On the Welfare Costs of Imperfect Information for Monetary Policy*

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Abstract

This paper considers a framework in which the central bank observes a potentially large set of noisy indicators but is uncertain about the state of the economy. We evaluate the welfare implications of exploiting all available information to assess the state of the economy. We show that it is possible to characterize in a unified state-space representation the equilibrium evolution of all model variables, whether the central bank sets its instrument following an arbitrary policy rule or commits to optimal policy, and whether the central bank has full information about the state, responds naively to observed indicators, or optimally estimates the state of the economy using available indicators. Using a stylized quantitative model, estimated on US data, we show that filtering out the noise in observable series is crucial to conduct policy appropriately, and argue that under current monetary arrangements, a policy that would systematically exploit all available information to assess the state of the economy is likely to result in substantial welfare gains.

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1 Introduction

Considerable research has sought to characterize desirable monetary policy in the context of particular models of the economy. In most cases, it is assumed that the central bank knows perfectly both the true model and the current and past states of the economy. By *state of the economy*, we mean the set of endogenous and exogenous variables needed to fully specify the realization of all current and expected future model variables. In reality, however, central banks do not know with certainty the state of the economy: they continuously wonder if, e.g., productivity is accelerating, inflationary pressures are building up, or employment is about to fall. These monetary policy authorities are forced to act on the basis of their understanding of the economy's functioning and using estimates of the state of the economy, which they can only infer from imperfect observable economic indicators. A central bank that would not understand that such indicators are noisy would likely let its policy instrument respond to noise, thereby introducing undesirable fluctuations in the economy. Studies abstracting from such difficulties may exaggerate the ability of central bank to conduct stabilization policies, and distort, e.g., the welfare evaluation of alternative policies.

There exist literally hundreds of indicators available containing imperfect but useful information about the state of the economy. Presumably, properly exploiting the information from this large set of indicators could mitigate the central banks' information problem. Evidence emerging from a recent strand of the empirical macroeconomic literature suggests that for forecasting and to properly capture the dynamics of the economy, significant gains may be obtained by moving beyond the handful of variables typically used in VAR analysis.[1] But is the information contained in large data sets necessarily important in practice for the optimal conduct of monetary policy? Or would a naive central bank, monitoring a few key indicators, perform equally well? These are open questions.

The goal of this paper is thus to provide an assessment of the importance of the imperfect information that central banks are facing when operating in a data-rich environment, and how important it is to determine the performance of monetary policy. Part of this goal consists of providing an evaluation — the first, as far as we know — of the welfare benefits associated with exploiting information beyond the handful of variables typically considered in the analysis of optimal monetary policy.

The empirical framework that we consider consists of a fully specified DSGE model. Both the central bank and the econometrician know the structure of the economy but they potentially do not observe the state of the economy. Instead what they each observe is a potentially different set of

[1] In a macroeconomic forecasting context, Stock and Watson (1999, 2002) and Forni, Hallin, Lippi and Reichlin (2000) among others find that factors estimated from large data sets of macroeconomic variables lead to considerable improvements over small scale VAR models. Bernanke and Boivin (2003) and Giannone, Reichlin and Sala (2004) show that this large information set appears to matter empirically to properly model monetary policy, and Bernanke, Boivin and Eliasz (2005) argue that inference based on small-scale VARs may be importantly distorted to the extent that it omits relevant information. These empirical models with large data sets remain however largely non-structural. This limits our ability to determine the source of economic fluctuations, to perform counterfactual experiments, or to analyze optimal policy.

economic indicators, providing an imperfect measure of the state of the economy. The discrepancies between the observed indicators and the underlying economic concepts may reflect measurement error, for instance due to the non-exhaustive coverage of a survey, or slight conceptual differences between the model variables and the data used to measure them.[2]

The possibility that the econometrician might not observe some of the theoretical concepts has already been investigated in the literature. Following Sargent (1989), some researchers have recognized explicitly the presence of measurement error in their empirical framework.[3] We extend this approach in two fundamental ways. First, once one acknowledges that the data provides only an imperfect indicator of the concept, it is plausible to think that many data series carry useful additional information. Yet, all existing studies that estimate structural models allowing for measurement errors are, to our knowledge, based on at most a *single* (and sometimes arbitrary), observable time series corresponding to each variable of the model. That is, whether or not one considers measurement error in the model estimation, it is typically assumed that a small number of data series contain *all* available information about concepts of the model such as output and inflation. In this paper, we propose to estimate the state of the economy exploiting the information from a potentially large panel of data series in a systematic fashion. We relax the common assumption that theoretical concepts are properly measured by a *single* data series, and instead treat them as unobserved common factors for which observed data series are merely imperfect *indicators*. We also include information from indicators that potentially have an unknown relationship with the state variables of the model. The resulting empirical framework can be seen as a dynamic factor model where the structure of a DSGE model is assumed to govern the dynamics of the factors. Given each indicator-specific idiosyncrasy, properly exploiting the information from several indicators — rather than from a single one — should help to better separate an estimate of the economic concept (such as employment or inflation) from the indicator-specific "measurement error."[4] This should also provide us with a better estimate of the underlying economic shocks.

The second key extension is that we endow the central bank with a similar informational problem as the econometrician. This is important since it implies that any mistake that the central bank makes in assessing the state of the economy can affect the dynamics of the economy. To characterize the economy's equilibrium for various policies under imperfect information and forward-looking behavior, we build on important advances made in particular by Pearlman, Currie and Levine (1986), Pearlman (1992), Svensson and Woodford (2003, 2004). Pearlman, Currie and Levine

[2] One could of course imagine macroeconomic models to be sufficiently detailed so as to specify a separate role for, e.g., each of the available price indices (such as the GDP deflator, PCE deflator, CPI, core-CPI, and so on). In practice, however, this distinction is rarely made, as there are advantages to analyzing relatively simple models. It follows that researchers often pick a particular price index in a more or less arbitrary way.

[3] See, e.g., Altuğ (1989), McGrattan (1994), Anderson, Hansen, McGrattan and Sargent (1996), McGrattan, Rogerson and Wright (1997), Schorfheide (2000), Fernández-Villaverde and Rubio-Ramírez (2004). Another practical motivation for adding measurement error is to avoid the stochastic singularity problem that arises when there are fewer theoretical shocks than observable series.

[4] In the same spirit, Prescott (1986) used these two indicators to calibrate the labor elasticity of output in his RBC model.

(1986) provide a solution to a class of linear forward-looking models with partial information. Pearlman (1992) uses this solution to characterize optimal policy under discretion and commitment in forward-looking models with partial information, and illustrates his results in a simple example.[5] Svensson and Woodford (2004) generalize the results on the characterization of optimal policy to the case in which there is asymmetric information between the private sector and the central bank. As they make clear, information asymmetry complicates the problem considerably: while the principle of certainty equivalence — according to which the optimal policy response to particular variables would be the same in the cases of full and imperfect information — holds in the case of symmetric information, it holds only for a particular representation of the policy reaction function in the case of asymmetric information. In addition, under asymmetric information (unlike the case of symmetric information), the separation principle fails: the determination of optimal policy responses cannot be separated from the signal extraction problem which aims at estimating the state of the economy.[6] In contrast to these studies, however, we focus on the information content contained a potentially large set of economic indicators.

This empirical framework allows us to estimate consistently the true state of the economy, the state of the economy as perceived by the central bank, and thus the discrepancies between the two. We then ask: What are the welfare consequences of imperfect information on the part of the central bank? Is it worth for the policy authority to invest resources in getting a more accurate assessment of the correct economic conditions or can it perform well by just responding to a small number of observed indicators? To provide an answer to these questions, we characterize the equilibrium of the economy under different assumptions about the behavior of the central bank and the information that it uses to conduct policy. This framework allows in particular to characterize the equilibrium under an arbitrary policy rule, under optimal policy, whether the central bank has full information about the state, responds naively to observed indicators, or optimally estimates the state of the economy using available observable indicators.

As an application, we use a stylized DSGE model based on microeconomic foundations. The model is based on Giannoni and Woodford (2004). It contains certain features which appear necessary to improve the fit of the data, but is sufficiently simple to allow for an analytical characterization of the social welfare function. We then use this model to analyze the welfare implication of different policies, and different information sets available to the central bank. The parameters of the model are calibrated and the estimation of the unobserved state of the economy in a data-rich environment involves a Markov-Chain Monte-Carlo (MCMC) algorithm that deals effectively with the dimensionality problem by working with marginal densities and avoiding gradient methods.

[5] Gerali and Lippi (2003) provide an algorithm to solve such models, using results from Svensson and Woodford (2003).

[6] Aoki (2003, 2006) studies optimal policy in a simple forward-looking model, and compares optimal policy responses in the case of full and partial information. He shows that increased uncertainty about available indicators can lead to smaller policy responses to indicators. Cukierman and Lippi (2005) argue that imperfect information about the US economy's potential output together with a policy that placed relatively little weight on inflation stabilization in the 1970s may explain both the inflation of the 1970s in the US and the price stability in the 1990s.

This paper distinguishes itself from Pearlman (1992) along several dimensions. First, our focus is on the role of information contained in large data sets for the conduct of policy. Second, we perform a quantitative exercise in an estimated model of the US economy to assess the welfare implications of information. Third, instead of assuming, as in Pearlman (1992), that the central bank and the private sector share the same information, we assume asymmetric information between the private sector and the central bank.

Our findings can be summarized as follows. First, we find that by responding naively to observable indicators, the central bank may perform very poorly. Indeed, by responding to indicators which provide an inaccurate assessment of the state of the economy, it introduces additional shocks to the economy which may be very costly in terms of welfare. Filtering out the noise in observable series is thus key to conduct policy appropriately. Second, even if the central bank understands that the available indicators are noisy, substantial welfare gains could be achieved by getting a more accurate estimate of the true state of the economy and reducing the measurement error. Doing so is generally possible by considering a larger amount of observable indicators. This implies that exploiting the information available in large macroeconomic data sets may be very valuable, from a welfare point of view, for the monetary authority to get a more accurate and precise assessment of the state of the economy.

The rest of the paper is structured as follows. Section 2 lays down the formal setup containing a large class of linear(ized) DSGE models, and presents in general terms the equilibrium evolution of all model variables. This setup includes in particular models in which monetary policy is conducted under full information or partial information about the state of the economy. It contains models in which policy is conducted optimally or alternatively follows a simple policy rule. Section 3 describes the estimation of the model using potentially a large number of data series, and discusses the advantages of using a large data set for the model estimation. Section 4 presents an application of this approach, which attempts to quantify the welfare gains from using a large data set in the conduct monetary policy. In that section, we present a stylized quantitative model of the US economy, its implied social welfare function, and estimate it using a large set of macroeconomic indicators. Performing a set of counterfactual exercises, we can then evaluate the welfare implications of adopting alternative policy rules and endowing the central bank with alternative information sets. Section 5 concludes.

2 Monetary policy under imperfect information

We now present formally the general framework which comprises a linearized private sector block and a monetary policy block. This setup includes as a particular case, the case in which all agents have full information, but it allows more generally for imperfect information about the state of the economy, on the part of the central bank.

We assume a certain asymmetry in the information available to the private sector and the central bank. As the general problem of optimal policy under asymmetric information is difficult, we simplify it by assuming that the private sector fully understands the economic conditions surrounding it. While some may find it unrealistic to assume that the central bank has less information than the private sector about the state of the economy, we view this assumption as a metaphor for the fact that in practice private individual and firms solve individual problems for which they may have considerable information, while central banks need to respond to aggregate economic conditions.

The assumption that the private sector has full information also simplifies the derivation of the structural equations characterizing the private sector behavior, and as stressed by Svensson and Woodford (2004, p. 663) a setup with full information for the private sector and imperfect information for the central bank is the "only case in which it is intellectually coherent to assume a common information set for all members of the private sector, so that the model's equations can be expressed in terms of aggregate equations that refer to only a single 'private sector information set', while at the same time, these model equations are treated as structural, and hence invariant under the alternative policies."

Monetary policy can be conducted either optimally, by minimizing some objective function subject to a set of constraints imposed by the private sector behavior and the available information set as in Svensson and Woodford (2004), or by following a given simple policy rule. In all cases considered, we can express the equilibrium evolution of the model's variables in a simple state-space form. The different specifications mentioned involve different variables or different matrices in that state space. We describe the equilibrium in each of these different specifications below.

2.1 Structural model

As in Pearlman (1992) and Svensson and Woodford (2004), the structural equations of the model describing the behavior of the private sector are given by

$$\begin{bmatrix} Z_{t+1} \\ \tilde{E} E_t z_{t+1} \end{bmatrix} = A \begin{bmatrix} Z_t \\ z_t \end{bmatrix} + B i_t + \begin{bmatrix} u_{t+1} \\ 0 \end{bmatrix} \tag{1}$$

where Z_t is a vector of n_Z predetermined variables, z_t is a vector of n_z forward-looking variables, i_t is a vector of the central bank's n_i policy instruments, u_t is a vector of n_Z iid shocks with mean zero and covariance matrix Σ_u, and $A, B, \tilde{E}$ are conformable matrices. Below, we will consider an example of a structural dynamic general equilibrium model based on microeconomic foundations that can be cast in the form (1). Models with additional lags, lagged expectations, or expectations of variables farther in the future can be written as in (1) by expanding the vectors z_t and Z_t appropriately.

A key feature of this system is that it is assumed to hold for any policy followed by the central bank. In particular, all parameters entering these matrices are assumed to be structural in the sense that they are invariant to alternative policy rules and to alternative information sets available to the monetary authority.

We denote by $E_t x_s$ the conditional expectation of any variable x_s in period s, given private-sector information I_t^f in period t, so that $E_t x_s \equiv E\left[x_s | I_t^f\right]$. The private sector is assumed to have full information, except for the realization of future exogenous disturbances and of future endogenous variables. The private sector information set is thus

$$I_t^f = \{Z_s, z_s, i_s, u_s, s \leq t; \Theta\}$$

where Θ is the set all model parameters, including those characterizing the distributions of the exogenous shocks, and the central bank's behavior.[7]

2.2 Optimal monetary policy under imperfect information

In the case that the central bank conducts optimal monetary policy, we assume that it has the quadratic objective function

$$E\left[\sum_{t=0}^{\infty} \beta^t \tau_t' W \tau_t | I_0^{cb}\right] \tag{2}$$

where W is a positive semidefinite weighting matrix, and the vector τ_t of n_τ target variables relates to the model variables according to

$$\tau_t = C \begin{bmatrix} Z_t \\ z_t \end{bmatrix} + C_i i_t, \tag{3}$$

where C, C_i are conformable matrices. Such a quadratic objective function may results from an approximation to the representative agent's expected utility. Alternatively, it may be an ad-hoc objective that policymakers have chosen to pursue. In any case, as shown in Benigno and Woodford (2007), the minimization of such a quadratic objective function subject to linearized equilibrium conditions is not very restrictive, as it yields a correct first-order approximation of the optimal equilibrium in a general class of nonlinear optimal policy problems, under relatively weak regularity conditions.

We assume that the central bank can commit to a plan for the indefinite future, and that it chooses a plan to minimize its objective function (2) subject to the constraints (1) imposed by the private sector behavior, and its information set I_t^{cb}.

In contrast to the private sector, the central bank does not generally observe the variables Z_t, z_t and u_t. Instead, we suppose that the central bank observes the vector of n_X indicators X_t which relate to the model variables according to

$$X_t^{cb} = \Lambda^{cb} \begin{bmatrix} Z_t \\ z_t \end{bmatrix} + v_t \tag{4}$$

[7] Formally, $\Theta \equiv \left\{A, B, C, C_i, \Lambda^{cb}, \tilde{E}, W, \beta, \phi, P, \Sigma_u, \Sigma_v\right\}$ where the remaining matrices will be defined below.

where the vector v_t is supposed to be iid with mean zero and covariance matrix Σ_v.[8] The central bank information set at date t is given by

$$I_t^{cb} = \left\{ X_s^{cb}, i_s, s \leq t; \Theta \right\}.$$

To the extent that the private sector and the central bank have different information sets, their forecasts will generally differ. We let $x_{s|t}$ denote the best estimate of x_s given the central-bank information in period t, so that $x_{s|t} \equiv E\left[x_s | I_t^{cb}\right]$. Note that when $\Lambda = I$ and $v_t = 0$, we obtain the special case in which the central bank observes all variables Z_t, z_t, so that both the central bank and the private sector have full information.

