Heterogeneous Quantal Response Equilibrium and Cognitive Hierarchies[1]

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Abstract

We explore an equilibrium model of games where behavior is given by logit response functions, but payoff responsiveness and beliefs about others' responsiveness are heterogeneous. We study two substantively different ways of extending quantal response equilibrium (QRE) to this setting: (1) Heterogeneus QRE, where players share identical correct beliefs about the distribution of payoff responsiveness; and (2) Truncated QRE, where players have downward looking beliefs, systematically underestimating others' responsiveness. We show that the congnitive hierarchy model is a special case of Truncated QRE. We conduct experiments designed to differentiate these approaches. We find significant evidence of payoff responsive stochastic choice, and of heterogeneity and downward looking beliefs in some games.

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1 Introduction

Finding a disciplined, empirically accurate way to incorporate limits on rationality has been a central challenge in game theory. One approach,[1] "quantal response equilibrium" (QRE, McKelvey and Palfrey 1995) maintains the assumption of equilibrium, in that beliefs are statistically accurate, but relaxes the assumption that players choose best responses.[2] A different approach, based on "cognitive hierarchy" (CH, Camerer, Ho, and Chong 2004) theories, relaxes the equilibrium assumption, by assuming that players do not correctly anticipate what others will do, but retains the assumption of best responding to beliefs. QRE and CH approaches both generate statistical predictions in which every strategy is played with positive probability (which obviates the need for perfection refinements), and have been used successfully to explain deviations from Nash equilibrium in many types of experiments. We introduce heterogeneity of "skill" to QRE, where skill (individual payoff responsiveness) and possibly beliefs about others' skill levels varies across players. We show that CH is a limiting case of this family of QRE models with skill heterogeneity.[3]

This paper makes a theoretical contribution and an empirical contribution. The theoretical contribution is the introduction of skill heterogeneity and identifying a formal and intuitve connection between QRE and CH. One version of these models is the natural and direct generalization of QRE, Heterogeneous Quantal Response Equilibrium (HQRE), where players share common and correct information about the distribution of skill-types. But we also consider more general specifications where the distribution of skill levels is not common knowledge. This is what provides the connection to CH. In truncated HQRE (TQRE), beliefs are downward-looking, in that

[1]Learning models explore a different kind of rationality limit than the static models considered here (see Camerer 2004, chapter 6).

[2]A purer interpretation is that players do best respond, but that their expected payoffs include a disturbance term which is unobserved by the econometrician, but whose distribution is commonly known.

[3]The first paper to combine notions of noisy best reply, central to QRE, and downward looking beliefs, central to CH, with the goal of explaining experimental data was Stahl-Wilson 1995.

players believe other other players are less skilled than themselves. A limiting version of TQRE, where all payoff responsive types perfectly best respond to their downward looking beliefs is behaviorally equivalent to CH.

The empirical contribution is new experimental data from a variety of games to analyze the differential predictions of QRE, HQRE, TQRE, and CH, comparing their relative ability to explain data from the different games. While the empirical fits are similar in the majority of games we study, there are some interesting differences that shed light on the strengths and weaknesses of the different approaches.We find significant evidence of heterogeneity (of both skill and cognitive levels) and significant evidence of downward looking beliefs (as opposed to rational expectations), at least in a subset of the games.

While the paper is nominally about two different behavioral theories of limited-rationality in games, the basic questions that are addressed are fundamental, especially with regard to questions about how to introduce heterogeneity in the analysis of behavioral data from games. In some areas of game theory it has proved useful to introduce special preference "types" (e.g., Kreps et al, 1982; Fudenberg and Maskin, 1986). The QRE and CH approaches are similar except the heterogeneity in types comes from either imperfect response or limited strategic thinking. Expanding the QRE approach to include heterogeneity creates a unified framework in which to compare these approaches and see their differences and similarities. Furthermore, introducing heterogeneity into QRE allows a concept of "gamesmanship" or skill which is absent in equilibrium analysis (since, in equilibrium, all players are equally accurate and rational). Allowing variation in skill opens up new questions: Why are some people more skilled? How do skills develop with experience and teaching? These questions are not the focus of our analysis, but are naturally raised by introducing heterogeneity.

The paper proceeds as follows. Section 2 defines SQRE. Section 3 defines and proves existence of the rational expectations version, HQRE. Section 4 defines and proves existence of TQRE, and provides an approximation theorem that shows CH is a limiting case of TQRE. Section 5 describes the experimental design, reports the experimental data, and contains an empirical analysis of the fit of the models across various games. An Appendix contains proofs of the theorems and a table summarizing part of our experimental design.

2 The Framework

We explore a logit quantal response equilibrium model where players' choice behavior follow logit quantal response functions but there is heterogeneity with respect to the responsiveness parameter.[4] The structure of this heterogeneity is not necessarily common knowledge. Instead, players hold subjective beliefs about the distribution of other players' types.

An important aspect of this framework, which we call Subjective Heterogeneous Quantal Response Equilibrium (SQRE), is that expectations about choice probabilities may be inconsistent with the actual choice frequencies of the other players. Models with this property could prove useful in explaining behavior in one-shot games, or complex games in which learning or other forces have not enabled beliefs to fully equilibrate to actual choices. However, the particular form of inconsistencies allowed in SQRE still permits it to be thought of as an equilibrium model: choice probabilities conditional on type are common knowledge; it is only the perceived distribution of types that varies across players.

More specifically, let $\lambda_i \in [0, \infty)$ denote the type of player i. We replace the rational expectations assumption by an assumption of *subjective expectations.* According to this model, the equilibrium strategies, which map types into choice probabilities, of all players are common knowledge in equilibrium, but players may have different beliefs

[4]Our approach and main theoretical results are easily extended beyond the logit specification to the general framework of regular quantal response equilibrium studied by Goeree, Holt and Palfrey (2005).

about the type distributions. Denote the conditional subjective beliefs of player k about the type of player i by $F_i^k(\lambda_i|\lambda_k)$, where we assume that each F_i^k has support contained in $[0,\infty)$, a smooth density function, f_i^k, and finite moments. For example, f_i^k could be the density function for an exponential or log normal distribution. Note that beliefs generally depend on a player's own type.[5] As we show below, this framework provides a way to link heterogeneous QRE approaches with cognitive hierarchy approaches, which share a similar feature of belief heterogeneity. This difference in beliefs results in equilibrium strategies (and induced mixed strategies) that in general are different from those that obtain with common beliefs in a heterogeneous logit QRE.

Let $\Gamma = [N, \{A_i\}_{i=1}^n, \{u_i\}_{i=1}^n]$ be a game in strategic form, where $N = \{1, ..., n\}$ is the *set of players*, $A_i = \{a_{i1}, \ldots, a_{iJ_i}\}$ is i's *action set* and $u_i : A \to \Re$ is i's *payoff function,* where $A = A_1 \times \cdots \times A_n$. Let ΔA_i denote the set of probability distributions over A_i and let $\Delta A = \Delta A_1 \times \cdots \times \Delta A_n$ denote the product set of probability distributions over A_i, $i = 1, ..., n$.

For any subjective belief about action profiles, $\widehat{\sigma} \in \Delta A$, player i's expected payoff is given by:

$$U_i(\widehat{\sigma}) = \sum_{a\in A} \left(\Pi_{k=1}^n \widehat{\sigma}_k(a_k)\right) u_i(a).$$

and the (subjective) expected payoff to player i from using action $a_{ij} \in A_i$ is:

$$U_{ij}(\widehat{\sigma}) = \sum_{a_{-i}\in A_{-i}} \left(\Pi_{k\neq i} \widehat{\sigma}_k(a_k)\right) u_i(a_{ij}, a_{-i}).$$

With logit response functions, choice probabilities are logit transformations of expected utilities, so if i has type λ_i and the actions by i have expected payoffs

[5] A special case is when $F_i^k(\lambda_i|\lambda_k) = F_i(\lambda_i)$ for all $i, k, \lambda_i, \lambda_k$. In this case subjectivity is absent and we are left with only heterogeneous QRE, discussed later.

$U_i = (U_{i1}, ..., U_{iJ_i})$, then the probability of i choosing action j as a function of λ_i is:

$$p_{ij}(\lambda_i; U_i) = \frac{e^{\lambda_i U_{ij}}}{\sum_{k=1}^{J_i} e^{\lambda_i U_{ik}}} \tag{1}$$

We call any measurable function $p_i : [0, \infty) \to \Delta A_i$ a *strategy* for player i.

We next turn to the beliefs that other players k have about i's choice probabilities without knowing λ_i, but with their subjective beliefs F_i^k about its distribution. Because of the different subjective beliefs about the distribution of λ_i, players k and k' can have different beliefs about player i's choice probabilities. However, we assume that any differences in their beliefs about i's mixed strategy are due only to differences in beliefs about the distribution of λ_i. That is, the strategy profile, $p = (p_1, \ldots, p_n)$, is assumed to be common knowledge (hence we refer to this as an equilibrium model), although the structure of beliefs departs from the conventional common prior assumption. We denote type λ_k of player k's belief about player i's choice probabilities by $\sigma_i^k(p_i)$. Therefore, given i's strategy, $p_i(\cdot)$, the belief of type λ_k of player k that player i will choose action j (i.e., before λ_i is drawn) is:

$$\sigma_{ij}^k(p_i|\lambda_k) = \int_0^\infty p_{ij}(\lambda_i) f_i^k(\lambda_i|\lambda_k) d\lambda_i \tag{2}$$

Given $\sigma_{-i}^i(p_{-i}|\lambda_i)$, the beliefs of type λ_i of player i about the profile of choice probabilities of all players other than i, type λ_i of player i's expected payoffs, $U_i^{\lambda_i}(\sigma_{-i}^i) = (U_{i1}^{\lambda_i}, ..., U_{iJ_i}^{\lambda_i})$, are simply:

$$U_{ij}^{\lambda_i}(\sigma_{-i}^i) = \sum_{a_{-i} \in A_{-i}} \left(\Pi_{k \neq i}^n \sigma_k^i(a_k|\lambda_i)\right) u_i(a_{ij}, a_{-i}). \tag{3}$$

In a SQRE with logit response functions, equations (1), (2), and (3) must all be satisfied simultaneously. This leads to the following:

Definition 1 *p^* is a Subjective Heterogeneous Logit Equilibrium if:*

$$p^*_{ij}(\lambda_i) = \frac{e^{\lambda_i U^{\lambda_i}_{ij}(\sigma^i_{-i}(p^*|\lambda_i))}}{\sum_{k=1}^{J_i} e^{\lambda_i U^{\lambda_i}_{ik}(\sigma^i_{-i}(p^*|\lambda_i))}} \textit{ for all } i = 1, ..., n, \ j = 1, ..., J_i \textit{ and } \lambda_i \in [0, \infty).$$

This definition reflects the idea that in SQRE players have rational expectations about strategies (that is, a player's behavior conditional on his type λ), but may have different beliefs about the distribution of mixed strategies, which are induced by different beliefs about the distribution of types λ. That is, the beliefs in SQRE are *subjective* and do not necessarily come from a common prior model. We refer to this an *equilibrium* in exactly the same sense as a Bayesian equilibrium. In standard definitions of Bayesian equilibrium (see for example Geanakoplos 1994, p. 1461), beliefs are type contingent and there is no requirement of a common prior model of beliefs.[6]

SQRE is a general framework in that there is little that constrains the extent of heterogeneity and subjective beliefs. In what follows, we consider a number of ways of imposing constraints on SQRE, which are relevant from an empirical standpoint. In addition, though, we are able to specialize SQRE in ways that reduce to more standard models of utility-based choice, namely Quantal Response Equilibrium and Cognitive Hierarchy. We also establish a connection between these models beyond simply taking some sort of "convex combination" of the two. More generally, the question of how to introduce heterogeneity into strategic models is a question of great concern. Since QRE and CH take very different approaches to this issue, the more general framework has the potential to benefit from ingredients of both models.

