Optimal Dynamic Nonlinear Income Taxes with No Commitment*

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Abstract

We wish to study optimal dynamic nonlinear income taxes. Do real world taxes share some of their features? What policy prescriptions can be made? We study a two period model, where the consumers and government each have separate budget constraints in the two periods, so income cannot be transferred between periods. Labor supply in both periods is chosen by the consumers. The government has memory, so taxes in the first period are a function of first period labor income, while taxes in the second period are a function of both first and second period labor income. The government cannot commit to future taxes. Time consistency is thus imposed as a requirement. The main results of the paper show that time consistent incentive compatible two period taxes involve separation of types in the first period and a differentiated

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lump sum tax in the second period, provided that the discount rate is high or utility is separable between labor and consumption. In the natural extension of the Diamond (1998) model with quasi-linear utility functions to two periods, an equivalence of dynamic and static optimal taxes is demonstrated, and a necessary condition for the top marginal tax rate on first period income is found.

1 Introduction

We wish to study optimal dynamic nonlinear income taxes. What do they look like? How do they change over time? Do real world taxes share some of their features? What policy prescriptions can be made? How do these prescriptions differ from those of the static model? In particular, must the top marginal tax rate be zero?

The public finance literature considers mainly static taxation. The macroeconomic literature considers mostly proportional taxes[1] (possibly on multiple income sources) over time, and thus is more closely related to the optimal commodity tax literature. For instance, information accumulated about the type of a particular taxpayer in one period typically cannot be used in the next, since the tax rate is unique (the same for all income from a source); thus differentiated lump sum taxes are excluded. See, for example, Persson and Tabellini (2002).

We study a two period model as a beginning. The consumers and the government each have independent budgets in each of the two periods, so wealth cannot be transferred over time. The government has memory, so first period tax liability is a function of first period income only, but second period tax liability can be a function of both first and second period income. Taxes are general (possibly nonlinear) functions of income. The government cannot commit to future tax functions, so time consistency is imposed as a restriction on taxes. Our main results involve analysis of the first and second order conditions for incentive compatibility in the consumer problem, followed by characterizations of optimal taxes under time consistency. The major theorem says that time consistent, incentive compatible income taxes typically involve separation of types in the first period followed by a differentiated lump sum tax in the second period, provided that the discount rate is high or utility is separable between labor and consumption. Thus, the second period tax rate as a function of second period income is constant. The separation of types in the first period is incentive compatible, in the sense that consumers know what's coming in the second period but choose to reveal their types anyway.

In the context of the natural extension of the Diamond (1998) model to dynamics, utilities are time separable, quasi-linear and involve discounting. We find an equivalence between optimal taxes in our dynamic extension and static optimal income taxes. In general, there is a continuum of optimal

[1] Often, taxes with a transfer (either positive or negative) at zero are excluded, so when the macroeconomic literature says "linear," it means "proportional."

dynamic taxes corresponding to a given optimal static tax. Moreover, we find that not only does the separation of types in the first period occur, followed by a differentiated lump sum tax in the second period, but this equivalence allows us to give a necessary condition on the marginal tax rate at the top of the income distribution for income in the first period.

The basic structure of this paper is to proceed from the most general to the most specific framework. Of course, as more assumptions are imposed, more results are found.

The two papers in the literature most closely related to our work are Brito, Hamilton, Slutsky and Stiglitz (1991), henceforth BHSS, and Roberts (1984). BHSS study a model with government commitment concerning future taxes, two types of taxpayers, and an infinite time horizon. One focus of their study is the relationship between static randomized taxes and nonstationary dynamic taxes. They find, for example, that under some conditions the nonstationary dynamic optimal income taxes are first best, but under other conditions, they are not. Revelation or separation of types occurs in the first period in this model, since the government has committed itself not to use this information in future periods. The possibility that pooling might occur in the first period, and the possibility that incentive constraints for periods beyond the first might bind, is not considered in this work. Roberts (1984) studies optimal income taxation under no commitment with discrete types and an infinite time horizon. He finds (see his Proposition 8) that separation of types will *never* occur over the infinite horizon. This work ignores the case where government revenue requirements are large and a pooling equilibrium (where all consumers earn the same income and pay the same tax) might bankrupt lower ability consumers. In that case, a pooling equilibrium is not feasible.

We feel that our assumptions are natural. We do not assume that government commitment is possible, because it usually isn't available. We use a finite time period approach, since actors (particularly taxpayers) are finite-lived.[2] And this assumption makes for a large contrast between our results and those of Roberts (1984).[3] Finally, we use a continuum of types, since

[2]One could conceive of an infinitely lived government with finitely lived taxpayers. We conjecture that this leads to results similar to ours, even in an overlapping generations framework. In particular, the government would impose a differentiated lump sum tax on the older workers, and an optimal income tax on the younger, whose types are currently unknown.

[3]In particular, consider the possibility that the Roberts results are only true if the time horizon is infinite.

this makes the analysis much easier by employing the first order approach to incentive compatibility.

In an interesting, related paper, Kapicka (2002) considers optimal nonlinear income taxation in an infinite horizon model. Steady states are examined when the time of consumers can be spent on schooling, leisure or labor. Human capital is accumulated through schooling. Kapicka finds that optimal tax rates are lower in this framework than in the static framework due to the additional inefficiencies caused by lower human capital accumulation in the dynamic context as opposed to the static context. A key assumption made by Kapicka is that current period tax liability depends only on current period income. Thus, it is assumed that the government has no memory.[4] In contrast, our two period model allows the government to use information on income gleaned from the first period tax when formulating the second period tax, so the government has memory. One of our main results says that when the government has memory and imposes a time consistent tax, then it will *not* be optimal for the government to forget the information it obtained in the first period when formulating the second period tax, though it has this option. In fact, it is precisely this dynamic information revelation question that makes analysis of our problem so difficult. Actually, in our model the government does not need to see a long history of incomes, but just one previous period's incomes, in order to separate types.

We note in passing that most of the literature also completely neglects the problem of *existence* of an optimal tax.

