Bankruptcy and Access to Credit in General Equilibrium
Job Market Paper

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Abstract

This paper develops a framework for the analysis of bankruptcy in an infinite time setting. With a market structure that is both anonymous and competitive, the model takes seriously the question of how a household's access to credit is endogenously determined following a bankruptcy declaration. Intended to model the realistic unsecured credit markets, any household can be a creditor and in the presence of bankruptcy, the asset payouts for these creditors must be appropriately diluted. To compensate, the market will set asset prices for the borrowers based on their expected repayment rates. Borrowers' expected repayment rates depend on their idiosyncratic income realizations. All borrowers are divided into pools based solely on their bankruptcy histories. The market does not know the optimal asset choices of borrowers, so can only set asset prices that vary across pools, but are linear in assets. Asset prices set in this manner are such that households will self select into bankruptcy pools and the resulting repayment rates of the pools will differ.

In this paper, the general existence of the described bankruptcy equilibrium is proven. Further, a theoretical result for a simple economy provides conditions such that households with different persistences of income states will self-select into different bankruptcy pools. This leads to an ordering in which the repayments rates are smaller for the pool of households with a more recent bankruptcy. Finally, for the same simple economy, normative impacts of a multi-asset structure are examined.

1 Introduction

The convergence of lack of commitment models from the macroeconomics literature and the default models from the general equilibrium literature (as pioneered by Dubey, Geanakoplos, and Shubik (2005)) offers the potential of models that are both competitive with incentives endogenously determined and tractable with obvious gains for both positive and normative quantitative analysis. This work is another step toward that potential. The model will capture the key features of chapter 7 bankruptcy by individuals holding debt in the unsecured credit markets. This model is not one of default (as in Dubey et. al. (2005)) in which a household chooses how much of its debt to repay on an asset-by-asset basis. Rather, this work considers bankruptcy, which results from a binary

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decision by the household not to repay debt over its entire portfolio of assets. With this binary decision, a continuum of households must be assumed in order to circumvent any non-convexity issues.

Several papers in the general equilibrium literature (Araujo and Pascoa (2002), Sabarwal (2003), and Hoelle (2009)) have addressed the question of bankruptcy in a 2-period general financial model. Sabarwal even extended the analysis to a longer, finite time horizon and suggested that investment constraints be set based on a household's repayment history. This only captures the "backward looking" effects of bankruptcy: how a prior bankruptcy declaration affects a household's access to credit in the current period. Faced with the binary bankruptcy decision in the current period, a successful model must incorporate the "forward looking" effects of bankruptcy: how a current bankruptcy declaration will affect future access to credit. With a finite time horizon, these "forward looking" effects would unravel from the final period. This necessitates the use of a discrete, infinite time process. Already working with a continuum of households, the natural model to develop will be an adaptation from the class of Bewley models (Bewley, 1986).[1]

In the anonymous setting, the terms of lending are set by the market rather than by individual creditors (though I will employ the general term "creditors" to refer to the market). All potential borrowers know their repayment possibilities, but the creditors can distinguish borrowers using only their credit scores. The FICO credit score is used by more than 75% of lending institutions and contains 5 components: payment history (35%), amounts owed (30%), length of credit history (15%), mix of credit (10%), and new credit (10%).[2] The credit score does not include a household's current income, so in my model I do not allow potential borrowers to be distinguished by income. However, this assumption is only to simplify the setup. All results in this paper remain valid if the market can distinguish households by current income. Concerning the second component of the credit score, the model considered here is not a model of debt holding. As assets can be traded every period without transaction costs, households would never choose to hold debt, preferring instead to pay off any negative dividend returns with the sale of new assets. In the basic model, a household's credit score will only depend on its payment (bankruptcy) history.

Households are ex-ante identical with heterogeneity resulting from the idiosyncratic realizations from a common Markov process over income states. I use the term "states" to encompass both the pecuniary income level as well as the persistence of that state. This process over states rather than over only levels prevents creditors for the most part from using borrowers' income levels to forecast their repayment rates. This is obvious because two households with identical income levels but difference persistences will optimize differently. With only borrowers' bankruptcy histories being used by creditors to forecast repayment rates, it is equivalent for creditors to face pools of borrowers, rather than individual borrowers. These pools are equivalence classes over borrowers' bankruptcy histories.

With a continuum of households, there are a continuum of creditors. The asset payouts for the creditors will be based on the repayment rates of the households to which the creditors lend. The creditors will set asset prices for each pool of borrowers and since all creditors have identical preferences, their collective actions match their individual ones. When setting asset prices, the creditors do not know the persistence of a potential borrower's income state. More importantly, creditors do not know the optimal asset choices of a potential borrower. Even if a potential borrower interacted with a single creditor and submitted a loan request, the creditor cannot verify that the same borrower did not submit a similar request to each of the continuum of other creditors. Thus, while a single creditor can set asset prices that are nonlinear in the size of the loan request, it is never optimal to do so without observing the *total* loan request of a borrower across all creditors.

[1] Bewley (1986) contains frequent references to his earlier works in this extended line of his research.
[2] FICO stands for Fair Isaac and Company and the information was lifted from the website: www.myfico.com.

A borrower will therefore face linear asset prices that differ by pools, that is, the prices are only conditioned on the borrower's bankruptcy history. For the same reason as above (borrowers can take out loans from a continuum of creditors), no investment constraints enter the household's optimization problem.

The bankruptcy history of a potential borrower appears to be an important indicator since the linear asset prices faced by borrowers can differ only based on this indicator. The bankruptcy history will be subject to 2 restrictions from the legal code with an immediate implication. The first restriction forbids households from declaring bankruptcy two periods in a row (in reality, twice in any 6-year period). The immediate implication is that households' repayment rates immediately after a bankruptcy declaration will be complete. With guaranteed full repayment, this is a desirable pool of borrowers for the creditors to lend to. The second restriction requires that a prior bankruptcy declaration must be removed from a household's credit report after 2 periods (in reality, 10 years). Thus, the bankruptcy history embodies both reputation losses and restrictions on future bankruptcy declarations, dynamic costs of bankruptcy that are endogenously determined in equilibrium.

In this setup, the repayment rates of pools will likely differ in equilibrium. This result is obtained mainly because creditors set linear asset prices and do not observe the optimal asset choices of borrowers. Borrowers will then self-select into pools through their bankruptcy decisions. These decisions will be based on how low their current income level is and how persistent that level will be. Creditors will set the asset prices using perfect foresight of the repayment rates of the pools. Thus different repayment rates imply different asset prices faced across pools. This anonymous framework differs noticeably from the macroeconomic literature in which creditors use the size of a borrower's loan request to forecast its repayment likelihood.

Without observing the optimal asset choices of borrowers, creditors must set identical asset prices for all borrowers with the same bankruptcy history. In a model with a single asset available for trade, this limited ability of prices to screen borrowers leads to the following natural inefficiency. Borrowers likely to declare bankruptcy face better (higher) asset prices than their repayment likelihoods suggest because they are being pooled together with households unlikely to declare bankruptcy. Those unlikely to declare bankruptcy face worse (lower) asset prices than their repayment likelihoods suggest and are subsidizing the borrowing of those households likely to declare bankruptcy. I formulate a general asset structure in which households can trade multiple assets in order to hedge against their idiosyncratic risk. The portfolio choices of the households with multiple assets then allows for prices to play a role in screening borrowers. This logic suggests that increasing the number of assets available for trade will lead to a Pareto improvement, a result that I prove in section 4.

While primary emphasis in this infinite-time model has been placed on the dynamic costs of bankruptcy, static costs are needed as well and these are written to reflect the legal code. Recent 2005 legislation[3] requires a household to pass the "means test" in order to declare bankruptcy. Roughly speaking, unless a large number of statutorily allowed expenses can be deducted, a household will fail the "means test" if its average monthly income is above the state median. Thus, if the transition over income states is such that a household's income next period is above the median for some state, then the financial decisions must incorporate the fact that the household cannot declare bankruptcy in that state. Additionally, when declaring bankruptcy, the household must pay a cost that is strictly increasing in the value of its asset purchases. This is a proxy for the cost of submitting (or hiring a lawyer to submit) detailed records to the bankruptcy court in order to take advantage of bankruptcy exemptions.

I would be remiss if I did not mention the key pieces of the large macroeconomic literature on bankruptcy. The standard may have been set by Chatterjee, Corbae, Nakajima, and Rios-Rull (2007) in which zero-profit intermediaries set the price for loans conditional on a borrower's

[3] 11 U.S.C. §707(b)(1) and the Bankruptcy Abuse Prevention and Consumer Protection Act of 2005

bankruptcy history and the size of its loan request. Following this, Chatterjee et. al. (2008) model similar intermediaries that use a household's bankruptcy history to update its beliefs about a household's private information. Finally, Krueger and Perri (2006) should be mentioned as the asset structure employed in their work (Arrow securities) is a special case of the asset structure introduced in this work.

This paper makes three contributions. First, the general model is introduced and the existence of a bankruptcy equilibrium is proven (section 2 with existence proofs in appendix A). Next, to justify the partition of borrowers into pools, I use theoretical results for a simple economy to show that the repayment rates differ across pools and further discuss how the repayment rates change with parameters (section 3 with proofs in appendix B). Finally, this framework allows for a multi-asset structure, so the normative impacts of increasing the number of assets available for trade are considered (section 4 with proof in appendix C). Section 5 concludes and discusses the next steps for this line of research.

2 The Model

Let the length of the model be described by an infinite-dimensional, discrete time process $t \in \{0, 1, .., T\}$ where $T \to \infty$. Let there be a continuum (with unit measure) of infinite-lived households $h \in \mathcal{H}\tilde{}[0, 1]$. As described by Judd (1985), there is a problem with this specification whereby a continuum of households draws realizations from an iid random process. The problem is that the set of sample realizations satisfying the property that the expectation of the sample function is equal to the theoretical one has outer measure one and inner measure zero (not measurable). Thus, I would not be able to apply the law of large numbers to state that the sample distribution is equal to the theoretical distribution of the process. As suggested by Judd and formally analyzed by Sun (1998), the solution lies in modeling the set of households as a hyperfinite process. The reasons to use this process are twofold. First, the asymptotic properties of finite processes are embedded in the limit setting. In particular, the exact law of large numbers applies. Second, the external cardinality of the index sets of a hyperfinite process is the same as the cardinality of the continuum. From this second point, my proof of lemma 2 (based on Aumann (1966)) remains valid. To avoid technical complexities, I will continue to refer to the set of households as a continuum and will apply the law of large numbers. In the back of one's mind, recall that the set of households is actually a hyperfinite process.

For simplicity, there is only one physical commodity (labor income) at each time period. Each household faces idiosyncratic risk in the form of an iid process over income states. The income states belong to the finite index set $\mathcal{E} = \{1, ..., E\}$. I use the term "states" because the state will include both the income level and the persistence of that state. Let the income level for any given state e be denoted as ω_e. I assume that $\omega_e > 0 \ \ \forall e \in \mathcal{E}$. Thus, at any given time period t, upon the realization of state e, household h has state $e^h(t) = e$ and income level $\omega^h(t) = \omega_e$. The sequence of state realizations up to and including time period t is given by $\left(e^h\right)^t = \left(e^h(0), ..., e^h(t)\right)$.

The process over income states is a Markov process governed by the transition matrix $\Pi \in M^{E,E}$. The terms of this transition matrix are $\pi(e'|e)$. Given realized state e, the probability that the realized income state is e' next period is given by $\pi(e'|e)$. I will employ the exact law of large numbers to state that the transition matrix is stationary and that not only is $\pi(e'|e)$ the probability of moving from state e to e', but also the fraction of households who make such a move in the sample (in *any* time period). The stationary measure across income states is given by $\mu \in \Delta^{E-1}$. By stationarity, $\mu(e') = \sum_e \pi\left(e'|e\right)$ is the fraction of households at *any* time period with income state e'. For consistency of Π, the sum $\sum_{e'} \pi\left(e'|e\right) = 1$. Due to the stationarity, the aggregate endowment (and the mean endowment since the set of households has unit measure) is given by

$$\bar{\omega} = \textstyle\sum_e \mu(e)\omega_e.$$

Further, the median endowment is given by $\omega_{med} = \omega_{e^*}$ where $\sum_e \mu(e)1\{\omega_e \leq \omega_{e^*}\} = \sum_e \mu(e)1\{\omega_e \geq \omega_{e^*}\} = 0.5$. As neither the aggregate, mean, nor median endowment vary over time, there is no aggregate uncertainty in the model.

Apart from the iid Markov process, the households will be assumed to be identical in all other aspects. Given the realizations of endowments $\{\omega^h(t)\}_{t\geq 0}$ and the choices of the household in the financial markets (to be introduced shortly), the household consumption is $c^h(t)$ at time period t and the entire sequence of consumption is given by $c^h = \{c^h(t)\}_{t\geq 0}$. Each household has identical, smooth preferences characterized by the utility function $U : l_\infty \to \mathbb{R}$

$$U(c^h) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c^h(t)\right).$$

Here, $\beta \in (0,1)$ is the time-preference parameter. I could have assumed that the Bernoulli utility function $u : \mathbb{R}_+ \to \mathbb{R}$ depends on the realized income state i and the results would go through with the only change being the added notation. I assume that $u : \mathbb{R}_+ \to \mathbb{R}$ is C^1, differentiably strictly increasing, differentiably strictly concave, and satisfies the Inada condition ($u'\left(c^h(t)\right) \to \infty$ as $c^h(t) \to 0$).

2.1 The asset structure

In order to hedge against the idiosyncratic risk, there will be an assortment of one-period lived financial instruments available to the households. The same financial instruments will be available in every time period. These instruments will be numeraire assets in zero net supply as I seek to model the modern bond markets. The asset structure will make use of the stationarity of the model. It is more general than the setup of Krueger and Perri (2006) in which only Arrow securities are considered.