There is a substantial body of research on which we build. On the one hand, a number of papers have considered various specific ways of extending QRE to allow for

[6]In fact, the entire theoretical analysis of this paper could be done using Bayesian equilibrium, because the logit QRE is itself a particular form of Bayesian equilibrium with iid payoff disturbances that follow an extreme value distribution. Heterogeneity in λ, simply means that the distribution of payoff disturbances, in particular, the variances of those distubances, can vary across players. However, we did not take this route because it would needlessly create more notation.

both heterogeneity and subjectivity (see footnote 6 below). In fact, the basic insight that QRE could be extended to to allow for heterogeneity across individuals was presented in the original formulation of McKelvey and Palfrey 1995. On the other hand, precursors of the CH model date at least to Stahl and Wilson 1994. Following their work, a number of authors developed "level-k" models which are closely related to CH (see footnote 10 below). An important difference between those models and CH is that CH assumes that players have accurate beliefs about the relative frequencies of those below themselves in the hierarchy, whereas most level-k formulations assume players look only one step down. Goeree and Holt 2004 develop a model of "noisy introspection" that can be viewed as incorporating the idea of downward-looking beliefs into a QRE-like model. However, the method and generality with which we combine QRE and CH is novel, and allows us to understand in a clearer way the relationships between the two approaches.

3 Eliminating Subjectivity and QRE

In this section we consider the special case of SQRE corresponding to the assumption of rational expectations of players' type distributions. That is, we eliminate the element of subjectivity from SQRE, leaving only heterogeneous QRE, or HQRE. From this model, it is straightforward to see that a limiting case in which the heterogeneity in types (which is common knowledge in the absence of subjectivity) vanishes reduces to the standard logit QRE.

In particular, we now require that $F_i^k = F_i$ for every $i, k \in N$, so that the distributions of each player's type is common knowledge. I.e., there is no subjectivity in beliefs.

Each player is independently assigned by nature a response sensitivity, λ_i, drawn from a commonly known distribution, $F_i(\lambda_i)$, obeying the same assumptions as above. The assumption in HQRE is that $F_i(\lambda_i)$ is common knowledge, but i's type, λ_i, is private information known only to i. Therefore, given i's profile of choice probability functions, $p_i(\cdot)$, the ex ante probability i chooses action j (i.e., before λ_i is drawn) is:

$$\sigma_{ij}(p) = \int_0^\infty p_{ij}(\lambda) f_i(\lambda) d\lambda \tag{4}$$

Following Harsanyi (1973), we call $\sigma_i = (\sigma_{i1}, \ldots, \sigma_{iJ_i})$ i's *induced mixed strategy*, which is common knowledge in HQRE. As in SQRE, strategies, p, are common knowledge, and now the distributions of types are common knowledge as well. Given σ_{-i}, the induced mixed strategy profile of all players other than i, i's expected payoffs, $U_i(\sigma_{-i}) = (U_{i1}, ..., U_{iJ_i})$, can be expressed as:

$$U_{ij}(\sigma) = \sum_{a_{-i} \in A_{-i}} \left(\Pi_{k \neq i}^{n} \sigma_k(a_k) \right) u_i(a_{ij}, a_{-i}). \tag{5}$$

In a heterogeneous quantal response equilibrium with logit response functions, equations (1),(4), and (5) must all be satisfied simultaneously. This leads to the following

Definition 2 *p^* is a Heterogeneous Logit Equilibrium if:*

$$p_{ij}^*(\lambda_i) = \frac{e^{\lambda_i U_{ij}(\sigma(p^*))}}{\sum_{k=1}^{J_i} e^{\lambda_i U_{ik}(\sigma(p^*))}} \text{ for all } i = 1, ..., n, \ j = 1, ..., J_i \text{ and } \lambda_i \in [0, \infty).$$

This captures the idea that in HQRE players have rational expectations about the distribution of mixed strategies, and these will then be self-fulfilling given the commonly known distribution of profiles of quantal response functions. Therefore, like Nash equilibrium, the solution to the problem is a fixed point of a mapping from choice probabilities to choice probabilities. The Appendix proves existence of HQRE for the logit case, using a fixed point theorem.

Theorem 1. *In finite games, a Heterogeneous Logit Equilibrium exists.*

Proof: See appendix.

Note that if each $F_i(\lambda_i)$ has a single mass point,[7] then the heterogeneity in types becomes degenerate. If, in addition $\lambda_i = \lambda_j$ for all i, j, HQRE collapses to standard (homogeneous) logit QRE.

4 Truncated Subjectivity, Best Response and CH

This section considers specializing the general framework of SQRE in a different direction. In particular, we begin by introducing a specific kind of subjectivity, called truncated expectations, which constrains the extent of subjectivity allowed in the general model. The idea is that players have accurate beliefs about the relative proportions of types that are lower than some bound, but are either unable to imagine the existence of higher types in the population or do not believe that such types exist. This "imagination bound" can be type-dependent. When the type distributions in SQRE take a discrete form, this truncation allows us to tie together heterogeneity in payoff sensitivities and in accuracy of beliefs. A limiting case in which all players (except for a random type) become perfectly payoff sensitive then reduces to the Cognitive Hierarchy (CH) model.

4.1 Truncated expectations and bounded imagination

Since SQRE is quite general, precision in applying it must come from additional restrictions on heterogeneity and subjective beliefs (preferably empirically-plausible ones).[8] We do this by introducing *truncated rational expectations*: Players act as if they are not aware of the existence of types who are more rational than some maximum upper bound, and this upper bound may depend on their own type. Given their truncated

[7]While the assumptions of the F_i above preclude this case, it can be approximated arbitrarily closely by F_i that do satisfy the assumptions.

[8]Earlier papers have considered variations of subjective HQRE. McKelvey, Palfrey, and Weber (2000) consider an HQRE model of self-centered subjective beliefs where players have different λ_i's and believe every one else is exactly like themselves. Weizsacker (2003) considers a version where the players still have point beliefs, but these beliefs are not necessarily self-centered.

beliefs, they form expectations by integrating over their perceived type distribution.

Denote the upper bound on player i's imagined types by $\theta_i(\lambda_i)$, where $\theta_i(\cdot)$ is commonly known. We assume that $\theta_i(\lambda_i)$ is strictly increasing and uniformly continuous in λ_i and for each i there exists $\bar{\theta}_i$ such that $\theta_i(\lambda_i) \leq \bar{\theta}_i \lambda_i$ for all λ_i. The beliefs of type λ_k of player k about λ_{-k} are rooted in the true distribution, F_{-k}, but normalized to reflect the missing density. That is, for $\lambda_k > 0$, the subjective beliefs of k about the type of player i is given by $F_i^k(\lambda_i|\lambda_k) = F_i(\lambda_i)/F_i(\theta_k(\lambda_k))$ for $\lambda_i \in [0, \theta_k(\lambda_k)]$ and $F_i^k(\lambda_i|\lambda_k) = 1$ for $\lambda_i \geq \theta_k(\lambda_k)$. This is truncated HQRE, or *TQRE.* Note that as $\theta_i \to \infty$ for all i, the upper bound on λ is lifted and the model converges to the standard HQRE model.

The truncation, $\theta_i(\lambda_i)$ can be interpreted as type λ_i of player i's *imagination.* Since $\theta_i(\lambda_i)$ is finite, this is a model of bounded imagination, in the sense that for any type λ_i of player i, all λ_{-i}-types beyond a certain threshold, $\theta_i(\lambda_i)$, are unimaginable in the sense that i assigns zero probability to all those higher types. Notice that since $\theta_i(\lambda_i)$ is increasing, then players who are more skillful in the sense of payoff responsiveness (i.e. higher λ_i) necessarily also have more accurate expectations, in the sense that their beliefs are closer to the true distribution F. Types for which $\theta_i(\lambda_i) \approx 0$ are almost completely unimaginative in the sense that they believe all other players are nearly random. Hence these very low types will act approximately as if they are applying the principle of insufficient reason to form expectations about the other players' strategy choices (as do the level-1 types in the cognitive hierarchy model), and then quantal respond to these beliefs. If $\theta_i(\lambda_i) \leq \lambda_i$, then we say that players are *self-limited,* because they cannot imagine types with higher λ than their own. Proving existence of TQRE requires a slightly different proof than HQRE because different λ-types have different beliefs about the other players.

Theorem 2. *In finite games, a Truncated Heterogeneous Logit Equilibrium exists.*
Proof: See appendix.

There are a number of reasons why truncated beliefs represent a reasonable manner of constraining belief heterogeneity. One rationale is that players with a low value of λ who can imagine players with higher λ, and compute what those other players will do, will generally want to switch to the higher-type behavior. There is also a significant body of evidence from the psychology literature indicating that people are often overconfident about their relative skill.[9] A third rationale is computational complexity: If there are cognitive costs to computing expected payoffs, those costs increase as players have more other types to consider. The benefits from more imagination—the expected payoff differential from imagining what a wider range of types will do—are likely to fall as λ rises, so the truncated expectations assumption can be seen as a reduced-form model of cost-benefit calculations which lead players to ignore information that is both hard to process and not too costly to ignore.

4.2 Discretized TQRE: A connection between QRE and CH

In this section we establish a formal equivalence between a version of TQRE and CH.

4.2.1 Truncation and Heterogeneity in CH

CH introduces heterogeneity of player types of a much different kind than HQRE. In CH there is a discrete distribution $f(k)$ of players who do k steps of thinking, so k indexes *strategic sophistication.* The choice probabilities for a k-step player i choosing strategy j are $p_{ij}(k)$. A 0-step player randomizes over her (finite) number of strategies J_i, so $p_{ij}(k) = 1/J_i$ for all j. Note that these players do not form beliefs or even attend to their payoffs; their presence is just assumed to get a hierarchical process started in a simple way.