2.2.1 Equilibrium with optimal monetary policy under imperfect information

The full characterization of optimal monetary policy under commitment with imperfect information on the part of the central bank is described in Svensson and Woodford (2004). We review it in Appendix A1, and summarize it here. The equilibrium evolution of all model variables can be expressed in the following state-space form:

$$\begin{bmatrix} i_t \\ \bar{z}_t \end{bmatrix} = DS_t \tag{5}$$

$$S_t = GS_{t-1} + H\varepsilon_t, \tag{6}$$

where i_t is the central bank's vector of instruments, $\bar{z}_t$ is an augmented vector of non-predetermined variables including the central bank's estimate of the nonpredetermined variables, S_t is the vector summarizing the state of the economy, which includes the vector of predetermined variables, a vector of Lagrange multipliers Ξ_{t-1} determined at date $t-1$, as well as their estimate by the central bank. The vector of iid disturbances ε_t contains both innovations to structural shocks and measurement errors. Specifically, we have

$$\bar{z}_t = \begin{bmatrix} z_t \\ z_{t|t} \end{bmatrix}, \quad S_t = \begin{bmatrix} \bar{Z}_t \\ \bar{Z}_{t|t} \end{bmatrix} \equiv \begin{bmatrix} Z_t \\ \Xi_{t-1} \\ Z_{t|t} \\ \Xi_{t-1|t} \end{bmatrix}, \quad \varepsilon_t = \begin{bmatrix} u_t \\ 0 \\ v_t \end{bmatrix}. \tag{7}$$

[8] In the case that the measurement error is serially correlated, we can rewrite the system so as to have serially uncorrelated measurement errors. For instance, when $v_t = \Psi_v v_{t-1} + \varepsilon_t^v$, with iid innovations ε_t^v, we can rewrite (4) as

$$\tilde{X}_t^{cb} = \tilde{\Lambda}^{cb} \begin{bmatrix} Z_t \\ z_t \end{bmatrix} + \varepsilon_t^v$$

where $\tilde{X}_t^{cb} \equiv X_t^{cb} - \Psi_v X_{t-1}^{cb}$, $\tilde{\Lambda}^{cb}$ is a matrix that depends on Λ^{cb} and Ψ_v, and Z_t is constructed so as to include the necessary lagged variables such as z_{t-1}.

The matrices D, G, and H are of the from

$$D = \begin{bmatrix} 0 & \bar{D}_1 \\ \bar{D}_2^{\dagger} & \left(\bar{D}_2 - \bar{D}_2^{\dagger}\right) \\ 0 & \bar{D}_2 \end{bmatrix}, \quad G = \begin{bmatrix} \bar{G}_1^{\dagger} & \left(\bar{G}_1 - \bar{G}_1^{\dagger}\right) \\ \bar{K}\bar{L}\bar{G}_1^{\dagger} & \left(\bar{G}_1 - \bar{K}\bar{L}\bar{G}_1^{\dagger}\right) \end{bmatrix}, \quad H = \begin{bmatrix} I & 0 \\ \bar{K}\bar{L} & \bar{K} \end{bmatrix}. \tag{8}$$

While the submatrices $\bar{D}_1$, $\bar{D}_2$, and $\bar{G}_1$, are independent of the central bank's information set, the matrices $\bar{D}_2^{\dagger}$, $\bar{G}_1^{\dagger}$, $\bar{L}$ depend on a Kalman gain matrix $\bar{K}$ which is affected by the information available to the central bank.

The system (5)–(6) allows us to characterize the response of all variables i_t, z_t, Z_t (and Ξ_{t-1}) as well as the forecasts by the central bank of these variables $z_{t|t}, Z_{t|t}$ (and $\Xi_{t-1|t}$) to all structural shocks u_t and all "measurement error" shocks v_t, for given initial values $Z_0, Z_{0|0}$, and the initial conditions $\Xi_{-1} = \Xi_{-1|0} = 0$.

2.2.2 Equilibrium with optimal monetary policy under full information

In the case that the central bank has the same full information as the private sector (which we obtain by setting $\Lambda = I$ and $v_t = 0$), the central bank's estimates coincide with those of the private sector. As we show in the Appendix A2, we have $z_{t|t} = z_t$, $Z_{t|t} = Z_t$, and $\Xi_{t-1|t} = \Xi_{t-1}$, and the matrix $\bar{K}\bar{L}$ reduces to the identity matrix. It follows that the equilibrium can again be described by a state space of the form (5)–(6), but this time with the reduced vectors

$$\bar{z}_t = z_t, \quad S_t = \begin{bmatrix} Z_t \\ \Xi_{t-1} \end{bmatrix}, \quad \varepsilon_t = u_t. \tag{9}$$

The matrices D, G, H in turn reduce to

$$D = \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \end{bmatrix}, \quad G = \bar{G}_1, \quad H = \begin{bmatrix} I_{n_Z} \\ 0 \end{bmatrix}, \tag{10}$$

where the blocks $\bar{D}_1$, $\bar{D}_2$, and $\bar{G}_1$ are the same as above. The fact that these blocks are invariant to the information available to the central bank will be useful to determine the welfare implications of alternative information sets.

2.3 Monetary policy under an arbitrary rule

While optimal policy refers to a particular type of behavior of the central bank, it is often argued that actual policy may be more accurately described by a different policy rule. Many authors have also studied the welfare implications of arbitrary (generally simple) policy rules to provide guidelines for monetary policy. Such studies have however generally been conducted in the context of models in the absence of uncertainty about the state of the economy. We here characterize the equilibrium resulting from an arbitrary policy, when the central bank faces again imperfect information.

We suppose that the central bank sets its policy instrument at each date according to the rule

$$i_t = \phi p_t \tag{11}$$

where p_t is a vector of variables entering the policy rule. We consider two cases.

2.3.1 Central bank responds naively to observable indicators

In the case that the central bank naively responds to observable indicators, not accounting for the possibility that those may constitute imperfect indicators of the model's variables, p_t is simply a subset of X_t^{cb}:

$$p_t = P_0 X_t^{cb} \tag{12}$$

where P_0 is a matrix that selects the appropriate elements of X_t^{cb}. Note that p_t may include lagged observable variables or lagged values of the policy instrument. Combining (12) with the central bank observation equation (4), we can express the policy variables p_t as a function of the true model variables and the measurement error:

$$p_t = P \begin{bmatrix} Z_t \\ z_t \end{bmatrix} + e_{pt} \tag{13}$$

where $P \equiv P_0 \Lambda^{cb}$, and $e_{pt} = P v_t$.

As we show in the Appendix A4, the resulting equilibrium can again be expressed in the state space form (5)–(6), with

$$\bar{z}_t = \begin{bmatrix} z_t \\ p_t \end{bmatrix}, \quad S_t = \begin{bmatrix} e_{p,t} \\ Z_t \end{bmatrix}, \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{p,t} \\ u_t \end{bmatrix}.$$

Note that S_t includes the combined measurement errors $e_{p,t}$ to which the central bank responds, and the vector ε_t contains innovations to these measurement errors as well as to structural shocks.

2.3.2 Central bank filters observable indicators

In the case that the central bank understands that the observables indicators are noisy, and that it knows the variance of the measurement error, it may respond to its best estimates of the model variables. The policy rule is again of the form (11), but with p_t being given by

$$p_t = P \begin{bmatrix} Z_{t|t} \\ z_{t|t} \end{bmatrix}. \tag{14}$$

Here, the central bank uses its observation equation (4) to determine these estimates. Such a case is not analyzed by Svensson and Woodford (2004), but in the Appendix A5, we show that the equilibrium is once again of the from (5)–(6) with

$$\bar{z}_t = \begin{bmatrix} z_t \\ z_{t|t} \end{bmatrix}, \quad S_t = \begin{bmatrix} Z_t \\ Z_{t|t} \end{bmatrix}, \quad \text{and} \quad \varepsilon_t = \begin{bmatrix} u_t \\ v_t \end{bmatrix},$$

and where the matrices D, G, H have blocks satisfying the same structure as in (8).

2.3.3 Arbitrary rule and full information

Finally, in the case that the central bank has the same full information as the private sector, the equilibrium is given by (5)–(6) with

$$\bar{z}_t = z_t, \quad S_t = Z_t, \quad \text{and} \quad \varepsilon_t = u_t,$$

and where the matrices D, G, H reduce again to expressions of the form (10).[9]

3 Econometrician's problem: Estimating the model parameters and states

As discussed in the previous section, whether the central bank has full or partial information, and whether it conducts policy optimally or following an arbitrary rule, the economy's equilibrium can be expressed in the state-space form (5)–(6), where the vector $\bar{z}_t$ contains non-predetermined variables, and S_t summarizes the state of the economy at date t. The econometrician's problem is then to estimate all model parameters summarized in $A, B, \tilde{E}$, the policy rule ϕ (if any), the variance of structural shocks Σ_u and of central bank's measurement errors, Σ_v, as well as the latent state of the economy, $\{S_t\}_{t=0}^{T}$, and the loading coefficients Λ^{cb}.

In many applications, the system (1) contains identities and Z_t includes redundant variables such as lags of variables in z_t. We will be interested in a subset F_t of the variables in $i_t, \bar{z}_t, S_t$ (all known at date t), which refers only to variables characterizing the economy in period t. The $(n_F \times 1)$ vector F_t will typically include endogenous variables of interest for which there exist observable indicators. Specifically, we define

$$F_t \equiv F \begin{bmatrix} i_t \\ \bar{z}_t \\ S_t \end{bmatrix}$$

where F is a matrix that selects the appropriate elements among all the model's variables which

[9] The derivation is the same as in Appendix A2.

are contained in the vector $[i_t', \bar{z}_t', S_t']'$. Given (5), we can rewrite the variables of interest as a linear combination of the state vector

$$F_t = \Phi S_t, \tag{15}$$

where

$$\Phi \equiv F \begin{bmatrix} D \\ I \end{bmatrix} \tag{16}$$

is entirely determined by the model parameters and the selection of variables in F_t. The evolution of F_t is given by (6) and (15).

3.1 Econometrician's observation equation

In order to estimate the model, we apply the procedure proposed in Boivin and Giannoni (2006). Specifically, we consider a vector X_t of n_X macroeconomic indicators observable by the econometrician. This vector may differ from that considered by the central bank, X_t^{cb}. While the econometrician may in some cases know how to relate the indicators to the model variables, this link may be less clear in other cases. We thus consider two parts to the observation equation which reflect these situations.

3.1.1 Observation equation with specified link

We collect in a $n_{XF} \times 1$ subvector $X_{F,t} = \left[x_{F,t}^1, ..., x_{F,t}^{n_{XF}}\right]'$ the indicators of the variables of interest $F_t = \left[f_t^1, ..., f_t^{n_F}\right]'$, where $n_{XF} \geq n_F$, and assume that the observed indicators relate to the variables of the model according to

$$x_{F,t}^i = \lambda_F^i f_t^j + e_{F,t}^i \tag{17}$$

for $i = 1, ..n_{XF}$, $j = 1, ...n_F$, where for each i, λ_F^i is a coefficient, and $e_{F,t}^i$ denotes a mean-zero indicator-specific component, which may be viewed as representing measurement error or conceptual differences between the theoretical concept f_t^j and the respective indicator $x_{F,t}^i$. We omit throughout a constant to simplify the notation. We assume that these indicator-specific components are potentially serially correlated, but that they are uncorrelated across indicators. The set of equations (17) can be rewritten in matrix form as

$$X_{F,t} = \Lambda_F F_t + e_{F,t}, \tag{18}$$

where $e_{F,t}$ is a $n_{XF} \times 1$ vector of mean-zero indicator-specific and potentially serially correlated components, and Λ_F is an $(n_{XF} \times n_F)$ matrix of coefficients. As each element of $X_{F,t}$ is supposed to be an indicator of one of the elements of F_t, each row of the matrix Λ_F will have at most one nonzero element. However, to the extent that each variable in F_t can be imperfectly measured by many indicators, each column of Λ_F can have many nonzero elements.

The observation equation (18) is appropriate in the case that several observable indicators relate directly to the same variable of interest, and that each of the indicator-specific components is uncorrelated with that of other indicators. For instance, if inflation based on the personal consumption expenditure deflator and the CPI correspond to the same concept of inflation in the model, then one may want to include both indicators in $X_{F,t}$. However, if these indicators refer actually to different concepts, then at least one of them should not be included in $X_{F,t}$. Such an indicator, even though it does not relate directly to any variable in F_t should still depend on the evolution of the state vector S_t.

3.1.2 Observation equation with unspecified link

More generally, to the extent that the theoretical model is true, a potentially very large number of indicators observed — e.g., asset prices, commodity prices, monetary aggregates and so on — should depend on the state vector S_t. Again, it may be useful to consider such indicators in the estimation, as they may be informative about the state of the model economy. To exploit the information provided by such indicators in the model estimation, we assume that the remaining data series of X_t which do not correspond to any particular variable of F_t are collected in a $n_{XS} \times 1$ vector $X_{S,t}$ and are related to the state vector according to

$$X_{S,t} = \Lambda_S S_t + e_{S,t}, \tag{19}$$

where $e_{S,t}$ is a $n_{XS} \times 1$ vector of mean-zero components that are not related to the model's state vector, and Λ_S is an $(n_{XS} \times n_S)$ matrix of coefficients. Equation (19) allows all indicators not associated with any particular variable of the model to potentially provide information about the state vector S_t. We propose to capture the information from the data in $X_{S,t}$ in a non-structural way, letting the weights in Λ_S be determined by the data.

While the weights Λ_F relating the variables of interest to their indicators can be interpreted as structural — i.e., policy invariant — the weights Λ_S relating the state vector to all other indicators do not need to be so.[10] Even though (19) may not be reliable to determine the effects of alternative policies on the variables in $X_{S,t}$, information about these variables can be very useful for the estimation of the state vector and model parameters under *historical* policy. Once the state vector and model parameters are correctly estimated — using the information provided by (19) — counterfactual exercises can legitimately be performed for all variables F_t, S_t, $X_{F,t}$, without using (19) any more.

[10] In fact the weights Λ_S mix the weights that the variables in $X_{S,t}$ would attribute to their theoretical counterpart, with the coefficients that relate these theoretical concepts to the state vector S_t.

3.1.3 Combined observation equation

Combining (18)–(19) and using (15), we obtain the *observation equation*

$$X_t = \Lambda S_t + e_t \tag{20}$$

where the vector X_t is of dimension $n_X = n_{XF} + n_{XS}$, and

$$X_t \equiv \begin{bmatrix} X_{F,t} \\ X_{S,t} \end{bmatrix}, \qquad e_t \equiv \begin{bmatrix} e_{F,t} \\ e_{S,t} \end{bmatrix}, \qquad \Lambda \equiv \begin{bmatrix} \Lambda_F \Phi \\ \Lambda_S \end{bmatrix}. \tag{21}$$

We assume that indicator-specific components $e_{F,t}$ and $e_{S,t}$ are uncorrelated across indicators but serially correlated, so that

$$e_{F,t} = \Psi_F e_{F,t-1} + v_{F,t} \tag{22}$$

$$e_{S,t} = \Psi_S e_{S,t-1} + v_{S,t} \tag{23}$$

where the vectors $v_{F,t}$ and $v_{S,t}$ are assumed to be normally distributed with mean zero and variance Σ_F and Σ_S, respectively, and where the matrices Σ_F, Σ_S and Ψ_F, Ψ_S are assumed to be diagonal.[11]

Our empirical model consists of the transition equation (6) — which is fully determined by the underlying DSGE model —, the selection equation (15), and the observation equation (20)-(23) which relates the model's theoretical concepts to the data. It is important to note that by expanding the vector X_t of indicators we are not facilitating the model's ability to fit the data. To the contrary, given the factor structure, the more indicators we have in X_t, the more we require the state variables to explain the common components in the data series, while at the same time satisfying their law of motion given by (6).