For those readers better acquainted with the principal-agent literature on incentives, it is useful to outline the comparisons between the (static) optimal income tax model and the standard principal-agent model. First, sometimes there are one or few agents in the principal-agent model, while there is often (but not always) a continuum in the optimal income tax model. Second, in the optimal income tax model, once an agent or taxpayer chooses their action (labor supply), there is no residual uncertainty for the agent. In the principal-agent model, sometimes there is residual uncertainty, specifically a non-degenerate distribution over outcomes. This makes a difference in the formal structure of the model (specifically in the second order conditions for incentive compatibility). Third, in the principal-agent literature, linear or quasi-linear utility is generally employed. The focus of the optimal income tax

[4]In other words, the model uses an infinitely repeated static optimal income tax framework, modified by the accumulation of human capital over time.

model is on the consumption-leisure trade-off, so more general utility functions are used. Fourth, the optimal income tax model has a revenue constraint, while the principal-agent model does not. Fifth, the principal-agent model has voluntary participation or individual rationality constraints, while the optimal income tax model does not. There is, however, a related problem in the optimal income tax literature. The income earning ability of each taxpayer is limited by the income they could earn if they worked all of the time and had no leisure. This "capacity constraint" is type-specific and is usually ignored in the literature; see Berliant and Page (2001) for a formal statement and analysis. In a model with quasi-linear utility but without voluntary participation or capacity constraints, one can achieve first best (i.e., the incentive constraints are not binding). Optimal income taxation often gives up quasi-linear utility and imposes capacity constraints; principal-agent models impose voluntary participation constraints. Each leads to interesting implications. However, if one replaces the Pareto criterion with a social welfare function, then one might not be able to attain its optimal value in a world with quasi-linear utility and no capacity or voluntary participation constraints.

In the next section, we give notation. In section 3 we write down the optimization problems of the consumers and the government. The first order approach to incentive compatibility is studied in section 4, while section 5 examines necessary conditions for a time consistent, incentive compatible tax; these conditions apply directly to time consistent optimal income taxes. Section 6 considers the Diamond (1998) example in our framework. Section 7 comments on conclusions and extensions, while an appendix contains two longer proofs.

2 Notation

Consumers differ by an ability parameter, w, often interpreted as a wage. Let $w \in [\underline{w}, \overline{w}] = W \subseteq \Re_+$. The types of individuals are completely specified by w. All references to measure-theoretic concepts are to Lebesgue measure m on $\Re$. There is a population density function $f : W \to \Re_{++}$, where f is integrable. The density function is common knowledge, but each agent's ability is private information. The only anonymous lump sum taxes that can be used are thus uniform, but even such taxes must be bounded by the earning capacity of the lowest ability individual.

We denote consumption by $c \in \Re_+$ and labor by $l \in [0, 1]$, where the total amount of labor that can be supplied in a period is 1. Leisure is given by $1 - l$. In this two period model, we denote time period by subscripts. All consumers are identical except for their wage. Their utility is given by a twice continuously differentiable function $U : (\Re_+ \times [0,1])^2 \to \Re$. We write $U(c_1, l_1, c_2, l_2)$. We sometimes assume that $\frac{\partial U}{\partial c_1} > 0$, $\frac{\partial U}{\partial l_1} < 0$, $\frac{\partial U}{\partial c_2} > 0$, $\frac{\partial U}{\partial l_2} < 0$. Often we will use special cases. We say that U is *time separable* if $U(c_1, l_1, c_2, l_2) = u(c_1, l_1) + \overline{u}(c_2, l_2)$, where the felicity functions of all consumers are the same twice continuously differentiable functions $u, \overline{u} : \Re_+ \times [0,1] \to \Re$. We say that U is *time separable with discounting* when it is time separable and $\overline{u}(c,l) = \rho \cdot u(c,l)$. In this special case, all consumers have a common discount factor $\rho \in \Re_{++}$ and time separable utility: $U(c_1, l_1, c_2, l_2) = u(c_1, l_1) + \rho \cdot u(c_2, l_2)$.

We define gross income in a period as $y = w \cdot l$. If there are no taxes, then $c = y$. Let $Y = [0, \overline{w}]$, the set of possible incomes.

An income tax (in a given period) is an indirect mechanism, since it is based on a revelation of income rather than type. It is not hard to map from a tax on types to an income tax and vice versa, provided that (endogenous) income is an increasing function of type. We use indirect mechanisms in this paper only because direct mechanisms would complicate notation.

Let a measurable function $t_1 : Y \to \Re$ denote a first period tax function, and let T_1 denote the set of all measurable maps from Y into $\Re$. Let a measurable function $t_2 : Y \times Y \to \Re$ denote a second period tax function, and let T_2 denote the set of all measurable maps from $Y \times Y$ into $\Re$. It accounts for both first and second period incomes, since information might be revealed in the first period. A *tax system* is a pair $(t_1, t_2) \in T_1 \times T_2$. The idea here is that the first period tax function t_1 is a (measurable) function of revealed first period income only. The second period tax function t_2 is a function of both revealed second period income y_2 *and* information (income) y_1 revealed in the first period.

3 Statement of the Problem

We assume that the government has memory, so that the problem is not simply a repeated one period optimization. In a two period model, there are many possible regimes. For instance, one could have a pooling equilibrium (all incomes are the same) in the first period, and a separating tax in the second period. More interesting is the case where one has a separating tax in the first period such that the consumers reveal their types even though they know that the government will impose a type specific (differentiated) lump sum tax in the second period. To find the optimal tax, one must find the optimum in each of these classes (and any others possible), and take the best among them.

We will sometimes assume (as is standard in the literature) that the government has a utilitarian objective:

$$\int_W U(c_1(w), l_1(w), c_2(w), l_2(w)) \cdot f(w)dw$$

where $c_1 : W \to \Re_+$, $l_1 : W \to [0,1]$, $c_2 : W \to \Re_+$, $l_2 : W \to [0,1]$ are all measurable functions. Alternatively, we will use the concept of second best Pareto optimality, which we will define formally below. The government also has revenue constraints. Let R_1 be the (exogenous) revenue to be raised in period 1 and let R_2 be the (exogenous) revenue to be raised in period 2. Perhaps this revenue is used to fund a public good that is additively separable in consumers' utility.

This brings up the issue of saving on the parts of either or both of the government and consumers. Can the consumers save, and can the government issue debt or buy bonds? These are issues peripheral to the one we are studying, namely sequential information revelation, and would only complicate the problem by adding more endogenous variables, namely the choice of consumption or saving.[5] We relegate these issues to future work.

Given a tax system, consumers of type w have the following optimization problem:

$$\max_{\substack{c_1, c_2 \in \Re_+ \\ l_1, l_2 \in [0,1]}} U(c_1, l_1, c_2, l_2) \tag{1}$$
$$\text{subject to}$$
$$l_1 \cdot w - t_1(l_1 \cdot w) \geq c_1$$
$$l_2 \cdot w - t_2(l_1 \cdot w, l_2 \cdot w) \geq c_2$$

[5] It could also complicate the problem because capital income might be treated differently from labor income by the income tax.

