An Approach to Asset-Pricing Under Incomplete and Diverse Perceptions*

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Abstract

We model a dynamic, competitive market, where in every period risk-neutral traders trade a one-period bond against an infinitely-lived asset, with limited short-selling of the long-term asset. Traders lack structural knowledge and use different "incomplete theories", all of which give statistically correct beliefs about next period's market price of the long-term asset. The more theories in the market, the higher is the equilibrium price of the long-term asset, which exceeds the most optimistic trader's expectation of its present-discounted value. When dividends are very persistent and the market includes traders with the least and most complete theories, additional theories decrease the range of market prices. Investors with more complete theories do not necessarily earn higher returns than those with less complete ones, who can earn above the risk-free rate. We provide two necessary conditions for a trader to earn above the risk-free rate. Prices are sub-linear in dividends because bundling dividend streams reduces heterogeneity in beliefs.

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1 Introduction

Recent economic events have brought into focus people's limited abilities to understand the *modus operandi* of financial markets and forecast their outcomes. In this paper, we explore the economic interaction of agents who have diverse yet limited degrees of sophistication in their abilities to recognise patterns and connections among economic variables affecting the financial market. Heterogeneity in beliefs arises in our model because traders who lack structural knowledge of the economy use different incomplete theories to forecast prices. Unlike models of asymmetric information, some traders simply neglect the relevance of observable variables. Unlike models of non-common priors, all traders have beliefs that are statistically correct.

The classical approach to modelling expectations in markets assumes that the theories of pricing upon which traders base their expectations are essentially complete and homogeneous; traders understand price formation just the same as the modeller, including which variables affect prices. In this paper, we attempt to dispense with the assumption that traders use complete models, whilst maintaining the assumption that their expectations are statistically correct given their limited perceptions of the environment. Just as we do not presume that physicists or other scientists have a complete understanding of all the connections and relationships among objects in the physical world, why should we assume that traders or economists do of markets?[1]

Our model of trade is basic. In each period, infinite-horizon, risk-neutral traders choose between holding a long-term asset with known current dividend and possibly uncertain future dividend and holding a one-period bond that yields the known current interest rate. Traders use potentially incomplete "theories" about which variables affect prices to form expectations about the price of the long-term asset in the subsequent period. Different traders may use different theories. All theories are "statistically correct" in that every trader's beliefs about the long-term asset's price in the subsequent period match its long-run frequency conditional upon the content of

[1] Indeed, Aragones, Gilboa, Postlewaite and Schmeidler (2005) show that determining whether a k-variable subset of a set of explanatory variables can achieve a given level of R^2 in a linear regression is an NP-complete problem, namely computationally difficult.

the trader's theory. A key behavioural assumption in our model is that traders are effectively oblivious to the incompleteness of theories, each trader operating as if her theory uses all the relevant data. Unlike models of asymmetric information, they do not attempt to invert market prices.

We begin by establishing the existence of a unique equilibrium pricing function, where short-sales constraints on the long-term asset and unlimited access to borrowing cause the market price of the long-term asset to equal the willingness to pay of the trader who is most optimistic about next period's price. We show that the richer the collection of theories in the market, the higher is the price of the long-term asset. We also demonstrate the converse: unless one collection of theories includes every "atom" of a second collection, then the first does not generate uniformly higher prices for all dividend and interest-rate processes.

Next, we explore how expanding the set of theories in the market affects the range of the price of the long term asset. We assume that dividends and interest rates depend upon affiliated economic variables and that states are persistent. With both the least and most complete theories present, adding new theories shrinks the range of market prices when persistence is sufficiently high or interest rates sufficiently stable; otherwise, the price range rises.

Our framework provides a natural taxonomy for ranking sophistication: one trader whose theory is less complete than second trader's theory is unambiguously less sophisticated. Although greater sophistication allows traders to better predict future market prices, it does not necessarily translate into higher market returns. Instead, we find that traders' sophistication *relative* to others in the marketplace determines their performance. One trader who is less sophisticated than another—but more sophisticated than most of the market—may lose out on the asset to a more sophisticated trader whenever its next-period price surpasses his expectation yet win out whenever its next-period price falls short of his expectation, a form of winner's curse driving his return below the interest rate. Meanwhile, a trader with a coarser theory may never buy the long-term asset, earning the interest rate. The real losers in asset markets may not be those with scant understanding of asset prices—they do not invest—but rather those utilising theories with real predictive power, albeit less than theories of other traders. In our model, not only might increased sophistication correlate negatively with portfolio return in the cross section, but it also can lower a trader's return holding fixed all other traders' theories.

Surprisingly, incomplete-theory traders may also enjoy a "loser's blessing" when trading against others who use theories neither coarser nor finer than their own: a trader may unwittingly buy the asset whenever its next-period price exceeds his expectation and not buy whenever its next-period price drops below his expectation. A trader who benefits from such "favourable selection" earns above the interest rate even in the face of competitive pressure. We provide two necessary conditions for a trader to earn above the interest rate. First, the market cannot host a trader more sophisticated than all others in the market. Second, because certain monotonicity conditions imply that all selection is adverse, the asset model must include some non-monotonicity.

The market price of the long-term asset exceeds the most optimistic trader's perceived value of holding the asset in perpetuity because it incorporates the option to sell in the future to a more optimistic trader. Equilibrium prices increase in dividends and decrease in the interest rate and are sub-linear in dividends: bundling two dividend processes into a single asset destroys value by producing an asset whose market price is below the sum of the market prices of its constituent elements. Intuitively, integrating assets reduces traders' heterogeneity in beliefs about individual assets.

Section 2 reviews the related literature. Section 3 presents a simple example in which the interest rate cycles deterministically and traders differ in their comprehension of this process. Section 4 introduces the primitives of the model. Section 5 defines equilibrium and proves existence. Section 6 presents comparative statics of the price function on the collection of theories in the market. Section 7 explores traders' equilibrium rates of return. Section 8 examines how prices compare to traders' expectations of the asset's present-discounted value. Section 9 describes how the long-term-asset price depends upon dividends and interest rates. Section 10 concludes.

2 Related Literature

Our model shares similarities with several recent models of boundedly-rational information processing, all of which maintain that agents hold statistically correct beliefs about the distribution of others' actions but depart from standard equilibrium conditions that agents understand the relationship between those actions and other variables of interest. In their "absent-minded driver's paradox", Piccione and Rubinstein (1997) model someone who cannot figure out which node in a information set she is at because she cannot remember whether she previously made a decision (to exit a freeway); her beliefs coincide the relative frequencies of reaching the different exits. Piccione and Rubinstein (2003) model consumers who, observing the entire history of a deterministic price process, understand it only partially and perceive it as non-deterministic: a consumer's beliefs about next period's price, given some most recent price realisations, equal the long-run frequencies conditional upon their perceived correlates. Eyster and Rabin (2005) introduce the concept of cursed equilibrium for Bayesian games, where players only partially appreciate the connection between other players' private information (types) and actions. They illustrate how such "cursedness" can produce information-based trade in no-trade settings.

The formal framework closest to ours is Jehiel's (2005) elegant analogy-based expectations equilibrium (ABEE), originally set in complete-information games and extended to incomplete-information games by Jehiel and Koessler (2008).[2] In an ABEE, players do not fully appreciate the history-contingent nature of their opponents' play; instead, they partition histories into analogy classes and best respond to beliefs that opponents' strategies are constant across each class. The idea that players have only a coarse understanding of how actions depend upon histories resembles our notion that players imperfectly understand how future prices depend upon relevant variables. Dekel, Fudenberg and Levine's (2004) self-confirming equilibrium in Bayesian games shares the feature that players have correct beliefs about the distribution of others' actions, yet allows for any (consistent) beliefs about the mapping

[2]Spiegler's (2011) insightful overview of this literature connects common features of the independent approaches in Piccione and Rubinstein (2003), Eyster and Rabin (2005) and Jehiel (2005).

from opponents' types to actions including misattributions that cannot occur in any cursed equilibrium or ABEE.

Our model uses a market-equilibrium approach instead of a game-theoretic one. In so doing, it overlaps with an area of the finance literature which studies traders with heterogeneous beliefs who cannot sell short. In Harrison and Kreps (1978), traders with non-common priors about an asset's dividend process can generate prices in excess of their most optimistic assessment of its long-run worth. In their model, traders have a complete understanding of price determination in that they have perfect knowledge of which variable affect prices. Because they have incorrect beliefs about the stochastic process governing dividends, they also have incorrect beliefs about prices. In contrast, traders in our model have incomplete knowledge of which variables affect prices but, conditional upon their understanding, correct beliefs about next-period prices. Scheinkman and Xiong (2003) focus on equities markets and assume heterogeneous beliefs as in Harrison and Kreps (1978). Heterogeneity arises from traders' overestimating the precision of Brownian signals. By putting more structure on traders' beliefs, these authors characterise how heterogeneity leads to high trading volume and price volatility. Xiong and Yan (2010) follow a similar approach for bond markets and show that it can explain excess volatility of bond prices. Morris (1996) models traders with non-common priors about an asset's dividend who learn the process over time and connects overpricing to IPO overvaluation. Whereas all these models feature biased beliefs about fundamentals and hence prices, ours requires that traders' beliefs about next-period prices be unbiased. Heterogeneity in our model derives not from biases in the traders' beliefs but instead from a diverse perception of which variables affect prices: traders may neglect conditioning variables and hence overlook correlations in the data despite having correct marginal beliefs.[3]

In our model, the degree of completeness of traders' theories provides a natural taxonomy of sophistication. In a model of non-common priors, Blume and Easley (2006) adopt a notion of closeness to the truth based on relative entropy. They show

[3]Models in which belief disagreement stems from informational asymmetries rather than non-common priors, such as Hong and Stein (1999), and that share the feature that traders do not infer unknown information from market prices are lucidly overviewed by Hong and Stein (2007).

that when one trader's priors are closer to the truth than another's, the second will be driven from the market. In contrast, we show that the relationship between sophistication and returns is not necessarily monotonic and that unsophisticated traders can earn market rates of return. A crucial difference between their work and ours is that their assumption of complete markets essentially allows traders to bet on differences in beliefs about any event; in our model, traders only "bet" about next-period prices.