Subsection 2.1.1 introduces the asset structure to be used throughout this paper. On the first pass, the reader may be worried that the asset structure is not consistent in the aggregate. Subsection 2.1.2 partially eases this worry by showing that with complete asset markets without bankruptcy, the equilibrium allocation is optimal. Subsection 2.1.3 completely eases this worry as it proves that even though the asset payouts differ by household, the aggregate asset payout is identical to standard GEI models.

2.1.1 Introduction of the asset structure

For any household $h : e^h(t) = e$, the risk faced by the household is a function of the total number of different income states that could be realized in the next time period. Thus, define $N(e) = \#\{e' : \pi(e'|e) > 0\}$ as this number. To hedge against this risk, the household has a vector of assets $(z_j(e))_{j\in\mathcal{J}(e)}$ to choose from. This vector has length $J(e) = \#\mathcal{J}(e) \leq N(e) - 1$. Given the current state of the household, the assets $(z_j(e,t))_{j\in\mathcal{J}(e)}$ allow the household to transfer wealth from time period t to some of the possible states $e' : \pi(e'|e) > 0$ that may occur in the next time period $t+1$. The payout of the asset $z_j(e,t)$ for the state realization $e^h(t+1) = e'$ is given by $r_j(e'|e)$. I assume that $r_j(e'|e)$ are nonnegative and nontrivial (that is, $r_j(e'|e) > 0$ for some (e,e') s.t. $\pi(e'|e) > 0$). This setup allows not only for the Arrow securities as modeled in Krueger and Perri (2006), but additionally any and all financial assets with linear payouts. The assumption that $J(e) \leq N(e) - 1$

is made without loss of generality as is the assumption that the assets $j \in \mathcal{J}(e)$ are not redundant (linearly independent payouts). Thus, the household h receives payout $\sum_{j\in\mathcal{J}(e)} r_j\left(e'|e\right) z_j(e,t)$ in time period $t+1$ from holding the vector of assets $\left(z_j(e,t)\right)_{j\in\mathcal{J}(e)}$ given that $e^h(t+1)=e'$.[4]

Now consider a different household k with $e^k(t)\neq e$. Though this household cannot hedge its own idiosyncratic risk by trading the assets $\left(z_j(e,t)\right)_{j\in\mathcal{J}(e)}$, it can still be the counterparty to transactions made by households $h: e^h(t)=e$. This is in line with the way modern bond markets pool insurance contracts over a large number of households. Thus, the household k can trade the assets $\left(z_j(e,t)\right)_{j\in\mathcal{J}(e)}$. Since it is not using these assets to hedge its own idiosyncratic risk, the payout of each asset $z_j(e,t)$ will be weighted by the measure of households $h: e^h(t)=e$. Further, the realization of income state e' next period given state e this period does not mean anything to k as these realizations are for households $h: \left(e^h(t), e^h(t+1)\right)=(e,e')$. Therefore, the payout will be risk-free (independent of e'). Define

$$r_j(e)=\textstyle\sum_{e'}\pi\left(e'|e\right) r_j\left(e'|e\right).$$

Then household k will receive the risk-free return $\sum_{j\in\mathcal{J}(e)} r_j\left(e\right) z_j(e,t)$ from holding the vector of assets $\left(z_j(e,t)\right)_{j\in\mathcal{J}(e)}$.[5]

Consider the simplest asset structure that can be imposed: the monetary equilibrium model espoused by Bewley (1986) and many others. Suppose $J(e)=1\ \ \forall e$ and that $r_1\left(e'|e\right)=1\ \ \forall(e,e')$ such that $\pi\left(e'|e\right)>0$. Households with any income state $e\in\mathcal{E}$ only have access to a risk-free bond. Any household $k: e^k(t)\neq e$ is indifferent between trading in asset $z_1(e,t)$ and $z_1(e^k(t),t)$ as these assets are identical.

Now suppose that for $e=1$ with $J(1)=1$ as above, the bond is now risky. In other words, $r_1\left(e'|1\right)\neq r_1\left(e''|1\right)$ for some $e'\neq e'': \pi\left(e'|e\right)\pi\left(e''|e\right)>0$. Households $k: e^k(t)\neq 1$ still only have access to a risk-free bond, but households $h: e^h(t)=1$ can trade in the risky bond $z_1(1,t)$ as well as any of the risk-free bonds $\left(z_1(e,t)\right)_{e\neq 1}$. Thus, when setting the number of independent assets $J(e)\leq N(e)-1$, I have implicitly assumed that no linear combination of the assets $\left(z_j(e,t)\right)_{j\in\mathcal{J}(e)}$ replicates a risk-free bond.

The total number of assets that can be traded in each time period t is given by $J=\sum_e J(e)$. Obviously with so many assets, there will be a continuum of asset choices by a household that will return the same value. A continuum of possible equilibrium asset choices is not a concern of mine so long as the aggregate asset payouts remain unchanged (no real consequences) and this will be the case.[6] I make the following two assumptions in order to endogenously bound the assets (as required for my fixed point argument).

A.1 $\forall e$, no linear combination of the asset payouts for $\left(z_j(e)\right)_{\forall j\in\mathcal{J}(e)}$ will replicate a risk-free bond.

Assumption $(A.1)$ dictates that a household $h: e^h(t)=e$ will have a total of $\sum_{\hat{e}\neq e} J(\hat{e})$ assets $\left(z_j^h(\hat{e})\right)_{\forall\hat{e}\neq e, j\in\mathcal{J}(\hat{e})}$ that are each a risk-free bond. I assume next that the household can only have

[4] I assume throughout this work that households cannot be distinguished by their realized income state. Thus, the timing of the model is as follows. A household reports to the market with its state realizations and receives its due asset payouts. The market instantaneously forgets the state realizations of the household (not a stretch with a continuum of households). When the household returns to sell assets, the market cannot observe the household's income state.

[5] As all households can transact in all the $\sum_e J(e)$ assets, a creditor gains no information about the realized state of a borrower by his choice of assets to trade.

[6] Additionally, I could have formulated the model to include a financial intermediary. The sole job of the intermediary would be to bundle all risk-free bonds for a household into a single asset. The intermediary would have to decide how the asset claims by one household for the single, combined risk-free asset would be split up amongst the asset markets for households using the particular assets to hedge risk. The resulting equilibrium would not change upon introducing such an intermediary.

asset purchases of one risk-free bond if it has asset purchases of all risk-free bonds (and identically for asset sales).

A.2 $\forall e$ and $\forall h : e^h(t) = e$, if $z_j^h(\hat{e}, t) \geq 0$ for some $\hat{e} \neq e$ and $j \in \mathcal{J}(\hat{e})$, then $z_{j'}^h(\hat{e}', t) \geq 0$ $\forall \hat{e}' \neq e$ and $\forall j' \in \mathcal{J}(\hat{e}')$. Likewise, if $z_j^h(\hat{e}, t) \leq 0$ for some $\hat{e} \neq e$ and $j \in \mathcal{J}(\hat{e})$, then $z_{j'}^h(\hat{e}', t) \leq 0$ $\forall \hat{e}' \neq e$ and $\forall j' \in \mathcal{J}(\hat{e}')$.

2.1.2 Complete markets

Markets are complete (all states of uncertainty are spanned) when for all income states $e \in \mathcal{E}$, $J(e) = N(e) - 1$ and no linear combination of the assets $\left(z_j^h(e, t)\right)_{j \in \mathcal{J}(e)}$ replicates a risk-free bond. I will denote $q_j(e, t)$ as the asset price of $z_j^h(e, t)$. The Lagrange multipliers of the household optimization problem are given by $\lambda^h\left(\left(e^h\right)^t\right) > 0$. Using dynamic programming, the infinite-dimensional household optimization problem can be reduced to a recursive structure (this is standard and discussed in a later subsection).

Complete markets and no aggregate uncertainty imply that the optimal consumption plan is stationary, $c^h\left(\left(e^h\right)^t\right) = c$ $\forall t$ and all realizations $\left(e^h\right)^t$. Take any $e \in \mathcal{E}$. For $h : e^h(t) = e$, the first order conditions with respect to $z_j^h(e, t)$ for some j are given by:

$$\left(\lambda^h\left(\left(e^h\right)^t\right), \lambda^h\left(\left(e^h\right)^t, 1\right), ..., \lambda^h\left(\left(e^h\right)^t\right), E\right) \begin{pmatrix} -q_j(e, t) \\ r_j(1|e) \\ \vdots \\ r_j(E|e) \end{pmatrix} = 0. \tag{2.1}$$

Facing a recursive problem, at each time period t, the household makes a contingency plan not knowing if $e^h(t+1) = 1, ..E - 1$, or E. Consider now the first order conditions with respect to $c^h\left(\left(e^h\right)^t\right), c^h\left(\left(e^h\right)^t, 1\right)$, ...and $c^h\left(\left(e^h\right)^t, E\right)$:

$$\begin{aligned} \beta^t \frac{1}{c^h\left(\left(e^h\right)^t\right)} - \lambda^h\left(\left(e^h\right)^t\right) &= 0 \\ \beta^{t+1} \frac{\pi(n|e)}{c^h\left(\left(e^h\right)^t, n\right)} - \lambda^h\left(\left(e^h\right)^t, n\right) &= 0 \quad \forall n \in \mathcal{E}. \end{aligned}$$

Then with the stationary consumption plan, I can rewrite equation (2.1) as:

$$q_j(e, t) = \beta\left(\pi(1|e), ..., \pi(E|e)\right) \begin{pmatrix} r_j(1|e) \\ \vdots \\ r_j(E|e) \end{pmatrix}.$$

By the definition of $r_j(e) = \sum_{e'} \pi(e'|e) r_j(e'|e)$, the asset price $q_j(e) = \beta r_j(e)$.

For $k : e^k(t) \neq e$, the first order conditions with respect to $z_j(e)$ for the same j are given by:

$$\left(\lambda^k\left(\left(e^k\right)^t\right), \lambda^k\left(\left(e^k\right)^t, 1\right), ..., \lambda^k\left(\left(e^k\right)^t\right), E\right) \begin{pmatrix} -q_j(e, t) \\ r_j(e) \\ \vdots \\ r_j(e) \end{pmatrix} = 0. \tag{2.2}$$

As above, the first order condition with respect to consumption yield that $\frac{\lambda^k\left(\left(e^k\right)^t,n\right)}{\lambda^k\left(\left(e^k\right)^t\right)} = \beta\pi\left(n|e^k(t)\right)$.
Thus, equation (2.2) can be rewritten as:

$$q_j(e,t) = \beta\left(\pi\left(1|e^k(t)\right), ..., \pi\left(E|e^k(t)\right)\right)\begin{pmatrix} r_j(e) \\ \vdots \\ r_j(e) \end{pmatrix}.$$

This is equivalent to $q_j(e) = \beta r_j(e)\sum_{e'}\pi\left(e'|e^k(t)\right)$ and by definition $\sum_{e'}\pi\left(e'|e^k(t)\right) = 1$.

This result is equivalent to that known to hold for GEI models and so the equilibrium allocation with complete markets is Pareto optimal.

2.1.3 Aggregate asset payouts

Consider the asset payouts

$$\int\limits_{h\in\mathcal{H}}\left\{\sum\nolimits_{\hat{e}\neq e}\left(\sum\nolimits_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e})z_j^h(\hat{e},t)\right) + \sum\nolimits_{j\in\mathcal{J}(e)} r_j(e'|e)z_j^h(e,t)\right\}d\Phi$$

where Φ is the measure of households to be defined shortly. Choose any $e\in\mathcal{E}$ and any $j\in\mathcal{J}(e)$. I can rewrite the equation for this asset (e,j) as:

$$r_j(e)\int\limits_{h:e^h(t)\neq e} z_j^h(e,t)d\Phi + \sum_{e'} r_j(e'|e)\int\limits_{h:(e^h(t),e^h(t+1))=(e,e')} z_j^h(e,t)d\Phi.$$

From the exact law of large numbers, $\pi\left(e'|e\right) = \frac{\int_{h:(e^h(t),e^h(t+1))=(e,e')} d\Phi}{\int_{h:e^h(t)=e} d\Phi}$ and since the integration is a linear operation, the above equation is equivalent to:

$$r_j(e)\int_{h:e^h(t)\neq e} z_j^h(e,t)d\Phi + \sum_{e'} r_j(e'|e)\pi\left(e'|e\right)\int_{h:e^h(t)=e} z_j^h(e,t)d\Phi.$$

By definition, $r_j(e) = \sum_{e'}\pi\left(e'|e\right)r_j\left(e'|e\right)$ and thus the asset payouts for $z_j(e)$ are reduced to:

$$r_j(e)\int z_j^h(e,t)d\Phi.$$

Thus, this general asset structure in which the assets are written to hedge against independent, idiosyncratic risk is consistent with the asset structure of GEI models. Moreover, it is actually easier to price assets in an incomplete markets model without bankruptcy as compared to GEI models. Once I know the asset price $q_j(e)$ for one asset $j\in\mathcal{J}(e)$, then all other assets are priced by no arbitrage:

$$q_{j'}(e',t) = q_j(e,t)\frac{r_{j'}(e')}{r_j(e)}\ \ \forall e' \text{ and } \forall j'\in\mathcal{J}(e').$$

I will remove bankruptcy from consideration in the next subsection and will prove the existence of a general financial equilibria with this asset structure.