This setup contains as an important special case the measurement error model proposed by Sargent (1989). In the latter model, each variable in F_t corresponds to a unique observable indicator in $X_{F,t}$, so that the observation equation reduces to $X_t = F_t + e_t = \Phi S_t + e_t$. In this case $n_{XS} = 0$, $\Lambda_F = I_{n_F}$, $\Lambda = \Phi$. A further trivial special case is one in which model variables are assumed to be directly measured, so that the observation equation reduces to $X_t = F_t = \Phi S_t$, as in most existing estimations of DSGE models.

The key innovation here is to generalize Sargent (1989)'s framework to the case where the vector

[11] We may allow the vector $e_{S,t}$ to be correlated across indicators, as we may want to include in the vector $X_{S,t}$ indicators that are driven by some common factors which are not included in the model's vector of state variables. This could happen for instance if several indicators included in $X_{S,t}$ are part of a same category of indicators, but that their theoretical counterpart is not fully fleshed out in the model. In this case we would assume that the component of these indicators which is not correlated with the model's state vector has the following factor structure

$$e_{S,t} = \Gamma S_{e,t} + \tilde{e}_{S,t}$$

where $\tilde{e}_{S,t}$ is a $n_{XS} \times 1$ vector of mean-zero indicator-specific (i.e., uncorrelated across indicators) and potentially serially correlated components, and $S_{e,t}$ is a vector of common components in the set of indicators $X_{S,t}$, which are uncorrelated with the model's state vector S_t.

of observables, X_t, may be much larger than the vector F_t of variables in the model, i.e. $n_X >> n_F$, and that their exact relationship, summarized by Λ, may be partially unknown. The interpretation is that this large number of macroeconomic variables are noisy indicators of model concepts and thus share some common sources of fluctuations. This implies an observation equation with a factor structure similar to the one assumed in the recent non-structural empirical literature which uses a large panel of macroeconomic indicators. However, an important difference with this literature is that, in the present framework, the evolution of the unobserved common components obeys the structure of a DSGE model.

3.2 Advantages of large information sets

The main reason to use large information sets in our framework is to obtain more precise estimates of the state of the economy.[12] The following proposition, which is well-known in the literature on empirical factor models, establishes this more formally.

Proposition 1 *Suppose that the true model implies a transition equation of the form*

$$S_t = GS_{t-1} + H\varepsilon_t \tag{24}$$

where S_t is a latent vector of state variables, ε_t is iid, G and H may contain restrictions, and suppose that the data at date t, is contained in a vector X_t of size $n_X \times 1$ that relates to S_t according to

$$X_t = \Lambda S_t + e_t, \tag{25}$$

where e_t is independent of ε_t and S_t, and Λ may also contain restrictions. Then, as the number of indicators in X_t tends to infinity (for a given sample size T), it is possible to obtain estimates of the states $\{S_t\}_{t=0}^T$ that have the properties:
1. The estimate of the state $\hat{S}_t$ converges to the true value S_t: $\lim_{n_X \to +\infty} \hat{S}_t = S_t$;
2. The variance of the estimator of $\hat{S}_t$ converges to 0.
It follows that one can recover the true state vectors $\{S_t\}_{t=0}^T$ in the case that $n_X \to +\infty$.

Bai and Ng (2006) provide a proof of this proposition under relatively weak regularity conditions, generalizing results first derived by Forni, Hallin, Lippi and Reichlin (2000), and Stock and Watson (2002). These authors all derive such results for reduced-form state-space models of the form (24)–(25). However, to the extent that the underlying DSGE model is correct, all cross-equation restrictions that it implies should be satisfied by its state-space representations (24)–(25), and so does not change the conclusions of the proposition.

[12] Boivin and Giannoni (2006) argue that another advantage of using large information sets for the estimation of DSGE models is that the cross-section of macroeconomic indicators allows one to identify a much richer pattern of "measurement errors," even in the presence of many structural shocks. This reduces the risk of biased estimation.

To illustrate these points, consider the following special case of the framework presented above. Suppose that, according to theory, a variable of interest, f_t, satisfies

$$f_t = \rho f_{t-1} + \eta_t, \tag{26}$$

where $|\rho| < 1$ and the exogenous disturbance η_t is iid.[13] Suppose moreover that we observe an indicator x_{1t} of f_t. In the case that x_{1t} constitutes a perfect measure of f_t, i.e., that the observation equation (20) is trivially $x_{1t} = f_t$, the variable of interest f_t is known, and the parameter ρ can easily be estimated by OLS or maximum likelihood. Suppose instead that x_{1t} is a noisy indicator of f_t and that the observation equation takes the form

$$x_{1t} = f_t + e_{1t} \tag{27}$$

where e_{1t} is iid.[14] In the case that $\rho \neq 0$, standard techniques such as proposed Sargent (1989) can be applied to estimate $\hat{f}_t$ and disentangle it from the "measurement error," using the Kalman filter. This requires however the stochastic process of f_t to be different from the one that drives the measurement error. Nonetheless, even if we have access to an inifinitely long time series for x_{1t}, we cannot fully recover the true f_t. Furthermore, when $\rho = 0$, standard techniques cannot be applied to recover the variable of interest f_t, as $x_{1t} = \eta_t + e_{1t}$ is the sum of two variables with the same stochastic process.[15] Instead, if one or more additional indicators

$$x_{it} = f_t + e_{it} \tag{28}$$

for $i = 2, .., n_X$ are available, then it is possible to consistently estimate f_t even if it is serially uncorrelated. In fact, f_t is a common factor that can be identified through the cross section, on the basis the observation equations (27)–(28), while the dynamic model (26) is used for identification of the shocks η_t.

More generally, when no more than one indicator is used for any concept of the model — i.e., when $n_X = n_F$, as in existing implementations — both the structural shocks and the unobserved variables have to be identified entirely from the restricted dynamics of the DSGE model, summarized by equations (6) and (15). In that case, having more structural shocks in the model limits the number of independent sources of measurement errors that can be contemplated and it is difficult to formally test whether the resulting model is properly identified or not. Typically, researchers avoid these problems by assuming either no measurement error or few structural shocks. But as argued in the introduction, measurement error or conceptual differences between the measured indicators and the theoretical variables might be quite prevalent, and if so, ignoring them would

[13] This is a special case of (6) and (15), where $f_t = F_t = S_t$, $\varepsilon_t = \eta_t$, $\Phi = 1$, $G = \rho$ and $H = 1$.

[14] This is a special case of (20) where $X_t = x_{1t}$, $\Lambda_F = 1$, $\Lambda = \Phi = 1$, and $e_t = e_{1t}$.

[15] The likelihood function in this case involves the sum of the variances of η_t and e_{1t}, so that each variance cannot be identified separately.

lead to biased inference.

In contrast, one key feature of factor models with multiple indicators is that the factors can be identified by the cross-section of macroeconomic indicators alone. This implies that in our framework with a factor structure, the large number ($n_X >> n_F$) of indicators provides enough restrictions to identify the latent variables, and the series-specific terms from the observation equation (20). As a result, we can allow for a large amount of measurement errors without restricting in any way the number of structural shocks that can be identified in the model. Rather than taking a stance on which source of variations should be part of the model, we can remain agnostic and determine empirically their importance.

Even when the factors can be identified solely from the model dynamics, as in Sargent (1989), considering the information from the large data set provides another important advantage, namely efficiency of the factor estimation. A key property of factor models is that the variances of the factor estimates are of order $1/n_X$ where n_X is again the number of indicators in X_t, so that as mentioned in Proposition 1, a consistent estimate of the factors can be obtained by letting $n_X \longrightarrow \infty$.

Equilibrium in the case that the central bank has access to a very large data set

The previous proposition states that in the limiting case that the number of indicators in X_t tends to infinity, an estimation of the above state-space model allows us to recover the true state S_t even if all indicators X_t involve measurement error. In particular, in the case that the central bank conducting monetary policy under imperfect information estimates the states using an infinite number of data series (of a given sample size), it can recover exactly the true state of the economy, S_t. As shown in (5)–(8), that state involves both the true predetermined variables $\bar{Z}_t$ (including lagged Lagrange multipliers) and the central bank's estimate of these predetermined variables $\bar{Z}_{t|t}$. Given that the central bank would know the true predetermined variables $\bar{Z}_t$ in that case, its estimate would satisfy $\bar{Z}_{t|t} = \bar{Z}_t$.

As discussed in section 2.2.2, when the central bank's observation equation has no measurement error ($\Sigma_v = 0$), the economy's equilibrium reduces to the full-information equilibrium.[16] The following proposition establishes that when the central bank observes an infinite number of data series, the equilibrium with optimal policy under partial information is given by the same the full-information equilibrium even when the central bank is confronted with measurement error ($\Sigma_v \neq 0$).

Proposition 2 *In the case that the central bank conducts optimal policy under imperfect information and that it estimates the economy's states using an infinite data set ($n_X \rightarrow +\infty$), the equilibrium is fully characterized by the state space characterizing the optimal equilibrium under*

[16] As shown in Appendix A2, the matrix product $\bar{K}\bar{L} = I$, which implies $Z_t = Z_{t|t}$ and $z_t = z_{t|t}$ for all t. It follows that the state space reduces to the expression (5)-(6) with vectors and matrices given by (9)-(10).

full information

$$\begin{bmatrix} i_t \\ z_t \end{bmatrix} = \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \end{bmatrix} \bar{Z}_t \tag{29}$$

$$\bar{Z}_{t+1} = \bar{G}_1 \bar{Z}_t + \bar{u}_{t+1}, \tag{30}$$

where all the matrices in that state space (i.e., $\bar{D}_1, \bar{D}_2, \bar{G}_1$) depend on the model in the absence of uncertainty and $\bar{\Sigma}_u$ depends only on the structural shocks, even if $\Sigma_v \neq 0$. (None of these matrices depend on the measurement error or the central bank's Kalman filter.) In addition

$$z_{t|t} = z_t, \quad \text{and} \quad \bar{Z}_{t|t} = \bar{Z}_t.$$

Proof. See Appendix A3. ■

As a result, the welfare loss in the case that the central bank conducts optimal policy, assessing the state from an infinite amount of noisy indicators is the same that the one that would obtain if it had perfect certainty about the state of the economy. Of course, our assumption that the central bank can observe an infinite amount of noisy indicators is extreme, as it allows us to resolve completely the uncertainty about the current state of the economy, but it provides a useful benchmark to assess the potential gains of exploiting additional information.

3.3 Estimation procedure

Given the objective of this paper, we restrict our attention to the estimation of the unobserved state of the economy. We thus treat the parameters of the equations (5), (6) and (15), which characterize the equilibrium dynamics of the economy and are contained in the matrices D, G, and H as known.[17] However, the variance of the structural shocks, ε_t, and the parameters of the observation equation, (20)–(23), need to be estimated, together with the state of the economy. We allow X_t to potentially contain a large number of macroeconomic indicators, and impose possibly few *a priori* restrictions on Λ. Doing so obviously comes at a cost. The high-dimensionality of the problem and the presence of unobserved variables considerably increase the computational burden of the estimation. In particular, methods that rely on explicitly maximizing the likelihood function or the posterior distribution appear impractical (see Bernanke, Boivin and Eliasz (2005)).

To circumvent this problem, we consider a variant of a Markov Chain Monte Carlo (MCMC) algorithm.[18] There are two key general features of these simulation-based techniques that help us in the present context. First, rather than working with the likelihood or posterior directly, these methods approximate the likelihood with empirical densities, thus avoiding gradient methods. Second, by exploiting the Clifford-Hammersley theorem, these methods sample iteratively from a

[17] In the application we consider below, these parameters will be estimated independently, using the standard estimation appoach for DSGE models.

[18] See Johannes and Polson (2004) for a survey of these methods and Geweke (1999).

complete set of conditional densities, rather than from the joint density of the parameters and the latent variables. This is particularly useful when the likelihood is not known in closed form, as it is the case in our application. Moreover, by judiciously choosing the break up of the joint likelihood or posterior distribution into the set of conditional densities, the algorithm deals effectively with the high dimensionality of the estimation problem.

In the case that the matrices D, G, and H are assumed to be known, the distribution of all parameters conditional on the states is known in closed-form. So the MCMC algorithm consists in fact of an application of the Gibbs sampling techniques developed by Geman and Geman (1984), Gelman and Rubin (1992), Carter and Kohn (1994) and surveyed in Kim and Nelson (1999). The particular algorithm we used closely mimics the one described in Bernanke, Boivin and Eliasz (2005). It consists of iterating over two steps. First, the factor loadings and the variances are drawn from their known distributions, conditional on the unobserved factors. Second, the unobservable states are drawn using Carter and Kohn (1994) forward-backward algorithm. The precise description of the algorithm is provided in Appendix B.

4 Welfare implications of imperfect information in a simple quantitative model

We now turn to our application, which involves assessing the welfare consequences of imperfect information in the conduct monetary policy. We do so in the context of a simple quantitative model presented in Giannoni and Woodford (2004). This model extends the basic New Keynesian model described in Rotemberg and Woodford (1997), Clarida, Galí, Gertler (1999) or Woodford (2003, chap. 3), by adding certain key features such as wage rigidities, habit formation in consumption and price and wage indexation to lagged inflation which are important to improve the fit of the data. It is possible to characterize the optimal policy problem as one that involves minimization of a loss function obtained as a second-order approximation to the expected utility of the representative household subject to a set of linearized conditions describing the behavior of the private sector. The model is sufficiently stylized for it to yield a simple and intuitive expression for the quadratic objective of the central bank.

Admittedly, the model abstracts from other features that have been argued to be important to describe the data. For instance, in contrast to larger models such as Christiano, Eichenbaum and Evans (2005) or Smets and Wouters (2007), we treat as nondurable consumption all domestic interest-rate sensitive expenditures, including what is commonly labeled as investment. As mentioned in Woodford (2003, chap. 5), to the extent that we are not interested in distinguishing consumption and investment, this should not affect importantly the model's predictions for the other variables.[19] However, properly modelling consumption and investment decisions may be im-

[19] In fact, macroeconomic models that successfully explain the behavior of investment often assume adjustment costs in the rate of investment spending (e.g., Basu and Kimball, 2003; Christiano, Eichenbaum and Evans, 2005).

portant for optimal monetary policy decisions (see, e.g., Edge, 2003). Rotemberg and Woodford (1997), Giannoni and Woodford (2004), or Christiano, Eichenbaum and Evans (2005) assume that consumption and pricing decisions involve at least a one-quarter delay in adjustment, as this improves the fit of the economy's response to monetary shocks. Another line of research has focused on modelling financial intermediaries and their effects on the monetary transmission mechanism (e.g., Christiano, Motto, Rostagno, 2007). We abstract from all of these issues here for simplicity, and leave such analysis for future work.

4.1 Model

The model involves a block describing the behavior of the private sector and one characterizing monetary policy.

4.1.1 Private sector

We now briefly describe the model underlying the behavior of the private sector. Additional details are provided in Giannoni and Woodford (2004), especially in that paper's appendix posted on the authors' webpages. As mentioned above, we assume that the private sector has full information about the current and past realization of all shocks.

We assume that there exists a continuum of households indexed by h and distributed uniformly on the $[0,1]$ interval. Each household h seeks, at date t, to maximize a lifetime expected utility of the form

$$U_t = (1-\beta) E_t \left\{ \sum_{T=t}^{\infty} \beta^{T-t} \left[u\left(C_T^h - \eta C_{T-1}^h\right) - v\left(H_T^h; \xi_T\right)\right]\right\} \tag{31}$$

where E_t denotes again the conditional expectation given private-sector information in period t, $\beta \in (0,1)$ is all households' discount factor, C_t^h is a Dixit and Stiglitz (1977) index of the household's consumption of differentiated goods involving an elasticity of substitution $\theta_p > 1$ between goods, and H_t^h is the amount of labor (of type h) that household h supplies at date t. We assume that each household specializes in the supply of one type of labor. The parameter $0 \leq \eta \leq 1$ represents the degree of internal habit formation. The function $u(\cdot)$ is assumed to be increasing and concave, while $v(\cdot;\xi)$ is increasing and convex for each value of ξ, where the vector ξ_t represents exogenous disturbances to the disutility of labor supply.