A nascent literature explores the effects of coarse thinking on asset prices. Bianchi and Jehiel (2010) show how ABEE can support bubbles, where traders know that an asset is overpriced but differ in their perceptions of when the bubble will burst. Fuster, Hebert and Laibson (2011, forthcoming) model a macroeconomy where a risky asset pays a dividend whose innovation follows an AR(40) process. All traders use a simpler "natural-expectations" model in which the innovation is an AR(p) process, for $p \leq 40$, causing them to overreact to dividend shocks and misperceive the riskiness of equities. In addition to making specific assumptions on the information structure, their model differs from ours by not incorporating heterogeneous theories. Steiner and Stewart (2012) use a model of coarse perceptions very similar to ours to explore high frequency trading. However, they study a price-setting equation very different than ours that averages all traders' price expectations. They show that in the limit, as trading frequency increases, prices converge in any pair of states bundled by some trader. All these models, including ours, take people's coarse thinking as exogenous. A literature on rational inattention (see Sims (2003), and Gul, Pesendorfer and Strzalecki (2011)) models agents who optimise over simple consumption plans.

Kurz (1994a,b) proposes a theory of expectations of traders who do not know the structural relations of a market. His approach complements ours by focusing on a different dimension of heterogeneity in beliefs. In the simplest variant of his model, the true data-generating process is stationary, but traders may hold beliefs that are non-stationary, as long as those beliefs generate the same asymptotic frequencies as the true data-generating process. In our model, we impose that beliefs are stationary and endogenous uncertainty arises from the agents' incomplete perception of the variables that determine asset prices. In contrast to our unique equilibrium, Kurz's model admits an infinity of equilibria, some of which involve excess volatility.

3 Introductory Examples

An asset that yields a dividend of 1 in every period is traded by risk-neutral traders. The interest rate takes on one of three possible values, $r_h > r_m > r_l > 0$, and cycles deterministically as follows: $r_h \longrightarrow r_m \longrightarrow r_l \longrightarrow r_m \longrightarrow r_h \ldots$ Equivalently, it follows a Markov process with transitions

$$(r_h, r_m) \longrightarrow (r_m, r_h) \longrightarrow (r_l, r_m) \longrightarrow (r_m, r_l) \longrightarrow (r_h, r_m) \ldots$$

where the first component in $(\cdot, \cdot)$ is the interest rate in the current period and the second component is the interest rate in the previous period. Each trader chooses between holding the asset and a short-term bond that lasts for one period and yields the current interest rate. The price of the short-term bond equals one. The asset's dividend and the current interest rate are known to all traders.

The following elementary examples illustrate our approach to modelling incomplete understanding of pricing, where p_{ij} denotes the price at state (r_i, r_j).

Example 1 *[Complete understanding]: Suppose that all traders understand the evolution of asset prices. In equilibrium the asset takes on four possible prices as follows:*

$$\begin{aligned} 1 + p_{mh} &= (1 + r_h)p_{hm} \\ 1 + p_{lm} &= (1 + r_m)p_{mh} \\ 1 + p_{ml} &= (1 + r_l)p_{lm} \\ 1 + p_{hm} &= (1 + r_m)p_{ml} \end{aligned}$$

When $r_h = 0.09$*,* $r_m = 0.06$ *and* $r_l = 0.03$*, the equilibrium prices are*

$$p_{mh} = 16.96, \quad p_{ml} = 16.49, \quad p_{hm} = 16.48, \quad p_{lm} = 16.98.$$

Example 2 *[Partial understanding]: Suppose that all traders only partially understand the relationship between the price of the asset and the interest rate. In particular, they understand the how the price at time* $t + 1$ *depends upon the interest rate at time* t *but fail to perceive its dependence on the interest rate at time* $t - 1$*. We* *assume that all traders' beliefs about next period's price are "statistically correct" in that they are determined by the long-run average conditional upon the current interest rate. The traders' beliefs about the behaviour of the next period price conditional upon the current interest rate may be summarised as follows:*

$$\begin{aligned}
&r_h \longrightarrow p_{mh} = p_{ml} \\
&r_m \longrightarrow p_{lm} \text{ with prob } \tfrac{1}{2},\ r_m \longrightarrow p_{hm} \text{ with prob } \tfrac{1}{2} \\
&r_l \longrightarrow p_{ml} = p_{mh}
\end{aligned}$$

Naturally, in equilibrium, $p_{mh} = p_{ml}$. The equilibrium prices are as follows:

$$\begin{aligned}
p_{mh} &= p_{ml} \\
1 + p_{mh} &= (1 + r_h)p_{hm} \\
1 + \frac{1}{2}p_{lm} + \frac{1}{2}p_{hm} &= (1 + r_m)p_{ml} \\
1 + p_{ml} &= (1 + r_l)p_{lm}
\end{aligned}$$

When $r_h = 0.09$, $r_m = 0.06$, $r_l = 0.03$, the equilibrium prices are

$$p_{hm} = 16.31, \quad p_{lm} = 17.26, \quad p_{ml} = p_{mh} = 16.78.$$

Equivalently, we can interpret traders as failing to understand the dynamic behaviour of the interest rate. Thus, they know that r_l and r_h precede r_m, yet fail to predict when r_m precedes r_l versus r_h. Under this interpretation, trader's incomplete understanding of the behaviour of the interest rate may be summarised as follows:

$$\begin{aligned}
&r_h \longrightarrow r_m \\
&r_m \longrightarrow r_h \text{ with prob } \tfrac{1}{2},\ r_m \longrightarrow r_l \text{ with prob } \tfrac{1}{2} \\
&r_l \longrightarrow r_m
\end{aligned}$$

Example 3 *[Heterogeneous understanding]: The market contains two types of traders. Some fully perceive the relationship between interest rate and price as in Example 1. Others share the same incomplete understanding of that relationship as presented in Example 2. Assume that the equilibrium price equals the reservation price of the most* *optimistic trader.*[4] *Intuitively, the traders with a partial understanding of prices purchase the asset when the interest rate is* r_m *and moving towards* r_h *since they believe that the next period price is equally likely to be either* p_l *or* p_h*. For symmetric reasons, the traders with complete understanding purchase the asset when the interest rate is* r_m *and moving towards* r_l *since they believe that the next period price is* p_l*. At all other interest rates, both types of traders predict the next-period price correctly. The equilibrium prices are as follows:*

$$\begin{aligned} 1+p_{mh} &= (1+r_h)p_{hm} \\ 1+p_{lm} &= (1+r_m)p_{mh} \\ 1+p_{ml} &= (1+r_l)p_{lm} \\ 1+\frac{1}{2}p_{lm}+\frac{1}{2}p_{hm} &= (1+r_m)p_{ml} \end{aligned}$$

When $r_h = 0.09$*,* $r_m = 0.06$*,* $r_l = 0.03$*, the equilibrium prices are*

$$p_{mh} = 18.45, \quad p_{ml} = 18.11, \quad p_{hm} = 17.84, \quad p_{lm} = 18.56$$

Prices exceed those in Examples 1 and 2. As in Harrison and Kreps (1978), heterogeneity of beliefs raises all prices. Note that traders with partial understanding fail to see the deterministic relationship between current and next-period prices.

Coarse-theory traders earn below the interest rate by not appreciating that they buy at r_m *only before a capital loss (incorrectly predicting a* 50% *chance of a gain).*[5]

4 The Model

In this section, we present the formal model.

[4] Our formal model includes a set of assumptions sufficient for this pricing rule.

[5] The seeming paradox that no trader outperforms and some traders under-perform the risk-free rate is explained by asset prices being "too high" relative to fundamental values; one interpretation is that the asset's issuer earned an excess return in some un-modelled initial period. Morris (1996) interprets prices above fundamental values in a Harrison-Kreps-style model as IPO overvaluation.

4.1 States, Dividends, and Interest Rates

In each of a countably infinite number of periods, traders on a financial market trade an infinitely-lived long-term asset and a one-period bond. The bond returns the principal plus the interest rate the following period. The long-term asset pays a dividend in every period. We adopt the following convention about the timing of dividends: the holder of the asset in period t receives the dividend in period $t+1$ (or, equivalently, after financial markets close in period t, precluding it from earning interest before period $t+1$).

Both dividend and interest rate are determined by the "state" of the financial world, which evolves over time. Let S be a finite state space and $\mathcal{S}$ the set of all its subsets.[6] The state of the world evolves according to a Markov process described by the transition function $Q : S \times \mathcal{S} \to [0,1]$, where $Q(s,A)$ is the probability that the next state is in $A \subseteq S$ when the current state is s. For simplicity, we assume that $Q(s,\{y\}) > 0$ for any $s,y \in S$. Given a function $f : S \to \mathbb{R}$, define the operator

$$Tf(s) = \sum_{y \in S} f(y) Q(s,\{y\}),\ s \in S,$$

which is the expectation of f in the next period conditional upon current state s. The mapping $T^k f(s)$ defines the k^{th} iteration of the above operator. Given a probability mass function (pmf) λ on S, define the operator $T^*\lambda$ such that

$$T^*\lambda(y) = \sum_{s \in S} Q(s,\{y\}) \lambda(s),$$

which is the next-period pmf on the state space given that the probability that the current state is s is $\lambda(s)$. By standard results (see Theorems 11.1 and 11.2 in Stokey and Lucas (1989)), there exists a unique invariant pmf μ, where $\mu(s) = T^*\mu(s) > 0$ for any $s \in S$.