2.2 Existence without bankruptcy

At every time period $t \geq 0$, each household must select a vector of assets $(z_j(e,t))_{j \in \mathcal{J}(e), \forall e \in \mathcal{E}}$. As mentioned above, assumptions $(A.1) - (A.2)$ are required so that the asset choices are bounded. In the recursive formulation using dynamic programming, it suffices to have the financial wealth (payouts from the financial markets only and not including endowments) as a state variable rather than the entire portfolio. As discussed shortly, the endogenous investment constraints will place a lower bound on the amount of wealth that can be transferred. Define the set $\mathcal{W} = [-b, \infty)$ as the set of potential wealth transfers where $-b < 0$ is the lower bound. Let $\mathcal{B}(\mathcal{W})$ be the Borel $\sigma-$algebra of $\mathcal{W}$ and $\mathcal{P}(\mathcal{E})$ to be the power set of the finite set $\mathcal{E}$. Then $\mathcal{M}$ is the set of all probability measures on the measurable space $M = (\mathcal{E} \times \mathcal{W}, \mathcal{P}(\mathcal{E}) \times \mathcal{B}(\mathcal{W}))$.

Thus, the measure across all households characterized by income state e and financial wealth w will be given by $\Phi \in \mathcal{M}$. The equilibrium prices will depend on the distribution Φ and therefore the optimal decisions will also. I will use primed variables to denote the variables in the next time period (and unprimed for the current time period).

The recursive formulation is given by:

$$V(e, w; \Phi) = \max_{c,z,w'} u(c) + \beta \sum\nolimits_{e'} \pi\left(e'|e\right) V(e', w'; \Phi') \tag{2.3}$$

$$s.t. \;\; c\left(e, w; \Phi\right) + q\left(\Phi\right) \circ z\left(e, w; \Phi\right) = \omega_e + w$$

$$w'(e') = \sum\nolimits_{\hat{e} \neq e} \left(\sum\nolimits_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}) \right) + \sum\nolimits_{j \in \mathcal{J}(e)} r_j(e'|e) z_j^h(e).$$

The initial wealth condition is $w_0 = 0$.

I will define the general financial equilibrium (without bankruptcy) and then prove its existence by (i) truncating the infinite-time equilibrium and proving the existence of the finite-length version and (ii) taking limits as the final time period of the truncated model becomes unbounded.

Definition 1 *A general financial equilibrium is a collection of mappings* $c : \mathcal{E} \times \mathcal{W} \times \mathcal{M} \to \mathbb{R}_+$ *(consumption),* $z : \mathcal{E} \times \mathcal{W} \times \mathcal{M} \to \mathbb{R}^J$ *(assets), and* $w' : \mathcal{E} \times \mathcal{W} \times \mathcal{M} \to \mathcal{W}$ *(next period wealth) identical across all households, and* $q : \mathcal{M} \to \mathbb{R}_+^J$ *(asset prices) such that the dynamic problem* (2.3) *is satisfied and* $\forall \Phi \in \mathcal{M}$:

1. $\displaystyle\int c\left(e, w; \Phi\right) d\Phi = \overline{\omega}.$

2. $\displaystyle\int z\left(e, w; \Phi\right) d\Phi = \overrightarrow{0}$ *for all* J *assets.*

3. $\displaystyle\int w'\left(e, w; \Phi\right) d\Phi = 0.$

4. *Given* $Q\left(\left(e, w\right), \left(\mathcal{E}', \mathcal{W}'\right)\right) = \sum\nolimits_{e' \in \mathcal{E}'} \left\{ \begin{array}{cc} \pi\left(e'|e\right) & \text{if } w'\left(e, w\right) \in \mathcal{W}' \\ 0 & \text{otherwise} \end{array} \right\}$, *then*

$$\Phi'\left(\mathcal{E}', \mathcal{W}'\right) = \int Q\left(\left(e, w\right), \left(\mathcal{E}', \mathcal{W}'\right)\right) \Phi\left(de \times dw\right).$$

Let the time horizon be discrete with $t \in \{0, ..., \bar{T}\}$ and $\bar{T} < \infty$. I will show in theorem 1 that a general financial equilibrium exists for this finite time horizon. The proof is contained in appendix A.

Theorem 1 *Under assumptions* $(A.1)-(A.2)$, *the truncated general financial equilibrium exists for all parameters* $\mathcal{E}$, Π, β, $u(\cdot)$, *and asset payouts* $(r_j(e'|e))_{\forall e, j\in\mathcal{J}(e)}$ *.*

Now, I will consider the steps required to show that an equilibrium with the discrete, infinite-length time horizon exists and that this infinite-length equilibrium is actually the limit of the appropriately defined truncated equilibrium as $\bar{T}\to\infty$.

The following two conditions are seemingly unrelated as they are derived from distinct lines of economic analysis, yet surprisingly equivalent. First, the debt constraint is a fundamental feature of applied work. Ideally, the debt constraint should be such that it is nonbinding in equilibrium so as not to introduce any additional inefficiencies into the model. It is written as:

$$\inf_t \textstyle\sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,t) z_j^h(e,t) > -\infty. \tag{2.4}$$

This condition requires that the value of a portfolio remains bounded for all possible realizations of the random Markov process.

Second, the transversality condition is a necessary condition of the household's optimization problem and is natural in theoretical work. Consider the household's spot budget constraints written for the realization $\left(e^h\right)^t = \left(e^h(0),...,e^h(t)\right)$:

$$c\left(e,w;\Phi\right) + q\left(\Phi\right)\circ z\left(e,w;\Phi\right) = \omega_e + w. \tag{2.5}$$

The Lagrange multiplier associated with each of the budget constraints in the household's problem (2.3) (and intuitively, the relative price specific to that household of transferring wealth into the potential state $\left(e^h\right)^t$) is defined as $\lambda^h\left(\left(e^h\right)^t\right)$. To write down a single budget constraint with the household making contingent transactions over the entire length of the model, I would multiply each budget constraint (2.5) by $\lambda^h\left(\left(e^h\right)^t\right)$ and sum. From the household's first-order conditions with respect to assets (necessary conditions for solutions to (2.3)), terms cancel and I am left with:

$$\begin{gathered}\sum_{t=0}^{\bar{T}} \sum_{(e^h)^t} \lambda^h\left(\left(e^h\right)^t\right)\left(c^h\left(\left(e^h\right)^t\right) - \omega_{e^h(t)}\right) + \\ \sum_{(e^h)^{\bar{T}}} \lambda^h\left(\left(e^h\right)^{\bar{T}}\right)\left(\textstyle\sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,\bar{T}) z_j^h(e,\bar{T})\right) = 0.\end{gathered} \tag{2.6}$$

With a finite time horizon ($\bar{T}<\infty$), in the final period, the assets $\left(z_j^h(e,\bar{T})\right)_{\forall e, j\in\mathcal{J}(e)}$ are not available for trade. Thus equation (2.6) is equivalent to the Arrow-Debreu budget constraint. Without a final time period ($\bar{T}\to\infty$), in the limit, the term $\sum_{(e^h)^{\bar{T}}} \lambda^h\left(\left(e^h\right)^{\bar{T}}\right)\sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,\bar{T}) z_j^h(e,\bar{T})$ must vanish. This is necessary for household optimization (equivalence with Arrow-Debreu budget constraint and the necessary condition of optimality that does not allow the optimal portfolio to "leave value" at infinity). Thus, the transversality condition can be written as:

$$\lim_{t\to\infty} \sum_{(e^h)^{t-1}} \sum_e \lambda^h\left(\left(e^h\right)^{t-1}, e\right) \textstyle\sum_{j\in\mathcal{J}(e)} q_j(e,t) z_j^h(e,t) = 0. \tag{2.7}$$

Magill and Quinzii prove the equivalence of the debt constraint (2.4) and the transversality condition (2.7) (though the proof of (2.4) $\Longrightarrow$ (2.7) requires the hard-to-justify assumption of *competitive perceptions* [the idea originally from Grossman and Hart, 1979]). I will use the transversality condition (a necessary condition for household optimization in this infinite-time, incomplete markets setup) to rule out Ponzi schemes and guarantee that the limits as $\bar{T}\to\infty$ are well-defined. The proof of theorem 2 is contained in appendix A.

Theorem 2 *If a truncated general financial equilibrium exists for any finite length* $\bar{T}$, *then the limit as* $\bar{T}\to\infty$ *is well defined (all equilibrium variables are uniformly bounded) and the equilibrium at the limit is the desired general financial equilibrium for the infinite-time horizon.*

2.3 Existence of bankruptcy equilibria

In the subsection 2.3.1, I will detail how the opportunity to declare bankruptcy (and the costs of doing so) are incorporated into the household budget set. Next, in subsection 2.3.2, I will discuss the aggregate consistency conditions required so that markets clear in the presence of bankruptcy. Then, in subsection 2.3.3, I will discuss how the creditors set the asset prices across pools of borrowers. Finally, in subsection 2.3.4, I will define a bankruptcy equilibrium and state the theorem of existence for this equilibrium. The proofs for this existence result will be contained in appendix A.

2.3.1 Bankruptcy budget set

As discussed in the introduction, I will be modeling the chapter 7 bankruptcy decisions of households in the unsecured credit markets. The bankruptcy problem is inherently a nonconvex one: either a household *is* or *is not* bankrupt. To circumvent this nonconvexity, I require a continuum of households. This fits perfectly into the framework of the Bewley model in which a continuum of agents face idiosyncratic endowment risk.

I wish to incorporate the following bankruptcy restrictions into the model:

1. A household cannot declare bankruptcy twice in any 6-year period.
2. According to recent legislation[7], households that fail the "means test" (roughly speaking, the income level is above the state median) *cannot* declare bankruptcy.
3. A bankruptcy declaration remains on a household's credit report for 10 years.

With these restrictions, it seems simplest to model a time period in my model as lasting for 5 years. As defined in section 2, the median income is given by ω_{med}. Thus, a household h cannot declare bankruptcy in time period t if $\omega^h(t) > \omega_{med}$.

The following two assumptions are required to obtain an endogenous bound on the asset sales of households:

A.3 $\forall e$, $r_j(e'|e) > 0 \ \ \forall (j,e)$ with $j \in \mathcal{J}(e)$ and $e' : \pi(e'|e) > 0$.

A.4 $\forall e$, $\exists e'$ s.t. $\omega_{e'} > \omega_{med}$ and $\pi(e'|e) > 0$.

Assumption $(A.3)$ says that all the relevant asset payouts are strictly positive. Assumption $(A.4)$ says that with strictly positive probability, all households may realize an income state in the next time period that will prevent them from declaring bankruptcy ($\omega^h(t) > \omega_{med}$).

I will introduce the bankruptcy indicator variable $b^h(t) \in \mathcal{B} = \{0,1,2\}$ such that $b^h(t) = 0$ if a household declares bankruptcy at time period t. According to restriction 1, if $b^h(t) = 0$, the household h cannot declare bankruptcy again in time period $t+1$. Thus, $b^h(t) = 0$ implies $b^h(t+1) = 1$. In time period $t+2$, if the household declares bankruptcy, the bankruptcy indicator resets $b^h(t+2) = 0$ and otherwise, $b^h(t+2) = 2$. When $b^h(t+2) = 2$, the bankruptcy flag has been removed from the household's credit history and so household h is indistinguishable from another household who has never declared bankruptcy. If $b^h(t+2) = 2$ and household h again decides to remain solvent in time period $t+3$, the bankruptcy indicator remains $b^h(t+3) = 2$.

Summarizing, the set $\mathcal{T}_h^*$ will be the set of time periods (endogenous) at which household h is legally barred from declaring bankruptcy. The set is defined as $\mathcal{T}_h^* = \left\{t : \omega^h(t) > \omega_{med} \text{ or } b^h(t-1) = 0\right\}$. The next step is to introduce the costs and benefits of bankruptcy to decide if a household would *want* to declare bankruptcy.

[7] Bankruptcy Abuse Prevention and Consumer Protection Act of 2005.

The declaration of bankruptcy forces the household to forfeit all of its nonexempt assets. I assume for simplicity that there are no bankruptcy exemptions, so a bankruptcy declaration would force a household to forfeit its entire portfolio of assets. Realistic legal exemptions can be included as an extension of my setup without changing the qualitative results.[8]

Define the set of bankrupt households as $\mathcal{H}'_t = \{h \in \mathcal{H} : b^h(t) = 0\}$. The financial payout to bankrupt households is given by the negative cost of bankruptcy. The cost of bankruptcy at time period $t \geq 1$ with $\left(e^h(t-1), e^h(t)\right) = (e, e')$ is parameterized by $\alpha >> 0$ and given by:

$$\alpha \circ r \circ z^h = \sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} \alpha_j(\hat{e}) r_j(\hat{e}) \left(z^h_j(\hat{e}, t-1) \right)^+ \right)$$

$$+ \sum_{j \in \mathcal{J}(e)} \alpha_j(e) r_j(e'|e) \left(z^h_j(e, t-1) \right)^+ .$$

A.5 The financial payout of a household $h \in \mathcal{H}'_t$ is given by $-\alpha \circ r \circ z^h$ with $\alpha >> 0$.

Thus, a household only pays a bankruptcy cost if it has long positions (asset purchases) in its portfolio at the time of the bankruptcy declaration. To receive a bankruptcy discharge, a household must submit a detailed listing of all assets with positive value that it maintains. In an actual bankruptcy proceeding with exemptions, a bankrupt household would typically hire a lawyer to file the necessary legal documents. The cost for the legal help would be proportional to the vale of the household's asset purchases. Thus, the assumption that the cost of bankruptcy is strictly increasing in asset purchases is in keeping with the actual workings of the bankruptcy process.