Given the assumed consumption index, an optimal allocation of consumption spending across differentiated goods for a given level of overall expenditure at any date t yields the household's conventional demand for the good z: $c_t^h(z) = C_t^h (p_t(z)/P_t)^{-\theta_p}$, where $p_t(z)$ is the price of good

As shown in Woodford (2003), such adjustment costs yield a log-linearized Euler equation for investment that is very similar to the one for consumption in the presence of internal habit formation. It follows that the intertemporal allocation of aggregate expenditures can be approximated by a similar Euler equation, in which the degree of habit formation also serves as a proxy for investment adjustment costs. Nonetheless, in treating investment similarly to non-durable expenditures, we do abstract from the effects of investment on future production capacities.

z and the aggregate price index is defined as $P_t \equiv \left[\int_0^1 p_t(z)^{1-\theta_p} dz\right]^{\frac{1}{1-\theta_p}}$.

We assume that financial markets are complete so that risks are efficiently shared. As a result, all households faces the same intertemporal budget constraint and choose identical state-contingent plans for consumption. The optimal intertemporal allocation of consumption requires

$$u_c(C_t - \eta C_{t-1}) - \beta\eta E_t[u_c(C_{t+1} - \eta C_t)] = \lambda_t \tag{32}$$

where the representative household's marginal utility of income λ_t satisfies

$$\lambda_t = \beta E_t[(1+i_t)\lambda_{t+1}P_t/P_{t+1}], \tag{33}$$

and i_t denotes the riskless one-period nominal interest rate.

We assume in addition that the government purchases a Dixit-Stiglitz aggregate G_t of all goods in the economy which it pays using lump-sum taxes. Aggregate demand thus satisfies the goods market equilibrium condition $Y_t = C_t + G_t$.

We will consider log-linear approximations of these relationships about the steady state equilibrium in which all exogenous disturbances take the value 0 and there is no inflation.[20] Log-linearizing (32), using the goods market equilibrium condition, and combining with a log-linear approximation of (33) yields

$$\tilde{y}_t = E_t\tilde{y}_{t+1} - \varphi^{-1}(\hat{\imath}_t - E_t\pi_{t+1}) + (\tilde{g}_t - E_t\tilde{g}_{t+1}), \tag{34}$$

where $\varphi^{-1} > 0$ reduces, in the case of no habit persistence, to the elasticity of intertemporal substitution, and

$$\tilde{y}_t \equiv (y_t - \eta y_{t-1}) - \beta\eta(E_t y_{t+1} - \eta y_t) \tag{35}$$

$$\tilde{g}_t \equiv (g_t - \eta g_{t-1}) - \beta\eta(E_t g_{t+1} - \eta g_t), \tag{36}$$

and where the variables refer to deviations from the deterministic steady state.[21]

It will be useful, for the welfare analysis below to express the equilibrium conditions in terms of the output gap

$$x_t \equiv y_t - y_t^n, \tag{37}$$

where y_t^n denotes log deviations of the natural rate of output from its steady state. That natural rate of output corresponds to the equilibrium level of output in the case of flexible prices and wages. Combining (34) and (37) yields

$$\tilde{x}_t = E_t\tilde{x}_{t+1} - \varphi^{-1}(\hat{\imath}_t - E_t\pi_{t+1} - r_t^n) \tag{38}$$

[20]We also assume that the government provides subsidies to bring the steady state level of output close to its efficient level.

[21]More specifically, we define $y_t \equiv \log(Y_t/\bar{Y})$, $g_t \equiv (G_t - \bar{G})/\bar{Y}$, $\hat{\imath}_t \equiv \log\left(\frac{1+i_t}{1+\bar{\imath}}\right)$, $\pi_t \equiv \log(P_t/P_{t-1})$, and $\varphi^{-1} \equiv -u_c(1-\beta\eta)/(u_{cc}\bar{Y})$.

where $\tilde{x}_t \equiv \tilde{y}_t - \tilde{y}_t^n$ and the fluctuations in the natural rate of interest r_t^n are given by

$$r_t^n = \varphi E_t \left[\left(\tilde{y}_{t+1}^n - \tilde{y}_t^n \right) - \left(\tilde{g}_{t+1} - \tilde{g}_t \right) \right]. \tag{39}$$

and $\tilde{y}_t^n$ satisfies an equation of the form (35).

On the production side, we assume that there is a single economy-wide labor market. The producers of all goods hire the same kinds of labor and face the same wages. Firm z is the monopolistic supplier of good z, which it produces according to the production function $y_t(z) = A_t f(H_t(z))$, where $f' > 0$, $f'' < 0$, the variable $A_t > 0$ is an exogenous technology factor, and the capital stock is implicitly assumed to be fixed. The labor used to produce each good z is a CES aggregate of all types of labor $H_t^h(z)$, involving a elasticity of substitution $\theta_w > 1$. The demand for labor of type h by firm z satisfies the conventional expression $H_t^h(z) = H_t(z) \left(w_t(h) / W_t \right)^{-\theta_w}$ where $w_t(h)$ is the nominal wage of labor of type h and W_t is a wage index. Each worker is in a situation of monopolistic competition, sets a wage $w_t(h)$, and stands ready to supply the amount of labor demanded at that wage. We assume that each wage is reoptimized with a fixed probability $1 - \alpha_w$ each period, but if it is not reoptimized, it is adjusted according to the indexation rule $\log w_t(h) = \log w_{t-1}(h) + \gamma_w \pi_{t-1}$, for some $0 \leq \gamma_w \leq 1$.

This setup yields, up to a first-order approximation, the following wage inflation equation

$$\pi_t^w - \gamma_w \pi_{t-1} = \xi_w \left(\omega_w x_t + \varphi \tilde{x}_t \right) + \xi_w \left(\omega_t^n - \omega_t \right) + \beta \left(E_t \pi_{t+1}^w - \gamma_w \pi_t \right), \tag{40}$$

where ω_t is the percent deviation of the real wage from its steady state, and ω_t^n is an exogenous variable representing the percent deviations of the "natural real wage," i.e., the equilibrium real wage that would obtain in the case of flexible prices and wages.[22] The parameter $\xi_w > 0$ depends on the degree of wage stickiness, α_w, the elasticity of marginal disutility of labor supply (i.e., the inverse of the Frisch elasticity of the labor supply), $\nu > 0$, and the elasticity of substitution θ_w. The parameter $\omega_w > 0$ measures the degree to which higher economic activity increases workers' desired wages.[23] Integrating equation (40) forward, we observe that nominal wages tend to increase with positive current and expected future output gaps, and when real wages are below the natural real wage. Note that that real wage and wage inflation relate to each other through the identity

$$\pi_t^w = \pi_t + \omega_t - \omega_{t-1}.$$

On the goods' supply side, we assume that firms are in monopolistic competition, that they reoptimize their price with a fixed probability $1 - \alpha_p$ each period, as in Calvo (1983), but if they don't reoptimize, they adjust their price according to the indexation rule $\log p_t(z) = \log p_{t-1}(z) + \gamma_p \pi_{t-1}$ for some $0 \leq \gamma_p \leq 1$. The first-order condition for the optimal pricing decision along with the

[22] One can show that $\omega_t^n \equiv (1 + \omega_p) a_t - \omega_p y_t^n$, where ω_p is defined below.

[23] Specifically, we have $\xi_w \equiv (1 - \alpha_w)(1 - \alpha_w \beta) / (\alpha_w (1 + \nu \theta_w))$, $\nu \equiv v_{hh} \bar{H} / v_h$ and $\omega_w \equiv \nu f / (\bar{H} f')$.

evolution of the aggregate price index yield a linearized aggregate supply equation of the form

$$\pi_t - \gamma_p \pi_{t-1} = \xi_p \omega_p x_t + \xi_p \left(\omega_t - \omega_t^n\right) + \beta \left(E_t \pi_{t+1} - \gamma_p \pi_t\right) \tag{41}$$

where $\omega_p > 0$ measures the degree to which higher economic activity increases producers' prices for given wages, and $\xi_p > 0$ is a function of the degree of price stickiness α_p, the elasticity of substitution θ_p, and ω_p.[24] This New Keynesian supply equation indicates that inflation tends to increase as current and expected future output gaps are positive and as current and future real wages lie above their natural rate.

As shown in the appendix of Giannoni and Woodford (2004), the natural rate of output y_t^n is then implicitly defined by

$$\left(\omega_p + \omega_w\right) y_t^n + \varphi \tilde{y}_t^n = \left(1 + \omega_p + \omega_w\right) a_t + \bar{h}_t + \varphi \tilde{g}_t \tag{42}$$

where $a_t = \log A_t$ denotes log deviations from steady state of total factor productivity and $\bar{h}_t \equiv -\frac{v_{h\xi}}{v_h} \xi_t$ summarizes exogenous disturbances to the disutility of labor supply. We assume that the exogenous shocks a_t, $\bar{h}_t$, and g_t follow AR(1) processes with iid innovations $\varepsilon_t^a, \varepsilon_t^g, \varepsilon_t^h$.

The linearized equations describing the behavior of the private sector can thus be summarized by (34)–(42) together with the exogenous shock processes, and can be cast in the general matrix form (1).

4.1.2 Historical monetary policy

We will be interested in quantifying the effects of alternative monetary policies, and alternative information sets for the central bank. For now, though, we specify an interest-rate rule that is designed to capture historical policy. This specification will be useful to estimate the model parameters, under historical policy. We suppose that the central bank acted naively in responding only to a couple of key observable indicators. Specifically, we assume that monetary policy has been conducted according to a generalized Taylor rule of the form

$$\hat{\imath}_t = \phi_{i1} \hat{\imath}_{t-1} + \phi_{i2} \hat{\imath}_{t-2} + \left(1 - \phi_{i1} - \phi_{i2}\right) \left(\phi_\pi \pi_t^* + \phi_y y_t^* / 4\right) + \varepsilon_t^i \tag{43}$$

where ε_t^i is an iid shock, and where π_t^* and y_t^* denote indicators observable by the central bank (but not necessarily to the econometrician), such as the growth rate of the GDP deflator and real GDP (expressed in deviations from their steady state).[25] These observable indicators are assumed

[24] The coefficients are defined as $\omega_p \equiv -f'' \bar{Y} / \left(f'\right)^2$ and $\xi_p \equiv \left(1 - \alpha_p\right)\left(1 - \alpha_p \beta\right) / \left(\alpha_p \left(1 + \omega_p \theta_p\right)\right)$.

[25] Note that in practice, the official statistics π_t^* and y_t^* are only published after the end of period t. We abstract from this issue for simplicity, though we note that by doing so, we endow the central with more information than it might have had.

to relate to the true concepts π_t and y_t according to

$$\pi_t^* = \pi_t + e_t^\pi \tag{44}$$
$$y_t^* = y_t + e_t^y \tag{45}$$

where e_t^π and e_t^y represents the central bank's measurement error. We will allow e_t^π to follow an AR(1) process but assume that e_t^y is iid to facilitate the identification of the model parameters in the estimation described below.

4.1.3 Equilibrium

The solution to the system (1) describing the behavior of the private sector (obtained from (34)–(42)), the historical policy rule (43) and the evolution of the observable indicators (44)–(45) can be expressed in the state space form (5)–(6) where the vectors of variables are given by

$$\begin{aligned}
\bar{z}_t' &= \left[z_t', p_t'\right] = [\pi_t, \pi_t^w, x_t, \tilde{x}_t, y_t, y_t^n, \tilde{y}_t^n, r_t^n, \omega_t, \omega_t^n, \tilde{g}_t, \pi_t^*, y_t^*] \\
S_t' &= \left[e_t^{p\prime}, Z_t'\right] = \left[e_t^\pi, e_t^y, a_t, g_t, \bar{h}_t, \varepsilon_t^i, \hat{\imath}_{t-1}, \hat{\imath}_{t-2}, \pi_{t-1}, \omega_{t-1}, x_{t-1}, y_{t-1}^n, g_{t-1}\right] \\
\varepsilon_t' &= \left[\varepsilon_t^{p\prime}, u_t'\right] = \left[\varepsilon_t^\pi, \varepsilon_t^y, \varepsilon_t^a, \varepsilon_t^g, \varepsilon_t^h, \varepsilon_t^i\right].
\end{aligned}$$

Note that the central bank's measurement errors e_t^π, e_t^y affect the state of the economy and hence the dynamics of the economy similarly to any other structural shock.

4.2 Estimation

4.2.1 Specification of observation equation

To estimate the state of the economy, $\{S_t\}_{t=0}^T$, for known values of the structural parameters, $A, B, \tilde{E}$ and ϕ, the econometrician writes down an observation equation of the form (20). As we want to use all available information in the estimation, i.e., both data whose link with the model's concepts is well known and unknown, we use the two components of the observation equation (18) and (19).

We first specify the observation equation with known link (18) as follows

$$X_{Ft} = \begin{bmatrix} \frac{\mathrm{FFR}_t}{4} - average \\ (\ln(\mathrm{GDP}_t/\mathrm{Pop}_t) - trend) * 100 \\ (\ln(\text{real compensation}_t) - trend) * 100 \\ \left(\ln\left(P_t^1/P_{t-1}^1\right) - average\right) * 100 \\ \left(\ln\left(P_t^2/P_{t-1}^2\right) - average\right) * 100 \\ \left(\ln\left(P_t^3/P_{t-1}^3\right) - average\right) * 100 \\ \left(\ln\left(P_t^4/P_{t-1}^4\right) - average\right) * 100 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & \lambda_3 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} i_t \\ y_t \\ \omega_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 0 \\ e_t^y \\ e_t^w \\ e_t^{\pi 1} \\ e_t^{\pi 2} \\ e_t^{\pi 3} \\ e_t^{\pi 4} \end{bmatrix}, \tag{46}$$

where $P_t^1, P_t^2, P_t^3, P_t^4$ denote respectively the GDP deflator, the deflator of personal consumption expenditures (PCE), the CPI and the CPI excluding food and energy. The growth rate of these latter indicators are all supposed to be noisy indicators of the underlying concept of inflation. The vector of variables F_t contains the policy instrument, i_t, which is assumed to observed perfectly, and the true concepts of output, the real wage and inflation, which are assumed to be observed with noise. The structure of this observation equation implies that the estimated concept of inflation is designed to capture common fluctuations in all of the inflation indicators considered.

In addition to the observation just described, we use all remaining available data $X_{S,t}$ to help us improve the estimate of the latent state of the economy, assuming that the latent state vector relates linearly to the data according the observation equation (19). The data considered involves 91 quarterly US macroeconomic indicators for the period 1982:1-2002:3, and listed in Appendix C.[26]

Both observation equations can be combined as to yield

$$X_t = \Lambda S_t + e_t$$

where again $X_t = \left[X'_{F,t}, X'_{St}\right]'$ contains our entire data set and the matrix Λ is defined in (21). As a result the estimated latent factors need to explain not only the indicators in X_{Ft} but also the common fluctuations among the indicators contained in X_{St}.

In summary, the econometrician uses this observation equation and the law of motion of all variables as characterized by the state-space solution (5)–(6) to estimate the latent state variables $\{S_t\}_{t=0}^{T}$. Provided that the model is correct, as $n_X \to \infty$, we should recover a "consistent" estimate of all latent variables and all model parameters.

4.2.2 Parameter "estimates": A short-cut

In principle, it would be possible to estimate jointly the state of the economy and the structural parameters using an MCMC algorithm.[27] However, we want to focus here on the role of the additional information for estimating the unobserved state of the economy, not the parameters of the model. Our strategy thus consists instead of calibrating the structural parameters and then investigating the sensitivity of our conclusions to changes in the values of some key parameters.

We calibrate the structural parameters using the values estimated via standard Bayesian estimation of the model on US data under the hypothesis that historical monetary policy has been conducted according to the rule (43).[28] The values chosen for the model's parameters are listed

[26] To reduce the dimension of the estimation problem, we include in X_{St} 30 principal components extracted from our large set of indicators.

[27] See Boivin and Giannoni (2006).