Truncation of beliefs in a similar way to TQRE (albeit relative to beliefs about the

[9]Kahneman and Tversky (1973) first studied overconfidence, and much work has followed, e.g. Camerer and Lovallo (1999) and Santos-Pinto and Sobel (2005).

distribution of a much different parameter) is the central feature of the cognitive hierarchy (CH) model of Camerer, Ho and Chong (2004). Players who do $k \geq 1$ steps of thinking form truncated beliefs about the fraction of $h-$step types according to $g_k(h) = f(h)/\sum_{n=0}^{k-1} f(n)$ for all $h < k$ and $g_k(h) = 0$ for all $h \geq k$. In this specification, players do not imagine that any others are at their level (or higher), so, in the notation of the TQRE, they effectively have $\theta < 1$. All positive-step thinkers best respond given their beliefs, so in a two-player game,[10] $p_{ij}(k) = 1$ iff $a_{ij} = argmax_a \sum_{h=0}^{k-1} g_k(h) \sum_{m=1}^{J_{-i}} p_{im}(h) u_i(a, a_{-im}))$.[11] The expected choice probabilities for player i implied by the CH model are given by $p_{ij} = \sum_{k=0}^{\infty} p_{ij}(k) f(k)$.

Camerer, Ho and Chong (2004) assume $f(k)$ is Poisson and estimate the mean of the distribution using data from more than 100 normal-form games. Other types of hierarchical models have been explored as well. Nagel (1995) and Stahl and Wilson (1995) were the first to use strategic hierarchies to study dominance-solvable "beauty contest" games and matrix games, respectively. In Nagel's approach $k-$step players think all others do $k-1$ steps of reasoning (i.e., $g_k(h) = I(h, k-1)$ where $I(x, y)$ is an identity function equalling one if $x = y$ and zero otherwise). Stahl and Wilson's limited-step types have the same one-step-below beliefs as in Nagel, but they also permit equilibrium types and "worldly" types who maximize against the empirical distribution of play. Players in these models are typically modeled as using quantal responses instead of best responses.[12]

[10]The expressions are more cumbersome to write out with n-player games because the probabilities of other players' types have a multinomial distribution with many terms. Roughly speaking, CH models become hard to compute as the number of players increases, while QRE models, which require finding a fixed point, become more difficult to compute as the number of strategies increases.

[11]If more than one action is a best response they are assumed to randomize equally across all best responses.

[12]Recent applications of this approach include Costa-Gomes and Crawford (2006), Crawford and Iriberri (2007b), and Crawford and Iriberri (2007a). The approach is also used to analyze Swedish lottery and experimental data by Ostling et al (2007) and box office reviews of unreviewed movies by Brown, Lovallo and Camerer (2007)

4.2.2 Differences and Similarities between CH and TQRE

The general form of TQRE is different from CH in three distinct ways. First, the maximum “imagined” type of other players could be equal to, greater than, or less than a player’s actual type (depending on θ_i), and this could be a second source of heterogeneity, whereas in all the CH and related approaches the imagination parameter for all players is strictly less than 1.[13] Second, levels of rationality are indexed by λ in TQRE, rather than k, so that types correspond to increasing payoff responsiveness rather than strategic sophistication. Third, in TQRE, all types exhibit some degree of randomness in response, reflecting the stochastic choice modeling. In CH all players with $k \geq 1$ best-respond, so the only source of stochastic choice behavior is buried in the $0-$level types.

In spite of these major differences between the two models, there are a number of important similarities between the TQRE and CH approaches. First, a central feature of both models is heterogeneity in types. Second, both models incorporate stochastic behavior. Third, they share an important type in common: the bottom of the food chain ($k = 0$ or $\lambda = 0$); and these lowest types are in the support of the beliefs of *all* (other) types. Fourth, both models assume there is a limit to the rationality of the other players, and this limit is monotonically increasing in type. Fifth, in both approaches, there is heterogeneity of beliefs as well as heterogeneity of types, and these are correlated: higher types have more accurate beliefs. These beliefs move in the direction of rational expectations about $f(\lambda)$ (or $f(k)$) as λ (or k) increases.[14] Finally, *all* players are overconfident in the sense that they underestimate the gamesmanship (be it sophistication or responsiveness) of the other players.

[13]The Stahl-Wilson (1995) and Costa-Gomes and Crawford (2006) specifications include other types that do not correspond to levels in the thinking hierarchy. If the maximum imagined type is always less that one’s own type, then the model’s solution can be computed recursively, as in CH. However, if $\theta = 1$, so that players are aware that others share their level of thinking, the model must be solved using fixed point methods.

[14]Full convergence (in λ) to rational expectations would require $\lim_{\lambda \to \infty} \theta_i(\lambda) = \infty$.

4.2.3 The formal connection between TQRE and CH

In this section we show that by placing two restrictions on TQRE, for any CH model there exist distributions of types in TQRE that lead to behavioral predictions that are essentially equivalent to CH. By *essentially* equivalent, we mean two things. First we mean that the the equivalence is in terms of approximations that can be made arbitrarily close; second, the approximating equilibria in TQRE are unique.

To make this approximation, we first consider distributions such that the set of λ values is discrete, $L_\gamma = \{0, \gamma, 2\gamma, ..., k\gamma, ...\}$, with grid size γ. A player of type k, is called a *level* k player, and has response parameter $\lambda = k\gamma$. We fix the distribution over k, so that the probabilities of types are $f = \{f(0), f(1), ...f(k), ...\}$. This is simply an HQRE specification with discrete λ-types.

The first restriction is $\theta_i(\lambda) = \frac{k}{k+1}\lambda$ for all i,$\lambda \in L_\gamma$. That is, players recognize only (and all) lower types, but otherwise have correct beliefs about the distribution. In this version of TQRE, level 0 players randomize uniformly, for any value of γ. Level 1 players quantal respond using $\lambda = \gamma \cdot 1$, assuming all other players are type 0. Level 2 players quantal respond (using $\lambda = \gamma \cdot 2$), assuming all other players are type 0 or type 1, with perceived probabilities $\frac{f(0)}{f(0)+f(1)}$ and $\frac{f(1)}{f(0)+f(1)}$, respectively. Higher-level types are defined analogously.

If we let $\gamma \to \infty$, then *all* $k > 0$ types have unboundedly large values of λ, so their choice behavior approaches best response. By effectively removing the stochastic choice component (except for the random $k = 0$ types), this converges to a generalized version of CH in which the type probabilities have the probabilitiy distribution $\{f(0), f(1), ...f(k), ...\}$. The second parametric assumption is that $f(k)$ follows a Poisson distribution.

The formal connection between TQRE and CH is asymptotic in γ, and we stress that the result applies only in the limit as the grid size becomes large.[15] In particular,

[15]For finite values of γ the model can be viewed alternatively as either a downward looking version of

for almost all games and almost all values of τ, the Poisson distribution parameter, the aggregate choice probabilties implied by the $\gamma - TQRE$ model converge to the aggregate choice probablilities of CH. This equivalence is stated more formally, as follows.

Fix τ. Denote the CH choice probability that level k of player i chooses action j by p^{τ}_{ijk}, and denote the $\gamma - TQRE$ choice probability (and f distributed Poisson with parameter τ) that type $\lambda = \gamma k$ of player i chooses action j by p^{γ}_{ijk}. Denote the expected CH choice probability of player i choosing action j by $\overline{p}^{\tau}_{ij} = \sum_{k=0}^{\infty} p^{\tau}_{ijk} f(k)$ and the expected $\gamma - TQRE$ choice probability of player i choosing action j by $\overline{p}^{\gamma}_{ij} = \sum_{k=0}^{\infty} p^{\gamma}_{ijk} f(k)$. Denote $\Delta^{\tau,\gamma} = \sum_{i=1}^{n} \sum_{j=1}^{J_i} (\overline{p}^{\tau}_{ij} - \overline{p}^{\gamma}_{ij})^2$.

Theorem 3: *Fix τ. For almost all finite games Γ and for any $\varepsilon > 0$, there exists $\overline{\gamma}$ such that $\Delta^{\tau,\gamma} < \varepsilon$ for all $\gamma > \overline{\gamma}$.*

Proof: See appendix.

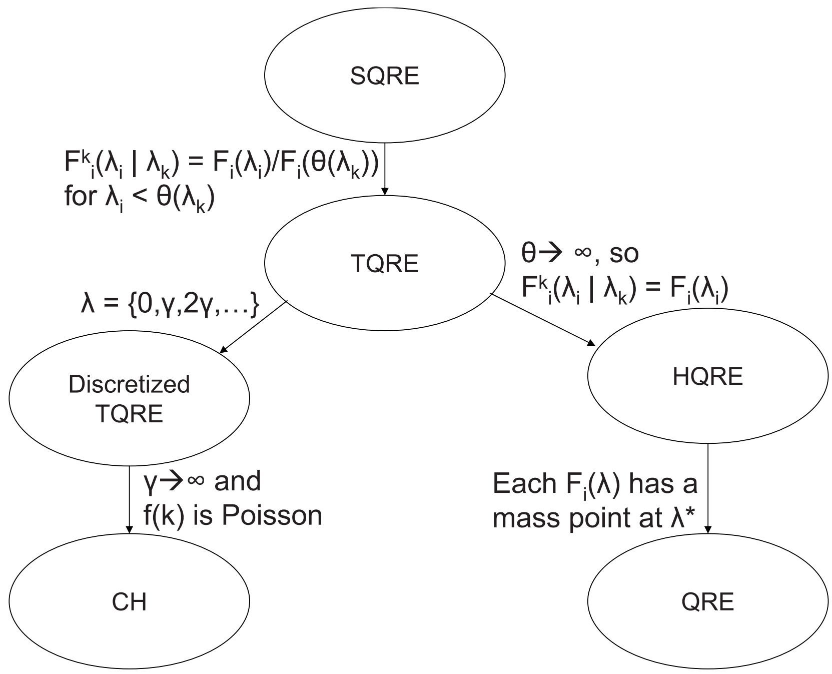

Figure 1: *All of the models considered are special or limiting cases of subjective HQRE. The relationships among the models are depicted in a "family tree."*

SQRE or a quantal response version of CH.

As illustrated in Figure 1, one can identify a "family tree" generated from SQRE by imposing additional restrictions. When all subjectivity takes the form of truncation at a player's $\theta_i(\lambda_i)$, we have TQRE. From TQRE there are two branches to follow. If we send $\theta \to \infty$, then subjectivity vanishes, producing the rational expectations version, HQRE. From there, a limiting distribution that places all mass at one value of lambda corresponds to standard QRE. Following the other branch from TQRE corresponds to discretizing TQRE so that λ takes on a countable set of values, $L_\gamma = \{0, \gamma, 2\gamma, ..., k\gamma, ...\}$. Sending $\gamma \to \infty$, and assuming a Poisson distribution on $F(k)$ then yields the standard CH model.