Hence, the government's problem is:

$$\max_{(t_1,t_2)\in T_1\times T_2} \int_W U(c_1(w), l_1(w), c_2(w), l_2(w)) \cdot f(w)dw \tag{2}$$

subject to

$$c_1(w), l_1(w), c_2(w), l_2(w) \text{ measurable and solving (1) almost surely in } w \in W,$$

$$\int_W t_1(l_1(w)\cdot w)\cdot f(w)dw \geq R_1$$

$$\int_W t_2(l_1(w)\cdot w, l_2(w)\cdot w)\cdot f(w)dw \geq R_2$$

A *utilitarian optimal tax system* $(t_1, t_2) \in T_1 \times T_2$ is defined to be a solution to this problem.

A tax system $(t_1, t_2) \in T_1 \times T_2$ is called *feasible* if there exist

$$c_1(w), l_1(w), c_2(w), l_2(w) \text{ measurable and solving (1) almost surely in } w \in W,$$

$$\int_W t_1(l_1(w)\cdot w)\cdot f(w)dw \geq R_1$$

$$\int_W t_2(l_1(w)\cdot w, l_2(w)\cdot w)\cdot f(w)dw \geq R_2$$

A (second best[6]) *optimal tax system* $(t_1, t_2) \in T_1 \times T_2$ is a feasible tax system (with associated $c_1(w), l_1(w), c_2(w), l_2(w)$) such that there is no other feasible tax system $(t_1', t_2') \in T_1 \times T_2$ (with associated $c_1'(w), l_1'(w), c_2'(w), l_2'(w)$) such that $U(c_1'(w), l_1'(w), c_2'(w), l_2'(w)) \geq U(c_1(w), l_1(w), c_2(w), l_2(w))$ almost surely in $w \in W$, with strict inequality holding for a measurable set $W' \subseteq W$, where $\int_{W'} f(w)dw > 0$. Notice that any utilitarian optimal tax system is necessarily Pareto optimal.

4 The First Order Approach to Incentive Compatibility

We examine problem (1) under the assumption of differentiability of tax and utility functions, using the definitions $y_1(w) = l_1(w)\cdot w$ and $y_2(w) = l_2(w)\cdot w$. For type w the problem reduces to:

$$\max_{y_1,y_2\in Y} U(y_1 - t_1(y_1), \frac{y_1}{w}, y_2 - t_2(y_1,y_2), \frac{y_2}{w})$$

Using subscripts on U to denote derivatives, the first order conditions are:

[6] The tax system is called second best due to the incentive compatibility constraints.

$$U_1 \cdot (1 - \frac{\partial t_1}{\partial y_1}) + U_2 \cdot \frac{1}{w} - U_3 \cdot \frac{\partial t_2}{\partial y_1} = 0 \quad (3)$$

$$U_3 \cdot (1 - \frac{\partial t_2}{\partial y_2}) + U_4 \cdot \frac{1}{w} = 0 \quad (4)$$

The first order conditions for the purely static (period 2 only) model are the second set of conditions, (4). This corresponds exactly to expressions obtained in the literature. In the standard static case, we obtain an ordinary first order differential equation for incentive compatible tax systems. Here we obtain a (nicely behaved) system of partial differential equations. The third term in equation (3) is an "extra term" in the system relative to the literature on the static case. It represents the effect of increased income in the first period on tax liability in the second period.

In the special case of time separability and discounting, using subscripts to denote partial derivatives of u, we obtain first order conditions for incentive compatibility:

$$u_1(y_1 - t_1(y_1), \frac{y_1}{w}) \cdot (1 - \frac{\partial t_1}{\partial y_1}) + u_2(y_1 - t_1(y_1), \frac{y_1}{w}) \cdot \frac{1}{w} - \quad (5)$$

$$\rho \cdot u_1(y_2 - t_2(y_1, y_2), \frac{y_2}{w}) \cdot \frac{\partial t_2}{\partial y_1} = 0$$

$$\rho \cdot u_1(y_2 - t_2(y_1, y_2), \frac{y_2}{w}) \cdot (1 - \frac{\partial t_2}{\partial y_2}) + \rho \cdot u_2(y_2 - t_2(y_1, y_2), \frac{y_2}{w}) \cdot \frac{1}{w} = 0 \quad (6)$$

Theorem 1 *(Second Order Conditions) Assume time separability and discounting in the utility function. Further assume that $u_1 \geq 0$, $u_{11} \leq 0$, $u_{22} < 0$, $u_{12} \leq 0$, $\frac{\partial^2 t_1}{(\partial y_1)^2} \geq 0$, $\frac{\partial^2 t_2}{(\partial y_1)^2} \geq 0$, $\frac{\partial t_1}{\partial y_1} \leq 1$, $\frac{\partial t_2}{\partial y_2} \leq 1$. Then there exists $\overline{\rho} > 0$ such that $\forall \rho < \overline{\rho}$, the second order condition for consumer optimization in y_1 holds, so the first order condition (5) characterizes optima. If, in addition, $u_2 \leq 0$, $\frac{\partial^2 t_2(y_1,y_2)}{\partial y_1 \partial y_2} \leq 0$ and $\frac{\partial t_2}{\partial y_1} \geq 0$, then if either u_{12} is sufficiently close to zero (or zero)* or *if ρ is sufficiently small, then $\frac{dy_1(w)}{dw} > 0$, and in particular y_1 is one to one.*

Proof: See Appendix.

These conditions are sufficient, but of course they are not necessary.

Theorem 2 *Suppose that $U(c_1, l_1, c_2, l_2) = V(c_1 + c_2, l_1 + l_2)$. Then any incentive compatible tax satisfies: $\frac{\partial t_1}{\partial y_1} = \frac{\partial t_2}{\partial y_2} - \frac{\partial t_2}{\partial y_1}$.*

Proof: Equations (3) and (4) reduce to:

$$\begin{aligned} V_1 \cdot (1 - \frac{\partial t_1}{\partial y_1}) + V_2 \cdot \frac{1}{w} - V_1 \cdot \frac{\partial t_2}{\partial y_1} &= 0 \\ V_1 \cdot (1 - \frac{\partial t_2}{\partial y_2}) + V_2 \cdot \frac{1}{w} &= 0 \end{aligned}$$

Simplifying, the result follows.■

5 Necessary Conditions for a Time Consistent Tax

As a preamble to the consideration of time consistent taxes, consider optimal income taxes in our framework. Two regimes of interest are:

- Nothing is revealed in the first period (all incomes are the same), and an optimal static income tax is imposed in the second period. Thus, $R_1 > \underline{w} \int_W f(w)dw$ implies this regime is impossible.
- A separating equilibrium occurs in the first period, and a differentiated (type-dependent) lump sum tax is imposed in the second period. In this case, t_2 is constant as a function of y_2. That is, $\frac{\partial t_2(y_1,y_2)}{\partial y_2} = 0 \; \forall y_1, y_2 \in Y$.