[28] We estimate the model parameters using US quarterly data on real GDP per capita for y_t^*, the GDP deflator for π_t^*, real hourly compensation for ω_t, and the Federal funds rate for $\hat{\imath}_t$, for the period 1982:1-2008:1. The estimation is performed using Dynare and involves samples of 100,000 draws, where the first 20,000 draws have been neglected. It uses the Metropolis-Hastings algorithm to generate posterior distributions of the parameters. Five Markov chains

in Table 1. The priors, and the confidence sets based on the posterior distributions are listed in Table B1 of the appendix. Based on this estimation, all parameters are found to be statistically significant.

The parameters reveal some noticeable differences with those estimated Giannoni and Woodford (2004), due to differences in the estimation approach, the sample considered and the model specification. Most prominently, the response of the output gap to interest rate movements, φ^{-1}, is lower here. This is in large part due to the fact that our model here does not contain decision lags.[29] The model involves significant degree of habit persistence, η, some indexation to past inflation, although that indexation is larger for wages than for prices. The elasticities ξ_p and ξ_w are combinations of several underlying parameters which are unidentified, but the fairly low value of ξ_p suggests a relatively flat slope of the New Keynesian Phillips curve, which is consistent with nominal price rigidities while the larger value of ξ_ω is consistent with more wage flexibility.

Table 1: Model parameters

"Calibrated" parameters						St. dev. estimated	
Structural parameters		*Historical policy rule*		*Persistence of shocks*		with large data set	
β	0.9900	ϕ_{i1}	0.9124	ρ_a	0.7975	σ_a	1.4995
φ	3.7719	ϕ_{i2}	-0.1012	ρ_g	0.5046	σ_g	0.0227
η	0.7759	ϕ_π	2.0438	ρ_h	0.6444	σ_h	0.9768
γ_p	0.1506	$\phi_y/4$	0.1058	$\rho_{e\pi}$	0.9245	$\sigma_{\varepsilon i}$	0.2589
γ_ω	0.6661					$\sigma_{e\pi}$	0.1880
ξ_p	0.0543					σ_{ey}	0.0222
ξ_ω	0.1923						
ω_p	0.6046						
ω_w	0.6718						

For the welfare analysis below, it will be necessary to calibrate the actual degrees of price and wage rigidities. We assume $\alpha_p = 2/3$, and $\alpha_w = 1/3$, so that the average interval between price and wage reoptimization is respectively of 3 and 1.5 quarters,[30] and assume an elasticity of output with respect to labor input of $\bar{H}f'/f = 3/4$. Such coefficients imply a gross markup of prices over

are computed. The diagnostic tests based on comparing the moments of within and between chains suggest that the Markov chains have converged. The parameter β is calibrated to 0.99.

[29] As reported in many VAR studies, unexpected interest rate changes have typically a very small contemporaneous effect on economic activity, but a larger effect one or two quarters following the shock. Since the coefficient φ^{-1} in Giannoni and Woodford (2004) measures the response of the predictable change in future output gaps due to predictable interest rate changes, it is natural that response be larger in that paper.

[30] Assuming a value of α_w as high as $2/3$ would be inconsistent with the requirement that the elasticity of substitution across labor types $\theta_w > 1$, thereby implying a negative markup on the labor market.

marginal costs of $\theta_p/(\theta_p-1)=1.36$ in the goods market, a gross markup of $\theta_w/(\theta_w-1)=1.09$ in the labor market, and a Frisch elasticity of the labor supply of $\nu^{-1}=1.98$.

The coefficients of the policy rule conform to typical estimates of generalized Taylor rules with a long run response of the (annualized) nominal interest rate to inflation of 2.04, a response to output fluctuations of 0.424, and substantial interest rate inertia.

Finally the structural shocks display substantial persistence, though somewhat less than in typically estimated DSGE models (e.g., Smets and Wouters, 2007). The measurement error in inflation is also persistent ($\rho_{e\pi}$).

4.3 Welfare analysis

With our estimated model in hands, we may perform a welfare analysis. We first describe the social objective function, and then proceed with an assessment of the welfare implications of alternative monetary policies, and alternative central bank information sets.

4.3.1 Social welfare loss function

A convenient benefit of using a structural model based on microeconomic foundations is that it provides us with a natural social welfare function: the expected utility of the representative household. Such a function then constitutes a natural objective for the central bank. As shown in Appendix B.2. of Giannoni and Woodford (2004), a second-order expansion of the expected utility of the representative household (31) at date 0 yields

$$E\left[U_0\right]=-\Omega E\left[\mathcal{L}_0\right]+tip+O\left(||\varepsilon||^3\right) \tag{47}$$

where $\Omega>0$, the welfare loss function is of the form

$$\begin{aligned}\mathcal{L}_0 \quad = \quad & E_0\left\{(1-\beta)\sum_{t=0}^{\infty}\beta^t\left[\lambda_p\left(\pi_t-\gamma_p\pi_{t-1}\right)^2+\lambda_w\left(\pi_t^w-\gamma_w\pi_{t-1}\right)^2+\lambda_x\left(x_t-\delta x_{t-1}-\hat{x}^*\right)^2\right.\right.\\ & \left.\left.+\lambda_i\left(\hat{\imath}_t-\hat{\imath}^*\right)^2\right]\right\}, \end{aligned} \tag{48}$$

where *tip* denotes terms independent of policy, and thus independent of the information available to the central bank, and where $O\left(||\varepsilon||^3\right)$ denotes terms of third order or smaller. While we characterize optimal policy by assuming that the central bank minimizes $\mathcal{L}_0$ conditional on its own information set, we evaluate welfare by taking expectations conditional on the full (private sector) information set.[31] The coefficients $\lambda_p, \lambda_w, \lambda_x$, $\hat{x}^*$ and $0\leq\delta\leq1$ are all positive and functions of the underlying model's parameters.[32] While the coefficient weighting interest-rate variability λ_i is zero

[31] The unconditional expectations in (47) integrate over all possible states of the initial exogenous variables, but assume that all variables dated prior to period 0 are fixed at the initial steady state.

[32] Specifically, $\Omega\equiv\frac{1}{2}\bar{Y}u_c(1-\beta\eta)\zeta>0$, $\zeta\equiv\left(\theta_p\xi_p^{-1}+\theta_w\phi^{-1}\xi_w^{-1}\right)>0$, $\phi\equiv f/\left(\bar{H}f'\right)>0$, and *tip* includes all terms independent of the policy adopted. The weights entering the loss function are given by $\lambda_p\equiv\theta_p\xi_p^{-1}\zeta^{-1}$, $\lambda_w\equiv$

in the model derived in section 4.1, it is positive in the case that the utility function (31) is extended to include a term involving real monetary balances.[33] As discussed in Woodford (2003, chap. 6) and Benigno and Woodford (2007), the minimization of such a quadratic objective function subject to our model's linearized equilibrium conditions yields a correct first-order approximation of the optimal equilibrium implied by nonlinear optimal policy problems.

The central bank's quadratic objective function involves the minimization of a weighted average of the discounted volatility of price inflation, wage inflation, the output gap, and interest rate. However, given that our model involves automatic indexation to past inflation, it is the deviations of price and wage inflation from a term proportional to past inflation that the central bank should try to minimize. Similarly, given the habit formation in consumption, the central bank should seek to minimize not the volatility of the output gap, but the deviations between the current and a term proportional to the past output gap. The weights in the central bank's loss function implied by our structural parameters are given in Table 2.

These weights suggest that the central bank should give a weight of 1 to a weighted average of price and wage inflation volatilities — with more emphasis given to the fluctuations in price inflation than in nominal wages — and a weight of 0.8 to the term involving the output gap volatility. (This contrasts with the results of Giannoni and Woodford (2004), who found a much smaller weight on the output gap variability; the difference is primarily due to the difference in the values of φ considered). We set the weight on interest rate variability, λ_i to be the same as in Woodford (2003, chap. 6) to account for transactions frictions, and set the target values $\hat{x}^* = \hat{\imath}^* = 0$.

Table 2: Loss function coefficients implied by structural parameters

λ_p	λ_w	$16\lambda_x$	λ_i	δ
0.596	0.404	0.800	0.077	0.501

4.4 Empirical Results

Having determined the structural parameters, those characterizing all shock processes and the weights in the welfare function, we may now compute the welfare loss under alternative assumptions about the type of policy conducted and the information available to the central bank. Table 3 reports the discounted expectation of the welfare losses (48), $\mathrm{E}[\mathcal{L}_0]$, as well as the statistics $V[z]$

$\theta_w \phi^{-1} \xi_w^{-1} \zeta^{-1}$, and $\lambda_x \equiv \vartheta \varphi \zeta^{-1}$. The coefficients δ and ϑ in turn satisfy $\delta = \eta \vartheta^{-1}$ where $\vartheta = \frac{\beta}{2}\left(\chi + \sqrt{\chi^2 - 4\eta^2\beta^{-1}}\right)$ and $\chi \equiv \left[(\omega_p + \omega_w) + \varphi\left(1 + \beta\eta^2\right)\right] / (\beta\varphi)$.

[33] As discussed in Woodford (2003, chap. 6), none of the model's equations are changed in the case that these transaction frictions enter the utility function in an additively separable way, though the approximated welfare loss function would include the term $\lambda_i \left(\hat{\imath}_t - \hat{\imath}^*\right)^2$ with a positive coefficients λ_i. Another motive for having a positive λ_i is to approximate for the nominal interest-rate lower bound.

measuring the discounted volatility

$$V[z] \equiv \left\{ (1-\beta) \sum_{t=0}^{\infty} \beta^t var(z_t) \right\}$$

of any variable z in five different cases.[34] In all cases, we set the monetary policy shock ε_t^i to zero, as our focus is on the comparison of alternative systematic policies rather than alternative path affected by random policy shocks.

In Case 1, we assume that the central bank naively sets its interest rate according to the historical rule (43), systematically responding to fluctuations in its inflation and output measures π_t^* and y_t^*, but not realizing that these measures generally differ from the actual inflation π_t and output y_t according to (44)–(45). By doing so, the central bank lets its instrument responds to noise (e_t^π and e_t^y), which in turn introduces additional disturbances in the economy, as discussed in section 2.3.1. As Table 3 reveals, such a policy causes very important welfare losses in the context of our model. Compared to the Case 3, in which the central bank conducts policy according to the same historical rule, but under full information (so that $\pi_t^* = \pi_t$ and $y_t^* = y_t$), the welfare losses are almost 4 times larger in the case of a naive policy. The volatility of the welfare-relevant measure of inflation, $\mathrm{V}\left[\pi_t - \gamma_p \pi_{t-1}\right]$ reaches 8.21 under a naive policy, compared to 1.85 under full information. The discounted volatility of the annualized inflation rate, $V[\pi]$, reaches 10.81 compared to 2.26 in the case of full information.

Considerable welfare improvements can be obtained, in Case 2, when the central bank realizes that the concepts of inflation and output it responds to (π_t^* and y_t^*) are noisy indicators of the private sector's corresponding variables. We furthermore let the central bank know the *true* standard deviations and persistence of the error terms e_t^π and e_t^y. With that knowledge, the central bank can then set its instrument according to the historical rule (43), but responding instead to its optimal estimates of the true inflation rate $\left(\pi_{t|t}\right)$ and output $\left(y_{t|t}\right)$, given its observation of π_t^* and y_t^*. The only information that the central bank misses in this case, to conduct policy, is the actual realization of the measurement errors e_t^π and e_t^y. By optimally filtering out the noise in the observable series, it manages to reduce importantly the welfare losses, from 7.70 to 2.54.

In our setup, the central bank knows almost everything, except for the realization of the actual measurement error e_t^π and e_t^y. Yet, this lack of information appears responsible for considerable welfare losses. Indeed, not knowing perfectly the true inflation and output accounts for welfare losses of 34% (2.74 instead of 2.05). As mentioned above, by exploiting all available information, the central bank would, in the limit, be able to recover perfectly the true underlying inflation and output. Thus by using all available information it could achieve a reduction of welfare losses of one third. In our model, this reduction in welfare losses would be due to substantial reductions in

[34] As in Woodford (2003, chap. 6), these statistics involve taking the expectation and the variance by integrating over all possible states of the initial exogenous variables (such as $\varepsilon_0^a, \varepsilon_0^g$...), but assuming that all variables dated prior to period 0 (such as π_{-1}...) are fixed at the initial steady state.

the volatility of the welfare-relevant measures of inflation, wage inflation and the output gap, but would require a slight increase in interest-rate volatility.

Cases 4 and 5 repeat the same comparison but in the case that the central bank commits to an optimal policy. Of course, under optimal policy the welfare losses are overall smaller than under an arbitrary policy rule. In Case 4, the central bank observes its indicators of inflation and output π_t^* and y_t^*, knows again the standard deviation and persistence of the respective measurement errors (e_t^π and e_t^y), and thus optimally assesses the state of the economy. Under optimal policy, the welfare gains obtained by exploiting the information from a very large data set are however smaller than under historical policy. In our benchmark model, these welfare gains are of the order of 4%. Optimal policy turns out to provide a greater robustness to imperfect information about the state of the economy than is the case under an arbitrary policy rule.[35]

Table 3: Welfare losses and statistics under alternative policies and information sets

		Welfare relevant statistics					Other statistics			
Case		$\mathrm{E}[\mathcal{L}_0]$	$\mathrm{V}[\pi-\gamma_p\pi_{-1}]$	$\mathrm{V}[\pi^w-\gamma_w\pi_{-1}]$	$\mathrm{V}[x-\delta x_{-1}]$	$\mathrm{V}[i]$	$\mathrm{V}[\pi]$	$\mathrm{V}[\pi^w]$	$\mathrm{V}[y]$	$\mathrm{V}[\pi^*]$
Historical policy										
1	naive	7.70	8.21	4.21	0.85	5.48	10.81	11.74	4.86	3.20
2	simple filtering	2.74	2.40	1.54	0.71	1.63	2.95	2.64	3.59	—
3	full info.	2.05	1.85	0.95	0.53	1.73	2.26	1.60	3.86	2.26
	Case 2/Case 3	1.34	1.30	1.62	1.32	0.94	1.31	1.65	0.93	—
Optimal policy										
4	simple filtering	0.98	0.61	0.85	0.21	1.28	0.71	0.49	6.29	—
5	full info.	0.94	0.58	0.75	0.22	1.45	0.68	0.32	6.32	—
	Case 4/Case 5	1.04	1.04	1.13	0.98	0.88	1.05	1.54	0.99	—

Notes: Statistics $\mathrm{E}[\mathcal{L}_0]$ and $V[z]$ are all expressed in annualized terms.

There are many reasons to believe that the welfare benefits of large information sets reported in this latest example underestimate the true benefit of exploiting large information set. First, actual policy is unlikely to be of the form prescribed by optimal policy, so that the calculations reported in Cases 1, 2, and 3 are probably more relevant for a characterization of the welfare benefits of exploiting information in large data sets, under current monetary arrangements.[36] Second, in the

[35] This is consistent with the finding of Giannoni (2007) according to which optimal policy, which fully exploits the dynamics implied by the private sector behavior, results in welfare outcomes that are robust to misspecifications of the shocks processes and of model parameters.

[36] Giannoni and Woodford (2004) report significant differences between the actual policy and optimal policy as prescribed by an expanded version of this model.

variant of the model considered here, the state of the economy can be assessed fairly precisely by looking only at inflation and output. This is mainly because total factor productivity shocks account for a large fraction of business cycle fluctuations in our estimated model. Since such shocks drive inflation and output in opposite directions while noise in inflation and output series are assumed to be uncorrelated, the central bank can disentangle fairly easily the noise in observed indicators. However extending the model to account for more shocks, such as shocks to desired markups, implies more tradeoffs for the central bank, and complicates the filtering problem. Such extensions should thus generally increase the benefits of exploiting information from large data sets, even when policy is conducted optimally. A discussion of these results pertaining to extensions of the model will be provided in a later version of this paper.

5 Conclusion

This paper considers a framework in which the central bank observes a potentially very large set of noisy indicators but is uncertain about the state of the economy. Such state of the economy effectively summarizes the realization of current and past shocks as well as endogenous variables which aggregate individual decisions of agents in the private sector. It is fundamentally latent for the central bank, yet its assessment plays a key role in the conduct of monetary policy. In this paper, we have evaluated the welfare implications of exploiting all available information to assess the state of the economy.