The long-term asset yields a dividend $d : S \to [0,\bar{d}]$. The one-period bond pays the interest rate $r : S \to [r_0, r_1]$, $r_0 > 0$. All traders know the realizations of d and r.

[6] Our result extend to the case in which S is a Polish space. See Eyster and Piccione (2011).

4.2 Theories

Traders in our model have only a limited understanding of how the next-period price of the long-term asset depends upon the state. For example, when $S = S_1 \times S_2$, a trader may fail to perceive that future prices depend on S_2 despite recognising their relationship to S_1. We model an incomplete *theory* through a partition $\mathcal{F}$ of S. For any element $F \in \mathcal{F}$, i.e., an *atom* of theory $\mathcal{F}$, and any $s, s' \in F$, a trader with theory $\mathcal{F}$, whom we dub an $\mathcal{F}$-trader, forms the same beliefs about next period's price when the current state is s as when it is s'. Under our interpretation, the trader may be able distinguish s from s' but has not discovered that next period's price may differ across the two states. With some abuse of terminology, we use $\mathcal{F}$ to denote both a partition and the algebra generated by it.

Unlike of models of imperfect information that use partitions to represent limits on the fineness of traders' observation of the state, in our model partitions capture the limits of the traders' understanding of the structure of price determination. In particular, traders may observe variables that they fail to incorporate into their theories.[7] Thus, a trader's theory may or may not include the partitions generated by $d(\cdot)$ and $r(\cdot)$, as traders who observe the dividend and the interest rate need not fully understand their effect on price determination. For example, a home buyer who does not understand how interest rates evolve over the business cycle may fail to perceive the relationship between the lagged interest rate and next period's house price (Example 2) despite recalling last period's interest rate.

Given the invariant pmf μ and a partition $\mathcal{F}$ of S, let $F(s)$ be the atom of $\mathcal{F}$ that contains $s \in S$. Given a function $g : S \to \mathbb{R}$, define the conditional expectation of g given $\mathcal{F}$ as the function $E_{\mathcal{F}}(g) : S \to \mathbb{R}$ such that

$$E_{\mathcal{F}}(g)(s) = \frac{\sum_{s' \in F(s)} g(s') \mu(s')}{\sum_{s' \in F(s)} \mu(s')}.$$

We refer to a collection Ψ of partitions of S as a *collection of theories.* We sometimes refer to the partition in which every element is a singleton, the finest possible theory, as the *complete theory.* The state space S might include lagged

[7] Indeed, in our model one can assume that every state is observable.

variables inessential to the Markov process that nevertheless, given the traders' partial understanding, can be of use in forming expectations.

4.3 Price Expectations

One of our main behavioural assumptions regards the formation of price expectations. Consider a pricing function $p : S \to \mathbb{R}_+$. We assume that when the current state is $s \in S$, an $\mathcal{F}$-trader forms an expectation of p in the next period equal to $E_{\mathcal{F}}(Tp)(s)$. This corresponds to the long-run empirical average of p given $\mathcal{F}$. To see why, let $p(S)$ be the range of p and define the probability measure $\Pi_{\mathcal{F}}$ on $S \times p(S)$ such that for any $B \subseteq p(S)$ and an element F of $\mathcal{F}$,

$$\Pi_{\mathcal{F}}(F \times B) = \sum_{s \in F} \sum_{y \in p^{-1}(B)} Q(s, \{y\}) \mu(s).$$

The probability measure $\Pi_{\mathcal{F}}$ describes the beliefs of an $\mathcal{F}$-trader about the next-period values of p. By standard results (see for example Theorem 14.7 in Stokey and Lucas (1989)) the frequency of $F \times B$ converges almost surely (with respect to the infinite-horizon process defined for any given initial state) to $\Pi_{\mathcal{F}}(F \times B)$. Hence, the trader's expectation corresponds to the empirical conditional average of the next-period price. In this sense, we can think of the trader's model as being the limit point of a statistical learning process.

Whereas our approach requires that traders have statistically correct expectations about the next-period price, it does not specify their perceived model of dynamic price formation. One way to complete the model of an $\mathcal{F}$-trader is as follows. Define the "perceived" transition function $\tilde{Q} : S \times \mathcal{F} \to [0, 1]$ to be

$$\tilde{Q}(s, F) = E_{\mathcal{F}}(Q(\cdot, F))(s),$$

for each atom F of $\mathcal{F}$. The transition function $\tilde{Q}$ specifies probabilities of the atoms of $\mathcal{F}$ in the next period conditional upon any atom of $\mathcal{F}$ in the current period by averaging over Q. The model of an $\mathcal{F}$-trader then consists of S, the Markov transition function $\tilde{Q}$, and the probability measure $\Pi_{\mathcal{F}}$ that describe the joint probability of states in any period and prices in the following period. The measure derived restricting μ to $\mathcal{F}$ is the invariant measure of $\tilde{Q}$, which distinguishes our model from those of non-common priors like Harrison and Kreps (1978), where the only trader whose model of dividends matches long-run frequencies is one having correct beliefs.

Iterating $\tilde{Q}$ and then applying $\Pi_{\mathcal{F}}$ yields beliefs about prices in all future periods. Note that expectations beyond the next period are not necessarily correct. For instance, in Example 2, a trader at period t facing interest rate r_h would assign probability one-half to p^{t+2} being p_{hm}, despite its true probability being zero. Nevertheless, such errors in perceived correlations do not affect market prices. From S, one can construct the space S^L, which includes redundant lagged variables spanning $L > 1$ periods, and derive the transition function for S^L from Q. By defining theories over S^L, in the "perceived" models constructed in the same way as above the joint probabilities of events spanning at most $L+1$ periods correspond to the empirical frequencies almost surely. The argument is analogous to the one above and is omitted. The number L can be interpreted as a bound on the ability of a trader to test her "perceived" model.

5 The Market

The financial market consists of risk-neutral traders who in every period trade the short-term bond and long-term asset. For simplicity, we assume that in each period the traders observe the current dividend and interest rate before trading. The bond is in infinite supply, and the long-term asset only in finite supply. Traders can borrow unlimited amounts at the interest rate for one period but there is no short-selling.[8] Because traders are risk-neutral, the only moment of next period's stochastic price

[8] Alternatively, we could allow unlimited short sales of the short-term bond and bounded short sales of the long-term asset. Short-sale constraints are widely used in the finance literature and have been justified by numerous authors. In some markets like housing, short sales simply do not exist. Other financial markets preclude "naked short-shelling", which effectively bounds short sales by the number of shares in the market. Shleifer and Vishny (1997) and Stein (2009) describe a multitude of obstacles to short-selling across different financial markets.

that concerns them is its expectation. Traders do not attempt to invert market prices since each treats her own theory as the best on offer.

5.1 Equilibrium

The price of the short-term asset is normalised to one. A stationary price function $p : S \to \mathbb{R}_+$ for the long-term asset maps realisations of the state to non-negative prices. In period t at state s, an $\mathcal{F}$-trader forms expectations about the price in period $t+1$ equal to $E_{\mathcal{F}}(Tp)(s)$, which are consistent with the long-run frequencies of prices conditional upon the relevant atom of $\mathcal{F}$. The price of the long-term asset is *cum dividend*, which purely for notational convenience is paid in period $t+1$.

Definition 1 *A stationary price function p_Ψ is a Ψ-equilibrium price function if*

$$\frac{d(s)}{p_\Psi(s)} + \frac{\max_{\mathcal{F}\in\Psi} E_{\mathcal{F}}(Tp_\Psi)(s)}{p_\Psi(s)} = 1 + r(s) \tag{1}$$

for any $s \in S$.

The equilibrium condition is analogous to the one in Harrison and Kreps (1978). When Ψ contains only the complete theory, then p_Ψ is a rational-expectations price function. Likewise, if the realisation of the state in each period is i.i.d., then p_Ψ is the same for any Ψ.

For a rationale for the equilibrium condition, consider first one-period buying/selling strategies. At prices below $p_\Psi(s)$, unboundedly large demands from $\mathcal{F}$-traders with the highest expectations of the price function exceed the finite supply of the long-term asset. At prices above $p_\Psi(s)$, no trader wishes to hold the long-term asset. Now consider a trader who at time t wishes to buy the long-term asset and hold it for $\tau > 1$ periods before selling. When he has the highest price expectations, he is indifferent between holding the long-term asset for τ periods and holding the short-term bond in period t before buying the long-term asset in period $t+1$ and holding it for $\tau - 1$ periods: both strategies yield the same expected wealth in period $t+1$. When he does not have the highest price expectations, he strictly prefers to hold the short-term bond in period t before buying the long-term asset in period $t+1$ and holding it for

$\tau - 1$ periods. Hence, regardless of the trader's beliefs about dividends, interest rates and prices in period $t+1$ and beyond, his behaviour in period t conforms to market clearing at $p_\Psi(s)$, and the pricing equation depends only upon one-period trade-offs. In the Appendix, we extend the "perceived" model of price formation in Section 4.3 to include dividends and interest rates and demonstrate that indeed no trader can expect the future discounted value of holding the long-term asset for an arbitrary number of periods to exceed the current price, as in Harrison and Kreps (1978).