In the first case, obviously the top marginal tax rate is zero in the second period for the usual reasons. In the second case, in the second period each individual is facing a lump sum tax, so their marginal rates are always zero. However, tax as a function of equilibrium income will not necessarily appear to have a top marginal rate of zero, since the individualized lump sum taxes could be increasing in type.

It is natural to try to advance an argument that when revelation of types occurs in the first period, and the government has memory, that the second period tax should be a differentiated lump sum tax. Here is how that argument, a proof by contradiction, would go. Suppose that the second period tax is not lump sum, i.e. it is a function of period 2 income as well as period 1 income. (Differentiated lump sum taxes will have zero derivative with respect to period 2 income.) Next we design a new tax system that Pareto dominates. Keep the first period tax the same. Now replace the second period tax with a differentiated lump sum tax that assigns each consumer (separated in the first period) the same tax liability as in the original second period tax, so there is no deadweight loss. This would clearly generate the same tax revenue, and would Pareto dominate the original tax. The problem is that the new tax might not be incentive compatible in the *first* period, since the incentive constraints are more severe. There is a trade-off between an efficiency gain in the second period from moving from a distorting to a non-distorting tax, but a possible efficiency loss in the first period since second period tax liability is now a function only of first period income (as is first period tax liability), so consumers have more of an incentive to pretend to be someone with lower income and ability, since it affects their second period tax liability. The result appears not to be true in general, and probably requires some very technical conditions concerning this trade-off. For instance, in the general case, it's possible that an optimum involves having the government (commit to) forget first period income when imposing the second period tax, that is, making the second period tax a function of second period income only. Then the problem reduces to a repeated static optimal income tax problem.

There is an entirely different argument for why the second period tax must be a differentiated lump sum tax. Suppose we impose subgame perfection or time consistency on the equilibrium concept (in particular, for the government). Suppose we impose the conditions of Theorem 1, so the first period tax separates. Will the government want to impose a second period tax that ignores revelation in the first period? It cannot credibly commit to do so, since once it gets to the second period decision node, given a Pareto or utilitarian objective, it will want to impose a non-distorting tax in the second period. So the use of this time consistency concept implies, in itself, that the second period tax will be a differentiated lump sum tax. And thus the second period tax will not be a function of second period income. It is possible, however, that a Nash equilibrium without time consistency Pareto dominates the one with time consistency.

Definition 1 *A tax system* $(t_1, t_2) \in T$ *is called* utilitarian time consistent *if* t_2 *solves the following optimization problem given* $t_1 \in T_1$ *and* $c_1(w)$, $l_1(w)$

measurable.

$$\max_{t_2' \in T_2} \int_W U(c_1(w), l_1(w), c_2'(w), l_2'(w)) df(w)$$

subject to

$$\int_W t_2'(l_1(w) \cdot w, l_2'(w) \cdot w) df(w) \geq R_2$$

and subject to

$$c_2'(w), l_2'(w) \text{ measurable and solving}$$

$$\max_{\substack{c_2' \in \Re_+ \\ l_2' \in [0,1]}} U(c_1(w), l_1(w), c_2', l_2')$$

subject to

$$l_2' \cdot w - t_2'(l_1 \cdot w, l_2' \cdot w) \geq c_2' \text{ almost surely in } w \in W.$$

Definition 2 *A tax system* $(t_1, t_2) \in T_1 \times T_2$ *is called* Pareto time consistent *if the following holds given* $t_1 \in T_1$ *and* $c_1(w)$, $l_1(w)$ *measurable. There is no* $t_2' \in T_2$ *such that* $U(c_1(w), l_1(w), c_2'(w), l_2'(w)) \geq U(c_1(w), l_1(w), c_2(w), l_2(w))$ *almost surely in* $w \in W$, *with strict inequality holding for a measurable set* $W' \subseteq W$, *where* $\int_{W'} f(w)dw > 0$, *and such that*

$$\int_W t_2'(l_1(w) \cdot w, l_2'(w) \cdot w) df(w) \geq R_2$$

with

$$c_2'(w), l_2'(w) \text{ measurable and solving}$$

$$\max_{\substack{c_2' \in \Re_+ \\ l_2' \in [0,1]}} U(c_1(w), l_1(w), c_2', l_2')$$

subject to

$$l_2' \cdot w - t_2'(l_1 \cdot w, l_2' \cdot w) \geq c_2' \text{ almost surely in } w \in W.$$

Equivalent definitions can be formulated using backward induction, but they are much messier. Notice that any utilitarian time consistent tax is necessarily Pareto time consistent.

It is very important to note that the concept of time consistency employed in a model must be logically related to the government objective function in the following way. The set of taxes generated using time consistency must be a set at least as small as the set generated by optimizing the government objective function in the second period. For instance, using a Pareto objective with Pareto time consistency is fine, as is using utilitarian consistency with a utilitarian objective. Employing a Pareto objective with utilitarian time consistency works fine as well. Consider, however, employing a utilitarian objective in conjunction with Pareto time consistency. In this case, the government may wish, when it reaches its decision node in the second period, to impose a utilitarian optimal income tax (given first period decisions) rather than one that is only Pareto optimal (but perhaps not utilitarian optimal). Thus, it is natural to require that the notion of time consistency employed in a model be compatible, in the sense we have given, with the objective function of the government. For otherwise "time consistency" does not mean that the government will hold to its decision when it reaches the second period.