We have shown that it is possible to characterize in a unified state-space representation the equilibrium evolution of all model variables, whether the central bank sets its instrument following an arbitrary policy rule or commits to optimal policy, and whether the central bank has full information about the state, responds naively to observed indicators, or optimally estimates the state of the economy using available indicators. We have then shown how an econometrician can efficiently exploit all available information for estimation.

Using a stylized quantitative model, estimated on US data, we have shown that by merely responding naively to observable but noisy indicators, the central bank may perform very poorly in terms of welfare. Filtering out the noise in observable series is thus key to conduct policy appropriately. In addition, even if the central bank understands that the available indicators are noisy, we have shown that substantial welfare gains could be achieved by getting a more accurate and precise estimate of the true state of the economy. Under the assumption that the model is correct, exploiting information from a very large data set allows the policy maker to obtain a very precise estimate of the state. Under historical monetary policy in the period starting in 1982 — which we model as a generalized Taylor rule in which the central bank responds to its best estimates of inflation and output — we determine that the welfare losses are about one third larger in the case that the central bank exploits only information from indicators of inflation and output, compared to the losses that would result from fully exploiting all available information.

Under optimal policy, these welfare gains are smaller than under historical policy, as optimal policy provides a greater robustness to imperfect information than is the case under a simple interest-rate rule rule. Nonetheless, under current monetary arrangements, a policy that would systematically exploit all available information to assess the state of the economy is likely to result in substantial welfare gains.

References

[1] Altuğ, Sumru (1989), "Time-to-Build and Aggregate Fluctuations: Some New Evidence," *International Economic Review* 30(4), 889-920.

[2] Anderson, Evan, Lars P. Hansen, Ellen R. McGrattan, and Thomas J. Sargent (1996), "Mechanics of Forming and Estimating Dynamic Linear Economies," in H. Amman, D. Kendrick, and J. Rust, ed. *Handbook of Computational Economics*, North-Holland.

[3] Aoki, Kosuke (2003) "On the Optimal Monetary Policy Response to Noisy Indicators", *Journal of Monetary Economics* 50(3): 501–523.

[4] Aoki, Kosuke (2006) "Optimal Commitment Plan under Noisy Information", *Journal of Economic Dynamics and Control* 30: 81–109.

[5] Bai, Jushan and Serena Ng (2006), "Confidence Intervals for Diffusion Index Forecasts and Inference for Factor-Augmented Regressions," *Econometrica* 74(4): 1133-1150.

[6] Basu, Susanto, and Miles S. Kimball (2003), "Investment Planning; Costs and the Effects of Fiscal and Monetary Policy," manuscript, University of Michigan.

[7] Bauwens, Luc, Michel Lubrano and Jean-Francois Richard (1999), *Bayesian Inference in Dynamic Econometric Models*, Oxford University Press.

[8] Benigno, Pierpaolo, and Michael Woodford (2007), "Linear-Quadratic Approximation of Optimal Policy Problems," manuscript, Columbia University.

[9] Bernanke, Ben, and Jean Boivin (2003), "Monetary Policy in a Data-Rich Environment," *Journal of Monetary Economics* 50(3), 525-546.

[10] Bernanke, Ben, Jean Boivin, and Piotr Eliasz (2005), "Measuring Monetary Policy: A Factor Augmented Vector Autoregressive (FAVAR) Approach,"*Quarterly Journal of Economics* 120(1), 387-422.

[11] Boivin, Jean, and Marc P. Giannoni (2006), "DSGE Models in a Data-Rich Environment," NBER wp 12772.

[12] Calvo, Guillermo A. (1983), "Staggered Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics* 12(3), 383-398.

[13] Carter, Christopher K., and Robert Kohn (1994), "On Gibbs Sampling for State Space Models," *Biometrika* 81(3), 541-53.

[14] Christiano, Lawrence J., Martin Eichenbaum, and Charles Evans (2005), "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy* 113(1), 1-45.

[15] Christiano, Lawrence J., Roberto Motto and Massimo Rostagno (2007), "Financial Factors in Business Cycles," Manuscript, Northwestern University.

[16] Clarida, Richard, Jordi Galí, and Mark Gertler (1999), "The Science of Monetary Policy: A New Keynesian Perspective," *Journal of Economic Literature* 37(4): 1661-1707.

[17] Cukierman, Alex and Francesco Lippi (2005), "Endogenous Monetary Policy with Unobserved Potential Output," *Journal of Economic Dynamics and Control* 29: 1951-1983.

[18] Dixit, Avinash K., and Joseph E. Stiglitz (1977), "Monopolistic Competition and Optimum Product Diversity," *The American Economic Review* 67(3): 297-308.

[19] Edge, Rochelle M. (2003), "A Utility-Based Welfare Criterion in a Model with Endogenous Capital Accumulation," FEDS Working Paper no. 2003-66.

[20] Fernández-Villaverde, Jesús, and Juan F. Rubio-Ramírez (2004), "Estimating Nonlinear Dynamic Equilibrium Economies: A Likelihood Approach," manuscript, University of Pennsylvania.

[21] Forni, Mario, Marc Hallin, Marco Lippi, and Lucrezia Reichlin (2000), "The Generalized Dynamic Factor Model: Identification and Estimation," *Review of Economics and Statistics* 82:4, 540–554.

[22] Gelman, A. and D. B. Rubin (1992), "A Single Sequence from the Gibbs Sampler Gives a False Sense of Security", in J. M. Bernardo, J. O. Berger, A. P. Dawid, and A. F. M. Smith, eds., *Bayesian Statistics*, Oxford: Oxford University Press.

[23] Geman, S. and D. Geman (1984), "Stochastic Relaxation, Gibbs Distributions and the Bayesian Restoration of Images," *IEEE Transactions on Pattern Analysis and Machine Intelligence* 6: 721-41.

[24] Gerali, Andrea and Francesco Lippi (2003), "Optimal Control and Filtering in Linear Forward-Looking Economies: A Toolkit," CEPR discussion paper no. 3706.

[25] Geweke, John F. (1999), "Using Simulation Methods for Bayesian Econometric Models: Inference, Development and Communication", *Econometric Reviews* 18, 1-126.

[26] Giannone, Domenico, Lucrezia Reichlin, and Luca Sala (2004), "Monetary Policy in Real Time," *NBER Macroeconomics Annual*, 161-200.

[27] Giannoni, Marc P. (2007), "Robust Optimal Monetary Policy in a Forward-Looking Model with Parameter and Shock Uncertainty," *Journal of Applied Econometrics* 22: 179-213.

[28] Giannoni, Marc P., and Michael Woodford (2004), "Optimal Inflation Targeting Rules," in Ben S. Bernanke and Michael Woodford (eds.), *The Inflation-Targeting Debate*, p. 93-162, Chicago: University of Chicago Press.

[29] Johannes, Michael, and Nicholas Polson (2004), "MCMC Methods for Continuous-Time Financial Econometrics," manuscript, Columbia University, September.

[30] Kim, Chang-Jin, and Charles R. Nelson (1999), *State-Space Models with Regime Switching*, Cambridge MA: MIT Press.

[31] McGrattan, Ellen R. (1994), "The Macroeconomics Effects of Distortionary Taxes," *Journal of Monetary Economics* 33, 573-600.

[32] McGrattan, Ellen R., Richard Rogerson and Randall Wright (1997), "An Equilibrium Model of the Business Cycle with Household Production and Fiscal Policy," *International Economic Review* 38, 267-290.

[33] Pearlman, Joseph G., David Currie and Paul Levine (1986), "Rational Expectations Models with Partial Information," *Economic Modelling* 3:90-105.

[34] Pearlman, Joseph G. (1992), "Reputational and Nonreputational Policies under Partial Information," *Journal of Economic Dynamics and Control* 16: 339-357.

[35] Prescott, Edward C. (1986), "Theory Ahead of Business-Cycle Measurement," *Carnegie-Rochester Conference Series on Public Policy* 25, 11-44.

[36] Rotemberg, Julio J., and Michael Woodford (1997), "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy," *NBER Macroeconomics Annual*, 297-346.

[37] Sargent, Thomas J. (1989), "Two Models of Measurements and the Investment Accelerator," *Journal of Political Economy* 97, 251-287.

[38] Schorfheide, Frank (2000), "Loss function-based evaluation of DSGE models," *Journal of Applied Econometrics* 15(6), 645-670.

[39] Smets, Frank, and Raf Wouters (2007), "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review* 97(3): 586-606.

[40] Stock, James H., and Mark W. Watson (1999), "Forecasting Inflation," *Journal of Monetary Economics* 44, 293-335.

[41] Stock, James H., and Mark W. Watson (2002), "Macroeconomic Forecasting Using Diffusion Indexes," *Journal of Business Economics and Statistics* 20(2), 147-162.

[42] Svensson, Lars E. O., and Michael Woodford (2003), "Indicator Variables for Optimal Policy," *Journal of Monetary Economics* 50, 691–720.

[43] Svensson, Lars E. O., and Michael Woodford (2004), "Indicator Variables for Optimal Policy under Asymmetric Information," *Journal of Economic Dynamics and Control* 28, 661–690.

[44] Woodford, Michael (2003), *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton, Princeton University Press.

A Appendix: Equilibrium characterization

A.1 Optimal policy with asymmetric information

In this section, we characterize the optimal equilibrium and the optimal filtering problem in the case of asymmetric information between the private sector and a central bank, following Svensson and Woodford (2004).

A.1.1 Optimal policy under commitment

The Lagrangian characterizing the optimal policy problem can be written as

$$\mathcal{L} = E\left[\sum_{t=0}^{\infty}\beta^t\left[L_t + \varphi_{t+1}'\left(Ay_t + Bi_t\right) - \beta^{-1}\varphi_t'\tilde{I}y_t\right]|I_0^{cb}\right]$$

where

$$L_t \equiv \tau_t'W\tau_t,, \qquad y_t \equiv \begin{bmatrix} Z_t \\ z_t \end{bmatrix}, \quad \varphi_t \equiv \begin{bmatrix} \varphi_{1t} \\ \Xi_{t-1} \end{bmatrix}, \quad \tilde{I} \equiv \begin{bmatrix} I & 0 \\ 0 & \tilde{E} \end{bmatrix}.$$

We decompose the vector of Lagrange multipliers φ_t in this way to emphasize that the n_Z first elements φ_{1t} are measurable with respect to the period t information set I_t^f whereas the last n_z elements are measurable with respect to the period $t-1$ information set I_{t-1}^f. The terms in square brackets includes $\beta^{-1}\Xi_{-1}z_0$ which is irrelevant as we add the initial condition $\Xi_{-1} = 0$.

Differentiating the Lagrangian with respect to y_t and i_t yields the first-order conditions

$$A'E_t\varphi_{t+1} + L_{yy}y_t + L_{yi}i_t - \beta^{-1}\tilde{I}\varphi_t = 0 \tag{49}$$

$$B'\varphi_{t+1|t} + L_{iy}y_{t|t} + L_{ii}i_t = 0 \tag{50}$$

where the matrices L_{jk} are second partial derivatives of the period loss function satisfying

$$L_t = \begin{bmatrix} y_t' & i_t \end{bmatrix}\begin{bmatrix} C' \\ C_i' \end{bmatrix}W\begin{bmatrix} C & C_i \end{bmatrix}\begin{bmatrix} y_t \\ i_t \end{bmatrix} \equiv \frac{1}{2}\begin{bmatrix} y_t' & i_t \end{bmatrix}\begin{bmatrix} L_{yy} & L_{yi} \\ L_{iy} & L_{ii} \end{bmatrix}\begin{bmatrix} y_t \\ i_t \end{bmatrix}.$$

Assuming that L_{ii} is full rank,[37] we can solve (50) for i_t and obtain

$$i_t = -L_{ii}^{-1}L_{iy}y_{t|t} - L_{ii}^{-1}B'\varphi_{t+1|t}. \tag{51}$$

Substituting (51) into (1) and (49) to eliminate i_t, and taking conditional expectations of both equations given the central bank information set I_t^{cb}, we obtain

$$\begin{bmatrix} 0 & R' \\ \tilde{I} & U \end{bmatrix}\begin{bmatrix} y_{t+1|t} \\ \varphi_{t+1|t} \end{bmatrix} = \begin{bmatrix} V & \beta^{-1}\tilde{I}' \\ R & 0 \end{bmatrix}\begin{bmatrix} y_{t|t} \\ \varphi_{t|t} \end{bmatrix} \tag{52}$$

where

$$R \equiv A - BL_{ii}^{-1}L_{iy}, \qquad U \equiv BL_{ii}^{-1}B', \quad \text{and} \quad V \equiv -L_{yy} + L_{yi}L_{ii}^{-1}L_{iy}.$$

As Svensson and Woodford (2004) show, for β sufficiently close to 1, this dynamic system has one half of the eigenvalues inside the unit circle, and the other half of them outside the unit circle. This

[37] Svensson and Woodford (2004) show that the approach described here can also be applied in the case that L_{ii} is of reduced rank.

system admits a single bounded solution in which $z_{s|t}$ and $\varphi_{1s|t}$, $Z_{s|t}$ $\Xi_{s|t}$ and can be expressed as linear functions of the initial conditions $Z_{t|t}$ and $\Xi_{t-1|t}$ for all $s \geq t$. Using standard techniques, we obtain a solution in the form

$$\begin{bmatrix} z_{t|t} \\ \varphi_{1t|t} \end{bmatrix} = \begin{bmatrix} \bar{D}_2 \\ \bar{D}_3 \end{bmatrix} \bar{Z}_{t|t} \tag{53}$$

$$\bar{Z}_{t+1|t} = \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} \bar{Z}_{t|t} \tag{54}$$

where

$$\bar{Z}_t \equiv \begin{bmatrix} Z_t \\ \Xi_{t-1} \end{bmatrix}$$

is a $n \times 1$ vector of predetermined variables, $n \equiv n_Z + n_z$, and $\tilde{G}_1$, $\tilde{G}_2$ have respectively n_Z and n_z rows. Using (51), we can write the solution as

$$i_t = \bar{D}_1 \bar{Z}_{t|t}, \tag{55}$$

where

$$\bar{D}_1 \equiv -L_{ii}^{-1} \left(L_{iy} \begin{bmatrix} [I \quad 0] \\ \bar{D}_2 \end{bmatrix} + B' \begin{bmatrix} \bar{D}_3 \\ [0 \quad I] \end{bmatrix} \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} \right).$$

It will be useful to realize that by taking expectations on both sides of the first row of (1) with respect to I_t^{cb} and using (53) and (55) to solve for $z_{t|t}$ and i_t, we obtain

$$Z_{t+1|t} = \left(A_{11} [I \quad 0] + A_{12} \bar{D}_2 + B_1 \bar{D}_1 \right) \bar{Z}_{t|t}$$

where $[A_{11}, \ A_{12}]$ and B_1 constitute respectively the first n_Z rows of the matrices A and B. It follows that $\tilde{G}_1$ must satisfy

$$\tilde{G}_1 = A_{11} [I \quad 0] + A_{12} \bar{D}_2 + B_1 \bar{D}_1. \tag{56}$$

Note that all the matrices $\bar{D}_1, \bar{D}_2, \bar{D}_3, \tilde{G}_1, \tilde{G}_2$ are independent of the matrices Λ^{cb} and Σ_v that define the partial information of the central bank and of Σ_u. As we show in the next subsection, these matrices are the same as in the case of the optimal plan with full information, or in the case of no uncertainty. This state-space representation thus satisfies a principle of certainty equivalence. This solution has also the same form as in the case of incomplete but symmetric information between the private sector and the central bank considered in Svensson and Woodford (2003). The only difference is that in the case of symmetric information, the Lagrange multipliers associated with the forward-looking variables satisfy $\Xi_{t-1|t} = \Xi_{t-1|t-1} = \Xi_{t-1}$.