Traders do not appreciate which variables their theories exclude. In particular, they do not try to extract information from market prices. Whether aware or unaware of the incompleteness of their theories, they do not attempt to better their understanding by incorporating additional variables. We view their theories as outcomes of a learning process that stalls having achieved empirical consistency.[9] Indeed, our model is not equivalent to one in which traders understand the structure of the environment but only partially observe relevant variables. Since partial observations on Markov processes do not necessarily have a Markov structure, this alternative model does not necessarily yield a stationary equilibrium.

The equilibrium equation (1) is a classic zero-profit condition. In a significant portion of the finance literature, this equation is construed as a partial-equilibrium requirement in the presence of a short-sale constraint on the long-term asset and infinitely elastic borrowing at the current interest rate. Naturally, one can construct general-equilibrium models with limited borrowing in which this condition fails. In particular, if the traders are budget-constrained consumers whose endowments include shares of the long-term asset, its equilibrium price is not necessarily determined by the most optimistic expectation and would depend on the distribution of shares and theories. Nevertheless, if endowments from sources other than the long-term asset are large, our equilibrium condition would hold.

Besides simplifying the analysis, short sale constraints and infinitely elastic borrowing imply that in a model with more than one long-term asset, the trader's problem

[9] Needless to say, historical examples of stalled learning abound. Note that, within the confines of our model, the weaker interpretation that the collection of theories stays still, even if some individual traders better their theories, suffices.

is separable across assets; the equilibrium price of one long-term asset does not depend upon any other long-term assets present in the market. Thus, it is without loss of generality that we include only one long-term asset.

The fact that traders's theories do not include prices does not imply that prices lack an informational role. Rather, traders use the known price, dividend and interest rate for instrumental purposes to compare the short- and long-term assets. Analogously, Walrasian-equilibrium prices convey information about mutual gains from trade that is fully exploited by traders. Information is aggregated unintentionally via the unmodelled equilibrating process.

5.2 Existence

The next result shows that an equilibrium price function exists and is unique for any collection of theories.

Proposition 1 *For any collection Ψ there exists a unique Ψ-equilibrium price function.*

Proof. Define the mapping

$$\mathcal{T}(p)(s) = \frac{1}{1+r(s)}\left(d(s) + \max_{\mathcal{F}\in\Psi} E_{\mathcal{F}}(Tp)(s)\right)$$

from the set of functions over S to itself. Note that $\mathcal{T}$ is monotone and that, given a constant $c \geq 0$

$$\mathcal{T}(z+c) \leq \mathcal{T}(z) + \frac{c}{1+r_0}.$$

Since the set of functions over S is a closed subspace of $\mathbb{R}^S$, Blackwell's sufficient conditions (see Theorem 3.3 in Stokey and Lucas (1989)) are satisfied. ■

Remark 1 *In Example 2, the state in period t can be expressed as $s^t = (r^t, r^{t-1})$. Both the equilibrium price and interest rate depend only on the state's first component and generate the same partition over the state space. Although we disallow theories from including prices, the incomplete-theory trader can be interpreted as using a theory in which p^t is the sole predictor of p^{t+1}. Similarly, the incomplete-theory trader in Example 3 can be interpreted as using a coarse theory of p^t as predictor* *that partitions current prices into the half-unit intervals* $[17.5, 18.0]$, $[18.0, 18.5]$ *and* $[18.5, 19.0]$, *giving the same price expectations after* p_{ml} *as* p_{mh}. *Likewise, traders could incorporate past returns in their theories. Although many examples can be interpreted as traders including limited information about prices in their theories, the general model precludes prices from entering theories for two reasons. First, when the pricing function is 1-1, a complete understanding of how current price affects future price leads to rational-expectations equilibrium, whereas we wish to model traders with coarse or incomplete understanding of the pricing process. Second, equilibrium may fail to exist. For example, let* $S = \{1, 2, 3\}$, *and begin with collection* Ψ_1 *that contains only the complete theory. Let* $Q(1, \{1\}) = 0.94$, $Q(1, \{2\}) = Q(1, \{3\}) = 0.03$ *and* $Q(2, \cdot) = Q(3, \cdot)$, *where* $Q(2, \{2\}) = Q(2, \{3\}) = 0.47$. *When* $r = 0.10$ *and* $d(1) = 1$, $d(2) = 0$, $d(3) = \frac{5}{3}$, $p_{\Psi_1}(1) = p_{\Psi_1}(3) = 9.5455$ *and* $p_{\Psi_1}(1) = 8.0303$. *Now consider adding theory* $\mathcal{F} = \{\{1, 3\}, \{2\}\}$ *to form the collection* Ψ_2. *Then, prices are* $p_{\Psi_2}(1) = 9.8052$, $p_{\Psi_2}(2) = 8.6364$, $p_{\Psi_2}(3) = 10.3250$. *Suppose now that traders refine their theories using prices. If prices at states* 1 *and* 3 *differ, then all traders acquire complete theories, and prices must coincide. If prices at states* 1 *and* 3 *are equal, then* $\mathcal{F}$*-traders have the same expectations at these states, and prices must differ.*

6 Properties of the Equilibrium Price Function

6.1 Monotonicity

The following result shows that enlarging at every state the set of "perceptions" in the market leads to uniformly higher prices. The equilibrium price distribution of a collection of theories with a larger set of atoms first-order stochastically dominates that of a collection of theories with a smaller set.

Proposition 2 *Let* Ψ *and* Ψ' *be two collections of theories. Then,* $\bigcup_{\mathcal{F} \in \Psi} \mathcal{F} \subset \bigcup_{\mathcal{F}' \in \Psi'} \mathcal{F}'$ *if and only if, for any transition function* Q, *dividend function* d, *and interest function* r, $p_{\Psi}(s) \leq p_{\Psi'}(s)$ *for any* $s \in S$.

Proof. Only if: For any function $p : S \to \mathbb{R}_+$, the contraction mapping in the proof of Proposition 1 has the property that $\mathcal{T}_{\Psi}(p)(s) \leq \mathcal{T}_{\Psi'}(p)(s)$ for any $s \in S$. Since the contraction mapping is monotonic, $\mathcal{T}_{\Psi'}(p)(s) \geq p_{\Psi}(s)$ for any p such that $p(s) \geq p_{\Psi}(s)$ for any $s \in S$. Thus, the unique fixed point $p_{\Psi'}(s)$ of $\mathcal{T}_{\Psi'}$ is such that $p_{\Psi'}(s) \geq p_{\Psi}(s)$ for any $s \in S$.

If: Let s_0 be a state for which an atom $F = \{s_0, s_1, ..., s_{k-1}\}$ of some theory $\mathcal{F} \in \Psi$ is not in $\bigcup_{\mathcal{F}' \in \Psi'} \mathcal{F}'$. We will show that, for some Q, d, and r, $p_{\Psi}(s_0) > p_{\Psi'}(s_0)$ By the first part of this proposition, it is sufficient to assume that $\Psi = \{\mathcal{F}\}$. Let $\#S$ be the number of states in S and suppose that for each $s, y \in S$,

$$Q(s, \{y\}) = \begin{cases} \dfrac{(1-\gamma)}{\#S - 1} & \text{if } y \neq s \\ \gamma & \text{if } y = s, \end{cases}$$

where $0 < \gamma < 1$, and

$$d(s) = \begin{cases} 1 & \text{if } s \in F \backslash \{s_0\} \\ 0 & \text{if } s \notin F \backslash \{s_0\}. \end{cases}$$

First note that $p_{\Psi}(s), p_{\Psi'}(s) \leq \dfrac{1}{r}$ for any $s \in S$. This implies that

$$p_{\Psi}(s), p_{\Psi'}(s) \leq \frac{1}{r(1+r)} \text{ for any } s \notin F \backslash \{s_0\}.$$

Obviously, the invariant pmf μ is such that $\mu(s) = \frac{1}{\#S}$ for any $s \in S$. Then, since $\Psi = \{\mathcal{F}\}$, $p_{\Psi}(s_i) = p_{\Psi}(s_1)$, $i = 1, ..., k-1$, and

$$p_{\Psi}(s_0) \geq \frac{\frac{1}{k}\gamma p_{\Psi}(s_0) + \frac{k-1}{k}\gamma p_{\Psi}(s_1)}{1+r}$$
$$p_{\Psi}(s_1) \geq \frac{1 + \frac{1}{k}\gamma p_{\Psi}(s_0) + \frac{k-1}{k}\gamma p_{\Psi}(s_1)}{1+r},$$

which implies that

$$p_{\Psi}(s_0) \geq \frac{(k-1)\gamma}{k(1+r)(1+r-\gamma)}$$

Now take any $F' \in \mathcal{F}' \in \Psi'$ which contains s_0. By construction, since $F \notin \bigcup_{\mathcal{F}' \in \Psi'} \mathcal{F}'$,

$$\frac{\sum_{y \in F' \cap F} \mu(y)}{\sum_{y \in F'} \mu(y)} < \frac{k-1}{k}$$

where the left-hand side is the probability of the state being in $F\backslash\{s_0\}$ conditional upon F' under the invariant pmf. Thus, if γ is sufficiently close to one, there exists $\beta > \frac{1}{k}$ such that

$$p_{\Psi'}(s_0) < \beta \frac{1}{r(1+r)^2} + (1-\beta)\frac{1}{r(1+r)}$$

After simplification, as $\gamma \to 1$, the difference between the lower bound on $p_{\Psi}(s_0)$ and the upper bound on $p_{\Psi'}(s_0)$ converges to

$$\frac{1}{kr(r+1)^2}(kr\beta - r - 1)$$

which is positive for sufficiently high r since $k\beta > 1$. ■

This result implies that enlarging the set of theories in the market leads to uniformly higher prices: the more theories present in the market, the more optimistic is the most optimistic trader about next period's price. It is stronger by virtue of holding at the level of atoms rather than theories. For example, if all the atoms of the complete theory are included in a collection of theories, the equilibrium price exceeds the rational-expectations equilibrium price even if no trader has the complete theory.