Theorem 3 *Let $(t_1^*, t_2^*) \in T_1 \times T_2$ be a Pareto time consistent tax system such that y_1^* is one to one. Then $\frac{\partial t_2^*(y_1^*(w), y_2^*(w))}{\partial y_2} = 0$ almost surely for $\{w \in W \mid f(w) > 0\}$.*

Proof: Suppose that there is a measurable set W' such that $\frac{\partial t_2^*(y_1^*(w), y_2^*(w))}{\partial y_2} \neq 0$ for $w \in W'$and $\int_{W'} f(w)dw > 0$. We claim that this tax is not Pareto time consistent. Consider the alternative tax system $t_2 \in T_2$ given by $t_2(y_1, y_2) = t_2^*(y_1, y_2^*(y_1^{*-1}(y_1)))$. Here, y_1^{*-1} denotes the inverse of the function y_1^*, which is well-defined by assumption. Notice that this alternative tax system does not depend on second period income, but only on first period income.

$$
\begin{aligned}
& \int_W t_2(l_1^*(w) \cdot w, l_2(w) \cdot w) df(w) \\
= & \int_W t_2^*(y_1(w), y_2^*(y_1^{*-1}(y_1(w)))) df(w) \\
= & \int_W t_2^*(y_1(w), y_2^*(w)) df(w) \geq R_2
\end{aligned}
$$

Incentive compatibility follows trivially from the definitions of $c_2(w)$ and $l_2(w)$. Fix $w \in W'$. Then evaluated at $(c_1^*(w), l_1^*(w), c_2^*(w), l_2^*(w))$, (4) tells us that offering tax t_2, where $t_2(y_1^*(w), y_2^*(w)) = t_2^*(y_1^*(w), y_2^*(w))$ and $\frac{\partial t_2}{\partial y_2} = 0$, leads to a local utility improvement for type w.■

The theorem implies that any time consistent tax system is a lump sum tax in the second period. In general, it will be a lump sum tax differentiated by consumer type, which is revealed in the first period. The consumers understand this when they make their first period labor supply decision.

Since any utilitarian time consistent tax system is also Pareto time consistent, the theorem applies to these tax systems as well.

Corollary 1 *Let* $(t_1^*, t_2^*) \in T_1 \times T_2$ *be a Pareto time consistent tax system satisfying the assumptions of Theorem 1. Then* $\frac{\partial t_2^*(y_1^*(w), y_2^*(w))}{\partial y_2} = 0$ *almost surely for* $\{w \in W \mid f(w) > 0\}$.

Definition 3 *A* utilitarian time consistent optimal tax *is a tax system* $(t_1, t_2) \in T_1 \times T_2$ *that solves problem (2) subject to the additional constraint that* (t_1, t_2) *is utilitarian time consistent.*

Definition 4 *A* Pareto time consistent optimal tax *is a tax system* $(t_1, t_2) \in T_1 \times T_2$ *is a feasible, Pareto time consistent tax system (with associated* $c_1(w), l_1(w), c_2(w), l_2(w)$*) such that there is no other feasible, Pareto time consistent tax system* $(t_1', t_2') \in T_1 \times T_2$ *(with associated* $c_1'(w), l_1'(w), c_2'(w), l_2'(w)$*) such that* $U(c_1'(w), l_1'(w), c_2'(w), l_2'(w)) \geq U(c_1(w), l_1(w), c_2(w), l_2(w))$ *almost surely in* $w \in W$*, with strict inequality holding for a measurable set* $W' \subseteq W$*, where* $\int_{W'} f(w)dw > 0$.

Since the set of utilitarian time consistent taxes could be a strict subset of Pareto time consistent taxes, it could be the case that a Pareto time consistent optimal tax Pareto (or even utilitarian) dominates a utilitarian time consistent optimal tax.

Corollary 2 *Presuming either the conditions of Theorem 1 or directly that* y_1^* *is one to one, any [utilitarian or Pareto] time consistent optimal tax is Pareto time consistent, so Theorem 3 applies and* $\frac{\partial t_2^*(y_1^*(w), y_2^*(w))}{\partial y_2} = 0$ *almost surely for* $\{w \in W \mid f(w) > 0\}$.

6 The Diamond Model

We extend the static Diamond (1998) model using time separable utility and discounting.

The utility function used by Diamond (1998) in our notation is:

$$u(c, l) = c + v(1 - l)$$

where $v : [0, 1] \to \Re$ and v is C^2. Let v' denote the derivative of v, and let v'' be its second derivative. Assume that $v' > 0$, $v'' < 0$. We refer to this specification as "the Diamond model."

Proposition 1 *For the Diamond model,* $u_{12} = 0$*, so any incentive compatible income tax satisfying the strict second order conditions locally, inequality (10), with* $\frac{\partial^2 t_1}{(\partial y_1)^2} \geq 0$, $\frac{\partial t_2}{\partial y_2} \leq 1$, $\frac{\partial^2 t_2(y_1, y_2)}{\partial y_1 \partial y_2} \leq 0$ *and* $\frac{\partial t_2}{\partial y_1} \geq 0$ *has separation (*$\frac{dy_1}{dw} > 0$*) in the first period, regardless of the discount rate* ρ.

The proof follows directly from the proof of Theorem 1. Alternatively, we could apply Theorem 1. We will generally assume that $\frac{dy_1}{dw} > 0$ in this section.

Equations (5) and (6) reduce to:

$$1 - \frac{\partial t_1}{\partial y_1} - v'(1 - \frac{y_1}{w}) \cdot \frac{1}{w} - \rho \cdot \frac{\partial t_2}{\partial y_1} = 0 \tag{7}$$

$$1 - \frac{\partial t_2}{\partial y_2} - v'(1 - \frac{y_2}{w}) \cdot \frac{1}{w} = 0 \tag{8}$$

Notice that in this case, equation (8) gives us

$$1 - v'(1 - \frac{y_2}{w}) \cdot \frac{1}{w} = \frac{\partial t_2}{\partial y_2}$$

Since the first period tax is separating (that is, $\frac{dy_1}{dw} > 0$), then time consistency implies $\frac{\partial t_2}{\partial y_2} = 0$ and the equation

$$v'(1 - \frac{y_2}{w}) = w \tag{9}$$

completely determines y_2 and l_2.

In order to prepare for the statement of the next result, it is important to inform the reader about some implicit assumptions. *For the remainder of the paper, we shall assume that consumers can transfer income between periods at interest rate ρ.* In other words, we assume that c_1 and c_2 can take on any real values, subject to $c_1 + \rho \cdot c_2 \geq 0$. The analog in the static Diamond model, which we will also use, is $c \geq 0$. The capacity constraint, mentioned in the introduction, has not required an explicit statement to this point. We give one now. Since total time for work and leisure for any consumer in each time period is 1, the capacity constraints in our two period model are $y_1 \leq w$, $y_2 \leq w$. The analog in the static Diamond model will be $y \leq w$.