Another interesting special case of TQRE is when $\theta_i(\lambda_i) = \lambda_i$ and $\gamma \to \infty$. The former restriction means that 1-step ($k = 1$) players are best-responding to a mixture of choices by their own types and some random (0-step, $\lambda = 0$) types. Under these restrictions, in games with strict Nash equilibria, if $F(0)$ is small enough (there are too few random types to induce the QRE types away from the Nash strategies), and $\gamma \to \infty$ (the QRE types best respond), the model is a "noisy Nash" model which has been used in previous applications as a benchmark that illuminates the empirical importance of quantal response behavior.[16]

5 Experimental evidence

5.1 Games and design

We explored the fit of different HQRE and CH models in 17 complete-information normal form games, and one game with information asymmetry (discussed separately in Section 5.3 below). Table V in the Appendix presents the payoff matrices of all 17 games and the relative choice frequencies from our data. The data from the row and

[16]See, for example, McKelvey and Palfrey (1992), El-Gamal, McKelvey and Palfrey (1993), and Fey, McKelvey and Palfrey (1996)

column roles are combined in the symmetric games.

Before describing the games from our experiments, we first highlight some qualitative features of how QRE and CH behave more generally. Even when Nash equilibrium makes a pure strategy prediction, it is nearly always the case with experimental data that many non-Nash equilibrium strategies are played with positive probability. The random component of response, which is the central feature of QRE, can help to describe such data. But more than simply adding a stochastic element to choice, the *equilibrium effects* of noisy response are what make QRE particularly interesting. This is what allows QRE to help explain, for instance, the behavior in asymmetric pennies games in which the row player plays "Up" more frequently when his payoff from ("Up","Left") increases (see, e.g., Ochs (1995) and Goeree, Holt and Palfrey (2003)). Another example is the Traveler's Dilemma, in which each of two players earns the minimum of their two announced strategies from an interval, plus a bonus, $R > 0$, for the lower announcer. Experimental data show a strong effect of lower announcements as R increases (see Capra, Goeree, Gomez and Holt (1999)), while Nash predicts no effect. QRE is consistent with this trend because higher R means that being the higher announcer is more costly, and so those strategies are played less frequently, which implies that the opponent has a further incentive to lower his strategy, and so forth. On the other hand, CH is not an equilibrium model, and does not incorporate random choice (except for level 0s). One case in which CH tends to predict well is in situations where it is natural to think in terms of iterated best response, such as "Beauty Contest" games, in which the goal is to announce nearest to, say, two thirds the average of all players' announcements. One reason to be interested in combining the theories of QRE and CH is to study a model that might benefit from both of these kinds of effects relative to Nash equilibrium.

We now outline the complete information games and highlight some of the aspects of QRE and CH in this setting. One game is an "unprofitable" game (Morgan and

Sefton, 2002) in which maximin strategies do not form an equilibrium, yet guarantee the same payoffs as equilibrium strategies. Twelve games are affine transformations of games created by Stahl and Wilson (1995) (hereafter SW) to fit models of iterated strategic thinking (which are precursors to CH that include more types). We changed some design details about how the games were presented, in part to see how robust the patterns of play were to such details, and to avoid focal points.[17]

These games were chosen because there is a high proportion of Nash play in some of the games in the original SW sample, but the CH model cannot fit those data because the Nash strategy is not reached by iterations of thinking steps with best response (see Camerer, Ho and Chong, 2004). These games are interesting to study since one of our goals is to identify strategic aspects in which some models make better predictions than others. Game 8 from SW is a good example.

S-W 8

	A	B	C	Data	QRE	CH
A	11, 11	59, 91	51, 51	0.17	0.11	0.33
B	91, 59	27, 27	51, 43	0.20	0.25	0.33
C	51, 51	43, 51	53, 53	0.63	0.64	0.33
					$\lambda = 1.05$	$\tau = 0.0$

Table I: *Game 8 from SW, along with empirical choice frequencies and the optimal predictions of QRE and CH.*

Table I shows the payoff matrix of our game based on SW 8. The three columns following the payoff matrix list the empirical choice frequencies and the predictions of

[17]The main difference is the payoff transformation. This was done to eliminate possible focal payoffs, such as 0 and 100, that appear in the original SW games. Instead, our payoffs are scaled so that all entries are two-digit numbers. We also included 3 games that were neither symmetric nor 3x3. Another difference is the matching protocol. We implemented a standard random matching procedure, whereas SW match each choice against the empirical distribution of others' choices. Also, we paid subjects exactly according to the payoff tables instead of using the lottery procedure of SW. Finally, our games were presented sequentially, without the possibility of changing choices in previous games, whereas SW allowed subjects to revise all decisions before submitting their choices.

QRE and CH, based on their fitted parameter values. We find an optimal $\lambda^* = 1.05$, which generates predictions that are close to the observed play. In contrast, the best fitting parameter for CH is $\tau^* = 0.0$, with corresponding uniformly random behavior. That is, no other parameterization of CH fits better than random choice, and the model is not consistent with the relative choice frequencies in our data in this sense. As stated, the reason for this relates to the fact that equilibrium strategies can not be reached through a process of iterated thinking, whereas the empirical choice probabilities show a strong tendency towards equilibrium, as evidenced by the relatively large value of λ that we estimate in QRE. Recall that QRE converges to a Nash equilibrium as λ increases and players become completely payoff responsive.

Four games involve "cloning"—presenting the same pure strategy more than once. These games are included because QRE and CH models can respond differently to the addition of cloned strategies. It is well-known that in stochastic choice models, splitting a single strategy into two equivalent strategies increases the predicted probability of play (the two split strategy frequencies are generally higher than the single strategy frequency) unless some hierarchical structure is imposed.[18] This property can lead to different predictions in QRE and CH approaches, since a cloned strategy does not necessarily receive more weight in CH (except for 0-level players) because players are assumed to best respond, rather than quantally respond, in CH.

One of our games with cloned strategies is asymmetric matching pennies, where "down" is cloned for the row player and "right" is cloned for the column player, creating a 3×3 game. The payoff matrix is given in Table II, along with observed choice

[18]In multinomial logit modeling this property is called the "red bus, blue bus" problem. This term comes from early transportation applications predicting whether commuters would drive or take a bus to work. The choice between {drive,bus} and {drive,red bus, blue bus} can be different if choice is stochastic. For example, if people choose randomly then there is a $\frac{1}{2}$ probability they will take the bus in the first choice set and a $\frac{2}{3}$ probability of taking the bus in the second choice set. A large literature on hierarchical models with nesting has emerged to take care of this problem, by treating the choice between {drive,bus} as a top-level choice and the choice between {red bus, blue bus}, conditional on choosing *bus*, as a second-level choice (where $P(bus) = P(redbus) + P(bluebus)$).

Cloned Matching Pennies						
	L	R	R	Data	QRE	CH
U	50, 10	10, 20	10, 20	0.78	0.51	0.73
D	10, 20	20, 10	20, 10	0.10	0.24	0.14
D	10, 20	20, 10	20, 10	0.12	0.24	0.14
Data	0.71	0.22	0.07			
QRE	0.33	0.34	0.34		λ = 0.12	
CH	0.50	0.25	0.25			τ = 0.90

Table II: *The matching pennies game where "bottom" is cloned for the row player and "right" is cloned for the column player, along with empirical choice frequencies and the optimal predictions of QRE and CH.*

frequencies, and the predictions from QRE and CH calculated at the best fitting parameter values. Notice the reversal in prediction quality of the two models relative to SW 8. QRE consistently overestimates the frequency of cloned strategy play.[19] In addition, the data show too much "up" and "left" play relative to Nash equilibrium, a phenomenon that the CH model does a better job of accounting for, due partly to the fact that these strategies are best responses to the uniform play of level 0's.

Another game with cloned strategies is the "Joker game" of O'Neill (1987), which was originally designed to allow a clean test of minimax play. The payoff matrix is depicted in the lower right of Table V, where the first strategy (the joker card) has been cloned for the row player. Notice that the row player's frequency of the joker strategy is 14%, and the column player's choice frequency of the joker strategy is 38%, both below the Nash equilibrium probability of 40%, which is predicted by QRE. The predicted change from Nash to CH depends on the value of τ. At the pooled maximum likelihood estimate, these probabilities are 23% and 32%, respectively. The empirical frequencies are also lower than what was observed in the original un-cloned O'Neill experiment (36% and 43%, respectively), where both QRE and CH correctly predict the column

[19]In this game, CH also overestimates the amount of cloned play, but it is not significant, and it is a much smaller magnitude than the error of QRE.

player's choice frequency to be *above* the Nash equilibrium level of 40%.

Before proceeding to a more comprehensive analysis of our data, we briefly summarize the design features. The experimental sessions took place at Caltech and UCLA in April, 2004. There were four sessions, each consisting of 25 rounds of the betting game and one shot each of the 17 matrix games, with each ordering of the two parts done twice. The sessions had 16, 18, 20, and 20 subjects each, resulting in a total of 1210 observations in the matrix games[20] as well as 1850 observations of the betting game, discussed later. Subjects consisted of undergraduate students in the two institutions, and were randomly selected to participate in the experiments. Upon arrival, students were seated at random locations in the lab. Once in the lab, instructions were read aloud for everyone to hear, and all subsequent interactions took place only through the computers. During the phase of 17 matrix games, subjects were randomly and anonymously rematched after each decision, and the same procedure was used during the 25 repetitions of the betting game in order to minimize possible repeated game effects. Average payoffs were \$7.50 for the matrix games and \$8.95 for the betting game, resulting in an average total payoff of \$21.45 after including a \$5.00 showup fee. Sessions lasted approximately 2 hours.

5.2 Complete information games

The first focus of our analysis is on estimation of the QRE, HQRE with a uniform distribution of λ_i, Poisson TQRE, and a Poisson CH model for the complete-information games. All four models are estimated separately for each normal-form game, as well as pooling data across games and constraining parameters to be the same for all games.

Table III summarizes the estimation results for the complete-information games. Each column lists the best-fitting parameter value(s) and negative log likelihood for a

[20] Due to technical problems, one session is missing data from games 3, 4 and 17, resulting in a reduction of $3 \cdot 16 = 48$ observations.

particular model. The parameterizations are as follows: QRE has a single λ^* (i.e., HQRE with a single mass point at λ^*); Poisson-CH has a mean number of levels, τ;[21] the distribution of types in HQRE has a uniform distribution over the interval $[\Lambda - \epsilon/2, \Lambda + \epsilon/2]$; TQRE is discretized with grid size γ and Poisson parameter τ (i.e., the probability that $\lambda_i = k\gamma$ is $f(k) = \frac{\tau^k e^{-\tau}}{k!}$, and a level k player has beliefs truncated at $k\gamma$). QRE is nested in HQRE, so we can test for significance of heterogeneity under the maintained hypothesis of rational expectations. CH is nested in TQRE, so we can test for rejection of the constrained model (CH).