The introduction of a new theory $\mathcal{A}$ into a market with preexisting theories Ψ affects prices only if in some state s, $\max_{\mathcal{F}\in\Psi} E_{\mathcal{F}}(Tp_{\Psi})(s) < E_{\mathcal{A}}(Tp_{\Psi})(s)$. In fact, either all prices remain unchanged or all prices increase.

Proposition 3 *Let Ψ and Ψ' be two collections of theories where $\bigcup_{\mathcal{F}\in\Psi}\mathcal{F} \subset \bigcup_{\mathcal{F}'\in\Psi'}\mathcal{F}'$. Then, either $p_{\Psi}(s) = p_{\Psi'}(s)$ for every $s \in S$, or $p_{\Psi}(s) < p_{\Psi'}(s)$ for every $s \in S$.*

Proof. Suppose that there exists some $s' \in S$ for which $p_{\Psi}(s') < p_{\Psi'}(s')$. Since $Q(s, \{s'\}) > 0$ for any $s \in S$, we have

$$Tp_{\Psi'}(s) > Tp_{\Psi}(s) \text{ for any } s \in S.$$

Hence, as $\mu(s) > 0$ for any $s \in S$,

$$E_{\mathcal{F}'}(Tp_{\Psi'})(s) > E_{\mathcal{F}'}(Tp_{\Psi})(s)$$

for any $\mathcal{F}' \in \Psi'$ and any $s \in S$. Thus, $p_{\Psi}(s) < p_{\Psi'}(s)$ for any $s \in S$. ■

The intuition for the above result is simple. When the price rises in state s, it also rises in any s' because s' transits to s with positive probability; price rise in one state propagates to all other states.

6.2 The Affiliated N-Variable Model

The remainder of the paper sometimes makes use of a special case where the state space is described by N "economic" variables. In the *N-variable model*, $S = \times_{n=1}^{N} X_n$, where each $X_n \subset \mathbb{R}$. Traders' theories correspond to sections of S; that is, each trader perceives some variables in their entirety but neglects all others altogether. In this case, we index the set of theories by the subsets of $\{1, 2, ..., N\}$.

An N-variable model is *affiliated* when for each $s, s', y, y' \in S$,

$$Q\left(\left(s,\{y\}\right) \vee \left(s',\{y'\}\right)\right) Q\left(\left(s,\{y\}\right) \wedge \left(s',\{y'\}\right)\right) \geq Q\left(s,\{y\}\right) Q\left(s',\{y'\}\right). \qquad (2)$$

When (2) holds, given any affiliated pmf $m(\cdot)$ on S, the pmf f on $S \times S$ defined as

$$f(s,y) = Q(s,\{y\})\, m(s)$$

is also affiliated. It follows that the unique invariant pmf $\mu(s)$ is affiliated.[10] Furthermore, if $p(s)$ is non-decreasing in s, $Tp(s)$ and $E_{\mathcal{F}}(Tp)(s)$ are also non-decreasing in s for any theory $\mathcal{F} \subset \{1, 2, ..., N\}$.

In the N-variable model, the only way that the union of atoms of Ψ' can strictly contain those of Ψ is if Ψ' includes all the theories in Ψ. Hence, in the N-variable model, Proposition 2 implies that Ψ' gives rise to higher prices than Ψ for every d, r, and Q if and only if it strictly contains all the theories in Ψ.

6.3 Price Range

In this section, we investigate how increases in heterogeneity affect the range of prices of the long-term asset. The fact that the price range in Example 2 (with one incomplete theory) contains that of Example 3 (with that incomplete theory as well as the

[10] When $m(s)$ and $f(s,y)$ are affiliated, so too is the marginal $\sum_{s \in S} f(s,y)$. Hence, the mapping T^* maps affiliated pmf's to affiliated pmf's. Since the set of affiliated pmf's is closed and T^* is a contraction, T^* has a fixed point in the set of affiliated pmf's.

complete theory), which in turn contains that of Example 1 (with only the complete theory) demonstrates that the price range cannot simply expand or contract in tandem with the collection of theories. By restricting attention to settings that include the finest and coarsest theories, and to a given class of transition functions, we are able to do comparative statics on the collection of theories.

Definition 2 *The transition function is* state persistent *if there exists* $0 < \gamma < 1$ *such that for each* $s, s' \in S$,

$$Q(s, \{s'\}) = \begin{cases} (1-\gamma)\,\mu\,(s') & \text{if } s' \neq s \\ (1-\gamma)\,\mu\,(s') + \gamma & \text{if } s' = s. \end{cases}$$

In the case of an affiliated N-variable model, we can choose any affiliated invariant pmf μ to obtain state-persistence. We refer to this model as the affiliated, state-persistent N-variable model.

To see how heterogeneity affects prices in the presence of persistence, first consider a state s such that atom of the trader who buys the long-term asset contains only s. When persistence is high, the price at s is close to $\dfrac{d(s)}{r\,(s)}$ as this trader believes that next period the price is likely to be the same. Thus, if the same trader buys the asset at state s when heterogeneity increases, the price at state s exhibits little variation, even if the price at other states increases by Proposition 2. Now consider a state in which the long term asset is bought by a trader whose theory is very coarse. Then, an increase in the price of the long term asset in different states will affect his expectations and thus propagate to this state. The next proposition makes use of this intuition to show that, in monotone environments that include the complete and the coarsest theory, low prices respond more than high prices to increased heterogeneity. Let $s^{max} = \arg\max_{s\in S} d(s)$ and $s^{min} = \arg\min_{s\in S} d(s)$.

Proposition 4 *Consider an affiliated, state-persistent N-variable model and two collections of theories Γ and Θ. Suppose that d is non-decreasing and r is non-increasing in s. Suppose that $\varnothing, \{1, \ldots, N\} \in \Gamma$, $\Gamma \subset \Theta$, and $p_\Theta \neq p_\Gamma$. Then*

$$\max_{s\in S} p_\Theta(s) - \min_{s\in S} p_\Theta(s) \leq \max_{s\in S} p_\Gamma(s) - \min_{s\in S} p_\Gamma(s)$$

if and only if

$$(1-\gamma)(1+r(s^{min})) \leq 1+r(s^{max})-\gamma.$$

Proof. For any $\Psi \subset 2^{\{1,\ldots,N\}}$, $p_\Psi(s)$ is non-decreasing in s. This follows from noting that

$$Tp_\Psi(s) = \gamma p_\Psi(s) + (1-\gamma)\sum_{s\in S} p_\Psi(s)\,\mu(s)$$

Thus, since $\mu(\cdot)$ is affiliated, the contraction mapping in Proposition 1 maps non-decreasing functions to non-decreasing functions and the claim follows. At s^{max}, complete-theory traders have the highest expectation and thus

$$d(s^{max}) + \gamma p_\Psi(s^{max}) + (1-\gamma)E[p_\Psi] = (1+r(s^{max}))\,p_\Psi(s^{max}) \tag{3}$$

where $E[p_\Psi]$ denotes the unconditional expectation of $p_\Psi(\cdot)$ under $\mu(\cdot)$. Given a theory $\mathcal{F}$,

$$E_{\mathcal{F}}(Tp_\Psi)\left(s^{min}\right) = \gamma E_{\mathcal{F}}(p_\Psi)\left(s^{min}\right) + (1-\gamma)E[p_\Psi].$$

Thus, from the affiliation of μ we have

$$E_{\mathcal{F}}(Tp_\Psi)\left(s^{min}\right) \leq E[p_\Psi]$$

Hence, the lowest price is set by the empty theory; that is,

$$d(s^{min}) + E[p_\Psi] = \left(1+r\left(s^{min}\right)\right) p_\Psi\left(s^{min}\right) \tag{4}$$

Substituting $E[p_\Psi]$ from Equation (4) into Equation (3) gives

$$d(s^{max}) - (1-\gamma)d(s^{min}) + (1-\gamma)(1+r(s^{min}))p_\Psi(s^{min}) = p_\Psi(s^{max})(1+r(s^{max})-\gamma),$$

By Proposition 3, $p_\Gamma(s) < p_\Theta(s)$ for any $s \in S$. Thus, if

$$(1-\gamma)(1+r(s^{min})) \leq 1+r(s^{max})-\gamma,$$

the highest price cannot increase by more than the lowest price. The opposite holds if the inequality is reversed. ■

For an intuition, first note that under the monotonicity restrictions of Proposition 4, complete-theory traders have the highest expectations and set the price at the top of the market, as they attach the highest probability that next period's price coincides with this period's price; empty-theory traders expect next period's price to equal the unconditional expected price and set the price at the bottom of the market. Then, with high persistence, the increase in the unconditional expectation of the price that, by Proposition 2, accompanies an increase in heterogeneity has little effect on the top price. Conversely, with low persistence, all traders' expectations of the next-period price are close to the unconditional expectation. When the interest rate exhibits large variation, the top price is more affected than the bottom price by a change in the unconditional expectation as in the former it gets weighted by a lower interest rate. Indeed, when the interest rate is constant, the price range increases regardless of persistence.

7 Rates of Return

The relationship between rates of return and traders' sophistication is complex: a better theory does not guarantee a higher return.