Theorem 4 *Consider the Diamond model. A tax system (t_1^*, t_2^*) is Pareto time consistent optimal in the two period model among tax systems with $\frac{dy_1}{dw} > 0$ if and only if there is no measurable $t : Y \to \Re$ such that $\frac{d(w \cdot l(w))}{dw} > 0$ and $\int_W t(w \cdot l(w))dw \geq R_1 + \rho R_2$, where $c(w), l(w)$ solve $\max_{c,l} c + v(1-l)$ subject to $w \cdot l - t(w \cdot l) \geq c$ a.s.(W), and such that $c(w) + v(1 - l(w)) \geq c_1^*(w) + \rho[c_2^*(w) - w \cdot l_2^*(w)] + v(1 - l_1^*(w))$, with strict inequality holding for a measurable set W' with $\int_{W'} f(w)dw > 0$.*

Thus, *the optima of the dynamic Diamond model are equivalent to those of a properly formulated static Diamond model.* Suppose that (t_1^*, t_2^*) is Pareto time consistent optimal. Then the static tax $t(y) = t_1^*(y) + \rho \cdot t_2^*(y)$ is optimal (note that t_2^* is a function of only period 1 income, since it is a differentiated lump sum tax in period 2).

Given an optimal income tax t^* in the static Diamond (1998) model, for example one computed by Diamond, any feasible tax system (t_1, t_2) satisfying $t^*(y) = t_1(y) + \rho \cdot t_2(y)$ and the conditions of Theorem 1 or Proposition 1 (or $\frac{dy_1}{dw} > 0$) will be Pareto time consistent optimal. There are many such tax systems. In fact, there are so many that neither the first nor the second period tax might look like the optimal tax in the static model.

Proof: See Appendix.

Corollary 3 *A Pareto time consistent optimal tax (t_1^*, t_2^*) for the Diamond model satisfying $\frac{dy_1^*}{dw} > 0$ also satisfies $[\frac{\partial t_1^*(y_1)}{\partial y_1} + \rho \frac{\partial t_2^*(y_1,y_2)}{\partial y_1}] \mid_{y_1=\overline{w}\cdot l_1(\overline{w}),\ y_2=\overline{w}\cdot l_2(\overline{w})}=$ 0.*[7]

Proof: Any Pareto time consistent optimal tax must generate a Pareto optimal tax t in the static model, as given by Theorem 4. The induced utility function in the static model is separable (and in fact, quasi-linear) in consumption and leisure, so consumption is noninferior. By Seade (1977, Theorem 1), $\frac{dt(y)}{dy} \mid_{y=\overline{w}\cdot l(\overline{w})}= 0$, and the result follows.■

7 Conclusions and Extensions

We have examined optimal income taxes in a two period model, beginning with the first and second order conditions for incentive compatibility. Then imposing time consistency of taxes, we find that if the discount rate is sufficiently high or utility is separable in labor and consumption, a time consistent tax has consumers revealing their types in the first period, so the second period tax is independent of second period income; it is essentially a type differentiated lump sum tax. Incentive constraints on first period consumer income are brutal, as the consumers know what's going to happen in the second period. In the special case of stationary, time separable, quasi-linear utility with discounting, we find an equivalence between static and dynamic optimal income taxes. In this case, an implication is that the present discounted value of the marginal tax rate on first period income at the top of the distribution must be zero. There is a huge number of optimal dynamic tax systems that correspond to a single optimal static tax; all that is required is that the present discounted

[7] In fact, we also know that $\frac{\partial t_2^*(y_1,y_2)}{\partial y_2} = 0$ everywhere.

value of the dynamic tax is equal to the static tax for any first period income. In this sense, the optimal one period tax in a two period model is not identified.

The next step is to examine time consistent taxes when the discount rate is low. We conjecture that all we will find is that at the top of an interval that has separation in the first period, the marginal tax rate on first period income must be zero.

Our long term goal is to integrate the theory of optimal income taxation with mechanism design. Each area brings useful ideas and techniques to the other. A first step in this direction is to examine the relationship between direct and indirect mechanisms in our context of dynamic revelation of information.

Time consistent, incentive compatible taxes might also be useful for purposes of positive political economy, where one replaces the Pareto or utilitarian objective with a voting mechanism.

We leave examination of the question of existence of optimal taxes to future work. Probably this issue can be addressed using the techniques of Berliant and Page (2001).

As mentioned in section 3, it would be interesting to examine the implications of allowing consumers or the government to transfer wealth across time, and to see how this alters our results.

Finally, we have examined the case where the correlation of a consumer's type in the first and second periods is perfect. The case when there is no correlation between types in the two periods is simply a repeated static optimal income tax problem. Intermediate cases are clearly of interest.

8 Appendix

Proof of Theorem 1: For the purpose of this proof, define $G(y_1, y_2, w) = u(y_1 - t_1(y_1), \frac{y_1}{w}) + \rho \cdot u(y_2 - t_2(y_1, y_2), \frac{y_2}{w})$. We remind the reader that subscripts on functions represent partial derivatives with respect to the appropriate arguments except for the function t, where a subscript denotes the time period. Note also that all terms with a ρ attached to them are evaluated at second period bundles, while all terms without a ρ attached to them are evaluated at first period bundles. The first order conditions are given by

$$
\begin{aligned}
G_1 &= u_1(y_1 - t_1(y_1), \frac{y_1}{w})(1 - \frac{\partial t_1(y_1)}{\partial y_1}) + \frac{1}{w} \cdot u_2(y_1 - t_1(y_1), \frac{y_1}{w}) - \rho \cdot u_1(y_2 - t_2(y_1, y_2), \frac{y_2}{w})\frac{\partial t_2}{\partial y_1} \\
&= 0 \\
G_2 &= \rho \cdot [u_1(y_2 - t_2(y_1, y_2), \frac{y_2}{w})(1 - \frac{\partial t_2}{\partial y_2}) + u_2(y_2 - t_2(y_1, y_2), \frac{y_2}{w}) \cdot \frac{1}{w}] = 0
\end{aligned}
$$