Based on the bankruptcy history (as given by the bankruptcy indicator $b^h(t)$), borrowers may be screened into credit groups by lenders. If a household h with bankruptcy history $b^h(t)$ seeks to sell an asset $z^h_j(e,t) < 0$ in time period t, the asset price is $q_j^{b^h(t)}(e,t)$. If the same household seeks to buy an asset $z^h_j(e,t) \geq 0$, the price is the market-clearing price $q_j(e,t)$, which does not depend on $b^h(t)$. Thus, I will define the household-specific asset prices $q^h_j(e,t)$ $\forall e$, $\forall j \in \mathcal{J}(e)$, and $\forall t \geq 1$ as:

$$\begin{array}{ll} q^h_j(e,t) = q_j(e,t) & \text{if } z^h_j(e,t) \geq 0 \\ q^h_j(e,t) = q_j^{b^h(t)}(e,t) & \text{if } z^h_j(e,t) < 0 \end{array} .^{9} \tag{2.8}$$

In the presence of bankruptcy, in order for markets to clear, the payouts to creditors must be appropriately diluted. The overall repayment rate $\rho_j(e,t) \geq 0$ will be the fraction of the total asset payout in period t that is expected given an asset purchase $z^h_j(e,t-1)$ in $t-1$. This repayment rate is endogenously determined. Recalling that all solvent households must completely fulfill their commitment, I define the household-specific divident payouts r^h as:

$$\begin{array}{ll} r^h_j(e'|e,t) = r_j(e'|e) & \text{if } z^h_j(e,t-1) < 0 \\ r^h_j(e'|e,t) = \rho_j(e,t) r_j(e'|e) & \text{if } z^h_j(e,t-1) \geq 0 \end{array} .$$

As before, $r^h_j(e,t) = \sum_{e'} \pi\left(e'|e\right) r_j\left(e'|e\right) = r_j(e)$ if $z^h_j(e,t-1) < 0$ and $r^h_j(e,t) = \sum_{e'} \pi\left(e'|e\right) r^h_j\left(e'|e,t\right) = \rho_j(e,t) r_j(e)$ if $z^h_j(e,t-1) \geq 0$.

Let $\mathcal{P}(\mathcal{B})$ be the power set of the finite set $\mathcal{B} = \{0,1,2\}$. Then $\mathcal{M}$ is the set of all probability measures on the measurable space $M = \left(\mathcal{E} \times \mathcal{W} \times \mathcal{B}, \mathcal{P}(\mathcal{E}) \times \mathcal{B}(\mathcal{W}) \times \mathcal{P}(\mathcal{B})\right)$.

[8] Quantitatively, one would expect the addition of bankruptcy exemptions to increase the number of bankruptcy declarations. This is exactly the case, but as with Zame (1993), the welfare effects of introducing exemptions is ambiguous.

[9] Setting asset prices by pools is equivalent to the price-spread assumption of Bisin and Gottardi (1999) that is a necessary condition for existence of equilibria under asymmetric information.

Thus, the measure across all households characterized by income state e, financial wealth w, and bankruptcy history b will be given by $\Phi \in \mathcal{M}$. The recursive formulation of the bankruptcy equilibria is given by:

$$V(e,w,b;\Phi) = \max_{c,z,w',b'} u(c) + \beta \sum\nolimits_{e'} \pi\left(e'|e\right) V(e',w',b';\Phi') \tag{2.9}$$

$$s.t. \;\; c\left(e,w,b;\Phi\right) + q^h\left(\Phi\right) \circ z\left(e,w,b;\Phi\right) = \omega_e + w$$

$$\text{Solvent } (b'>0):$$

$$w'(e') = \sum\nolimits_{\hat{e}\neq e} \left(\sum\nolimits_{j\in\mathcal{J}(\hat{e})} r_j^h(\hat{e}) z_j^h(\hat{e}) \right) + \sum\nolimits_{j\in\mathcal{J}(e)} r_j^h(e'|e) z_j^h(e)$$

$$\text{Bankrupt } (b'=0):$$

$$w'(e') = \max\left\{ r^h \circ z^h, -\alpha \circ r \circ z^h \right\}$$

$$\begin{aligned} \text{where } r^h \circ z^h &= \sum\nolimits_{\hat{e}\neq e} \left(\sum\nolimits_{j\in\mathcal{J}(\hat{e})} r_j^h(\hat{e}) z_j^h(\hat{e}) \right) + \sum\nolimits_{j\in\mathcal{J}(e)} r_j^h(e'|e) z_j^h(e) \\ \text{and } \alpha \circ r \circ z^h &= \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} \alpha_j(\hat{e}) r_j(\hat{e}) \left(z_j^h(\hat{e})\right)^+ \right) + \sum_{j\in\mathcal{J}(e)} \alpha_j(e) r_j(e'|e) \left(z_j^h(e)\right)^+ . \end{aligned}$$

Recall, that a household can only legally declare bankruptcy if $t \notin \mathcal{T}_h^*$. The financial wealth of a bankrupt household is defined such that the household can never be forced to repay more than it initially owed.[10]

2.3.2 Aggregate consistency

Choose any time period $t \geq 1$. There are two types of bankrupt households $h \in \mathcal{H}_t'$ to consider: those such that $w'(e') = r^h \circ z^h$ and those such that $w'(e') = -\alpha \circ r \circ z^h$. I consider the latter case first (the household repays less than what is owed). Assume that for a bankrupt household $h \in \mathcal{H}_t'$ with $\left(e^h(t-1), e^h(t)\right) = (e, e')$, the total value

$$\begin{aligned} num = & \sum_{\hat{e}\neq e} \sum_{j\in\mathcal{J}(\hat{e})} \left(r_j^h(\hat{e},t) + \alpha_j(\hat{e}) r_j(\hat{e}) \right) \left(z_j^h(\hat{e},t-1) \right)^+ + ... \\ & ... \sum_{j\in\mathcal{J}(e)} \left(r_j^h(e'|e,t) + \alpha_j(e) r_j(e'|e) \right) \left(z_j^h(e,t-1) \right)^+ \geq 0 \end{aligned}$$

is split up among the J asset pools. Define

$$den = \sum_{\hat{e}\neq e} \sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e}) \left(z_j^h(\hat{e},t-1) \right)^- + \sum_{j\in\mathcal{J}(e)} r_j(e'|e) \left(z_j^h(e,t-1) \right)^- .$$

Then the fraction

$$\frac{r_j(\hat{e}) \left(z_j^h(\hat{e},t-1) \right)^-}{den}$$

of the positive value num will be returned to asset pool $(\hat{e}, j)$ with $\hat{e} \neq e$ and $j \in \mathcal{J}(\hat{e})$ and the fraction

$$\frac{r_j(e'|e) \left(z_j^h(e,t-1) \right)^-}{den}$$

[10] The question of why a household would choose to declare bankruptcy even though the wealth from bankruptcy is less than the wealth from solvency will be discussed later (intuitively, it must be that the access to credit is actually enhanced by having a bankruptcy flag on a household's credit report).

of the positive value num will be returned to asset pool (e,j) with $j \in \mathcal{J}(e)$. The sum of these fractions over all J assets equals 1.

Define the individual repayment rate $\delta^h(t) \geq 0$ as

$$\delta^h(t) = -\frac{num}{den}$$

for $h \in \mathcal{H}'_t$ where num and den are defined above. The variable satisfies $\delta^h(t) \geq 0$ trivially. For this type of bankrupt household, $-\alpha \circ r \circ z^h \geq r^h \circ z^h$ and thus $\delta^h(t) \leq 1$.

Now consider bankrupt households such that $w'(e') = r^h \circ z^h$. For these households, the individual repayment rate $\delta^h(t) = 1$ as the households pay back exactly the amount owed. Likewise, the individual repayment rate $\delta^h(t) = 1$ for solvent households $h \notin \mathcal{H}'_t$.

Necessary and sufficient conditions for existence of equilibria are the aggregate consistency (AC) conditions. These conditions state that the expectations held by creditors about the overall repayment rates ρ must be equal to the actual weighted individual repayment rates across all borrowers:

$$\rho_j(e,t)\int_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^+ d\Phi = -\int_{h\in\mathcal{H}} \delta^h(t)\left(z_j^h(e,t-1)\right)^- d\Phi \tag{2.10}$$

$\forall t \geq 1, \forall e$, and $\forall j \in \mathcal{J}(e)$. From (2.10), the overall repayment rate $\rho_j(e,t) \geq 0$. If $\int_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^+ d\Phi = 0$ (no trade in this asset), the rate $\rho_j(e,t)$ can be anything. This keeps open the possibility that undue pessimism about the creditor payouts can be self-fulfilling in equilibrium (as in Dubey et. al. (2005)).[11] However, if $\int_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^+ d\Phi > 0$, the overall repayment rate $\rho_j(e,t) \leq 1$. This is best seen by using the market clearing condition $\int_{h\in\mathcal{H}} z_j^h(e,t-1)d\Phi = 0$ to rewrite equation (2.10) as:

$$\rho_j(e,t) = 1 + \frac{\displaystyle\int_{h\in\mathcal{H}'_t} (1-\delta^h(t))\left(z_j^h(e,t-1)\right)^- d\Phi}{\displaystyle\int_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^+ d\Phi}. \tag{2.11}$$

2.3.3 Asset prices by pools

By no arbitrage, the market clearing asset price $q_j(e,t-1)$ is some positive scalar of the expected payout $\rho_j(e,t)r_j(e)$. Choose any element $b \in \mathcal{B} = \{0,1,2\}$. Consider the resulting equilibrium prices when creditors only lend to $h : b^h(t-1) = b$. For $b^h(t-1) = 0$, the creditors know that these households will repay the entire amount because they cannot (by law) declare bankruptcy in period t. Thus, we only need to consider $b \in \{1,2\}$. Define $\mathcal{H}(b)$ as the set of households with $b^h(t) = b$ (where the time period is known given the context). The repayment rate for pool b, $\rho_j^b(e,t)$, is defined by:

$$\rho_j^b(e,t)\int_{h\in\mathcal{H}(b)} -\left(z_j^h(e,t-1)\right)^- d\Phi = -\int_{h\in\mathcal{H}(b)} \delta^h(t)\left(z_j^h(e,t-1)\right)^- d\Phi. \tag{2.12}$$

As with the overall repayment rate, the repayment rate for pool b is nonnegative, $\rho_j^b(e,t) \geq 0$. If $\int_{h\in\mathcal{H}(b)} -\left(z_j^h(e,t-1)\right)^- d\Phi = 0$ (that is, there are no debtors in pool b), then the repayment rate for

[11] This undue pessimism is in fact irrational (though still possible as an equilibrium). With the iid Markov process, there will always exist some realization of the endowment process in the next time period such that a subset of households cannot declare bankruptcy (by assumption ($A.4$)). This subset will have unity repayment rates and keep the overall repayment rate ρ bounded above 0.

pool b, $\rho_j^b(e,t)$, can take on any value (in particular, $\rho_j^b(e,t) = 0$ meaning that this pool of borowers cannot obtain loans).

The asset price $q_j^b(e,t-1)$ faced by the pool b is calculated as:

$$q_j^b(e,t-1) = q_j(e,t-1) \cdot E_{t-1}\left(\frac{\rho_j^b(e,t)}{\rho_j(e,t)}\right) \quad \forall t \geq 1, \forall b, \forall e, \text{ and } \forall j \in \mathcal{J}(e) \tag{2.13}$$

where $\rho_j^0(e,t) = 1\ \forall t, \forall e$, and $\forall j \in \mathcal{J}(e)$ as discussed above.[12]

Adding up (2.12) over all pools $b \in \mathcal{B}$ and using market clearing shows how the overall repayment rate $\rho_j(e,t)$ is related to the repayment rates across pools $\rho_j^b(e,t)$:

$$\rho_j(e,t) = \frac{\sum_{b\in\mathcal{B}} \rho_j^b(e,t) \int_{h\in\mathcal{H}(b)} \left(z_j^h(e,t-1)\right)^- d\Phi}{\int_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^- d\Phi}.$$

From (2.13), the same relation will hold for the market clearing price $q_j(e,t-1)$ and the pool asset prices $q_j^b(e,t-1)$:

$$q_j(e,t-1) = \frac{\sum_{b\in\mathcal{B}} q_j^b(e,t-1) \int_{h\in\mathcal{H}(b)} \left(z_j^h(e,t-1)\right)^- d\Phi}{\int_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^- d\Phi}. \tag{2.14}$$

Equation (2.14) is essential in the proof of existence. This is because equation (2.14) implies

$$\int_{h\in\mathcal{H}} q_j^h(e,t-1) z_j^h(e,t-1) d\Phi = q_j(e,t-1) \int_{h\in\mathcal{H}} z_j^h(e,t-1) d\Phi. \tag{2.15}$$

The term $\int_{h\in\mathcal{H}} q_j^h(e,t-1) z_j^h(e,t-1) d\Phi$ n the left-hand side of (2.15) comes from the summed budget constraints of households (Walras' law). The summed budget constraints must be used to show that the equilibrium price $q_j(e,t-1)$ will be such that markets clear: $\int_{h\in\mathcal{H}} z_j^h(e,t-1) d\Phi = 0$. The equality of (2.15) maintains this connection between Walras' law and market clearing that is required in the proof of existence.

2.3.4 Existence of bankruptcy equilibria

I am now prepared to define a bankruptcy equilibrium. The distribution Φ of households across labor endowment e, wealth w, and bankruptcy history b is allowed to vary over time.

Definition 2 *A bankruptcy equilibrium is a collection of mappings* $c : \mathcal{E} \times \mathcal{W} \times \mathcal{B} \times \mathcal{M} \to \mathbb{R}_+$ *(consumption),* $z : \mathcal{E} \times \mathcal{W} \times \mathcal{B} \times \mathcal{M} \to \mathbb{R}^J$ *(assets),* $w' : \mathcal{E} \times \mathcal{W} \times \mathcal{B} \times \mathcal{M} \to \mathcal{W}$ *(next period wealth), and* $b' : \mathcal{E} \times \mathcal{W} \times \mathcal{B} \times \mathcal{M} \to \mathcal{B}$ *(bankruptcy) identical across all households, and* $q : \mathcal{M} \to \mathbb{R}_+^J$ *(asset prices) and* $\rho : \mathcal{M} \to \mathbb{R}_+^J$ *(repayment rates) such that the dynamic problem* (2.9) *is satisfied and* $\forall \Phi \in \mathcal{M}$:

[12] The expectation at $t-1$ is equal to the realized repayment rates at t in equilibrium.