A.1.2 Optimal filtering

Taking expectations on both sides of the structural equations (1), with respect to I_t^f and I_t^{cb}, we obtain

$$\begin{aligned} \tilde{I} E_t y_{t+1} &= A y_t + B i_t \\ \tilde{I} y_{t+1|t} &= A y_{t|t} + B i_t \end{aligned}$$

which implies

$$\tilde{I}\left(E_t y_{t+1} - y_{t+1|t}\right) = A\hat{y}_t$$

where $\hat{y}_t \equiv y_t - y_{t|t}$. Similarly, the first-order conditions (49) imply

$$A'\left(E_t \varphi_{t+1} - \varphi_{t+1|t}\right) = \beta^{-1}\tilde{I}\hat{\varphi}_t - L_{yy}\hat{y}_t$$

where we define $\hat{\varphi}_t \equiv \varphi_t - \varphi_{t|t}$. Combining the last two equations yields the system

$$\tilde{A}\left(\begin{bmatrix} E_t y_{t+1} \\ E_t \varphi_{t+1} \end{bmatrix} - \begin{bmatrix} y_{t+1|t} \\ \varphi_{t+1|t} \end{bmatrix}\right) = \tilde{B}\begin{bmatrix} \hat{y}_t \\ \hat{\varphi}_t \end{bmatrix} \tag{57}$$

where

$$\tilde{A} \equiv \begin{bmatrix} \tilde{I} & 0 \\ 0 & A' \end{bmatrix}, \qquad \tilde{B} \equiv \begin{bmatrix} A & 0 \\ -L_{yy} & \beta^{-1}\tilde{I}' \end{bmatrix}.$$

To solve this dynamic system, we must determine the evolution of the central bank's conditional expectations. Svensson and Woodford (2004) show that the Kalman filter for the central bank problem can be written as

$$\bar{Z}_{t+1|t+1} = \bar{Z}_{t+1|t} + \bar{K}\left[\bar{L}\left(\bar{Z}_{t+1} - \bar{Z}_{t+1|t}\right) + v_{t+1}\right] \tag{58}$$

where $\bar{K}$ is a $(n \times n_X)$ matrix and $\bar{L}$ is a $(n_X \times n)$ matrix to be determined below, and n_X is the number of series in the central bank data set X_t^{cb}.

Note that using (53), we can write $z_{t+1|t+1} = \bar{D}_2\bar{Z}_{t+1|t+1}$ and $\varphi_{1t+1|t+1} = \bar{D}_3\bar{Z}_{t+1|t+1}$. Using this, and premultiplying on both sides of (58) by the $(2n \times n)$ matrix

$$\tilde{D} \equiv \begin{bmatrix} [I_{n_Z} \quad 0] \\ \bar{D}_2 \\ \bar{D}_3 \\ [0 \quad I_{n_z}] \end{bmatrix},$$

we obtain the Kalman filter for all variables

$$\begin{bmatrix} y_{t+1|t+1} \\ \varphi_{t+1|t+1} \end{bmatrix} = \begin{bmatrix} y_{t+1|t} \\ \varphi_{t+1|t} \end{bmatrix} + \tilde{D}\bar{K}\left[\bar{L}\left(\bar{Z}_{t+1} - \bar{Z}_{t+1|t}\right) + v_{t+1}\right].$$

Taking expectations on both sides with respect to I_t^f we have

$$\begin{bmatrix} E_t y_{t+1|t+1} \\ E_t \varphi_{t+1|t+1} \end{bmatrix} = \begin{bmatrix} y_{t+1|t} \\ \varphi_{t+1|t} \end{bmatrix} + \tilde{D}\bar{K}\bar{L}\left(E_t\bar{Z}_{t+1} - \bar{Z}_{t+1|t}\right).$$

Adding $\left[E_t y'_{t+1}, E_t \varphi'_{t+1}\right]'$ to both sides and rearranging, we have

$$\begin{aligned}
\left[\begin{array}{c} E_t y_{t+1} \\ E_t \varphi_{t+1} \end{array}\right] - \left[\begin{array}{c} y_{t+1|t} \\ \varphi_{t+1|t} \end{array}\right] &= \left[\begin{array}{c} E_t \hat{y}_{t+1} \\ E_t \hat{\varphi}_{t+1} \end{array}\right] + \tilde{D}\bar{K}\bar{L}\left(E_t \bar{Z}_{t+1} - \bar{Z}_{t+1|t}\right) \\
&= \left[\begin{array}{c} E_t \hat{y}_{t+1} \\ E_t \hat{\varphi}_{t+1} \end{array}\right] + \tilde{D}\bar{K}\bar{L}\bar{I}\left(\left[\begin{array}{c} E_t y_{t+1} \\ E_t \varphi_{t+1} \end{array}\right] - \left[\begin{array}{c} y_{t+1|t} \\ \varphi_{t+1|t} \end{array}\right]\right) \\
&= \left(I - \tilde{D}\bar{K}\bar{L}\bar{I}\right)^{-1}\left[\begin{array}{c} E_t \hat{y}_{t+1} \\ E_t \hat{\varphi}_{t+1} \end{array}\right], \qquad (59)
\end{aligned}$$

where we suppose that $\left(I - \tilde{D}\bar{K}\bar{L}\bar{I}\right)$ is invertible, and the $(n \times 2n)$ matrix $\bar{I}$ selects the elements

$$\bar{Z}_t \equiv \bar{I}\left[\begin{array}{c} y_t \\ \varphi_t \end{array}\right].$$

Substituting (59) into (57) yields the dynamic system

$$\tilde{A}\left(I - \tilde{D}\bar{K}\bar{L}\bar{I}\right)^{-1}\left[\begin{array}{c} E_t \hat{y}_{t+1} \\ E_t \hat{\varphi}_{t+1} \end{array}\right] = \tilde{B}\left[\begin{array}{c} \hat{y}_t \\ \hat{\varphi}_t \end{array}\right]. \qquad (60)$$

We assume that the eigenvalues of that system are such that there exists a single bounded solution. Using standard techniques, we obtain a solution in the form

$$\left[\begin{array}{c} \hat{z}_t \\ \hat{\varphi}_{1t} \end{array}\right] = \left[\begin{array}{c} \bar{D}_2^{\dagger} \\ \bar{D}_3^{\dagger} \end{array}\right]\left(\bar{Z}_t - \bar{Z}_{t|t}\right) \qquad (61)$$

$$\left[\begin{array}{c} \hat{Z}_{t+1} \\ \hat{\Xi}_t \end{array}\right] = \left[\begin{array}{c} \tilde{G}_1^{\dagger} \\ \tilde{G}_2^{\dagger} \end{array}\right]\left(\bar{Z}_t - \bar{Z}_{t|t}\right). \qquad (62)$$

Combining (61) with (53) to solve for $z_{t|t}$, we get

$$z_t = \bar{D}_2^{\dagger}\bar{Z}_t + \left(\bar{D}_2 - \bar{D}_2^{\dagger}\right)\bar{Z}_{t|t}. \qquad (63)$$

Similarly, combining (62) with (54) to solve for $\Xi_{t|t}$, we get

$$\Xi_t = \tilde{G}_2^{\dagger}\bar{Z}_t + \left(\tilde{G}_2 - \tilde{G}_2^{\dagger}\right)\bar{Z}_{t|t}. \qquad (64)$$

To determine the evolution of $\bar{Z}_t$, we use the first row of (1) together with (55) and (63) to solve for i_t and z_t:

$$Z_{t+1} = \left(A_{11}\left[I \quad 0\right] + A_{12}\bar{D}_2^{\dagger}\right)\bar{Z}_t + \left(A_{12}\left(\bar{D}_2 - \bar{D}_2^{\dagger}\right) + B_1\bar{D}_1\right)\bar{Z}_{t|t} + u_{t+1}$$

where $[A_{11},\ A_{12}]$ and B_1 constitute respectively the first n_Z rows of the matrices A and B. Combining this with (64), and using (56) yields

$$\bar{Z}_{t+1} = \bar{G}_1^{\dagger}\bar{Z}_t + \left(\bar{G}_1 - \bar{G}_1^{\dagger}\right)\bar{Z}_{t|t} + \bar{u}_{t+1} \qquad (65)$$

where

$$\bar{u}_t \equiv \begin{bmatrix} u_t \\ 0 \end{bmatrix}$$

and

$$\bar{G}_1 \equiv \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix}, \quad \bar{G}_1^\dagger \equiv \begin{bmatrix} A_{11}\left[I \quad 0\right] + A_{12}\bar{D}_2^\dagger \\ \tilde{G}_2^\dagger \end{bmatrix}.$$

Next, using (63), we rewrite the measurement equation (4) as follows:

$$X_t^{cb} = \bar{L}\bar{Z}_t + \bar{M}\bar{Z}_{t|t} + v_t \tag{66}$$

where

$$\bar{L} \equiv \Lambda_1\left[I \quad 0\right] + \Lambda_2\bar{D}_2^\dagger, \qquad \bar{M} \equiv \Lambda_2\left(\bar{D}_2 - \bar{D}_2^\dagger\right)$$

and where Λ_1, Λ_2 are matrices of the observation equation satisfying

$$\Lambda^{cb} \equiv \left[\Lambda_1 \quad \Lambda_2\right].$$

Equations (65) and (66) represent a system containing a linear transition equation for $\bar{Z}_t$ and a measurement equation. It follows that the optimal estimates of $\bar{Z}_t$ are given by a Kalman filter of the form (58) with a Kalman gain matrix satisfying

$$\bar{K} = \bar{P}\bar{L}'\left(\bar{L}\bar{P}\bar{L}' + \Sigma_v\right)^{-1}, \tag{67}$$

where the covariance matrix of prediction errors $\bar{P} \equiv Cov\left[\bar{Z}_t - \bar{Z}_{t|t-1}\right]$ satisfies the Riccati equation

$$\bar{P} = \bar{G}_1^\dagger\left[\bar{P} - \bar{P}\bar{L}'\left(\bar{L}\bar{P}\bar{L}' + \Sigma_v\right)^{-1}\bar{L}\bar{P}\right]\bar{G}_1^{\dagger\prime} + \bar{\Sigma}_u \tag{68}$$

where

$$\bar{\Sigma}_u \equiv \begin{bmatrix} \Sigma_u & 0 \\ 0 & 0 \end{bmatrix}.$$

Using (65) to form the forecast $\bar{Z}_{t+1|t}$, we may rewrite (58) as

$$\bar{Z}_{t+1|t+1} = \left(I - \bar{K}\bar{L}\right)\bar{G}_1\bar{Z}_{t|t} + \bar{K}\left(\bar{L}\bar{Z}_{t+1} + v_{t+1}\right). \tag{69}$$

To summarize, for a given Kalman gain matrix $\bar{K}$, the complete system of equations describing the evolution of the endogenous and estimated variables is given by (55), (53), (63), (65) and (69) can be written in matrix form as

$$\begin{bmatrix} i_t \\ z_t \\ z_{t|t} \end{bmatrix} = \begin{bmatrix} 0 & \bar{D}_1 \\ \bar{D}_2^\dagger & \left(\bar{D}_2 - \bar{D}_2^\dagger\right) \\ 0 & \bar{D}_2 \end{bmatrix} \begin{bmatrix} \bar{Z}_t \\ \bar{Z}_{t|t} \end{bmatrix} \tag{70}$$

$$\begin{bmatrix} \bar{Z}_{t+1} \\ \bar{Z}_{t+1|t+1} \end{bmatrix} = \begin{bmatrix} \bar{G}_1^\dagger & \left(\bar{G}_1 - \bar{G}_1^\dagger\right) \\ \bar{K}\bar{L}\bar{G}_1^\dagger & \left(\bar{G}_1 - \bar{K}\bar{L}\bar{G}_1^\dagger\right) \end{bmatrix} \begin{bmatrix} \bar{Z}_t \\ \bar{Z}_{t|t} \end{bmatrix} + \begin{bmatrix} I & 0 \\ \bar{K}\bar{L} & \bar{K} \end{bmatrix} \begin{bmatrix} \bar{u}_{t+1} \\ v_{t+1} \end{bmatrix}, \tag{71}$$

which corresponds to the state space (5)–(8). This allows us to characterize the response of all variables i_t, z_t, Z_t (and Ξ_{t-1}) as well as the forecasts by the central bank of these variables $z_{t|t}, Z_{t|t}$

(and $\Xi_{t-1|t}$) to all structural shocks u_t and all "measurement error" shocks v_t, for given initial values $Z_0, Z_{0|0}$, and $\Xi_{-1} = \Xi_{-1|0} = 0$.

A.1.3 Computing Kalman gain matrix $\bar{K}$ and $\bar{L}$

The above calculations assume knowledge of the Kalman gain matrix $\bar{K}$ and of $\bar{L}$ in order to determine matrices such as $\bar{D}_2^\dagger$ and $\bar{G}_1^\dagger$. To find these matrices, we proceed numerically as follows.

1. Conjecture an initial value for the $n \times n$ matrix $\bar{K}\bar{L}$, which we denote by $\Pi^{(j)}$
2. Compute the matrices $\tilde{A}\left(I - \tilde{D}\Pi^{(j)}\bar{I}\right)^{-1}$ and $\tilde{B}$ of the dynamic system (60)
3. Solve (60) to obtain a solution of the form (61)-(62), and compute matrices $\bar{D}_2^{\dagger(j)}$, $\bar{G}_1^{\dagger(j)}$, $\bar{L}^{(j)}$
4. Solve the Riccati equation (68) to obtain $\bar{P}^{(j)}$
5. Compute the implied Kalman gain $\bar{K}^{(j)}$ using (67)
6. Compute the product $\Pi^{(j+1)} = \bar{K}^{(j)}\bar{L}^{(j)}$.

We then keep iterating through the steps 1-6 until we converge to $\Pi^{(j+1)} = \Pi^{(j)}$.

A.2 Special case: Optimal policy with no measurement error

The case of optimal policy under full information, i.e., no measurement error on the part of the central bank is a special version of the case considered in the previous subsection when we set $\Lambda^{cb} = I$, $v_t = 0$ and $\Sigma_v = 0$. In this case, the central bank has the same full information set as the private sector, as $X_t^{cb\prime} = [Z_t', z_t']$, and so its expectations are the same as those of the private sector: $E_t x_{t+j} = x_{t+j|t}$, for any variable x_{t+j} and for all j. Repeating the derivations of section A.1.1 in this context of full information, we obtain again a solution of the form (53)–(55), and thus

$$\begin{aligned}
\begin{bmatrix} i_t \\ z_t \end{bmatrix} &= \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \end{bmatrix} \bar{Z}_t \\
\Xi_{t-1} &= \tilde{G}_2 \bar{Z}_t.
\end{aligned} \tag{72}$$

Substituting these solutions in the first row of (1), and using (56), we obtain

$$Z_{t+1} = \tilde{G}_1 \bar{Z}_t + u_{t+1}.$$

By combining these last two expressions, we see that the state space (70)–(71) reduces to (72) and

$$\bar{Z}_{t+1} = \bar{G}_1 \bar{Z}_t + \bar{u}_{t+1} \tag{73}$$

where $\bar{D}_1$, $\bar{D}_2$, and $\bar{G}_1$ are the same matrices as in the previous subsection.

A.3 Proof of Proposition 2

In section A.2 we show that in the case that the central banks observation involves no measurement error, i.e., $v_t = 0$, and $\Lambda^{cb} = I$, the model's equilibrium can be characterized by the state space (72)–(73). Proposition 2 states that even if there is measurement error, the equilibrium is characterized by the same state space — where all the matrices in that state space (i.e., $\bar{D}_1, \bar{D}_2, \bar{G}_1$) are the same as in the absence of uncertainty — in the case that the central bank has access to an infinite data set ($n_X \to +\infty$), and that the assumption of Proposition 1 are satisfied.