Useful second derivatives are

$$
\begin{aligned}
G_{11} = u_{11} \cdot (1 - \frac{\partial t_1(y_1)}{\partial y_1})^2 + \frac{1}{w} \cdot u_{12} \cdot (1 - \frac{\partial t_1(y_1)}{\partial y_1}) - u_1 \cdot \frac{\partial^2 t_1}{\partial (y_1)^2} + \frac{1}{w} \cdot u_{21} \cdot (1 - \frac{\partial t_1(y_1)}{\partial y_1}) + \\
\frac{1}{w^2} \cdot u_{22} + \rho \cdot u_{11} \cdot [\frac{\partial t_2}{\partial y_1}]^2 - \rho \cdot u_1 \cdot \frac{\partial^2 t_2}{\partial (y_1)^2} \\
= u_{11} \cdot (1 - \frac{\partial t_1(y_1)}{\partial y_1})^2 + \frac{2}{w} u_{12} \cdot (1 - \frac{\partial t_1(y_1)}{\partial y_1}) - u_1 \cdot \frac{\partial^2 t_1}{\partial (y_1)^2} + \frac{1}{w^2} \cdot u_{22} \\
+ \rho \cdot u_{11} \cdot [\frac{\partial t_2}{\partial y_1}]^2 - \rho \cdot u_1 \cdot \frac{\partial^2 t_2}{\partial (y_1)^2}
\end{aligned}
$$

$$
G_{12} = G_{21} = -\rho \cdot [u_{11} \cdot (1 - \frac{\partial t_2}{\partial y_2}) \cdot \frac{\partial t_2}{\partial y_1} + u_{12} \cdot \frac{1}{w} \cdot \frac{\partial t_2}{\partial y_1} + u_1 \cdot \frac{\partial^2 t_2}{\partial y_1 \partial y_2}]
$$

$$
\begin{aligned}
G_{22} &= \rho \cdot [u_{11} \cdot (1 - \frac{\partial t_2}{\partial y_2})^2 + \frac{1}{w} \cdot u_{12} \cdot (1 - \frac{\partial t_2}{\partial y_2}) - u_1 \cdot \frac{\partial^2 t_2}{\partial (y_1)^2} + \frac{1}{w} \cdot u_{21} \cdot (1 - \frac{\partial t_2}{\partial y_2}) + \frac{1}{w^2} \cdot u_{22}] \\
&= \rho \cdot [u_{11} \cdot (1 - \frac{\partial t_2}{\partial y_2})^2 + \frac{2}{w} \cdot u_{12} \cdot (1 - \frac{\partial t_2}{\partial y_2}) - u_1 \cdot \frac{\partial^2 t_2}{\partial (y_1)^2} + \frac{1}{w^2} \cdot u_{22}]
\end{aligned}
$$

$$
G_{13} = -\frac{y_1}{w^2} \cdot u_{12} \cdot (1 - \frac{\partial t_1(y_1)}{\partial y_1}) - \frac{1}{w^2} \cdot u_2 - \frac{y_1}{w^3} \cdot u_{22} + \rho \cdot \frac{y_2}{w^2} \cdot u_{12} \cdot \frac{\partial t_2}{\partial y_1}
$$

$$
G_{23} = \rho \cdot [-\frac{y_2}{w^2} \cdot u_{12} \cdot (1 - \frac{\partial t_2}{\partial y_2}) - \frac{y_2}{w^3} \cdot u_{22} - \frac{1}{w^2} \cdot u_2]
$$

Checking term by term, under the stated assumptions, $G_{11} < 0$, $G_{22} < 0$. Let $A = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$. For the second order conditions, it is sufficient to prove that A is negative definite, or

$$
\mid A \mid = G_{11} \cdot G_{22} - (G_{12})^2 > 0 \tag{10}
$$

The result follows by noticing that the first few (negative) terms in G_{11} are the only ones in the expression without ρ in them, whereas all terms in G_{22}

and G_{12} have ρ in them. Hence $(G_{12})^2$ tends to zero with ρ^2 while $G_{11} \cdot G_{22}$ tends to zero at rate ρ.

For the second part of the theorem, notice that $A^{-1} = \frac{1}{G_{11}G_{22}-(G_{12})^2} \begin{bmatrix} G_{22} & -G_{12} \\ -G_{12} & G_{11} \end{bmatrix}$.

By the implicit function theorem, $\begin{bmatrix} \frac{\partial y_1(w)}{\partial w} \\ \frac{\partial y_2(w)}{\partial w} \end{bmatrix} = -A^{-1} \cdot \begin{bmatrix} G_{13} \\ G_{23} \end{bmatrix} = \frac{-1}{|A|} \cdot \begin{bmatrix} G_{22} \cdot G_{13} - G_{12} \cdot G_{23} \\ -G_{12} \cdot G_{13} + G_{11}G_{23} \end{bmatrix}$. We're actually only interested in the top part of the vector. $G_{22} < 0$, $G_{23} > 0$. Under the additional assumptions of the theorem, $G_{12} \geq 0$. If u_{12} is sufficiently close to zero (or is zero) or if ρ is sufficiently small, then $G_{13} > 0$ and we have $\frac{\partial y_1(w)}{\partial w} > 0$. ■

Proof of Theorem 6: Suppose that (t_1^*, t_2^*) is Pareto time consistent optimal in the two period model and there is a measurable $t : Y \to \Re$ such that $\int_W t(w \cdot l(w))dw \geq R_1 + \rho R_2$, $\frac{d(w \cdot l(w))}{dw} > 0$, where $c(w), l(w)$ solve $\max_{c,l} c + v(1-l)$ subject to $w \cdot l - t(w \cdot l) \geq c$ a.s.(W), and such that $c(w) + v(1 - l(w)) \geq c_1^*(w) + \rho[c_2^*(w) - w \cdot l_2^*(w)] + v(1 - l_1^*(w))$, with strict inequality holding for a measurable set W' with $\int_{W'} f(w)dw > 0$. We show a contradiction, in that there is a feasible (t_1, t_2) that Pareto time consistent dominates (t_1^*, t_2^*). Define $t_1(y_1) = \frac{R_1}{R_1+\rho R_2} \cdot t(y_1)$ and $t_2(y_1, y_2) = \frac{R_2}{R_1+\rho R_2} \cdot t(y_1)$.[8] In fact, second period labor supply is the same for both tax systems, and determined by (9). Evidently, the first order condition for incentive compatibility of the system (t_1, t_2) is, from (7),

$$\begin{aligned}
& 1 - \frac{\partial t_1}{\partial y_1} - v'(1 - \frac{y_1}{w}) \cdot \frac{1}{w} - \rho \cdot \frac{\partial t_2}{\partial y_1} \\
= \; & 1 - \frac{R_1}{R_1 + \rho R_2} \cdot \frac{dt}{dy} - \rho \cdot \frac{R_2}{R_1 + \rho R_2} \cdot \frac{dt}{dy} - v'(1 - \frac{y_1}{w}) \cdot \frac{1}{w} \\
= \; & 1 - \frac{dt}{dy} - v'(1 - \frac{y_1}{w}) \cdot \frac{1}{w} \\
= \; & 0
\end{aligned}$$

The last line follows from the first order conditions for incentive compatibility of t, from the static model, so $y_1(w) = y(w)$. In fact, it is clear from these calculations that the second order conditions for incentive compatibility are the same for (t_1, t_2) (namely $\frac{dy_1}{dw} > 0$) and t (namely $\frac{dy}{dw} > 0$).[9] Since

[8] There are actually many ways to define (t_1, t_2).

