), (225) and (227) are necessary and sufficient for $\left(\lambda^{\cdot -\theta}\bar{p}, \lambda\bar{t}, \lambda^{\cdot -(1+\theta)}\bar{M}\right)$ to be a $\rho-$competitive equilibrium in the scaled economy $\mathcal{E}^\lambda$. ■

Proof. (Lemma 2: Inconsistency of $M^j > 0$ with constant marginal cost c.)

Consider an economy with J firms with technology characterized by a constant marginal cost. The consistency condition for the price impacts and symmetry across firms imply

$$\left(\bar{M}^j\right) = \left((J-1)\left(\bar{M}^j\right)^{-1} + \sum_{i\in I}\left(\bar{M}^i + \bar{\gamma}^i\right)^{-1}\right)^{-1}, \tag{228}$$

which is equivalent to

$$\left(\bar{M}^j\right)^{-1}(2-J) = \sum_{i\in I}\left(\bar{M}^i + \bar{\gamma}^i\right)^{-1}. \tag{229}$$

If $J \geq 2$, then $(2-J) \leq 0$ and hence the left hand side of the equation is less than zero. But this is a contradiction since the right hand side is always positive. ■

Proof. (Proposition 5: Existence of a smooth tax rule.)

With taxes, the budget constraint becomes

$$p_{\bar{p},\bar{t}^i,\bar{M}^i}\left(t^i\right)\cdot t^i + m^i \ - T^i\left(\bar{t}^i\right) = 0. \tag{230}$$

In the case of an interior stable demand $\bar{t}^i$, the necessary condition for stability is given by

$$Du^i(\cdot) + DT^i(\cdot) = \bar{p} + \bar{M}^i\bar{t}^i. \tag{231}$$

Consequently, the Jacobian of a consumers stable trade consists of three elements

$$D_p t^i\left(\bar{p}, \bar{M}^i\right) = -\left(M^i - D^2u^i(\cdot) - D^2T^i(\cdot)\right)^{-1}. \tag{232}$$

Similarly, the Jacobian of the firms stable trade

$$D_p t^j\left(\bar{p}, \bar{M}^i\right) = -\left(M^i + D^2f^j(\cdot) + D^2T^i(\cdot)\right)^{-1}.$$

For all i define augmented convexity as

$$\bar{V}^i = -D^2u^i\left(\bar{t}^i_{PE} + e^i\right) - D^2T^i\left(t^i\right), \tag{233}$$

and for all j

$$\bar{V}^j = D^2f^j\left(-\bar{t}^i_{PE}\right) + D^2T^i\left(\bar{t}^i_{PE}\right).$$

Observe that by assumption of Proposition 5, $\bar{V}^i$ and $\bar{V}^j$ are positive definite. Let $\bar{V}$

$$\bar{V} = \left(\bar{V}^1, \ldots, \bar{V}^I, \ldots, \bar{V}^N\right).$$

The existence of a system of $\rho-$consistent matrices $\bar{M}$ at t_{PE} is equivalent to the existence of a fixed point of the mapping $H\left(M, \bar{V}\right)$ where

$$H^i\left(M, \bar{V}\right) = \left(1-\rho^n\right)\left(\sum_{i' \in \mathcal{I}_{-i}} \left(M^{i'} + \bar{V}^{i'}\right)^{-1}\right)^{-1}. \tag{234}$$

By the same arguments as those made in Claim 3, $H\left(, \bar{V}\right)$ maps a non-empty convex and compact set of positive semi-definite diagonal matrices into itself in a continuous way. Therefore, there exists a fixed point of $H\left(, \bar{V}\right)$, $\bar{M}_T$ and $\bar{M}_T$ is positive definite and diagonal. Then define T'^n,

$$T'^n \equiv -\bar{M}_T^n t^n \tag{235}$$

for all n, and observe that with such definition $\Psi^n = 0$. ■

Proof. (Proposition 6: De-convexification of utility.)
The assumption that

$$\lim_{k\to\infty} T_k''^i = D^2 u^i\left(\bar{t}_{PE}^i + e^i\right) \tag{236}$$

$$\lim_{k\to\infty} T_k''^i = D^2 f^j\left(-\bar{t}_{PE}^i\right) \tag{237}$$

is equivalent to

$$\lim_{k\to\infty} \bar{V}_k = 0. \tag{238}$$

This fact, with the continuity of a norm $\overline{\|\cdot\|}$ implies that

$$\lim_{k\to\infty} \bar{\gamma}_k = \lim_{k\to\infty} \overline{\|V_k\|} = \overline{\left\|\lim_{k\to\infty} V_k\right\|} = 0, \tag{239}$$

and hence the value of $\bar{\lambda}_k$

$$\lim_{k\to\infty} \bar{\lambda}_k = \lim_{k\to\infty} \frac{1-\rho}{I+\rho-2}\bar{\gamma}_k = 0.$$

For any k, $\rho-$consistent system of matrices $\bar{M}_k$ belongs to $\mathcal{M}_0^{\bar{\lambda}_k}$ and hence each diagonal entry of M_k^n for any n is bounded above by $\bar{\lambda}_k$. Consequently,

$$0 \le \lim_{k\to\infty} \bar{M}_{(k)\ l,l}^n \le \lim_{k\to\infty} \bar{\lambda}_k = 0.$$

Therefore

$$\lim_{k\to\infty} \bar{M}_k^n = 0,$$

and hence

$$\lim_{k\to\infty} \bar{M}_k = 0.$$

■