We use maximum likelihood to estimate the parameters of the models and assess their qualities of fit. Each model makes a unique statistical prediction in each game as a function of its parameter(s). At the aggregate level, our data consists of group-level choice counts for each strategy of each game. Denote the empirical choice count of strategy j for player i in game g by c_{ijg}, and denote model M's prediction of the frequency at parameter value ρ by $f_{ijg}^M(\rho)$. We can express the log likelihood of model M as a function of ρ by

$$\ln L^M(\rho) = \sum_{g=1}^{17} \sum_{i \in N^g} \sum_{j=1}^{J_i^g} c_{ijg} \ln f_{ijg}^M(\rho). \tag{6}$$

Maximizing $L^M(\rho)$ allows us to estimate the parameter(s) for each model. The results from this exercise appear near the bottom of Tabe 3, in the row marked "Pooled," to indicate that a single parameterization is estimated across all games. In addition, we estimate the parameters separately for each of the 17 games, simply by taking the likelihood function to consist only of the terms corresponding to strategies from a particular game. That is,

$$\ln L^M(\rho, g) = \sum_{i \in N^g} \sum_{j=1}^{J_i^g} c_{ijg} \ln f_{ijg}^M(\rho). \tag{7}$$

[21] In Camerer, Ho and Chong (2002), more flexible distributions with up to six free parameters were estimated. This led to only very slightly better fits than the Poisson distribution.

These results occupy the bulk of the table.

The first two columns of Table III are important for assessing the fits of the models. The "random" log likelihood is the the likelihood that results from a model that assumes every player randomizes uniformly in every game, so that the choice frequencies for player i in game g are simply $1/J_i^g$. This number represents a lower bound on the quality of fit (recall all models include uniformly random behavior as a special case). The "empirical" log likelihood results from a (hindsight) model that assigns to every strategy its empirical frequency, that is, $f_{ijg}^e = \frac{c_{ijg}}{\sum_{j' \leq J_i^g} c_{ij'g}}$. This is the model that results in the best possible fit to the aggregate data.

Matrix Game Estimates

Game	Description	Random neg Log L	Empirical -log L	CH τ	CH -log L	QRE λ	QRE -log L	TQRE γ	TQRE τ	TQRE -log L	HQRE Λ	HQRE ε	HQRE -log L
1	Unprofitable	81.30	73.12	10.19	73.65	0.06	75.97	0.35	3.73	73.12	0.07	0.13	75.96
2	Cloned MP	81.30	52.34	0.90	58.39	0.12	71.72	∞	0.90	58.39	0.12	0	71.72
3	Cloned SH (Low	51.96	46.50	0.44	46.72	0.15	46.74	0.50	0.62	46.67	0.15	0.3	46.73
4	Cloned SH (Hi)	51.96	40.18	1.00	41.91	0.18	41.88	0.89	1.40	40.65	0.19	0.29	41.88
5	SW 1	81.30	39.55	3.10	40.92	0.13	41.51	0.17	2.66	39.93	0.15	0.23	40.51
6	SW 2	81.30	46.00	5.59	46.03	0.13	46.75	∞	5.59	46.03	0.13	0	46.75
7	SW 3	81.30	53.26	0.72	63.26	0.07	73.94	∞	0.72	63.26	0.08	0.16	73.63
8	SW 4	81.30	55.02	1.96	56.11	∞	56.40	∞	1.96	56.11	∞	—	56.40
9	SW 5	81.30	79.27	0.00	81.30	0.14	80.50	0.00	0.00	81.30	0.15	0.3	80.40
10	SW 6	81.30	79.54	1.27	79.69	0.05	80.86	0.11	2.81	79.54	0.05	0.1	80.85
11	SW 7	81.30	73.69	0.67	73.73	∞	73.73	∞	0.67	73.73	∞	—	73.73
12	SW 8	81.30	70.76	0.00	81.30	0.98	73.25	0.02	51.30	78.30	1	1.5	73.23
13	SW 9	81.30	54.30	1.01	60.94	0.08	66.56	∞	1.01	60.94	0.08	0	66.56
14	SW 10	81.30	73.12	13.90	77.35	0.42	73.17	0.15	7.13	73.31	0.44	0.88	73.15
15	SW 11	81.30	73.81	0.83	73.82	0.17	74.55	0.17	0.93	73.81	0.18	0.36	74.52
16	SW 12	81.30	66.81	0.94	67.02	0.03	72.94	0.16	0.87	66.81	0.04	0.07	72.70
17	Cloned Joker	86.88	77.28	0.83	81.08	0.16	83.62	∞	0.82	81.07	0.17	0	83.62
	Sum	1329.00	1054.55	—	1103.21	—	1134.08	—	—	1092.96	—	—	1132.34
	Pooled	—	—	0.53	1239.26	0.10	1192.00	0.11	1.57	1164.41	0.12	0.24	1182.95

Table III: *Game 8 from SW, along with empirical choice frequencies and the optimal predictions of QRE and CH.*

There are several findings implied by the estimates in Table III. First, while the parameter values for each model are reasonably similar across games, all models have one or more games with outlying values. For example, two QRE λ estimates are infinite

(Nash equilibrium)[22] and four CH τ values are either zero (corresponding to random choice) or implausibly high (above 10).

QRE will produce large estimates of λ for games where play is close to Nash equilibrium, and in fact looking at the QRE estimates in Table III gives a way to quantify how well Nash equilibrium performs, if that is a question of interest. Except for the outliers where $\lambda = \infty$, the difference in λ estimates across games is hard to interpret directly, since a given change in λ will produce changes in behavior that depend on the spread of game-specific payoffs. The game-dependent interpretation is different for CH, where τ represents the mean number of iterated steps of thinking carried out in the population when computing best responses. A priori, one might expect the estimates to be more stable across games, if players are indeed "types" that apply the same reasoning to all these games, but this is clearly not the case. This is partly due to the fact that the likelihood surface is often discontinuous in τ, since a small change in the population can induce the best response for some types to simultaneously switch to a different action.

Second, and related to the existence of "outlier" estimates of λ and τ, we reject the constrained model, where the parameters are the same across all 16 games. This can be seen in the table by comparing the likelihoods in the last two rows, marked "Sum" (unconstrained) and "Pooled" (constrained). The decline in fit is substantially worse for CH relative to QRE. The log of the the likelihood ratio versus the random model declines from 199 to 149 for both QRE and HQRE. This decline is much greater for CH, where it falls from 226 to 90. Thus, while CH provides the best fit when game-specific parameters are estimated, it provides the worst fit if the parameters are constrained to be constant across games. The large improvement in fit for CH when allowing τ to be game-specific is puzzling. On the one hand it suggests that differences in levels of reasoning *across games* may be important behaviorally, but on the other hand it

[22]For these two cases, we cannot estimate the Uniform HQRE model. Instead we estimate a two point distribution, with probability q at 0 and $1 - q$ at λ. In both cases we estimate $\widehat{q} = 0$, $\widehat{\lambda} = \infty$.

appears that for some games (such as SW8) it is simply a poor model of behavior.

Third, once rational expectations is assumed, type-heterogeneity appears to be only a minor factor in these games. Comparing QRE and HQRE, we fail to reject the QRE model for *every* game. The separately estimated mean of the uniform distribution in the HQRE models, Λ, are within .02 of the estimated $\widehat{\lambda}$ in the homogeneous QRE model in all games, and the estimated width of the uniform is exactly 0 (no variance) in 4 cases and equal to $\widehat{\Lambda}$ (maximum variance, conditional on $\widehat{\Lambda}$) in 8 cases. However, in none of the games where we estimate positive variance is the fit significantly different from 0 at the $p = 0.05$ level. The bottom line is somewhat different in the pooled estimation, where we can reject homogeneity of λ_i at very high significance ($p < 0.001$). However, even here the improvement in the value of the log likelihood function is quite small (less than one-half of one percent improvement).

A fourth observation, also related to heterogeneity, concerns the TQRE model. While the comparison of QRE and HQRE estimates suggest heterogeneity is a minor factor under the maintained assumption of rational expectations, heterogeneity emerges as an important factor if we assume a structure of "downward looking" beliefs. In 10 of the 17 games, TQRE improves over CH, and in 15 of 17 games TQRE improves over QRE. We also find that in 7 of the games we estimate $\gamma = \infty$. This is interesting because TQRE incorporates both QRE and CH features, and at least in some cases the estimates show that there is no positive effect (as far as likelihoods) to adding a stochastic choice element to the standard CH model. However, for 3 of these 7 games, the homogeneous QRE model fits as well as or better than CH. Overall, our results indicate that a combination of "strategic" heterogeneity and stochastic choice are significant factors in explaining behavior in these games, and a downward looking belief structure helps explain behavior in some of the games.

Fifth, despite their important structural differences, the differences in game-by-game fits of the different models are small in magnitude for most games. The QRE and CH fits differ by five or more likelihood points in only five of 17 games. Not surprisingly, TQRE also fits about as well as both QRE and CH, and slightly better in many cases, since it contains structural elements of both models. Our prior expectation was that the models would be widely separated in many of these games, but they are generally not. The surprise here is not that the models differ, but that they differ relatively little in most of these games. This is in stark contrast to the pooled estimation, where the models based on stochastic choice outperform the CH model quite dramatically. This suggests a robustness of the stochastic choice models that is absent from pure best-reponse models.

Table III shows that in many of the games we studied QRE and CH have surprisingly small differences in their qualities of fit. To see this relationship in another way, consider Figure 2. Each point corresponds to a single strategy from one of the 17 matrix games. The horizontal axis plots empirical choice frequencies for the strategies, and the vertical axis plots the predicted choice frequencies from the models at the pooled parameter estimate. QRE predictions are shown in black and CH in gray. For a perfect fitting model, like the "empirical" model shown in Table III, all points would fall on the 45^0 line, shown in heavy black. Of course, both QRE and CH show substantial deviations from this line. Both models are also "biased" in the direction of under-predicting extreme frequencies. That is, the models put too much weight on strategies that are empirically played the least often, and too little weight on strategies that are played the most often. This can be seen by looking at the solid and dashed lines, which show the best fitting lines to the scatter plots from tho models. Both lines have positive intercept and slope less than unity. The R^2 for the QRE scatter is 0.56, while for CH it is 0.46, so that the simple linear relationship explains more of the variance in predicted frequencies for QRE. Perhaps most interestingly, the fitted lines are almost identical, and can barely be distinguished in the figure (they differ only slightly toward the upper right).