[9] Although we have not proved formally that these are, in fact, the second order conditions, notice that in a rather trivial way (due, in part, to separability of the utility function), agents facing either tax system are actually solving the same optimization problem. So the solutions are the same.

$\frac{dy}{dw} > 0$ by assumption, $\frac{dy_1}{dw} > 0$.

$$\begin{aligned}
& \int_W t_1(y_1(w)) \cdot f(w)dw \\
= \; & \frac{R_1}{R_1 + \rho R_2} \cdot \int t(y_1(w)) \cdot f(w)dw \\
\geq \; & R_1
\end{aligned}$$

$$\begin{aligned}
& \int_W t_2(y_1(w)) \cdot f(w)dw \\
= \; & \frac{R_2}{R_1 + \rho R_2} \cdot \int t(y_1(w)) \cdot f(w)dw \\
\geq \; & R_2
\end{aligned}$$

So (t_1, t_2) is feasible. Now from incentive compatibility, $y(w) = y_1(w)$. So

$$\begin{aligned}
& c_1(w) + \rho \cdot c_2(w) + v(1 - l_1(w)) + \rho \cdot v(1 - l_2(w)) \\
= \; & y_1(w) - t_1(y_1(w)) + \rho \cdot [y_2(w) - t_2(y_1(w), y_2(w))] + v(1 - \frac{y_1(w)}{w}) + \rho \cdot v(1 - \frac{y_2(w)}{w}) \\
= \; & y(w) - [t_1(y(w)) + \rho \cdot t_2(y_1(w), y_2(w))] + \rho \cdot y_2(w) + v(1 - \frac{y(w)}{w}) + \rho \cdot v(1 - \frac{y_2(w)}{w}) \\
= \; & y(w) - t(y(w)) + v(1 - \frac{y(w)}{w}) + \rho \cdot y_2(w) + \rho \cdot v(1 - \frac{y_2(w)}{w}) \\
= \; & c(w) + v(1 - \frac{y(w)}{w}) + \rho \cdot y_2(w) + \rho \cdot v(1 - \frac{y_2(w)}{w}) \\
\geq \; & c_1^*(w) + \rho c_2^*(w) + v(1 - l_1^*(w)) + \rho \cdot v(1 - \frac{y_2^*(w)}{w})
\end{aligned}$$

with strict inequality holding on W'. So (t_1, t_2) Pareto time consistent dominates (t_1^*, t_2^*) with $\frac{dy_1}{dw} > 0$.

Next suppose that (t_1, t_2) is feasible and Pareto time consistent dominates (t_1^*, t_2^*). Note that since by definition, t_2 satisfies (8) and (9), t_2 is independent of y_2. Define $t(y) = t_1(y) + \rho \cdot t_2(y)$. The optimization problem for consumers in the static model is thus

$$\max_y y - t_1(y) - \rho \cdot t_2(y) + v(1 - \frac{y}{w})$$

The first order condition for incentive compatibility in the static model is

$$1 - \frac{dt_1(y)}{dy} - \rho \cdot \frac{dt_2(y)}{dy} - v'(1 - \frac{y}{w}) = 0$$

This is the same as (7). Moreover, since (t_1, t_2) satisfies the second order conditions for incentive compatibility (namely $\frac{dy_1}{dw} > 0$), so does t (hence $\frac{dy}{dw} > 0$).[10] So $y(w) = y_1(w)$.

$$\begin{aligned}
& \int_W t(y(w)) \cdot f(w)dw \\
= \; & \int_W t_1(y(w)) \cdot f(w)dw + \rho \cdot \int_W t_2(y(w)) \cdot f(w)dw \\
\geq \; & R_1 + \rho \cdot R_2
\end{aligned}$$

$$\begin{aligned}
& c(w) + v(1 - l(w)) \\
= \; & y(w) - t(y(w)) + v(1 - \frac{y(w)}{w}) \\
= \; & y_1(w) - t_1(y_1(w)) - \rho \cdot t_2(y_1(w)) + v(1 - \frac{y_1(w)}{w}) \\
= \; & c_1(w) + \rho[c_2(w) - w \cdot l_2(w)] + v(1 - l_1(w))
\end{aligned}$$

Now (t_1, t_2) Pareto time consistent dominates (t_1^*, t_2^*), so

$$\begin{aligned}
& c_1(w) + \rho \cdot c_2(w) + v(1 - l_1(w)) + \rho \cdot v(1 - l_2(w)) \\
\geq \; & c_1^*(w) + \rho \cdot c_2^*(w) + v(1 - l_1^*(w)) + \rho \cdot v(1 - l_2^*(w))
\end{aligned}$$

with strict inequality holding for a measurable set W' with $\int_{W'} f(w)dw > 0$. Since by incentive compatibility of (t_1, t_2) and (t_1^*, t_2^*), (8) holds for both, $y_2(w) = y_2^*(w)$ and $l_2(w) = l_2^*(w)$. So

$$\begin{aligned}
& c_1(w) + \rho[c_2(w) - w \cdot l_2(w)] + v(1 - l_1(w)) \\
\geq \; & c_1^*(w) + \rho[c_2^*(w) - w \cdot l_2^*(w)] + v(1 - l_1^*(w))
\end{aligned}$$

Hence

$$\begin{aligned}
& c(w) + v(1 - l(w)) \\
\geq \; & c_1^*(w) + \rho[c_2^*(w) - w \cdot l_2^*(w)] + v(1 - l_1^*(w))
\end{aligned}$$

with strict inequality holding for a measurable set W' with $\int_{W'} f(w)dw > 0$.■

[10] Please see the previous footnote.

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