1. $\int c(e,w,b;\Phi)\,d\Phi = \overline{\omega}$.

2. $\int z(e,w,b;\Phi)\,d\Phi = \overrightarrow{0}$ *for all* J *assets.*

3. $\int w'(e,w,b;\Phi)\,d\Phi = 0$.

4. $\rho_j(e,t)$ *satisfies* (2.10) $\forall t, \forall e,$ *and* $\forall j \in \mathcal{J}(e)$.

5. $q_j^b(e,t-1)$ *is related to* $q_j(e,t-1)$ *using* (2.13) *and* (2.12) $\forall b \in \{1,2\}, \forall t, \forall e,$ *and* $\forall j \in \mathcal{J}(e)$.

6. *Given* $Q\left((e,w,b),(\mathcal{E}',\mathcal{W}',\mathcal{B}')\right) =$
$\sum_{e'\in\mathcal{E}'}\left\{\begin{array}{ll}\pi(e'|e) & \text{if } w'(e,w,b)\in\mathcal{W}' \text{ and } b'(e,w,b)\in\mathcal{B}' \\ 0 & \text{otherwise}\end{array}\right\}$, *then*

$$\Phi'(\mathcal{E}',\mathcal{W}',\mathcal{B}') = \int Q\left((e,w,b),(\mathcal{E}',\mathcal{W}',\mathcal{B}')\right)\Phi(de\times dw\times db).$$

Let the time horizon be discrete with $t \in \{0,...,\bar{T}\}$ and $\bar{T} < \infty$. I will show in theorem 3 that a bankruptcy equilibrium exists for this time horizon. The proof is contained in appendix A.

Theorem 3 *Under assumptions* $(A.1)-(A.5)$, *the truncated bankruptcy equilibrium exists for all parameters* $\mathcal{E}$, Π, β, $u(\cdot)$, α, *and asset payouts* $(r_j(e'|e))_{\forall e, j\in\mathcal{J}(e)}$.

The transversality condition for the 'no-bankruptcy' model needs to be updated since the value of a household's portfolio depends on the asset prices of each asset and the prices depend on whether a household is a creditor or a debtor on a particular asset. Recall the definition of household-specific asset prices in equations (2.8) :

$$\begin{array}{ll} q_j^h(e,t) = q_j(e,t) & \text{if } z_j^h(e,t)\geq 0 \\ q_j^h(e,t) = q_j^{b^h(t)}(e,t) & \text{if } z_j^h(e,t) < 0 \end{array}. \tag{2.8}$$

The transversality condition, which is a necessary condition for the household optimization problem (2.9), will thus be given as:

$$\lim_{t\to\infty}\sum_{(e^h)^{t-1}}\sum_e \lambda^h\left(\left(e^h\right)^{t-1},e\right)\sum_{j\in\mathcal{J}(e)} q_j^h(e,t)z_j^h(e,t) = 0. \tag{2.16}$$

In the general financial model without bankruptcy, the transversality served to rule out Ponzi schemes. In this bankruptcy model, one may wonder if households will run up unbounded debt as in a Ponzi scheme and then simply declare bankruptcy to unload this debt. The assumption $(A.4)$ plays an important role in ruling out this possibility. This assumption keeps open the possibility that the household will realize an unending sequence of income states with an income level above the median level in all of them. As the summation in the transversality condition (2.16) is over all possible realizations of the Markov process, even one realization $\left(e^h\right)^t$ with $\lambda^h\left(\left(e^h\right)^t\right)\sum_{j\in\mathcal{J}(e)} q_j^h(e,t)z_j^h(e,t) < 0$ would prevent the necessary condition (2.16) from binding.

I will use the transversality condition (a necessary condition for household optimization in this infinite-time, incomplete markets setup) to rule out Ponzi schemes and guarantee that the limits as $\bar{T}\to\infty$ are well-defined. The proof of theorem 4 is identical to the proof of theorem 2 and is therefore omitted.

Theorem 4 *If a truncated bankruptcy equilibrium exists for any finite length $\bar{T}$, then the limit as $\bar{T} \to \infty$ is well defined (all equilibrium variables are uniformly bounded) and the equilibrium at the limit is the desired bankruptcy equilibrium for the infinite-time horizon.*

3 The Ordering of Repayment Rates by Pools

This section will give conditions on the parameters (mostly the transition matrix) such that the repayment rates will be ordered by the bankruptcy indicator $b^h(t)$. That is, the pool of households with indicator $b^h(t) = 1$ will have a lower expected repayment rate (and thus face lower asset prices) than the pool of households with indicator $b^h(t) = 2$. This result will then be extended for a bankruptcy model in which $b^h(t) \in \{0, 1, 2, ..., I\}$ and in which a household's bankruptcy flag is not removed until I periods have elapsed since the bankruptcy was declared.

3.1 Two period bankruptcy effects

In the standard model, $b^h(t) \in \{0, 1, 2\}$. Let there be 3 income states, $E = 3$. To illustrate the key point that a creditor's observation of a borrower's pecuniary income does not imply that it can forecast its repayment likelihood, consider $\omega_1 = \omega_2 \leq \omega_{med} < \omega_3$. Income states $e = 1$ and $e = 2$ differ only in the persistence of that low income level. Households with current income state $e = 1$ and $e = 2$ can declare bankruptcy, while households with current income state $e = 3$ cannot (from assumption $(A.4)$).

The ordering of repayment rates will be different for each asset that is traded. To obtain a precise result, I must limit the model so that only one asset, a risk-free bond, is traded in every period. The parameters under consideration for this simple economy are $\mathcal{E} = \{\omega_1, \omega_2, \omega_3\}$, Π, β, and $u(\cdot)$. To obtain the result requires joint conditions on all these parameters. However, I am able to write the conditions so that they directly depend on Π and implicitly on the rest.[13] Let the transition matrix Π be sparse with terms given by:

$$\begin{bmatrix} \pi(1|1) = \gamma_1 & & \pi(3|1) = 1 - \gamma_1 \\ & \pi(2|2) = \gamma_2 & \pi(3|2) = 1 - \gamma_2 \\ \pi(1|3) = \frac{1-\gamma}{2} & \pi(2|3) = \frac{1-\gamma}{2} & \pi(3|3) = \gamma \end{bmatrix}. \tag{3.1}$$

The parameters $\gamma_1, \gamma_2, \gamma \in (0, 1)$. The first income state will be more persistent than the second: $\gamma_1 > \gamma_2$.

Recall the household's maximization problem as given in (2.9). In each time period t, the household enters with income state $e^h(t)$ and wealth $w^h(t)$. Suppose that $e^h(t) = 1$ or $e^h(t) = 2$. For either state, the household has the following binary decision to make. It can either make the optimal asset choice while planning to remain solvent in the following period or it can make the optimal asset choice while planning to declare bankruptcy given a realization of a state in the next period that allows for bankruptcy. For $e^h(t) = 1$, the only one (of two) state realizations allowing for bankruptcy is $e^h(t + 1) = 1$ (and analogously for $e^h(t) = 2$). The decision about whether to plan to declare bankruptcy or not is conditional on the household's current wealth level $w^h(t)$ and the persistence of its income state (either γ_1 or γ_2). I will define the bankruptcy cutoff $\bar{\gamma}(w)$ as a function of wealth so that a household will plan to declare bankruptcy at time period $t + 1$ iff $\gamma_i \geq \bar{\gamma}(w^h(t))$ where $i = e^h(t)$. This bankruptcy cutoff $\bar{\gamma}(w)$ is a strictly increasing function of wealth w. To see this, notice that for a fixed persistence, a larger wealth will decrease the chances of a bankruptcy declaration.

[13] Obviously, the endogenous financing and bankruptcy decisions of households are functions of all parameters, but I will be able to write the conditions for the result in terms of the transition matrix Π.

A household having just declared bankruptcy ($b^h(t-1)=0$) will have wealth $w^h(t-1)=0$. Suppose that $e^h(t-1)=e^h(t)=1$. From the bankruptcy setup, the household cannot declare bankruptcy in time period t ($b^h(t)=1$). The wealth brought into the period t will be denoted as $w(\gamma_1)=w\left(t : e(t-1)=e(t)=1, w(t-1)=0, b(t)=1\right)$. The first condition to obtain the ordering of repayment rates is that:

$$\gamma_1 > \bar{\gamma}(w(\gamma_1)) > \gamma_2. \tag{3.2}$$

This condition states that a household with $e^h(t-1)=1$ and $b^h(t-1)=0$ will declare bankruptcy in time period $t+1$ if $e^h(t)=e^h(t+1)=1$, while a household with $b^h(t-1)=0$ and $e^h(t-1)=e^h(t)=e^h(t+1)=2$ will not declare in time period $t+1$.

The second condition places an additional restriction of (γ_1,γ_2). This condition states that the weighted repayment rates of all households that declare from pool $b^h(t)=1$ (the households declaring from this pool are only households with $e^h(t-1)=e^h(t)=e^h(t+1)=1$) will be strictly less than the expected repayment rates of all households that declare from pool $b^h(t)=2$. The description of the households that declare from pool $b^h(t)=2$ will be any household with $e^h(t+1)=1$ or $e^h(t+1)=2$, but further specification requires a detailed accounting of households' possible state realizations. The endogenous bankruptcy decisions of households from pool $b(t)=2$ are governed by the (potentially) infinite history of income state realizations up to that time period. The second condition is given as:

$$\frac{(1-\gamma_2)(\gamma_1)^4}{(1-\gamma_2)(\gamma_1)^3+(1-\gamma_1)(\gamma_2)^3} > \frac{(\gamma_2)^3+\gamma_1\frac{2(1-\gamma_2)}{(\gamma_2)^2+2(1-\gamma_2)}}{(\gamma_2)^2+\frac{1}{2}\gamma_2+\frac{1}{3}+\frac{2(1-\gamma_2)}{(\gamma_2)^2+2(1-\gamma_2)}}. \tag{3.3}$$

The values for (γ_1,γ_2) that satisfy both $\gamma_1>\gamma_2$ and condition (3.3) are plotted in figure 1 below (only the values for $\gamma_2 \geq 0.6$ are displayed in order to focus on values of interest for the persistence parameters).

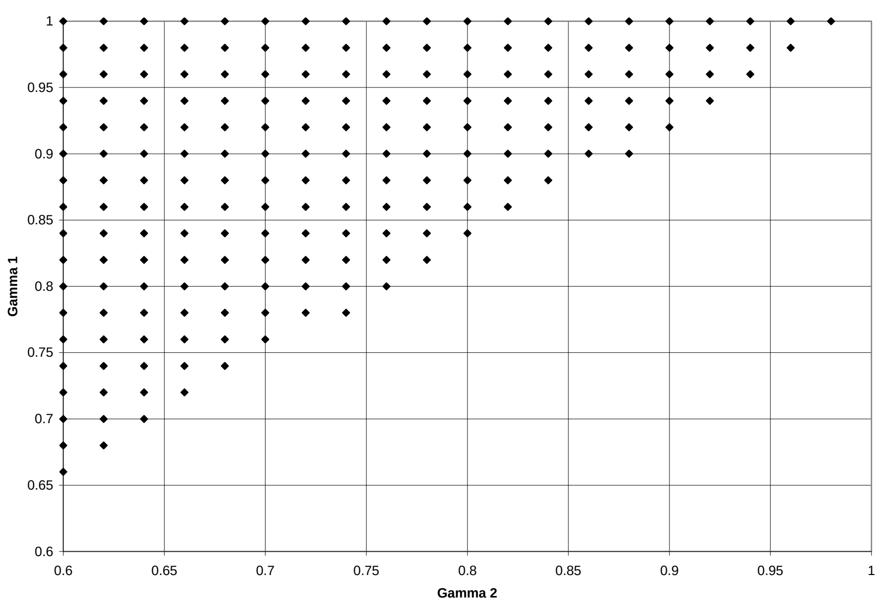

Figure 1: Values of (γ_1,γ_2) satisfying (3.3) and $\gamma_1>\gamma_2$.

The proof of theorem 5 is contained in appendix B.

Theorem 5 *Under the simple economy presented with $E = 3$, $\omega_1 = \omega_2 \leq \omega_{med} < \omega_3$, and Π given by (3.1), then if (γ_1, γ_2) such that $\gamma_1 > \gamma_2$ additionally satisfy*

- $\gamma_1 > \bar{\gamma}(w(\gamma_1)) > \gamma_2$ *and*
- $\frac{(1-\gamma_2)(\gamma_1)^4}{(1-\gamma_2)(\gamma_1)^3+(1-\gamma_1)(\gamma_2)^3} > \frac{(\gamma_2)^3+\gamma_1\frac{2(1-\gamma_2)}{(\gamma_2)^2+2(1-\gamma_2)}}{(\gamma_2)^2+\frac{1}{2}\gamma_2+\frac{1}{3}+\frac{2(1-\gamma_2)}{(\gamma_2)^2+2(1-\gamma_2)}}$,

the repayment rates are ordered by pools: $\rho^1(t) < \rho^2(t) \quad \forall t$.

From equation (2.13), this theorem in particular implies that $q^1(t-1) < q^2(t-1) \quad \forall t$. Therefore, households in pool $b^h(t) = 1$ face lower asset prices when borrowing than households in pool $b^h(t) = 2$.