Example 4 $S = \{a, b, c, d\}$, *with all states transiting back to themselves with probability* α *close to one, and otherwise to the uniform distribution. Trader* i *holds the theory* $\mathcal{F}_i$, *where*

$$\begin{aligned} \mathcal{F}_1 &= \{\{a\}, \{b\}, \{c\}, \{d\}\}, \mathcal{F}_2 = \{\{a, b\}, \{c\}, \{d\}\} \\ \mathcal{F}_3 &= \{\{a, c\}, \{b\}, \{d\}\}, \mathcal{F}_4 = \{\{a, b, d\}, \{c\}\} \\ \mathcal{F}_5 &= \{\{a, b, c, d\}\}, \end{aligned}$$

$d(a) = 1$ *and otherwise* $d(\cdot) = 0$*;* r *is constant. Trader 1 buys in state* a*; Trader 2 buys in state* b*; Trader 3 buys in state* c*; Trader 4 buys in state* d*; Trader 5 does not buy and earns the rate of return* r*, more than the more sophisticated Traders 2,3,4.*

Traders 2,3, and 4 suffer the winner's curse because when they purchase the long-term asset they mispredicts that next period the price might equal the high price of state a. Because Trader 5 is never the most optimistic one in the market, she never purchases the long-term asset and enjoys higher returns than the more sophisticated Traders 2,3 and 4. To earn below market returns in this example, a trader must neither be too well nor too poorly informed.[11]

Now refine Trader 5's theory to $\hat{\mathcal{F}}_5 = \{\{a, d\}, \{b\}, \{c\}\}$, making her more sophisticated than Trader 4. This time, she buys in state d and suffers the winner's curse. Comparing across Trader 5's two theories, not only can more sophisticated investors earn lower equilibrium rates of return than less sophisticated ones, but increased sophistication can worsen a trader's return *holding all other traders' theories fixed*.[12]

Although competition drives the price of the long-term asset up to the maximum willingness to pay, some traders can earn a return above the interest rate.

Example 5 *Suppose that $N = 2$ and $X_i = \{0, 1\}$, $i = 1, 2$. The transitions are such that each state transits to itself with probability α and to all other states uniformly. Obviously, the invariant distribution is uniform. The interest rate is constant and equal to $r = 0.05$. The dividend function is*

$$d(x_1, x_2) = 0.9 \cdot x_1 + 0.95 \cdot x_2 - x_1 x_2$$

The market has three theories, namely $\{1\}$, $\{2\}$, and $\varnothing$. The equilibrium, as $\alpha \to 1$ is characterised by the following equations, where $p_{x_1 x_2}$ is the price at state (x_1, x_2):

$$\begin{aligned} 0.9 - 0.05 + \tfrac{1}{2}(p_{01} + p_{11}) &= (1 + r)\, p_{11} \\ 0.95 + \tfrac{1}{2}(p_{01} + p_{11}) &= (1 + r)\, p_{01} \\ 0.9 + \tfrac{1}{2}(p_{10} + p_{11}) &= (1 + r)\, p_{10} \\ \tfrac{1}{4}(p_{01} + p_{00} + p_{10} + p_{11}) &= (1 + r)\, p_{00} \end{aligned}$$

In particular, a trader with theory $\{2\}$ buys the asset in states $(1, 1)$ and $(0, 1)$, a trader with theory $\{1\}$ buys the asset in state $(1, 0)$, and a trader with theory $\varnothing$ buys the asset in state $(0, 0)$. The equilibrium prices are, as α approaches 1,

$$p_{00} = 15.48,\ p_{01} = 16.62,\ p_{10} = 16.58,\ p_{11} = 16.38$$

[11] Ettinger and Jehiel (2010) make the related observation that in order to be deceived a person must be perceptive enough to see the bait but not perceptive enough to see through it.

[12] It is easy to construct examples where increasing a trader's sophistication without altering the ranking of traders lowers this trader's returns.

When $\alpha \to 1$*, a trader with theory* $\{1\}$ *earns a return* $\frac{0.9}{16.58} = 0.054 > r$ *in state* $(1,0)$ *and* r *in all other states. This trader's return exceeds* r *due to "favourable selection". The dividend at* $(1,1)$ *is lower than the dividend at* $(1,0)$*. A trader with theory* $\{1\}$ *believes that the state is equally likely to be* $(1,0)$ *or* $(1,1)$*. However, his expectations are too pessimistic because he does not factor in that at state* $(1,1)$ *the asset is acquired by the trader with theory* $\{2\}$*.*[13]

Two ingredients are needed for Trader i to earn above-market returns. First, some Trader j with a theory neither finer nor coarser than her own must create "favourable selection" by buying the asset away from Trader i when its next-period price falls below i's expectation. Second, no trader can identify a set of states (as an atom or union of atoms) over which Trader i earns above-market returns.

Although some traders can earn returns in excess of the interest rate, it is impossible that all traders do so, for the return of holding one unit of the long-term asset in every state

$$\sum_{s\in S} \left(d(s) + Tp_{\Psi}\left(s\right) - p_{\Psi}\left(s\right)\right)\mu(s)$$

cannot exceed

$$\sum_{s\in S} p_{\Psi}\left(s\right) r(s)\mu(s).$$

To see this, note that the equilibrium equation implies that for any theory $\mathcal{F}$,

$$d\left(s\right) + E_{\mathcal{F}}\left(Tp_{\Psi}\right)\left(s\right) - p_{\Psi}\left(s\right) \leq p_{\Psi}\left(s\right) r(s)$$

and that, by definition,

$$\sum_{s\in S} Tp_{\Psi}\left(s\right)\mu(s) = \sum_{s\in S} E_{\mathcal{F}}\left(Tp_{\Psi}\right)(s)\mu(s)$$

For each theory $\mathcal{F}$ in a collection of theories Ψ, let $R\left(p_{\Psi};\mathcal{F}\right)$ be the expected equilibrium return of an $\mathcal{F}$-trader from allocating an amount equal to the price of

[13] By adding irrelevant states, it is straightforward to show that a trader with theory $\{1\}$ can earn above r even in the presence of traders with theories finer than her own yet no finer than traders with theory $\{2\}$.

one unit of the long-term asset to the purchase of either the long-term or the short-term asset. Define

$$B_{\mathcal{F}} = \{s \in S : E_{\mathcal{F}}\left(Tp_{\Psi}\right)(s) \geq E_{\mathcal{G}}\left(Tp_{\Psi}\right)(s), \forall \mathcal{G} \in \Psi\},$$

the set of states in which an $\mathcal{F}$-trader perceives the long-term asset to pay a return at least as large as the interest rate. Then

$$R\left(p_{\Psi};\mathcal{F}\right) = \sum_{s \in B_{\mathcal{F}}^c} p_{\Psi}\left(s\right) r(s)\mu(s) + \sum_{s \in B_{\mathcal{F}}} \left(Tp_{\Psi}\left(s\right) + d(s) - p_{\Psi}(s)\right)\mu(s).$$

One condition preventing any trader from earning above market returns is that some trader uses a theory more refined than any others.

Proposition 5 *Suppose that $\mathcal{G} \in \Psi$ refines every other theory $\mathcal{F} \in \Psi$. Then*

$$R\left(p_{\Psi};\mathcal{F}\right) \leq \sum_{s \in S} p_{\Psi}\left(s\right) r(s)\mu(s)$$

for any $\mathcal{F} \in \Psi$. Moreover,

$$R\left(p_{\Psi};\mathcal{G}\right) = \sum_{s \in S} p_{\Psi}\left(s\right) r(s)\mu(s)$$

Proof. Define $\hat{\mathcal{G}}$ to be the set of unions of elements of $\mathcal{G}$: $T \in \hat{\mathcal{G}}$ iff $T = \cup_i G_i$ for $G_i \in \mathcal{G}$. Because for each $\mathcal{F}$ in Ψ, $E_{\mathcal{F}}\left(Tp_{\Psi}\right)(s)$ is constant across elements of $\mathcal{G}$, since $\mathcal{F}$ is a coarsening of $\mathcal{G}$, either $B_{\mathcal{F}} = \emptyset$ or $B_{\mathcal{F}} \in \hat{\mathcal{G}}$. The equilibrium equation implies that for any $G \in \hat{\mathcal{G}}$,

$$\sum_{s \in G} \left(d\left(s\right) + E_{\mathcal{G}}\left(Tp_{\Psi}\right)(s) - p_{\Psi}\left(s\right)\right)\mu(s) \leq \sum_{s \in G} p_{\Psi}\left(s\right) r(s)\mu(s). \tag{5}$$

Because G is the union of elements of $\mathcal{G}$,

$$\sum_{s \in G} Tp_{\Psi}\left(s\right)\mu(s) = \sum_{s \in G} E_{\mathcal{G}}\left(Tp_{\Psi}\right)(s)\mu(s),$$

which implies

$$\sum_{s \in G} \left(Tp_{\Psi}\left(s\right) + d(s) - p_{\Psi}(s)\right)\mu(s) \leq \sum_{s \in G} p_{\Psi}\left(s\right) r(s)\mu(s).$$

The first statement follows from the fact that for each $\mathcal{F} \in \Psi$, either $B_{\mathcal{F}} = \emptyset$ or $B_{\mathcal{F}} \in \hat{\mathcal{G}}$. The second statement follows from (5) holding with equality on $B_{\mathcal{G}}$. ■

If Trader 1 is more sophisticated than all others, then every trader's price expectations are constant on any atom of Trader 1's theory. Thus, the $B_{\mathcal{F}}$ sets where trader $\mathcal{F}$ holds the asset are union of atoms of Trader 1's theory. Since Trader 1's price expectations are correct on such sets, the market cannot clear on any set $B_{\mathcal{F}}$ where holding the long-term asset yields above the interest rate. The statistical correctness of the most sophisticated theory is crucial for this result. Whereas when one trader is more sophisticated than all others, she earns a weakly higher return that all others, it is not the case that when all traders can be ordered by sophistication, returns increase (weakly) in sophistication.[14]

A second type of sufficient condition that prevents any trader from earning above-market returns is that the dividend, interest rate and states obey certain monotonicity conditions that cause all selection to be adverse. Consider an N-variable model. Given a theory $\mathcal{F} \subset \{1, 2, ..., N\}$, write s as $(s_{-\mathcal{F}}, s_{\mathcal{F}})$, where $s_{-\mathcal{F}}$ includes the components of s not in $\mathcal{F}$ and $s_{\mathcal{F}}$ those in $\mathcal{F}$; let $S(s_{\mathcal{F}})$ be the section of S in which the components in $\mathcal{F}$ are equal to $s_{\mathcal{F}}$.