Proof. In the case that the central bank conducts optimal policy, the state space characterizing the equilibrium is given by (5)–(8). As stated in Proposition 1 (under some regularity conditions), an econometrician that has access to an infinite number of data series can perfectly recover the true state of the economy S_t for all t. In particular, a central bank estimating the model recovers $S_t = \left[\bar{Z}_t', \bar{Z}_{t|t}'\right]'$. Such a central bank thus observes $\bar{Z}_t$, so that its estimate $\bar{Z}_{t|t} \equiv E\left[\bar{Z}_t | I_t^{cb}\right] = \bar{Z}_t$ at all dates. It then follows that the evolution of the state, given by (65), reduces to (29). Similarly, the solutions for i_t and z_t, (55) and (63), reduce to (30). Comparing this last expression with (53) implies $z_t = z_{t|t}$. Finally, we note that the matrices involved $\bar{D}_1, \bar{D}_2, \bar{G}_1$ are the same as in the equilibrium with out uncertainty, and thus do not depend on the data available to the central bank. ■

A.4 Equilibrium when central bank responds naively to observable indicators

We now characterize the equilibrium in the case that the central bank follows an arbitrary policy rule (11) according to which it responds naively to observable indicators (12). By combining the structural equations (1), the policy rule (11) to eliminate i_t, (13), and allowing the measurement error e_{pt} to follow an AR(1) process, $e_{pt} = \Psi_p e_{pt-1} + \varepsilon_{p,t}$, we obtain the dynamic system

$$\begin{bmatrix} I_{n_p} & 0 & 0 & 0 \\ 0 & I_{n_Z} & 0 & 0 \\ 0 & 0 & \tilde{E} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{p,t+1} \\ Z_{t+1} \\ E_t z_{t+1} \\ E_t p_{t+1} \end{bmatrix} = \begin{bmatrix} \Psi_p & 0 & 0 \\ 0 & A & B\phi \\ I_{n_p} & P & -I_{n_p} \end{bmatrix} \begin{bmatrix} e_{pt} \\ Z_t \\ z_t \\ p_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{p,t+1} \\ u_{t+1} \\ 0 \\ 0 \end{bmatrix}.$$

When this system admits a single bounded solution, the solution can be expressed in state space form as

$$\begin{aligned} \begin{bmatrix} z_t \\ p_t \end{bmatrix} &= D_0 \begin{bmatrix} e_{pt} \\ Z_t \end{bmatrix} \\ \begin{bmatrix} e_{pt} \\ Z_t \end{bmatrix} &= G \begin{bmatrix} e_{p,t-1} \\ Z_{t-1} \end{bmatrix} + H \begin{bmatrix} \varepsilon_{p,t+1} \\ u_{t+1} \end{bmatrix} \end{aligned}$$

Using this solution and combining with (11), we can express the i_t as a linear function of e_{pt} and Z_t. We thus have obtained a solution of the form (5)–(6).

A.5 Equilibrium with arbitrary policy rule and optimal filtering

In this section, we characterize the equilibrium resulting from an arbitrary policy rule and the optimal filtering problem in the case of asymmetric information between the private sector and a central bank. The central bank is assumed to set its instrument according to the following policy rule

$$i_t = \tilde{\phi} \begin{bmatrix} Z_{t|t} \\ z_{t|t} \end{bmatrix} \tag{74}$$

at all dates, where $\tilde{\phi} \equiv \phi P$ and ϕ contains the coefficients of the policy rule in (11) while P is the selection matrix in (14).

A.5.1 Equilibrium

Substituting (74) into (1) to eliminate i_t, and taking conditional expectations with respect to I_t^{cb}, we obtain

$$\tilde{I}y_{t+1|t} = \left(A + B\tilde{\phi}\right) y_{t|t} \tag{75}$$

where

$$y_t \equiv \begin{bmatrix} Z_t \\ z_t \end{bmatrix}, \quad \text{and} \quad \tilde{I} \equiv \begin{bmatrix} I & 0 \\ 0 & \tilde{E} \end{bmatrix}.$$

When this system admits a single bounded solution, the solution can be expressed as

$$z_{t|t} = D_2 Z_{t|t} \tag{76}$$

$$Z_{t+1|t} = \tilde{G}_1 Z_{t|t}. \tag{77}$$

Using (74), we can express the instrument as

$$i_t = D_1 Z_{t|t} \tag{78}$$

where

$$D_1 \equiv \tilde{\phi} \begin{bmatrix} I \\ D_2 \end{bmatrix}.$$

Equations (76)–(78) fully determine policy instrument and the conditional expectations of the predetermined variables and non-predetermined variables as a function of the current estimates of the predetermined variables. Note that the matrices D_1, D_2, and $\tilde{G}_1$ are the same as in the case of full information, as they are independent of the matrices Λ^{cb} and Σ_v that define the partial information of the central bank.

A.5.2 Optimal filtering

Taking expectations on both sides of the structural equations (1), with respect to I_t^f and I_t^{cb}, we obtain

$$\begin{aligned} \tilde{I}E_t y_{t+1} &= Ay_t + Bi_t \\ \tilde{I}y_{t+1|t} &= Ay_{t|t} + Bi_t \end{aligned}$$

which implies

$$\tilde{I}\left(E_t y_{t+1} - y_{t+1|t}\right) = A\hat{y}_t \tag{79}$$

where $\hat{y}_t \equiv y_t - y_{t|t}$.

We now augment this system with implications of the policy rule (74) which can be written as $i_t = \tilde{\phi} y_{t|t}$. Note that this implies

$$\hat{\imath}_t \equiv i_t - i_{t|t} = 0.$$

The policy rule implies also

$$\begin{aligned} E_t i_{t+1} &= \tilde{\phi} E_t y_{t+1|t+1} \\ i_{t+1|t} &= \tilde{\phi} y_{t+1|t} \end{aligned}$$

so that

$$E_t i_{t+1} - i_{t+1|t} = \tilde{\phi}\left[\left(E_t y_{t+1|t+1} - E_t y_{t+1}\right) + \left(E_t y_{t+1} - y_{t+1|t}\right)\right]$$

which implies in turn

$$\tilde{\phi}\left(E_t y_{t+1} - y_{t+1|t}\right) - \left(E_t i_{t+1} - i_{t+1|t}\right) - \tilde{\phi} E_t \hat{y}_{t+1} = 0. \tag{80}$$

Combining (79) and (80) yields the system

$$\begin{bmatrix} \tilde{I} & 0 \\ \tilde{\phi} & -1 \end{bmatrix} \begin{bmatrix} E_t y_{t+1} - y_{t+1|t} \\ E_t i_{t+1} - i_{t+1|t} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \tilde{\phi} & 0 \end{bmatrix} E_t \begin{bmatrix} \hat{y}_{t+1} \\ \hat{\imath}_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{\imath}_t \end{bmatrix} \tag{81}$$

To solve this dynamic system, we must specify how the central bank's conditional expectations are updated. We follow Svensson and Woodford (2004), assuming again (and verifying below) that the Kalman filter for the central bank problem can be written as

$$Z_{t+1|t+1} = Z_{t+1|t} + K\left[L\left(Z_{t+1} - Z_{t+1|t}\right) + v_{t+1}\right] \tag{82}$$

where K is a $(n_Z \times n_X)$ matrix and L is a $(n_X \times n_Z)$ matrix to be determined.

Note that using (76), we can write $z_{t+1|t+1} = D_2 Z_{t+1|t+1}$. Similarly, the policy rule implies $i_{t+1|t+1} = \tilde{\phi} y_{t+1|t+1}$. Using this, we have

$$\begin{bmatrix} y_{t+1|t+1} \\ i_{t+1|t+1} \end{bmatrix} = \begin{bmatrix} I_n \\ \tilde{\phi} \end{bmatrix} y_{t+1|t+1} = \tilde{D} Z_{t+1|t+1}$$

where

$$\tilde{D} \equiv \begin{bmatrix} I_n \\ \tilde{\phi} \end{bmatrix} \begin{bmatrix} I_{n_Z} \\ D_2 \end{bmatrix}.$$

Premultiplying on both sides of (82) by the $((n+1) \times n_Z)$ matrix $\tilde{D}$, we obtain the Kalman filter for all variables

$$\begin{bmatrix} y_{t+1|t+1} \\ i_{t+1|t+1} \end{bmatrix} = \begin{bmatrix} y_{t+1|t} \\ i_{t+1|t} \end{bmatrix} + \tilde{D} K\left[L\left(Z_{t+1} - Z_{t+1|t}\right) + v_{t+1}\right].$$

Adding $\left[E_t y_{t+1}', \; E_t i_{t+1}\right]'$ to both sides, rearranging, and taking expectations on both sides with respect to I_t^f we have

$$\begin{aligned} \begin{bmatrix} E_t y_{t+1} - y_{t+1|t} \\ E_t i_{t+1} - i_{t+1|t} \end{bmatrix} &= \begin{bmatrix} E_t \hat{y}_{t+1} \\ E_t \hat{\imath}_{t+1} \end{bmatrix} + \tilde{D} K L\left(E_t Z_{t+1} - Z_{t+1|t}\right) \\ &= \begin{bmatrix} E_t \hat{y}_{t+1} \\ E_t \hat{\imath}_{t+1} \end{bmatrix} + \tilde{D} K L \bar{I} \begin{bmatrix} E_t y_{t+1} - y_{t+1|t} \\ E_t i_{t+1} - i_{t+1|t} \end{bmatrix} \\ &= \left(I - \tilde{D} K L \bar{I}\right)^{-1} \begin{bmatrix} E_t \hat{y}_{t+1} \\ E_t \hat{\imath}_{t+1} \end{bmatrix}, \end{aligned} \tag{83}$$

where we suppose that $\left(I - \tilde{D} K L \bar{I}\right)$ is invertible, and the $(n_Z \times (n+1))$ matrix $\bar{I}$ selects the elements $Z_t \equiv \bar{I}\,[y_t' \quad i_t]'$. Substituting (83) into (81) yields

$$\tilde{A} \begin{bmatrix} E_t \hat{y}_{t+1} \\ E_t \hat{\imath}_{t+1} \end{bmatrix} = \tilde{B} \begin{bmatrix} \hat{y}_t \\ \hat{\imath}_t \end{bmatrix}. \tag{84}$$

where

$$\tilde{A} \equiv \left(\begin{bmatrix} \tilde{I} & 0 \\ \tilde{\phi} & -1 \end{bmatrix} \tilde{I} \left(I - \tilde{D} K L \bar{I} \right)^{-1} - \begin{bmatrix} 0 & 0 \\ \tilde{\phi} & 0 \end{bmatrix} \right), \qquad \text{and} \qquad \tilde{B} \equiv \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

We assume that the eigenvalues of that system are such that there exists a single bounded solution. Given that $\hat{\imath}_t = 0$ regardless of the realization of the shocks in period t, and that $E_{t-1}\hat{\imath}_t = \hat{\imath}_t = 0$, the variable $\hat{\imath}_t$ is predetermined. Provided that n_z eigenvalues of the system lie outside of the unit circle, we have a solution of the form

$$\hat{z}_t = D_2^{\dagger} \left(Z_t - Z_{t|t} \right) \tag{85}$$

$$\hat{Z}_{t+1} = \tilde{G}_1^{\dagger} \left(Z_t - Z_{t|t} \right). \tag{86}$$

We can then combine (85) with (76) to solve for $z_{t|t}$, we get

$$z_t = D_2^{\dagger} Z_t + \left(D_2 - D_2^{\dagger} \right) Z_{t|t}. \tag{87}$$

To determine the evolution of Z_t, we use the first row of (1) together with (78) and (87) to solve for i_t and z_t and get

$$Z_{t+1} = G_1^{\dagger} Z_t + \left(G_1 - G_1^{\dagger} \right) Z_{t|t} + u_{t+1} \tag{88}$$

where

$$G_1 \equiv A_{11} + A_{12} D_2 + B_1 D_1, \quad \text{and} \quad G_1^{\dagger} \equiv A_{11} + A_{12} D_2^{\dagger}.$$

Next, using (87), we rewrite the measurement equation (4) as follows:

$$X_t^{cb} = L Z_t + M Z_{t|t} + v_t \tag{89}$$

where

$$L \equiv \Lambda_1 + \Lambda_2 D_2^{\dagger}, \qquad M \equiv \Lambda_2 \left(D_2 - D_2^{\dagger} \right)$$

and where Λ_1, Λ_2 are matrices of the observation equation satisfying

$$\Lambda^{cb} \equiv [\Lambda_1 \quad \Lambda_2].$$

Equations (88) and (89) represent a system containing a linear transition equation for Z_t and a measurement equation. As shown in Svensson and Woodford (2003), this implies that the optimal estimates of Z_t are given by a Kalman filter of the form (82) with a Kalman gain matrix satisfying

$$K = \tilde{P} L' \left(L \tilde{P} L' + \Sigma_v \right)^{-1}$$

where the covariance matrix of prediction errors $\tilde{P} \equiv Cov\left[Z_t - Z_{t|t-1} \right]$ satisfies the Riccati equation

$$\tilde{P} = G_1^{\dagger} \left[\tilde{P} - \tilde{P} L' \left(L \tilde{P} L' + \Sigma_v \right)^{-1} L \tilde{P} \right] G_1^{\dagger\prime} + \Sigma_u.$$

Using (88) to form the forecast $Z_{t+1|t}$, we may rewrite (82) as

$$Z_{t+1|t+1} = (I - KL)\, G_1 Z_{t|t} + K\,(L Z_{t+1} + v_{t+1})\,. \tag{90}$$

To summarize, for a given Kalman gain matrix K, the complete system of equations describing the evolution of the endogenous and estimated variables is given by (78), (76), (87), (88) and (90) can be written in the state space form (5)–(8), where the matrices $\bar{D}_1$, $\bar{D}_2$, $\bar{G}_1$, $\bar{D}_2^{\dagger}$, $\bar{G}_1^{\dagger}$, $\bar{K}$, and $\bar{L}$ are replaced respectively by the smaller matrices D_1, D_2, G_1, $D_2^{\dagger}$, $G_1^{\dagger}$, K, and L. This allows us to characterize the response of all variables i_t, z_t, Z_t as well as the forecasts by the central bank of these variables $z_{t|t}, Z_{t|t}$ to all structural shocks u_t and all "measurement error" shocks v_t, for given initial values $Z_0, Z_{0|0}$.

To compute the Kalman gain matrices K and L, we proceed iteratively as described in section A.1.3.

B Estimates of model parameters and priors

Table B.1: Priors and estimates of structural parameters and shocks' persistence

	Prior distribution			Posterior		
	Type	Mean	St.Err.	Mean	95%	5%
Structural parameters						
φ	Normal	0.75	1.00	3.7719	2.6602	4.8026
η	Beta	0.70	0.10	0.7759	0.6725	0.9056
γ_p	Beta	0.50	0.20	0.1506	0.0533	0.2639
γ_ω	Beta	0.50	0.20	0.6661	0.4359	0.9301
ξ_p	Gamma	0.01	0.05	0.0543	0.0267	0.0976
ξ_ω	Gamma	0.01	0.05	0.1923	0.0812	0.4095
ω_p	Gamma	0.33	0.10	0.6046	0.4182	0.8625
ω_w	Gamma	1.00	0.25	0.6718	0.4310	1.0233
Historical policy rule						
ϕ_{i1}	Beta	0.75	0.10	0.9124	0.8511	0.9736
ϕ_{i2}	Normal	0.00	0.25	-0.1012	-0.1752	-0.0343
ϕ_π	Normal	1.50	0.25	2.0438	1.7676	2.3522
ϕ_y	Normal	0.125	0.05	0.1058	0.0319	0.1562
Persistence of shocks						
ρ_a	Beta	0.5	0.2	0.7975	0.6990	0.8955
ρ_g	Beta	0.5	0.2	0.5046	0.1890	0.8433
ρ_h	Beta	0.5	0.2	0.6444	0.5555	0.8407
ρ_π	Beta	0.5	0.2	0.9245	0.8799	0.9584

The parameter estimates are given by the mean of the posterior distribution
Results are based on 100,000 replications.

Table B.1 (continued): Priors and estimates of standard deviations of shocks' innovations

	Prior distribution			Posterior		
	Type	Mean	Deg. of freedom	Mean	95%	5%
σ_a	invGam	0.1	2	1.3853	0.9132	1.8387
σ_g	invGam	0.1	2	0.0940	0.0279	0.1901
σ_h	invGam	0.1	2	5.3059	2.4484	6.9662
σ_π	invGam	0.1	2	0.2346	0.1657	0.3100
σ_y	invGam	0.1	2	0.0832	0.0279	0.1327
σ_i	invGam	0.1	2	0.1691	0.1459	0.1914

The parameter estimates are given by the mean of the posterior distribution
Results are based on 100,000 replications.