3.2 Finite period bankruptcy effects

Now consider an extension of the model in which the bankruptcy indicator $b^h(t)$ can take on $I + 1$ values $b^h(t) \in \{0, 1, ..., I\}$. If a household declares bankruptcy in time period t, the indicator takes value $b^h(t) = 0$. For every period thereafter at which the household does not declare bankruptcy, the indicator $b^h(t+\tau) = b^h(t+\tau-1)+1\{b^h(t+\tau-1) < I\}$. When $b^h(t+\tau-1) = I$, the bankruptcy flag has been removed and the household is indistinguishable from any household who has never declared bankruptcy. As in the standard model, a household may not declare bankruptcy two periods in a row. As a result, the repayment rates can only be less than unity for pools $b^h(t) \in \{1, ..., I\}$.

There will now be $I+1$ income states, $E = I+1$. To highlight the point that the observation of a household's income level is immateral without also observing the persistence of that income level, set $\omega_1 = \omega_2 = ... = \omega_I \leq \omega_{med} < \omega_E$. Let the transition matrix Π be sparse with terms given by:

$$\begin{bmatrix} ... & 0 & 0 & ... \\ 0 & \pi(i|i) = \gamma_i & 0 & \pi(E|i) = 1 - \gamma_i \\ 0 & 0 & ... & ... \\ ... & \pi(i|E) = \frac{1-\gamma}{I} & ... & \pi(E|E) = \gamma \end{bmatrix}. \tag{3.4}$$

The persistence of the income states is ordered as (without loss of generality): $\gamma_1 > \gamma_2 > ... > \gamma_I$.

As above, I will define $w(\gamma_i)$ as the amount of wealth that a household with a string of realizations $e(\cdot) = i$ (where $i \in \{1, ..., I\}$) will bring into period t given that the household is planning to declare bankruptcy in time period $t + 1$ upon realization $e(t + 1) = i$. Therefore,

$$w(\gamma_i) = w\left(t : (e(t-i) = .. = e(t) = i), (w(t-i) = 0), (b(t-i) = 0, ..b(t) = i)\right)$$

for all $i \in \{1, ..., I\}$. As above, the bankruptcy cutoff $\bar{\gamma}(w)$ is a function of wealth so that a household will plan to declare bankruptcy at time period $t + 1$ iff $\gamma_i \geq \bar{\gamma}(w(t))$ where $i = e^h(t)$.

With these definitions, the following theorem is a generalization of theorem 5 for the case of any finite number of bankruptcy pools $b^h(t) \in \{0, 1, ..., I\}$. The proof of this theorem 6 is contained in appendix B.

Theorem 6 *Under the extended economy presented with $E = I + 1$, $\omega_1 = \omega_2 = ... = \omega_I \leq \omega_{med} < \omega_E$, and Π given by (3.4), then if $(\gamma_1, ..., \gamma_I)$ such that $\gamma_1 > .. > \gamma_I$ additionally satisfy*

- $\gamma_1 > \bar{\gamma}(w(\gamma_1)) > \gamma_2 > \bar{\gamma}(w(\gamma_2)) > ... > \gamma_I$ *and*

- $\frac{(1-\gamma_I)\left(\gamma_{I-1}\right)^{2I}}{(1-\gamma_I)\left(\gamma_{I-1}\right)^{2I-1}+(1-\gamma_{I-1})(\gamma_I)^{2I-1}} > \frac{\frac{(\gamma_I)^{2I+1}}{I}+(1-\gamma_I)\left(\gamma_1+\sum_{i=2}^{I}(\gamma_i)^{i+1}\right)}{\frac{(\gamma_I)^{2I}}{I}+(1-\gamma_I)\left(\gamma_1+\sum_{i=2}^{I}\sum_{j=0}^{i}\frac{1}{j+1}(\gamma_i)^{i-j}\right)}$,

  the repayment rates are ordered by pools: $\rho^1(t) < \rho^2(t) < ... < \rho^I(t) \quad \forall t$.

From equation (2.13), this theorem in particular implies that $q^1(t-1) < q^2(t-1) < .... < q^I(t-1)\ \forall t$. Therefore, households in a lower pool $b^h(t) = i$ face lower asset prices when borrowing than households in a higher pool $b^h(t) = j$ for all $i < j$.

4 Multiple Assets and Normative Predictions

[to be completed]

5 Conclusion

This work has contributed a framework for the analysis of bankruptcy in the class of Bewley models. If creditors observed the asset choices of potential borrowers, then the asset prices could be conditioned on the expected repayment rate of each borrower. However, in this anonymous and competitive setting, creditors can only set asset prices conditioned on the pool to which a borrower belongs, where the pools are distinguished only by bankruptcy history. In this setup, general existence of a bankruptcy equilibrium was shown and theoretical results for a simple economy demonstrated that the different pools would be subject to different endogenous asset prices for borrowing.

The next steps in this line of research are first quantitative and then theoretical (again). On the quantitative side, the asset structure of the model allows for equilibria to be computed using standard dynamic programming techniques for any set of calibrated parameters. The normative impacts of bankruptcy, both allowing for its possibility and then regulating its consequences, can then be studied. Can the introduction of bankruptcy lead to a welfare gain for most/all households? Is screening borrowers into pools by bankruptcy history inefficient and would creditors prefer to eliminate credit reports? Though the intuitive answer to both of these questions is "no", a complete answer is only possible with a rigorous quantitative exercise.

On the theoretical side, it is natural to ask how the bankruptcy effects translate to the set of secured credit markets. As developed by Geanakoplos and Zame (2002), any asset sales in these markets must be backed by some specified amount of the durable good (collateral). Creditors have a claim on the collateral of any bankrupt households that they lend to. This will be referred to as the primary repayment market. Following this, if still owed funds, creditors can file claims on the secondary repayment markets, an unsecured credit market. Finally, quite novel will be the setup in which pools of borrowers do not face different asset prices when borrowing, but different collateral requirements. Thus, the collateral levels can be endogenized without the insurance/collateral intertemporal trade-off of Araujo et. al. (2000).

Appendix A

Proof of Theorem 1

Over the truncated horizon $t \in \{0, ..., \bar{T}\}$ with $\bar{T} < \infty$, define the sequence of asset prices $q_j(e) = \{q_j(e,t)\}_{0 \le t < \bar{T}}$ and $q = (q_j(e))_{\forall e, \forall j \in \mathcal{J}(e)}$. Likewise, define the sequence of household consumption and asset choices as $c^h = \{c^h(t)\}_{0 \le t \le \bar{T}}$ and $z_j^h(e) = \{z_j^h(e,t)\}_{0 \le t < \bar{T}}$ with $z^h = \left(z_j^h(e)\right)_{\forall e, \forall j \in \mathcal{J}(e)}$. Define the upper bound $\bar{c} = 2 \max\{\omega_1, .., \omega_E\}$. Define the bounded budget set for each household $h \in \mathcal{H}$ as:

$$\bar{B}^h(q) = \left\{ \begin{array}{l} \left(c^h, z^h\right) \in \mathbb{R}_+^{\bar{T}+1} \times \mathbb{R}^{J\bar{T}} : c^h(t) \le \bar{c} \ \ \forall t \\ \omega^h(0) - c^h(0) - \sum_e \sum_{j \in \mathcal{J}(e)} q_j(e,0) z_j^h(e,0) \ge 0, \\ \omega^h(t) - c^h(t) + \sum\limits_{\hat{e} \ne e} \left(\sum\limits_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t-1) \right) + .. \\ ... + \sum\limits_{j \in \mathcal{J}(e)} r_j(e'|e) z_j^h(e, t-1) - \sum\limits_e \sum\limits_{j \in \mathcal{J}(e)} q_j(e,t) z_j^h(e,t) \ge 0, \\ \omega^h(\bar{T}) - c^h(\bar{T}) + \sum\limits_{\hat{e} \ne e} \left(\sum\limits_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, \bar{T}-1) \right) + ... \\ ... + \sum\limits_{j \in \mathcal{J}(e)} r_j(e'|e) z_j^h(e, \bar{T}-1) \ge 0 \end{array} \right\}. \tag{a.1}$$

The realized income states $(e, e') = \left(e^h(t-1), e^h(t)\right)$ where the time period t is understood from context. The middle budget constraint holds $\forall t : 0 < t < \bar{T}$.

In equilibrium, the constraints $c^h(t) \le \bar{c}$ will be nonbinding. Since the objective function is quasi-concave and continuous, it is innocuous to add the constraints to the budget set as the optimal solutions to the household problem will *not* be affected. $\bar{B}^h(q)$ is nonempty. The proofs of both the following lemmas are contained at the completion of the *proof of theorem 1*.

Lemma 1 *With $q_j(e,t) > 0 \ \ \forall t \ge 0, \forall e,$ and $\forall j \in \mathcal{J}(e)$, then $\bar{B}^h(q)$ is compact.*

Though $q_j(e,t) > 0 \ \ \forall t, \forall e$, and $\forall j \in \mathcal{J}(e)$ is an assumption in lemma 1, it will be shown that this is a necessary condition of equilibrium.

Although the set $\bar{B}^h(q)$ as written is convex, when introducing bankruptcy the convexity will not be preserved. Even so, lemma 2 will still hold.

Lemma 2 *$\int \bar{B}^h(q) d\Phi$ is convex.*

Define the price space as:

$$\Delta^* = \left\{ (p,q) : p(t) + \sum_e \sum_{j \in \mathcal{J}(e)} q_j(e,t) = 1 \ \forall t \right\}.$$

Since all the payouts are nonnegative, the equilibrium asset prices $q_j(e,t) \ge 0$. The price $p(t)$ will be the price of the physical commodity in each time period t. In equilibrium, it is normalized to 1, but when defining the price space, it is more convenient to restrict $\left(p(t), (q_j(e,t))_{\forall e, \forall j \in \mathcal{J}(e)}\right) \in \Delta^J$ $\forall t \ge 0$. Δ^* is nonempty, convex, and compact.

Next, I will write down the household's truncated optimization problem and define the household demand. The household problem (H) is given by

$$\max_{c^h, z^h} E_0 \sum_{t=0}^{\bar{T}} \beta^t u\left(c^h(t)\right) \tag{H}$$

$$subj\ to\ \left(c^h, z^h\right) \in \bar{B}^h(q).$$

I will define the household demand correspondence as

$$\Upsilon^h : \Delta^* \rightrightarrows \bar{B}^h(q)$$

such that given $q \in \Delta^*$, $\left(\tilde{c}^h, \tilde{z}^h\right) \in \Upsilon^h(q)$ iff $\left(\tilde{c}^h, \tilde{z}^h\right)$ solves (H). As $u(\cdot)$ is continuous and $\bar{B}^h(q)$ is compact, the demand correspondence Υ^h is well-defined. From lemma 1, since $\int \bar{B}^h(q)d\Phi$ is convex and all households face the same objective function, the overall household demand correspondence Υ, defined such that $\Upsilon(q) = \int \Upsilon^h(q)d\Phi \quad \forall q \in \Delta^*$, is convex-valued. The proof of the following lemma is located at the completion of the *proof of theorem 1.*

Lemma 3 Υ^h *is an upper hemicontinuous (uhc) correspondence.*

I will now write down the price correspondence

$$\Psi : \int \bar{B}^h d\Phi \rightrightarrows \Delta^*.$$

Given $\left(c^h, z^h\right)_{h\in\mathcal{H}}$, then $(p,q) \in \Psi\left(\left(c^h, z^h\right)_{h\in\mathcal{H}}\right)$ iff

$$(p,q) \in \arg\max \left\{ \begin{array}{l} p(0)\left(\int c^h(0)\,d\Phi - \overline{\omega}\right) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,0) \int z_j^h(e,0)d\Phi + \\ \sum_{t=1}^{\bar{T}} p(t)\left(\int c^h(t)\,d\Phi - \overline{\omega}\right) + \sum_{t=1}^{\bar{T}-1}\sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,t)\int z_j^h(e,t)d\Phi - \\ \sum_{t=1}^{\bar{T}} \int_{h\in\mathcal{H}} \left[\sum_{\hat{e}\neq e}\sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t_{-1}) + \sum_{j\in\mathcal{J}(e)} r_j(e'|e) z_j^h(e, t_{-1})\right] d\Phi \end{array} \right\}. \tag{a.2}$$

As the objective function is continuous and Δ^* is compact, Ψ is well-defined. As the objective function is quasi-concave and Δ^* is convex, Ψ is convex-valued. Finally, since the constraint set Δ^* is independent of $\left(c^h, z^h\right)_{h\in\mathcal{H}}$ and is compact, it can be viewed as the values of a continuous correspondence. Using the maximum principle, the correspondence Ψ is uhc.

Define the overall equilibrium correspondence as the Cartesian product $\Upsilon \times \Psi$. The overall correspondence is well-defined, convex-valued, and uhc. It maps from the Cartesian product $\int \bar{B}^h d\Phi \times \Delta^*$ into itself. The set $\int \bar{B}^h d\Phi \times \Delta^*$ is nonempty, convex, and compact. Applying Kakutani's fixed point theorem yields a fixed point of this overall equilibrium correspondence. By definition, the fixed points is such that $\left(c^h, z^h\right)$ satisfies the household optimization problem (H) $\forall h \in \mathcal{H}$.

In equilibrium, since $r_j(e) = \sum_{e'} \pi\left(e'|e\right) r_j\left(e'|e\right) > 0$ holds $\forall e$ and $\forall j \in \mathcal{J}(e)$ by definition, no arbitrage conditions (necessary conditions of household optimization) imply $q_j(e,t) > 0 \ \ \forall t, \forall e$, and $\forall j \in \mathcal{J}(e)$. All that remains to complete the *proof of theorem 1* is to verify that the markets clear (both commodity and asset markets) for the fixed point given by Kakutani.