Proposition 6 *Consider an affiliated, N-variable model. If d is non-decreasing and r non-increasing in s, then*

$$R(p_\Psi; \mathcal{F}) \leq \sum_{s \in S} p_\Psi(s) \, r(s) \mu(s),$$

for any $\mathcal{F} \in \Psi$.

Proof. By standard properties of affiliated variables, the contraction mapping $\mathcal{T}$ in Proposition 1 maps non-decreasing functions to non-decreasing functions. Hence, there exists a unique and non-decreasing Ψ-equilibrium price function $p_\Psi(\cdot)$. Since $E_{\mathcal{F}}(Tp_\Psi)(s)$ is non-decreasing in s, so is $\max_{\mathcal{F} \in \Psi} E_{\mathcal{F}}(Tp_\Psi)(s)$. Consider $s = (s_{-\mathcal{F}}, s_{\mathcal{F}})$ and $s' = \left(s'_{-\mathcal{F}}, s_{\mathcal{F}}\right)$, $s' \geq s$. Obviously, $E_{\mathcal{F}}(Tp_\Psi)(s) = E_{\mathcal{F}}(Tp_\Psi)(s')$, so if $s \notin B_{\mathcal{F}}$, then $s' \notin B_{\mathcal{F}}$. Thus, $B^c_{\mathcal{F}} \cap S(s_{\mathcal{F}})$ is an increasing set in $S(s_{\mathcal{F}})$. If $B_{\mathcal{F}}$ is empty, the

[14] This can be verified by eliminating state $\{d\}$ and Traders 3 and 4 from Example 4.

claim follows trivially. If not, by Milgrom and Weber (Theorem 22 (iii), 1982), for any $(s_{-\mathcal{F}}, s_{\mathcal{F}}) \in S(s_{\mathcal{F}})$,

$$E_{\mathcal{F}}(Tp_{\Psi})(s_{-\mathcal{F}}, s_{\mathcal{F}}) \geq \frac{\sum_{z \in B_{\mathcal{F}} \cap S(s_{\mathcal{F}})} Tp_{\Psi}(z)\,\mu(z)}{\sum_{z \in B_{\mathcal{F}} \cap S(s_{\mathcal{F}})} \mu(z)};$$

that is, the expected, next-period price conditional upon $s_{\mathcal{F}}$ and purchasing the asset cannot exceed the expected price conditional upon $s_{\mathcal{F}}$ alone. By definition and Equation (1),

$$\sum_{s \in B_{\mathcal{F}}} (E_{\mathcal{F}}(Tp_{\Psi})(s) + d(s) - p_{\Psi}(s))\,\mu(s) = \sum_{s \in B_{\mathcal{F}}} p_{\Psi}(s) r(s)\,\mu(s).$$

Since

$$\sum_{s \in B_{\mathcal{F}}} \frac{\sum_{z \in B_{\mathcal{F}} \cap S(s_{\mathcal{F}})} Tp_{\Psi}(z)\,\mu(z)}{\sum_{z \in B_{\mathcal{F}} \cap S(s_{\mathcal{F}})} \mu(z)} \mu(s) = \sum_{s \in B_{\mathcal{F}}} Tp_{\Psi}(s)\,\mu(s),$$

it follows from simple substitutions that for any $\mathcal{F} \in \Psi$,

$$R(p_{\Psi}; \mathcal{F}) \leq \sum_{s \in S} p_{\Psi}(s)\, r(s) \mu(s)$$

■

When traders' conditioning variables are affiliated, dividends are increasing, and interest rate decreasing in these variables, all selection is adverse: Trader i does not buy the asset precisely when Trader j's theory provides positive news about its value, unbeknownst to Trader i.

8 Prices and Fundamental Values

One important question is how prices relate to real and perceived expected values of holding the long-term asset in perpetuity. We define the *fundamental value* of the long-term asset to be the expected present-discounted value of holding it in perpetuity, where expectations are taken with respect to the complete theory. Recall that $T^k f$ is the k^{th} iteration of the operator T on the function f. For notational simplicity, we assume in this section that the interest rate is deterministic and equal to r.

Definition 3 *The* fundamental value *of the long-term asset in state s is*

$$V(s) := \frac{1}{1+r}\left(d(s) + \sum_{k=1}^{\infty} \frac{T^k d(s)}{(1+r)^k}\right).$$

The fundamental value in state s equals the correct expected present-discounted value of the infinite stream of dividend payments that ownership confers upon the asset holder. Obviously,

$$V(s) = \frac{1}{1+r}\left(d(s) + TV(s)\right)$$

Agents with coarser theories perceive fundamental values to differ from $V(s)$. We define the *$\mathcal{F}$-perceived fundamental value* of the long-term asset recursively where expectations are taken with respect to the theory $\mathcal{F}$.

Definition 4 *The* $\mathcal{F}$-perceived fundamental value *of the long-term asset is the function $V_{\mathcal{F}} : S \to \mathbb{R}_+$ such that*

$$V_{\mathcal{F}}(s) := \frac{1}{1+r}\left(d(s) + E_{\mathcal{F}}(TV_{\mathcal{F}})(s)\right) \tag{6}$$

Such function $V_{\mathcal{F}}$ exists as the mapping defined by (6) is a contraction. The following lemma shows that the perceived long-run value of holding the asset is equal to ex-ante value of receiving the expected dividend in perpetuity

Lemma 7 *For any theory $\mathcal{F}$,*

$$\sum_{s\in S} V_{\mathcal{F}}(s)\mu(s) = \frac{1}{r}\sum_{s\in S} d(s)\mu(s) = \sum_{s\in S} V(s)\mu(s)$$

Proof. It follows from (6), the definition of fundamental value, and that, since μ is invariant,

$$\sum_{s\in S} f(s)\mu(s) = \sum_{s\in S} Tf(s)\mu(s),$$

for any $f : S \to \mathbb{R}$. ■

With a single theory in the market, the price equals its perceived fundamentals.

Corollary 8 *If $\Psi = \{\mathcal{F}\}$, then in each s, $p_{\Psi}(s) = V_{\mathcal{F}}(s)$.*

Proof. It follows by the definition of $V_{\mathcal{F}}$, since p_Ψ is unique by Proposition 1. ■

When the market includes more than one theory, Corollary 8 together with Proposition 2 tell us that prices must be at least as high as the maximum perceived fundamental values in every state. The following results shows that, unless all $\mathcal{F}$-perceived fundamental values are identical, prices must exceed the highest perceived fundamental value.

Proposition 9 *For any collection of theories Ψ and $s \in S$,*

$$p_\Psi(s) \geq \max_{\mathcal{F}\in\Psi} V_{\mathcal{F}}(s).$$

Moreover

(i) if $V_{\mathcal{G}}(s) \neq V_{\mathcal{G}'}(s)$ for some $\mathcal{G}$ and $\mathcal{G}'$ in Ψ and $s \in S$ then

$$p_\Psi(s) > \max_{\mathcal{F}\in\Psi} V_{\mathcal{F}}(s) \text{ for every } s \in S.$$

(ii) if $V_{\mathcal{G}}(s) = V_{\mathcal{G}'}(s)$ for any $\mathcal{G}$ and $\mathcal{G}'$ in Ψ and $s \in S$, then

$$p_\Psi(s) = V_{\mathcal{F}}(s) \text{ for every } s \in S.$$

Proof. The first statement follows trivially from Corollary 8 and Proposition 2.
To prove (i), we first show that if two collection of theories, Ψ'' and Ψ', are such that $\Psi'' = \Psi' \cup \{\mathcal{H}\}$ for some partition $\mathcal{H}$, and

$$p_{\Psi'}(s) > \max_{\mathcal{F}\in\Psi'} V_{\mathcal{F}}(s) \text{ for every } s \in S$$

then

$$p_{\Psi''}(s) > \max_{\mathcal{F}\in\Psi''} V_{\mathcal{F}}(s) \text{ for every } s \in S,$$

To see this note that, by Lemma 7, it is impossible that $V_{\mathcal{H}}(s) > \max_{\mathcal{F}\in\Psi'} V_{\mathcal{F}}(s)$ for every $s \in S$. Since $p_{\{\mathcal{H}\}} = V_{\mathcal{H}}$, by Proposition 2, it is impossible that

$$p_{\{\mathcal{H}\}}(s) = p_{\Psi''}(s)$$

for every s. Thus, by Proposition 3,

$$p_{\Psi''}(s) > V_{\mathcal{H}}(s)$$

for every $s \in S$ and the claim follows. To conclude the proof of (i), it suffices to show that if $\Psi = \{\mathcal{F}, \mathcal{G}\}$ and $V_{\mathcal{F}}(s) \neq V_{\mathcal{G}}(s)$ for some $s \in S$, then

$$p_{\Psi}(s) > \max\{V_{\mathcal{F}}(s), V_{\mathcal{G}}(s)\}, \text{ for every } s \in S.$$