Finally, another regularity in the data, which is discussed in more length in Camerer

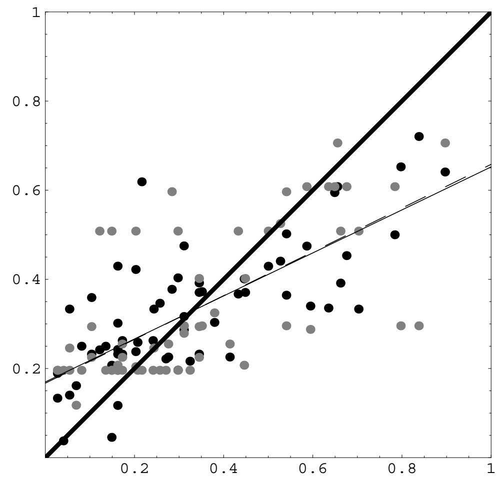

Figure 2: *Each point represents one strategy from one of the complete information games. Empirical choice frequencies are plotted on the horizontal axis, and predicted frequencies from pooled estimates of the models are on the vertical axis. QRE is shown in black, and CH in gray. The heavy black line is the* 45^0*, which corresponds to a perfect fit, and the solid and dashed lines are the fits to the QRE and CH scatters. The fits are almost identical, and in both cases are flatter than the* 45^0*.*

et al. (2006), concerns the "better response" feature commonly associated with the QRE family of models. Besides rational expectations, the essence of the QRE approach (and its HQRE and SQRE relatives) is payoff-responsive stochastic choice.[23] We find that *fitted* choice probabilities of the CH model also tend to exhibit this property empirically although it is not guaranteed (for example, it is violated in SW game 8) because it is not part of the model specification. Given that the data exhibit stochastic payoff responsiveness, one would conjecture that a necessary condition for a behavioral model to fit behavioral data from a specific game is that this monotonicity in choice probabilities must arise for some parameter values. Indeed the CH model satisfes this feature for at least some parameters for almost all the games we studied here, and fit data badly only in games where this was lacking. On the other hand, the parameter values for which this property holds in CH can vary widely across games. This might help explain why our attempt to fit the CH model to the pooled data was so much less

[23]QRE models with this property are called "regular". See Goeree et al. (2005).

successful than the pooled estimation of the models that incorporate quantal response behavior directly.

5.3 The betting game and learning

We also studied a zero-sum betting game with asymmetric information over four states used by Sonsino, Erev and Gilat (2001) and replicated by Sovik (2004).[24] The game is shown in Table IV. Player 1 has two information sets, $\{A, B\}$ and $\{C, D\}$. Player 2 has three information sets, $\{A\}$, $\{B, C\}$, and $\{D\}$. Note that if the state is A or D, player 2 knows the state with certainty. The prior on the states is uniform. Players choose whether to "bet" or "not bet". If both players bet, their payoffs are determined as in the top panel of Table IV. If at least one player opts out, then there is an outside option yielding an expected payoff of 36.

Betting Game		A	B	C	D	
Payoffs	Player 1	67	7	55	19	
	Player 2	3	63	15	51	
Empirical Betting Frequencies	Player 1	28.5% (A–B)		41.2% (C–D)		
	Player 2	5.5%	44.7% (B–C)		96.7%	
CH Betting Frequencies	Player 1	16.3% (A–B)		60.4% (C–D)		τ = 3.09
	Player 2	2.3%	38.0% (B–C)		97.7%	neg Log L = 1073.5
QRE Betting Frequencies	Player 1	24.7% (A–B)		37.1% (C–D)		λ = 0.23
	Player 2	12.9%	46.7% (B–C)		78.7%	neg Log L = 1051.8
TQRE Betting Frequencies	Player 1	19.1% (A–B)		43.4% (C–D)		τ = 2.34 γ=1.1
	Player 2	4.8%	50.9% (B–C)		95.2%	neg Log L = 1024.3
HQRE Betting Frequencies	Player 1	24.7% (A–B)		37.1% (C–D)		Λ=0.23 ε=0
	Player 2	12.9%	46.7% (B–C)		78.7%	neg Log L = 1051.8

Table IV: *The betting game payoffs, empirical betting frequencies, and model estimates.*

[24]Because neither of those papers have been published, we use data from new experiments we conducted.

The Betting Game allows us to move beyond the scope of the games studied above by considering settings with private information.[25] In fact, the informational structure of the game allows it to be solved via iterated steps of eliminating dominated strategies. In this sense the game is one in which the different levels of CH-type agents will have easily-classified decision rules. Moreover, looking ahead to Figure 3, which plots the QRE and CH predictions of betting frequencies at the various information sets, it is clear that this game generates substantial separation in these two theories. In particular, notice that QRE predicts at least as much betting in information set BC as in CD for *all* parameter values, whereas the opposite is true of CH.

This game tests the "Groucho Marx Theorem" (Milgrom and Stokey, 1982)—the idea that privately-informed players should never agree to a zero-sum bet in equilibrium. With these payoffs, player 2 loses by betting on A, and wins by betting on D. As a result, although a CH 1-step risk-neutral player 1 will bet if her information is $\{A, B\}$ (thinking she is equally likely to win 31 or lose 29, relative to the expectation of the outside option), in equilibrium she will never win since a rational player 2 will know the state if it is A, and won't bet. Hence if player 1 guesses that player 2 is rational, she won't bet if her information is $\{A, B\}$ because she deduces that she will never win the 31 and might lose 29. By similar logic, if player 2 is rational, thinks player 1 is rational, and thinks that player 2 thinks she (player 1) is rational, she can deduce that player 1 won't bet if player 1's information is $\{A, B\}$; player 2 therefore will not bet if her information is $\{B, C\}$, since she can only lose by so doing. One more step of iterated reasoning leads player 1 to not bet if her information is $\{C, D\}$. So there will be no state in which *both* players agree to bet, if players are sufficiently confident about rationality of others, and about others' perceptions of rationality.

However, Sonsino et al (2001) and Sovik (2004) find that players *do* bet, even after

[25] Carrillo and Palfrey 2007 analyze a related game with asymmetric information and study the implications of QRE and CH, among other models, for behavior in their setting.

many periods of experience. In most of the Sonsino treatments, however, the marginal incentive is quite low; because they ran many periods, they used a low per-period conversion rate from experimental currency to Israeli Shekels (at stake was roughly 2.4 US cents per observation). In early periods a surprising fraction of player 2's bet when they are sure to lose in $\{A\}$ (around 20%) or don't bet in $\{D\}$ when they are sure to win (around 20% do not bet). This game was therefore included with some design changes to test the robustness of betting to higher incentives and other changes.

The main design change is that players who choose not to bet play a mixed-equilibrium game with expected value of 36 instead. This helped control for possible demand effects favoring betting, and also approximates the psychological value of betting with playing a mixed-equilibrium game in which the outcome is also uncertain.[26]

The second panel of Table IV presents aggregate betting rates in each information set for our data. Notice first that in all non-degenerate information sets, there is a substantial amount of betting, in contrast to the Nash prediction of no betting, reached through iterated deletion of dominated strategies. Second, in the two information sets where Player 2 has a dominant strategy, that strategy is selected about 95% of the time. This contrasts with previous results in which, as mentioned, the error rate in these information sets is closer to 20%. Third, the betting rates tend to increase when going from information set A to AB to BC to CD to D, reflecting the higher levels of sophistication required to eliminate betting in the latter information sets.[27] Given the observed behavior, betting is empirically suboptimal in every information set other than D, where betting is a dominant strategy. Moreover, the observed betting rates are strictly decreasing in the empirical expected cost of betting across information sets. For

[26] Sonsino et al. (2001) also included one treatment in which there was a small fixed payment for not betting, which did not reduce betting rates. A fixed payment treatment is a step in the right direction but does not control for a taste for gambling or risk-preference.

[27] The only exception is the comparison between BC and CD, which is not statistically significant.

instance, we observe the most betting in BC, about 45%, and betting here has an expected cost of less than one point (each point is worth one cent).[28]

The bottom five panels of Table IV show that maximum likelihood estimates for the four models we consider, and the corresponding predicted betting frequencies for each information set.[29] Looking at the estimates for CH and QRE (third and fourth panels, $\widehat{\lambda} = 0.23, \widehat{\tau} = 3.09$) it is clear that their qualitative predictions for the betting game are much different. While both models can account for positive amounts of betting in all information sets, QRE is able to match Player 1's behavior closely, but predicts too much noise for Player 2, in the sense that QRE predicts too much betting in state A and not enough betting in state D. CH, on the other hand, captures behavior of Player 2 at information sets with dominant strategies, but under-predicts betting in AB and over-predicts betting in CD for Player 1. The correspondences for QRE and CH are plotted in Figure 3. It is clear from the figure that QRE is unable to capture the high rate of betting in state D, whereas in CH this frequency converges to 1. However, QRE does a better job of predicting the relative ordering of betting frequencies. In particular, the betting rate in BC is always higher than the betting rate in CD for QRE, a feature that holds in our data. Yet the CD betting rate is higher in CH for all values of τ. This reason that QRE is able to capture this feature of the data is that for low λ, players optimize against a nearly uniform distribution of play, in which case betting in BC has higher expected payoff then betting in CD. As λ increases, the betting rate in BC is determined by the relative betting rates in AB and BC. Since that difference is relatively small, the betting rate in BC remains higher than that for CD, which is characterized by the larger difference in betting rates between BC and D, the latter of

[28]An anomaly in our data is that, by chance, the percentage of times state A was realized was 19.8% which is significantly less than 25%.

[29]The estimation is done representing the Betting Game in behavior strategies. We also estimated the game in mixed strategies where, in all cases, the fit is (weakly) worse. The representation does not affect the predictions of CH because of the best response property, but in all other models, where quantal-response is an element, the representation does make a difference.

which is quite large. TQRE is able to retain this feature since it incorporates QRE-like features, although pure CH cannot since relative betting rates are closely related to orders of thinking in the hierarchy. It is also interesting to note that QRE selects an equilibrium where Player 2 bets with strictly positive probability in all information sets, whereas CH selects the equilibrium corresponding to iterative deletion of dominated strategies.

The best-fitting model in terms of likelihood is TQRE, the model that nests both QRE and CH. Notice that TQRE retains most of the goodness-of-fit to Player 1's behavior from QRE, while allowing the extreme predictions of CH for Player 2's dominant strategies. Both CH and QRE are rejected in favor of the hybrid model. Finally, we estimated the HQRE with a uniform distribution of types, as with the complete information game. The maximum likelihood estimate is $\widehat{\Lambda} = 0.23, \widehat{\epsilon} = 0$, so the constrained (homogeneous λ) QRE model is not rejected.