Lemma 4 *Markets clear.*

Proof. From Walras' law:

$$\begin{array}{l} p(0)\left(\int c^h\left(0\right)d\Phi-\overline{\omega}\right)+\sum_e\sum_{j\in\mathcal{J}(e)}q_j(e,0)\int z_j^h(e,0)d\Phi=0 \\ p(t)\left(\int c^h\left(t\right)d\Phi-\overline{\omega}\right)+\sum\limits_e\sum\limits_{j\in\mathcal{J}(e)}q_j(e,t)\int z_j^h(e,t)d\Phi-.. \\ ...-\int\limits_{h\in\mathcal{H}}\left\{\sum\limits_{\hat{e}\neq e}\left(\sum\limits_{j\in\mathcal{J}(\hat{e})}r_j(\hat{e})z_j^h(\hat{e},t-1)\right)+\sum\limits_{j\in\mathcal{J}(e)}r_j(e'|e)z_j^h(e,t-1)\right\}d\Phi=0 \\ p(\bar{T})\left(\int c^h\left(\bar{T}\right)d\Phi-\overline{\omega}\right)-... \\ ..-\int\limits_{h\in\mathcal{H}}\left\{\sum\limits_{\hat{e}\neq e}\left(\sum\limits_{j\in\mathcal{J}(\hat{e})}r_j(\hat{e})z_j^h(\hat{e},\bar{T}-1)\right)+\sum\limits_{j\in\mathcal{J}(e)}r_j(e'|e)z_j^h(e,\bar{T}-1)\right\}d\Phi=0 \end{array} \quad . \tag{a.3}$$

The convention is that the middle equality holds for all $t: 0 < t < \bar{T}$.

I will first prove a general property about the asset payouts and then show that markets clear using an induction argument.

Consider the asset payouts

$$\int\limits_{h\in\mathcal{H}}\left\{\sum_{\hat{e}\neq e}\left(\sum_{j\in\mathcal{J}(\hat{e})}r_j(\hat{e})z_j^h(\hat{e},t)\right)+\sum_{j\in\mathcal{J}(e)}r_j(e'|e)z_j^h(e,t)\right\}d\Phi.$$

Choose any $e \in \mathcal{E}$ and any $j \in \mathcal{J}(e)$. I can rewrite the equation for this asset (e, j) as:

$$r_j(e)\int\limits_{h:e^h(t)\neq e}z_j^h(e,t)d\Phi+\sum_{e'}r_j(e'|e)\int\limits_{h:(e^h(t),e^h(t+1))=(e,e')}z_j^h(e,t)d\Phi.$$

By definition, $\pi\left(e'|e\right)=\dfrac{\int_{h:(e^h(t),e^h(t+1))=(e,e')}d\Phi}{\int_{h:e^h(t)=e}d\Phi}$ and since the integration is a linear operation, the equation is equivalent to:

$$r_j(e)\int_{h:e^h(t)\neq e}z_j^h(e,t)d\Phi+\sum_{e'}r_j(e'|e)\pi\left(e'|e\right)\int_{h:e^h(t)=e}z_j^h(e,t)d\Phi.$$

By definition, $r_j(e)=\sum_{e'}\pi\left(e'|e\right)r_j\left(e'|e\right)$ and thus the asset payouts for $z_j(e)$ are reduced to:

$$r_j(e)\int z_j^h(e,t)d\Phi.$$

<u>Initialization</u>: $t = 0$

If $\left(\int c^h\left(0\right)d\Phi-\overline{\omega}\right)>0$, the definition of the price correspondence Ψ requires $p(0) = 1$ and $\left(q_j(e,0)\right)_{\forall e,\forall j\in\mathcal{J}(e)}=\overrightarrow{0}$. The first of the equalities of $(a.3)$ is violated. Thus $\left(\int c^h\left(0\right)d\Phi-\overline{\omega}\right)\leq 0$.

If $\left(\int z_j^h(e,0)d\Phi\right)>0$ for some e and some $j\in\mathcal{J}(e)$, then $q_j(e,0)=1$ for that (e,j) and $\left(p(0),(q_{j'}(e',0))_{\forall(e',j')\neq(e,j)}\right)=\overrightarrow{0}$. This is a violation of the first equality of $(a.3)$. Thus $\left(\int z_j^h(e,0)d\Phi\right)\leq 0\ \ \forall e$ and $\forall j\in\mathcal{J}(e)$.

In order for the first equality of $(a.3)$ to hold, it must be that

$$\begin{aligned}\left(\int c^h(0)\,d\Phi-\overline{\omega}\right) &= 0. \qquad\qquad (\text{a.4})\\ \left(\int z_j^h(e,0)d\Phi\right) &= 0 \;\; \forall e \text{ and } \forall j\in\mathcal{J}(e).\end{aligned}$$

where the equalities follows since $p(0)>0$ and $q_j(e,0)>0\;\;\forall e$ and $\forall j\in\mathcal{J}(e)$.

Induction: $t:0<t\leq\bar{T}$

Suppose that the following equations $(a.5)$ hold for some time period $t-1$:

$$\begin{aligned}\left(\int c^h(t-1)\,d\Phi-\overline{\omega}\right) &= 0. \qquad\qquad (\text{a.5})\\ \left(\int z_j^h(e,t-1)d\Phi\right) &= 0 \;\; \forall e \text{ and } \forall j\in\mathcal{J}(e).\end{aligned}$$

The aggregate asset payouts defined simply as $r_j(e)\int z_j^h(e,t-1)d\Phi$ from the analysis above would have value $r_j(e)\int z_j^h(e,t-1)d\Phi=0\;\;\forall e$ and $\forall j\in\mathcal{J}(e)$.

If $\left(\int c^h(t)\,d\Phi-\overline{\omega}\right)>0$, the definition of the price correspondence Ψ requires $p(t)=1$ and $(q_j(e,t))_{\forall e,\forall j\in\mathcal{J}(e)}=\overrightarrow{0}$. One of the equalities of $(a.3)$ is violated. Thus $\left(\int c^h(t)\,d\Phi-\overline{\omega}\right)\leq 0$.

If $\left(\int z_j^h(e,t)d\Phi\right)>0$ for some e and some $j\in\mathcal{J}(e)$, then $q_j(e,t)=1$ for that (e,j) and $\left(p(t),(q_{j'}(e',t))_{\forall(e',j')\neq(e,j)}\right)=\overrightarrow{0}$. This is a violation of one of the equalities of $(a.3)$. Thus $\left(\int z_j^h(e,t)d\Phi\right)\leq 0\;\;\forall e$ and $\forall j\in\mathcal{J}(e)$.

For the equalities in $(a.3)$ to hold, it must be that (since $p(t)>0$ and $q_j(e,t)>0\;\;\forall e$ and $\forall j\in\mathcal{J}(e)$):

$$\begin{aligned}\left(\int c^h(t)\,d\Phi-\overline{\omega}\right) &= 0.\\ \left(\int z_j^h(e,t)d\Phi\right) &= 0 \;\; \forall e, \text{ and } \forall j\in\mathcal{J}(e).\end{aligned}$$

Proceeding by induction, I have verified that the market clearing conditions are satisfied. ■

Proof of Lemma 1

To show this result, I recognize that the vector of consumption c^h is bounded (by definition). Then, beginning in time period $\bar{T}$ and working by backwards induction, I will prove that the assets are bounded as well (relying on assumptions $(A.1)$ and $(A.2)$).

Final period: $t=\bar{T}$

The term

$$\textstyle\sum_{\hat{e}\neq e}\left(\sum_{j\in\mathcal{J}(\hat{e})}r_j(\hat{e})z_j^h(\hat{e},\bar{T}-1)\right)+\sum_{j\in\mathcal{J}(e)}r_j(e'|e)z_j^h(e,\bar{T}-1) \qquad (\text{a.6})$$

is bounded according to the budget constraint at time period $t = \bar{T}$. Normalizing $r^* = 1$, then there exists $z^{h*}(\bar{T}-1) \in \mathbb{R}$ s.t. $r^* z^{h*}(\bar{T}-1) = \sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, \bar{T}-1) \right)$ ($z^{h*}(\bar{T}-1)$ will be a cumulative risk-free bond with payoff $r^* = 1$). Therefore, the sum

$$r^* z^{h*}(\bar{T}-1) + \sum_{j \in \mathcal{J}(e)} r_j(e'|e) z_j^h(e, \bar{T}-1)$$

is bounded and by assumption $(A.1)$, the vector $\left(z^{h*}(\bar{T}-1), \left(z_j^h(e, \bar{T}-1) \right)_{\forall j \in \mathcal{J}(e)} \right)$ is bounded. Thus, $r^* z^{h*}(\bar{T}-1)$ is bounded and so is the equivalent expression $\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, \bar{T}-1) \right)$. Suppose without loss of generality that $\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, \bar{T}-1) \right) \geq 0$. By assumption $(A.2)$, the assets $z_j^h(\hat{e}, \bar{T}-1) \geq 0 \ \ \forall \hat{e} \neq e$ and $\forall j \in \mathcal{J}(\hat{e})$. As $\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, \bar{T}-1) \right)$ is bounded and $r_j(\hat{e})$ is fixed, then each of the assets $\left(z_j^h(\hat{e}, \bar{T}-1) \right)_{\forall \hat{e} \neq e, \forall j \in \mathcal{J}(\hat{e})}$ is bounded as well.

Backward induction: $t < \bar{T}$

With $\left(\left(z_j^h(e,t) \right)_{\forall j \in \mathcal{J}(e)}, \left(z_j^h(\hat{e}', t) \right)_{\forall \hat{e} \neq e, j \in \mathcal{J}(\hat{e})} \right)^T$ bounded (this is the inductive hypothesis), the budget constraint at time period $t < \bar{T}$ dictates that the following term is bounded:

$$\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t-1) \right) + \sum_{j \in \mathcal{J}(e)} r_j(e'|e) z_j^h(e, t-1). \tag{a.7}$$

Exactly as above, define $r^* = 1$ and $z^{h*}(t-1) \in \mathbb{R}$ s.t. $r^* z^{h*}(t-1) = \sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t-1) \right)$. Therefore, the sum

$$r^* z^{h*}(t-1) + \sum_{j \in \mathcal{J}(e)} r_j(e'|e) z_j^h(e, t-1)$$

is bounded and by assumption $(A.1)$, the vector $\left(z^{h*}(t-1), \left(z_j^h(e, t-1) \right)_{\forall j \in \mathcal{J}(e)} \right)$ is bounded. Thus, $\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t-1) \right)$ is bounded. Suppose without loss of generality that $\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t -$ 0. By assumption $(A.2)$, the assets $z_j^h(\hat{e}, t-1) \geq 0 \ \forall \hat{e} \neq e$ and $\forall j \in \mathcal{J}(\hat{e})$. As $\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t-1) \right)$ is bounded, then each of the assets in the vector $\left(z_j^h(\hat{e}, t-1) \right)_{\forall \hat{e} \neq e, \forall j \in \mathcal{J}(\hat{e})}$ is bounded as well.

This completes the backward induction argument and the proof of the lemma.

Proof of Lemma 2

$\bar{B}^h(q)$ is a set-valued function (terminology of Aumann, 1966) or correspondence. From Aumann (1966), $\int \bar{B}^h(q) d\Phi$ is convex provided that $\bar{B}^h(q)$ is a set-valued function defined on the set of households $\mathcal{H}\tilde{}\ [0,1]$ (recall that the set of households is actually a hyperfinite process in order to be able to apply the law of large numbers) and the values of $\bar{B}^h(q)$ are subsets of $\mathbb{R}_+^{\bar{T}+1} \times \mathbb{R}^{J\bar{T}}$.

Proof of Lemma 3

I will define the budget correspondence

$$\bar{B}^h : \Delta^* \rightrightarrows \bar{B}^h(q)$$

such that given $q \in \Delta^*$, the values of the correspondence $\bar{B}^h$ are the entire budget set $\bar{B}^h(q)$. This correspondence is trivially uhc. The following proof will show that $\bar{B}^h$ is also lhc. Using the maximum principle (with the continuous utility function from the household optimization problem), Υ^h is a uhc correspondence.

Claim 1 *$\bar{B}^h$ is an lhc correspondence.*

Proof. Consider a sequence $q^\nu \to q$ with $\left(c^h, z^h\right) \in \bar{B}^h(q)$ for some h (I will drop the superscript now). I will define a scaling factor $(\theta^\nu(t))_{0 \le t \le \bar{T}}$ such that for the scaled consumption and assets

$$\begin{aligned} c^\nu(t) &= \left(\min_{0 \le t \le \bar{T}} \theta^\nu(t)\right) c(t) \ \ \forall t \\ z_j^\nu(e,t) &= \left(\min_{0 \le t \le \bar{T}} \theta^\nu(t)\right) z_j(e,t) \ \ \forall t, \forall e, \text{ and } \forall j \in \mathcal{J}(e), \end{aligned}$$

$\exists N$ s.t. $\forall \nu \ge N$, $(c^\nu, z^\nu) \in \bar{B}^h(q^\nu)$ and $(c^\nu, z^\nu) \to (c, z)$. For simplicity, define $\theta^\nu = \left(\min\limits_{0 \le t \le \bar{T}} \theta^\nu(t)\right)$.

The budget set $\bar{B}^h(q)$ has the so-called scaling propety (so called by Dubey et. al. (2005)) meaning that it is fairly straightforward to define the sequence of scaling fractions $\theta^\nu(t) \in [0,1]$ for $0 \le t \le \bar{T}$. This is done by induction.