By Lemma 7, if $V_{\mathcal{F}}(s) \neq V_{\mathcal{G}}(s)$ for some $s \in S$, the sets

$$\{s \in S : V_{\mathcal{F}}(s) > V_{\mathcal{G}}(s)\}, \quad \{s \in S : V_{\mathcal{F}}(s) < V_{\mathcal{G}}(s)\}$$

are both non-empty. When the market includes only the theory $\mathcal{F}$, Corollary 8 gives that $p_{\{\mathcal{F}\}}(s) = V_{\mathcal{F}}(s)$ in every state s. The claim then follows by Proposition 3.
To prove (ii), first note that, since $V_{\mathcal{F}}(s) = V_{\mathcal{G}}(s)$ for any $s \in S$

$$E_{\mathcal{G}}(TV_{\mathcal{G}})(s) = E_{\mathcal{G}}(TV_{\mathcal{F}})(s) = E_{\mathcal{F}}(TV_{\mathcal{F}})(s)$$

for every $\mathcal{F}, \mathcal{G} \in \Psi$ and $s \in S$. Thus, for any $s \in S$ and any $\mathcal{F} \in \Psi$

$$V_{\mathcal{F}}(s) = \frac{1}{1+r}\left(d(s) + \max_{\mathcal{G} \in \Psi} E_{\mathcal{G}}(TV_{\mathcal{F}})(s)\right)$$

Hence, $V_{\mathcal{F}}(\cdot)$ is the unique Ψ-equilibrium price function. ■

Either all theories generate the same perceived fundamental values in every state or the price exceeds the perceived fundamental values of every theory in every state. Prices can strictly exceed the highest perceived fundamental value due to speculative motives. The owner of the asset in state s may sell it in state $s' \neq s$ next period for a price in excess of her own perceived fundamental values in s'; recognising this possibility, her willingness to pay exceeds her own perceived fundamental value now. Once more, prices higher than the highest perceived fundamental values in some state propagate to all other states. In a similar way, Harrison and Kreps (1978) and Morris (1996) show that prices exceed the highest perceived fundamental values in models of non-common priors. By assuming statistical correctness, we gain the conclusion that average prices exceed the true average value of holding the asset in perpetuity.[15]

Our model also allows us to address the relationship between prices and fundamental values, which may not be monotone. In Example 4, the price in state b exceeds

[15]This follows from combining Lemma 7 with Proposition 9.

that in state d despite equal fundamental values; a slight increase in the probability of the good state a following d would raise the fundamental value in d above that in b whilst preserving the ordering of prices across states. In the affiliated N-variable model where dividends increase and interest rates decrease in the state, however, the relationship between prices and fundamentals is monotone.[16]

9 Prices and Dividends

The relationship between prices and dividends is more straightforward than that between prices and fundamental values.

Proposition 10 *Given a collection of theories Ψ, consider the dividend functions d, d', and d'' and the corresponding Ψ-equilibrium price functions p_Ψ, p'_Ψ, p''_Ψ.*
(i) If $d' \geq d$, then $p'_\Psi \geq p_\Psi$. If, in addition, $d' \neq d$, then $p'_\Psi \neq p_\Psi$.
(ii) If for some $\lambda \in \mathbb{R}_+$, $d'' = \lambda d$, then $p''_\Psi = \lambda p_\Psi$.
(iii) If $d'' = d + d'$, then $p''_\Psi \leq p_\Psi + p'_\Psi$.

Proof. Let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the contraction mappings derived, as in the proof of Proposition 1, from the equilibrium condition for dividend functions d, d', and d''.
(i) The first part follows from the fact that since$\mathcal{T}'(p) \geq \mathcal{T}(p)$ for any p, $\mathcal{T}'$ maps any $p \geq p_\Psi$ into the same space. The second part follows because $d'(s) > d(s)$ in some state s implies that

$$p_{\Psi'}(s) = \mathcal{T}'(p_{\Psi'})(s) > \mathcal{T}(p_\Psi)(s) = p_\Psi(s).$$

(ii) If p_Ψ is a fixed point of $\mathcal{T}$, λp_Ψ is a fixed point of $\mathcal{T}''$.
(iii) First note that

$$\mathcal{T}''(p_\Psi + p'_\Psi) \leq \mathcal{T}(p_\Psi) + \mathcal{T}'(p'_\Psi) = p_\Psi + p'_\Psi.$$

[16]The proof of Proposition 4 shows that under affiliation the contraction $\mathcal{T}$ maps non-decreasing functions to non-decreasing functions, so both equilibrium prices and fundamental values (prices with only the complete theory in the market) are non-decreasing in the state.

Since $\mathcal{T}''$ is monotone, $\mathcal{T}''(p) \leq p_\Psi + p'_\Psi$ for any $p \leq p_\Psi + p'_\Psi$. Thus, the unique fixed point p''_Ψ of $\mathcal{T}''$ is such that $p''_\Psi \leq p_\Psi + p'_\Psi$. ■

When dividends are higher, the price of the long-term asset is also higher. Likewise, for fixed dividends, prices are non-increasing in the interest rate. For brevity, we omit the formal statement. Because prices are homogenous of degree one in dividends, the market value of a company is not affected by a stock split.

Finally, prices are sub-additive in dividends. Since equilibrium pricing equations are separable across assets, our model can encompass multiple assets. With this interpretation, subadditivity implies that bundling equities by creating a mutual or index fund destroys market value. The intuition for this is that the one investor who is most optimistic about the value of the sum of N assets cannot have a higher willingness to pay than the sum of the willingness to pay of the N investors most optimistic about each of the N assets. That integrating assets lowers price is broadly consistent with the finding that closed-end mutual funds typically sell at a discount relative to the value of the underlying equities (Lee, Shleifer and Thaler, 1990).

10 Conclusion

In this paper, we develop a framework where traders use incomplete models of the myriad connections among economic variables relevant for dividends and interest rates: each trader's model uses a subset of the predictive variables and is statistically correct about the next-period price. We have aimed to establish some broad results about asset pricing and returns when traders have incomplete theories. We conclude by remarking on some implications that our framework has for particular contexts.[17]

Consider a setting in which the current dividend is a sufficient statistic for the current state in predicting next period's state. Traders whose theories exclude the dividend can form expectations based on partial aspects of the state, which has two consequences. First, prices are noisy functions of dividends (and, hence, of fundamental values), consistent with Shiller's (1981) finding that prices are more volatile than

[17] These appear as examples in Eyster and Piccione (2011).

dividends. Second, prices may become noisier as traders become more sophisticated.

Our model embodies a natural asymmetry between optimism and pessimism, for it takes only one trader's optimism to support a high price but all traders' pessimism to support a low price. This suggests that where all variables are complements in determining dividends, bad news must percolate through the entire financial market in order that prices decline and thus that high prices linger longer than low ones.

We hope to explore the scope of such observations in future research.

11 Appendix

We first extend the "perceived" model of Section 4.3 to include the dividend and the interest rate processes. For notational convenience, we define the stochastic process over a different but equivalent state space. Consider any $\mathcal{F} \in \Psi$. For any $\left(\hat{p}, \hat{r}, \hat{d}\right) \in p_{\Psi}(S) \times r(S) \times d(S)$, define

$$I\left(\hat{p}, \hat{r}, \hat{d}\right) = p_{\Psi}^{-1}(\hat{p}) \cap r^{-1}(\hat{r}) \cap d^{-1}\left(\hat{d}\right).$$

Given the atoms F^t and F^{t+1} of $\mathcal{F}$, define a Markov process for the set

$$\mathcal{F} \times p_{\Psi}(S) \times r(S) \times d(S),$$

whose the transition function is

$$\bar{Q}\left(F^t, p^t, r^t, d^t, \{F^{t+1}, p^{t+1}, r^{t+1}, d^{t+1}\}\right) = E_{\mathcal{F}}\left(Q\left(\cdot, F^{t+1} \cap I\left(p^{t+1}, r^{t+1}, d^{t+1}\right)\right)\right)(s)$$

for any $s \in F^t$. Note that the probability of an event at time $t+1$ depends solely on the realization of the atom of $\mathcal{F}$ at time t. Now consider the random variable

$$X^t = \begin{cases} \dfrac{d^0 + p^1}{(1 + r^0)} - p^0 & \text{for } t = 0 \\[2ex] X^{t-1} + \dfrac{d^t + p^{t+1} - (1 + r^t)\, p^t}{\prod_{\tau=0}^{t} (1 + r^{\tau})} & \text{for } t \geq 1. \end{cases}$$

The variable X^t is the net discounted return from holding the long-term asset for t periods over the amount p^0 paid initially. Obviously, for any $s \in F \in \mathcal{F}$,

$$\begin{aligned} E\left(d(s) + p^{t+1} - (1 + r(s)) p_\Psi(s) \mid F, p_\Psi(s), r(s), d(s)\right) &= \\ d(s) + E_{\mathcal{F}}(Tp_\Psi)(s) - (1 + r(s)) p_\Psi(s) &\leq 0 \end{aligned}$$

Thus, the process $\{X^t\}$ is a supermartingale. Applying the Optional Stopping Theorem (see, e.g., Brzezniak and Zastawniak (2005), Theorem 3.1) yields the conclusion.

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