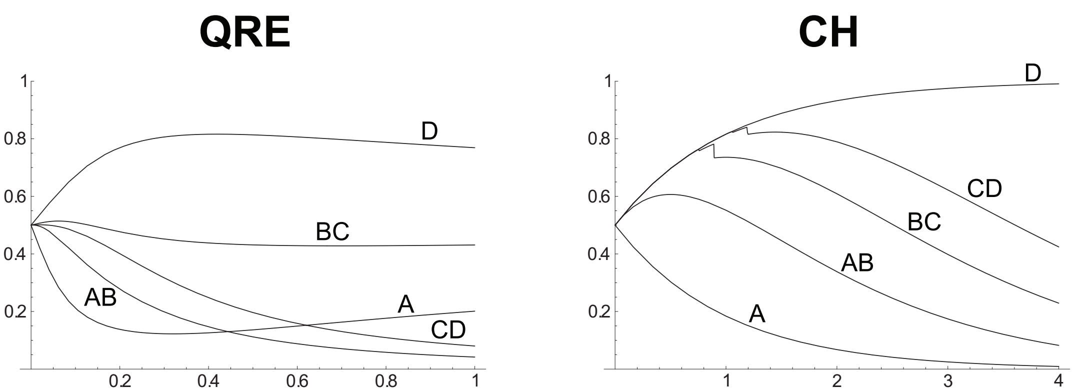

Figure 3: *Correspondences for betting frequencies by information set for QRE (left) and CH (right).*

One advantage of running the Betting Game in 25 repeated rounds per session is that we are able to look at changes in behavior over time (as do Sonsino et al (2001) and Sovik (2004)). Figure 4 shows betting rates across time for both players in a four-period moving average. As with the Sonsino et al. (2001) and Sovik (2004) results, our data show that betting is common and is slow to be extinguished by learning. However, our

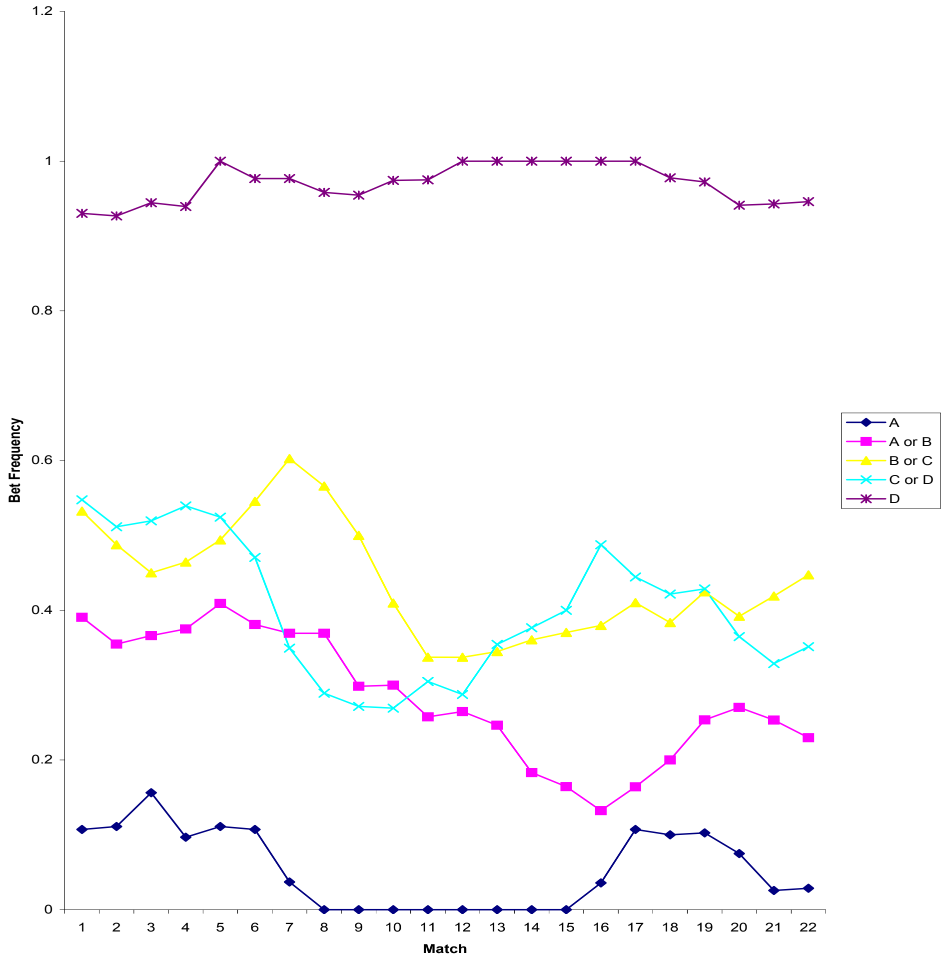

Figure 4: *Betting percentages for the different information sets of the two roles in the betting game. Each point represents a 4-period moving average.*

initial betting rates are significantly lower than Sonsino et al (2001), showing more levels of sophisticated reasoning, a difference due perhaps to our attempt to balance the design. We fit versions of QRE and CH in which the parameters λ and τ drift up over time, as a reduced-form way of characterizing learning, since greater parameter values correspond to play that is closer to Nash equilibrium. In QRE we estimate an initial $\lambda^0 = 0.55$ with a time trend of 0.012, which results in a negative log likelihood of 1083.5, an improvement of less than two points over the fixed λ model. Allowing for the time trend generates a larger improvement in CH. We estimate $\widehat{\tau}^0 = 2.5$ with a time trend of 0.017, which has a corresponding negative log likelihood of 1061.6, an improvement of about twelve points. These results reinforce the central conclusion above, that despite their structural difference the QRE and CH share some basic commonalities.

6 Conclusion

The QRE approach combines Nash equilibrium with Luce's (1959) stochastic utility model. Players have rational expectations, but "better-respond"—choosing strategies with higher expected payoffs more often—rather than necessarily best-responding with the highest expected-payoff strategy. The CH approach goes in a different direction; players iterate reasoning in discrete steps with some players doing more steps of iterated reasoning than others. In contrast to QRE, in CH (and related level-k models), players of all (positive) levels always choose best responses given their beliefs, but beliefs are incorrect. Both models have had success explaining deviations from Nash equilibrium in many experimental data sets, and are also generally consistent in those cases where the Nash model fits well (Goeree and Holt, 2001; Camerer, Ho and Chong, 2004).

This paper generalizes QRE by incorporating skill-hetereogeneity: some players better respond better than other players. With rational expectations, this leads to HQRE, existence is established, and it is applied to data. By relaxing rational expectations, and allowing subjective beliefs about the distribution of skill, one generates a wide class of models, referred to here as SQRE, of which CH turns out to be a special case. This provides an intuitive link between two models that were developed from two much different perspectives about behavior – one with stochastic choice, and the other based heterogeneous cognitive limitations.. The link is as follows: if SQRE beliefs are downward looking this leads to truncated SQRE, or TQRE. If skill levels of all but the lowest level type increase without bound in TQRE, this converges to CH.

The identification of these distinct ways to model heterogeneity in games raises deeper questions about how to build heterogeneity into structural models of behavior in games. What lies at the heart of the comparison between QRE and CH is that there are two fundamentally different ways to introduce heterogeneity: first, by assuming that types fully understand the distribution of heterogeneity (rational expectations); second, by letting players be self-centered or otherwise ignorant of the full heterogeneity. The former is HQRE; the latter is some version of TQRE (or CH or level-k). The former are more challenging to analyze from a technical perspective, because they require solving fixed-point problems, while the latter are defined recursively. The former has the property that the classical Nash model is nested, while this is not generally true for the recursive models with subjective beliefs. Therefore, if one wants to introduce heterogeneity, then in principle there is a modeling choice to be made. This paper, by characterizing a parent model (TQRE) than includes both kinds of approaches as special cases, offers the opportunity of settling such questions empirically, rather than being forced to choose one approach or the other. As we find in our analysis of the data, for some games hierarchical thinking seems to capture the key features of behavior better than rational expectations, and the opposite for some other games. Introducing heterogeneity while assuming rational expectations does not significantly change the coefficients for specific games, but leads to a significantly better overall fit of data pooled across all games.

Two challenges for future research are to identify additional games where the models make sharply different predictions, and to explore further why their predictions are often so similar. The findings from this paper offer some insights about how to go about constructing such games. Another important area for research is extension of these ideas to extensive form games. QRE has been applied successfully to extensive form games, typically in an "agent normal form" in which the response function at each information set is controlled by a separate agent (e.g., McKelvey and Palfrey, 1998). Extending hierarchical or recursive approaches to extensive form games is less straightforward, because there is some ambiguity about the correct way to construct the hierarchy.

While the direct extension of CH to the extensive form is simply to assume level zero players are completely random, level ones best respond to this, etc., an alternative approach is to model agents low in the hierarchy as also acting more myopically than higher types (Camerer, et al. 2002b). Linking hierarchical reasoning as in TQRE with differences in look-ahead could generate valuable insights and provide a disciplined way to think about "chain store paradox" -type anomalies in which players do not appear to use backward induction, even in relatively simple games (e.g., Johnson et al, 2002).

Finally, the preliminary understanding of these models derived by comparing them on experimental data is just meant to get a sense of where the models fit well and badly, and to see which restrictions are most plausible. The eventual hope is that these behavioral theories can be applied to the economic analysis of naturally-occurring games, just as conventional equilibrium concepts are now applied. The finding that these models can reliably explain behavioral departures from Nash equilibrium in one-shot games offers some promise that they can be useful in explaining anomalies in more complex games and field data. Recent examples are Crawford and Iriberri (2007b), which applies level-k models to auctions, Östling et al's (2007) study of lowest-number lotteries, and Brown, Lovallo and Camerer's (2007) study of reactions to undisclosed qualities of movies. Models like these can also be used for economic design. Proposed institutions in which behavior predicted by models of limited rationality is far from what the designer intends (even if equilibrium behavior is ideal) might be bad designs in practice. Thus, applying these models is one approach to study in a disciplined way about the robustness of mechanisms to mistakes.

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7 Appendix

7.1 Proof of existence of HQRE

Theorem 1: *In finite games, a Heterogeneous Logit Equilibrium exists.*

Proof: To define the fixed point mapping, we take a slightly different approach from the standard one. Rather than identify a mapping, the fixed points of which are equilibria, we consider a fixed point in the induced mixed strategies, and then an equilibrium is constructed from the induced mixed strategies using (1). This simplifies the existence theorem because we are finding a fixed point in ΔA, a compact convex subset of $\Re^m$ rather than in a function space.

Let $\alpha \in \Delta A$. We construct the mapping $\Sigma : \Delta A \to \Delta A$ in the following way. Using (5), $U : \Delta A \to \Re^m$ maps α into $U(\alpha)$, where $m = \sum_{i=1}^{n} J^i$. Using (1), for each i, $P_i : \Re^{J^i} \times [0, \infty) \to \Delta A_i$ maps U_i into ΔA_i for each $\lambda_i \in [0, \infty)$. Finally, using (4), for each i, σ_i maps $P_i(U_i(\alpha))$ into ΔA_i by taking expectations over λ_i according to the distribution F_i. We define $\Sigma = \Sigma_1 \times ... \times \Sigma_n$ by the composed mapping $\Sigma_i = \sigma_i \circ P_i \circ U_i \circ \alpha$. To see that this has a fixed point, observe that U_i is a single-valued, bounded continuous function on ΔA. Furthermore, P_i is single valued, continuous and uniformly bounded and hence $\int_0^\infty P_i(\lambda_i; U_i) dF_i(\lambda_i)$ exists for all U_i. Therefore, $\sigma_i(P_i)$ is well defined, and continuous by Lebegue's dominated convergence theorem. Hence Σ is a continuous function from ΔA into itself and has a fixed point $\sigma^* \in \Sigma$. For each i and each $\lambda_i \in [0, \infty)$, let:

$$p^*_{ij}(\lambda_i) = \frac{e^{\lambda_i U_{ij}(\sigma *)}}{\sum_{k=1}^{J_i} e^{\lambda_i U_{ik}(\sigma *)}}.$$

so p^* is a Heterogeneous Logit Equilibrium. *QED*

Theorem 2: *In finite games, a TQRE exists.*

Proof: To define the fixed point mapping, we take a slightly different approach than above, because player i's beliefs about other players strategies depends on λ_i. Rather than finding a fixed point in ΔA, a compact convex subset of $\Re^m$, we find a fixed point in distributional strategies, where a distributional strategy for i, σ_i, is a probability measure on the subsets of $[0, \infty) \times A_i$, the type-action product space, since in our approach $i's$ type is $\lambda_i \in [0, \infty)$. The proof is a straightforward adaptation of Milgrom and Weber (1985). The only two differences are: (1) players have truncated expectations rather than rational expectations; and (2) players quantal respond according to the logit rule instead of best responding.

Payoffs are equicontinuous because each A_i is finite. Because of the truncated distribution of beliefs, the (ex ante) expected payoff to player i is then defined slightly differently from Milgrom and Weber (p. 624), the difference being that the integral with respect to the distribution of *other* player types (λ_{-i}) is truncated at $\theta_i(\lambda_i)$ for each type λ_i. Since our distribution of types is independent and a density function exists for each f_i, and because $\theta_i(\lambda_i)$ varies continously in λ_i, absolute continuity is satisfied, so we can express expected payoffs almost exactly as in Milgrom and Weber (1982, p. 625, expression 3.1), except for the well-behaved dependence of the upper bound for types λ_{-i} on λ_i. Consequently, using the topology of weak convergence for the distributional strategies, strategy sets are convex compact metric spaces and payoff functions are continuous and linear, so a fixed point exists by Glicksberg's theorem (1952). The fact that we are considering quantal responses rather than best responses is of no consequence. It simply means that the fixed-point correspondence is single-valued and continuous rather than being multi-valued and upper hemicontinuous. *QED*

Theorem 3: *Fix τ. For almost all finite games Γ and for any $\varepsilon > 0$, there exists $\bar{\gamma}$ such that $\Delta^{\tau,\gamma} < \varepsilon$ for all $\gamma > \bar{\gamma}$.*

Proof: Fix τ and let Γ^τ denote the set of finite games with the property that in the CH model with parameter τ there is a unique best reply for all levels $k \geq 1$. It is straightforward to show that for each n and for each J, where J is the maximum size of any of the n players' strategy sets, the set of games without these properties has Lebesgue measure 0. Since the countable union of measure 0 sets has measure 0, this implies that Γ^τ consists of almost all finite games, in the generic sense. Let $g \in \Gamma^\tau$. Denote the unique maximizing action of a level k type of player i by a_{ik}^τ, and let δ_k^τ be the smallest difference in expected utility for a level k type of player i between choosing a_{ik}^τ and any other pure strategy. Fix $\varepsilon > 0$ and let L be an integer sufficiently large such that $\sum_{k=L}^{\infty} \frac{\tau^k}{k!} e^{-\tau} < \frac{\varepsilon}{3IJ}$. Denote $\bar{p}_{ijL}^\tau = \sum_{k=0}^{L} p_{ijk}^\tau \frac{\tau^k}{k!} e^{-\tau}$ and $\bar{p}_{ijL}^\gamma = \sum_{k=0}^{L} p_{ijk}^\gamma \frac{\tau^k}{k!} e^{-\tau}$. We immediately obtain that $|\bar{p}_{ijL}^\tau - \bar{p}_{ij}^\tau| < \frac{\varepsilon}{3IJ}$ for all i, j. Hence $\sum_{i=1}^{n} \sum_{j=1}^{J^i} (\bar{p}_{ij}^\tau - \bar{p}_{ijL}^\tau)^2 < \frac{\varepsilon}{3}$.

Simlarly, $\sum_{i=1}^{n}\sum_{j=1}^{J^i}(\bar{p}_{ij}^{\gamma}-\bar{p}_{ijL}^{\gamma})^2<\frac{\varepsilon}{3}$ for any γ.Note that for each i and k, $p_{ijk}^{\tau}=0$ except if j is the index corresponding to action a_{ik}^{τ}. Next, we wish to examine $\bar{p}_{ij}^{\gamma}$, for large γ. First, we show that there exists a number $\bar{\gamma}(L)$ such that for all $\gamma\geq\bar{\gamma}(L)$, a_{ik}^{τ} is the unique maximizing action for *all* types $1\leq k\leq L$ and $p_{ia_{ik}^{\tau}L}^{\gamma}>1-\frac{\varepsilon}{3ILJ}$ for all $k\leq L$. That is, if $\gamma\geq\bar{\gamma}(L)$ then for all types L or lower types of player i, $|\bar{p}_{ijL}^{\tau}-\bar{p}_{ijL}^{\gamma}|<\frac{\varepsilon}{3LJ}$ for all $j\in S_i$. The proof is recursive. It is true for level 1 types because they have the same beliefs about other players that the CH-level-1 players have, and therefore have the same unique maximizing strategy a_1^{τ}. Therefore, by choosing a large enough γ we can make the probability a level 1 type of player i chooses a_{i1}^{τ}, as close to 1 as we wish. In particular, we can find some $\bar{\gamma}(1)$ so that it is greater than $1-\frac{\varepsilon}{3ILJ}$ for all $\gamma\geq\bar{\gamma}(1)$. Level 2 (and higher) types are only slightly more complicated. For the level 2 types, their optimal strategy will be a_2^{τ} as long as the probability level 1's of the players other than i play a_{-i1}^{τ} is sufficiently close to 1. This will be true for all γ greater than some number, call it $\widehat{\gamma}(1)$. Hence, we can find a $\bar{\gamma}(2)$ such that the probability a level 1 type of player i chooses a_{i2}^{τ} is greater than $1-\frac{\varepsilon}{3ILJ}$ for all $\gamma\geq\bar{\gamma}(2)$. Proceeding recursively, we can do the same for level 3 and higher types, and so forth all the way to level L types. By construction, $\sum_{i=1}^{n}\sum_{j=1}^{J^i}(\,\bar{p}_{ijL}^{\tau}-\bar{p}_{ijL}^{\gamma})^2<\frac{\varepsilon}{3}$ for all $\gamma\geq\bar{\gamma}(L)$. Finally, by the triangle inequality:

$$\begin{aligned}
\Delta^{\tau,\gamma} &= \sum_{i=1}^{n}\sum_{j=1}^{J^i}(\bar{p}_{ij}^{\tau}-\bar{p}_{ij}^{\gamma})^2\\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{J^i}(\bar{p}_{ij}^{\tau}-\bar{p}_{ijL}^{\tau})^2+\sum_{i=1}^{n}\sum_{j=1}^{J^i}(\bar{p}_{ijL}^{\tau}-\bar{p}_{ijL}^{\gamma})^2+\sum_{i=1}^{n}\sum_{j=1}^{J^i}(\bar{p}_{ijL}^{\gamma}-\bar{p}_{ij}^{\gamma})\\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\text{ for all }\gamma\geq\bar{\gamma}(L)\\
&< \varepsilon
\end{aligned}$$

QED

Unprofitable

40, 40	60, 10	10, 40	0.33
10, 60	10, 10	60, 40	0.24
40, 10	40, 60	40, 40	0.43

Cloned Matching Pennies

50, 10	10, 20	10, 20	0.78
10, 20	20, 10	20, 10	0.10
10, 20	20, 10	20, 10	0.12
0.71	0.22	0.07	

Cloned Stag Hunt (Low)

21, 21	10, 20	0.24
21, 21	10, 20	0.17
20, 10	20, 20	0.59
0.34	0.66	

Cloned Stag Hunt (Hi)

31, 31	10, 20	0.45
31, 31	10, 20	0.34
20, 10	20, 20	0.21
0.90	0.10	

S-W 1

35, 35	39, 47	95, 40	0.11
47, 39	51, 51	67, 15	0.81
40, 95	15, 67	47, 47	0.07

S-W 2

79, 79	51, 59	55, 59	0.80
59, 51	31, 31	99, 67	0.15
59, 55	67, 99	19, 19	0.06

S-W 3

73, 73	13, 77	49, 93	0.15
77, 13	41, 41	49, 41	0.17
93, 49	41, 49	46, 46	0.69

S-W 4

42, 42	58, 50	98, 46	0.61
50, 58	54, 54	26, 66	0.35
46, 98	66, 26	18, 18	0.04

S-W 5

21, 21	93, 13	45, 29	0.33
13, 93	69, 69	53, 53	0.28
29, 45	53, 53	61, 61	0.39

S-W 6

35, 35	39, 63	95, 91	0.44
63, 39	40, 40	56, 39	0.26
91, 95	39, 56	15, 15	0.30

S-W 7

37, 37	93, 45	53, 53	0.59
45, 93	13, 13	85, 73	0.13
53, 53	73, 85	36, 36	0.28

S-W 8

11, 11	59, 91	51, 51	0.17
91, 59	27, 27	51, 43	0.20
51, 51	43, 51	53, 53	0.63

S-W 9

50, 50	98, 44	70, 82	0.65
44, 98	38, 38	70, 18	0.04
82, 70	18, 70	70, 70	0.31

S-W 10

47, 47	51, 44	28, 43	0.52
44, 51	11, 11	43, 91	0.19
43, 28	91, 43	11, 11	0.30

S-W 11

41, 41	97, 45	35, 58	0.50
45, 97	17, 17	53, 57	0.17
58, 35	57, 53	33, 33	0.33

S-W 12

50, 50	30, 36	74, 42	0.24
36, 30	82, 82	18, 98	0.17
42, 74	98, 18	62, 62	0.59

Cloned Joker

30, 10	10, 30	10, 30	10, 30	0.07
30, 10	10, 30	10, 30	10, 30	0.07
10, 30	10, 30	30, 10	30, 10	0.17
10, 30	30, 10	10, 30	30, 10	0.28
10, 30	30, 10	30, 10	10, 30	0.41
0.38	0.34	0.17	0.10	

Table V: *Payoff matrices for the 17 normal form games, along with empirical choice frequencies (the row and column roles are combined in symmetric games).*