<u>Initialization:</u> $t = 0$

If $\omega^h(0) - c(0) - \sum_e \sum_{j \in \mathcal{J}(e)} q_j(e,0) z_j(e,0) > 0$, then $\exists N$ s.t.

$$\omega^h(0) - c(0) - \textstyle\sum_e \sum_{j \in \mathcal{J}(e)} q_j^\nu(e,0) z_j(e,0) > 0$$

holds $\forall \nu \ge N$. Define $\theta^\nu(0) = 1$ for this case.

Otherwise, $\omega^h(0) - c(0) - \sum_e \sum_{j \in \mathcal{J}(e)} q_j(e,0) z_j(e,0) = 0$ and thus $c(0) + \sum_e \sum_{j \in \mathcal{J}(e)} q_j(e,0) z_j(e,0) > 0$ using the assumption that $\omega^h(0) > 0$.

Define $\theta^\nu(0) \in [0,1] \ \ \forall \nu$ as:

$$\theta^\nu(0) = \begin{array}{cc} \frac{\omega^h(0)}{c(0) + \sum_e \sum_{j \in \mathcal{J}(e)} q_j^\nu(e,0) z_j(e,0)} & \text{if } num < den \text{ and } den \neq 0 \\ 1 & \text{otherwise} \end{array}$$

where $num = \omega^h(0)$ and $den = c(0) + \sum_e \sum_{j \in \mathcal{J}(e)} q_j^\nu(e,0) z_j(e,0)$. With $num \to den$ and knowing that for some N_0, $\forall \nu \ge N_0$, $den > 0$, then $\theta^\nu(0) \to 1$. The following equations verify that the scaled $\left(c^\nu(0), \left(z_j^\nu(e,0)\right)_{\forall e, j \in \mathcal{J}(e)}\right)$ satisfy the budget constraint $\forall \nu \ge N_0$ at time period $t = 0$:

$$\begin{array}{l} c^\nu(0) + \sum_e \sum_{j \in \mathcal{J}(e)} q_j^\nu(e,0) z_j^\nu(e,0) = \\ \theta^\nu c(0) + \theta^\nu \sum_e \sum_{j \in \mathcal{J}(e)} q_j^\nu(e,0) z_j(e,0) \le \\ \theta^\nu(0) \left[c(0) + \sum_e \sum_{j \in \mathcal{J}(e)} q_j^\nu(e,0) z_j(e,0)\right] \le \omega^h(0). \end{array}$$

<u>Induction:</u> $0 < t \le \bar{T}$

Pick any time period $t : 0 < t \le \bar{T}$ and suppose that the budget constraints are satisfied for all prior time periods (induction hypothesis). Define the asset payouts for this time period as

$$ap(t) = \textstyle\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j(\hat{e}, t-1)\right) + \sum_{j \in \mathcal{J}(e)} r_j(e'|e) z_j(e, t-1).$$

If $\omega^h(t) - c(t) - \sum_e \sum_{j \in \mathcal{J}(e)} q_j(e,t) z_j(e,t) + ap(t) > 0$, then $\exists N$ s.t.

$$\omega^h(t) - c(t) - \textstyle\sum_e \sum_{j \in \mathcal{J}(e)} q_j^\nu(e,t) z_j(e,t) + ap(t) > 0$$

holds $\forall \nu \geq N$. Define $\theta^{\nu}(t) = 1$ for this case.

Otherwise, $\omega^h(t) - c(t) - \sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,t)z_j(e,t) + ap(t) = 0$ and thus $c(t) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,t)z_j(e,t) - ap(t) > 0$ using the assumption that $\omega^h(t) > 0$.

Define $\theta^{\nu}(t) \in [0,1] \ \ \forall \nu$ as:

$$\theta^{\nu}(t) = \begin{array}{ll} \frac{\omega^h(t)}{c(t)+\sum_e \sum_{j\in\mathcal{J}(e)} q_j^{\nu}(e,t)z_j(e,t)-ap(t)} & \text{if } num < den \text{ and } den \neq 0 \\ 1 & \text{otherwise} \end{array}$$

where $num = \omega^h(t)$ and $den = c(t) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j^{\nu}(e,t)z_j(e,t) - ap(t)$. With $num \to den$ and knowing that for some N_t, $\forall \nu \geq N_t$, $den > 0$, then $\theta^{\nu}(t) \to 1$. The following equations verify that the scaled $\left(c^{\nu}(t), \left(z_j^{\nu}(e,t-1), z_j^{\nu}(e,t)\right)_{\forall e, j\in\mathcal{J}(e)}\right)$ satisfy the budget constraint $\forall \nu \geq N_t$ at time period t :

$$\begin{array}{l} c^{\nu}(t) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j^{\nu}(e,t)z_j^{\nu}(e,t) - .. \\ .. - \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e})z_j^{\nu}(\hat{e},t-1)\right) + \sum_{j\in\mathcal{J}(e)} r_j(e'|e)z_j^{\nu}(e,t-1) = \\ \theta^{\nu}c(t) + \theta^{\nu}\sum_e \sum_{j\in\mathcal{J}(e)} q_j^{\nu}(e,t)z_j(e,t) - \theta^{\nu}ap(t) \leq \\ \theta^{\nu}(t)\left[c(t) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j^{\nu}(e,t)z_j(e,t) - ap(t)\right] \leq \omega^h(t). \end{array}$$

Thus, $\exists N = \max\{N_0, ..., N_{\overline{T}}\}$ s.t. $\forall \nu \geq N$, $(c^{\nu}, z^{\nu}) \in \bar{B}^h(q^{\nu})$ and also $(c^{\nu}, z^{\nu}) \to (c, z)$. This completes the proof. ■

Proof of Theorem 2

The proof of this result should follow from Magill and Quinzii (1994), Bewley (1972), and possibly Bewley (1986). [to be completed]

Proof of Theorem 3

Over the truncated horizon $t \in \{0, ..., \overline{T}\}$ with $\overline{T} < \infty$, define the sequence of asset prices $q_j(e) = \{q_j(e,t)\}_{0 \le t < \bar{T}}$ and $q = (q_j(e))_{\forall e, \forall j \in \mathcal{J}(e)}$ and the sequence of repayment rates $\rho_j(e) = \{\rho_j(e,t)\}_{0 < t \le \bar{T}}$ and $\rho = (\rho_j(e))_{\forall e, \forall j \in \mathcal{J}(e)}$. Further, for each bankruptcy history $b \in \mathcal{B}$, there exists a sequence of asset prices $q_j^b(e) = \{q_j^b(e,t)\}_{0 \le t < \bar{T}}$ and $q^b = (q_j^b(e))_{\forall e, \forall j \in \mathcal{J}(e)}$. Define $q^* = (q, (q^b)_{b \in \mathcal{B}})$ as the vector containing all asset price sequences. Define the sequence of household consumption and asset choices as $c^h = \{c^h(t)\}_{0 \le t \le \bar{T}}$ and $z_j^h(e) = \{z_j^h(e,t)\}_{0 \le t < \bar{T}}$ with $z^h = (z_j^h(e))_{\forall e, \forall j \in \mathcal{J}(e)}$. Define the upper bound $\bar{c} = 2\max\{\omega_1, .., \omega_E\}$. Unless specified, the simple notation (e, e') will replace $(e^h(t-1), e^h(t))$ where the time period t is clear from context. It is also clear from context that the second set of middle equations is for a household declaring bankruptcy. Define the bounded budget set for each household $h \in \mathcal{H}$ as:

$$\bar{B}^h(q^*, \rho) = \left\{ \begin{array}{l} \left(c^h, z^h, b^h\right) \in \mathbb{R}_+^{\bar{T}+1} \times \mathbb{R}^{J\bar{T}} \times \mathcal{B}^{\bar{T}} : c^h(t) \le \bar{c} \ \ \forall t \\ \omega^h(0) - c^h(0) - \sum\limits_e \sum\limits_{j \in \mathcal{J}(e)} q_j(e,0) z_j^h(e,0) \ge 0, \\ \omega^h(t) - c^h(t) - \sum\limits_e \sum\limits_{j \in \mathcal{J}(e)} q_j^h(e,t) z_j^h(e,t) + \max\{S^h(t), B^h(t)\} \ge 0, \\ \text{where } S^h(t) = \sum\limits_{\hat{e} \ne e} \sum\limits_{j \in \mathcal{J}(\hat{e})} r_j^h(\hat{e},t) z_j^h(\hat{e}, t_{-1}) + \sum\limits_{j \in \mathcal{J}(e)} r_j^h(e'|e,t) z_j^h(e, t_{-1}) \\ \text{and } B^h(t) = -\sum_{\hat{e} \ne e} \left(\sum_{j \in \mathcal{J}(\hat{e})} \alpha_j(\hat{e}) r_j(\hat{e}) \left(z_j^h(\hat{e}, t-1)\right)^+ \right) - ... \\ ... - \sum_{j \in \mathcal{J}(e)} \alpha_j(e) r_j(e'|e) \left(z_j^h(e, t-1)\right)^+ \\ \omega^h(\bar{T}) - c^h(\bar{T}) + \sum_{\hat{e} \ne e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, \bar{T}-1) \right) + ... \\ ... + \sum_{j \in \mathcal{J}(e)} r_j(e'|e) z_j^h(e, \bar{T}-1) \ge 0 \end{array} \right\}. \tag{a.8}$$

Notice that in the truncated budget set given in $(a.8)$, it is not permissible for a household to declare bankruptcy in the final period $\bar{T}$. As a result, overall repayment rate at $\bar{T} \in \mathcal{T}_h^*$ is unity. That is, $r_j^h(\hat{e}, \bar{T}) = r_j(\hat{e}) \ \forall \hat{e} \ne e, \forall j \in \mathcal{J}(\hat{e})$ and $r_j^h(e'|e, \bar{T}) = r_j(e'|e) \ \forall j \in \mathcal{J}(e)$. As there is no final period in the limit, this restriction is innocuous in the infinite horizon equilibrium.

In equilibrium, the constraints $c^h(t) \le \bar{c}$ will be nonbinding. As the objective function is quasi-concave and continuous, it is innocuous to add the constraints to the budget set as the optimal solutions to the household problem will *not* be affected. $\bar{B}^h(q^*, \rho)$ is nonempty and from lemma 2, $\displaystyle\int \bar{B}^h(q^*, \rho) d\Phi$ is convex. The proof of the following lemma is located after the *proof of theorem 2.*

Lemma 5 *If* $q_j^b(e,t) > 0 \ \forall b \in \mathcal{B}, \forall t \ge 0, \forall e,$ *and* $\forall j \in \mathcal{J}(e)$, *then* $\bar{B}^h(q^*, \rho)$ *is compact.*

Though $q_j^b(e,t) > 0 \ \forall b \in \mathcal{B}, \forall t \ge 0, \forall e$, and $\forall j \in \mathcal{J}(e)$ is an assumption in lemma 5, it will be shown that this is a necessary condition of equilibrium.

Define the price space as:

$$\Delta^* = \left\{ \begin{array}{l} (p, q^*, \rho) : p(t) + \sum_e \sum_{j \in \mathcal{J}(e)} q_j(e,t) = 1 \quad \forall t. \\ \rho_j^b(e,t+1) \int_{h \in \mathcal{H}(b)} - \left(z_j^h(e,t)\right)^- d\Phi = -\int_{h \in \mathcal{H}(b)} \delta^h(t+1) \left(z_j^h(e,t)\right)^- d\Phi. \\ q_j^b(e,t) = q_j(e,t) \cdot E_t \left(\frac{\rho_j^b(e,t+1)}{\rho_j(e,t+1)} \right) \quad \forall t, \forall b, \forall e, \text{ and } \forall j \in \mathcal{J}(e). \\ 0 \le \rho_j(e,t+1) \le 1 \text{ where } \forall t, \forall e, \text{ and } \forall j \in \mathcal{J}(e) : \\ \rho_j(e,t+1) \int_{h \in \mathcal{H}} \left(z_j^h(e,t)\right)^+ d\Phi = -\int_{h \in \mathcal{H}} \delta^h(t+1) \left(z_j^h(e,t)\right)^- d\Phi. \end{array} \right\}.$$

Since all the asset payouts are nonnegative, $q_j(e,t) \ge 0$ in equilibrium. The price $p(t)$ will be the price of the single commodity in each time period t. In equilibrium, it is normalized to 1, but when defining the price space, it is more convenient to restrict $\left(p(t), (q_j(e,t))_{\forall e, j \in \mathcal{J}(e)}\right) \in \Delta^J$. Since $q_j(e,t)$ is bounded, then $q_j^b(e,t)$ is bounded $\forall b$. Finally, by definition, when $\int_{h \in \mathcal{H}} \left(z_j^h(e,t)\right)^+ d\Phi > 0$, the repayment rate $\delta^h(t+1) \le 1 \, \forall h \in \mathcal{H}$ implying that $\rho_j(e,t+1) \le 1$. However, if $\int_{h \in \mathcal{H}} \left(z_j^h(e,t)\right)^+ d\Phi = 0$, then any value for $\rho_j(e,t+1)$ will solve the conditions (AC). The upper bound 1 is added to bound $\rho_j(e,t+1)$ and it is clear that this upper bound is only hit if the determination of prices is irrelevant for market clearing (since markets are closed anyway). The price space Δ^* is nonempty, convex, and compact.

Next, I will write down the household's truncated optimization problem and define the household demand. The household problem (H) is given by

$$\max_{c^h, z^h, b^h} E_0 \sum_{t=0}^{\bar{T}} \beta^t u\left(c^h(t)\right) \tag{H}$$

$$subj \ to \ \left(c^h, z^h, b^h\right) \in \bar{B}^h(q^*, \rho).$$

I will define the household demand correspondence as

$$\Upsilon^h : \Delta^* \rightrightarrows \bar{B}^h(q^*, \rho)$$

such that given $(q^*, \rho) \in \Delta^*$, $\left(\tilde{c}^h, \tilde{z}^h, \tilde{b}^h\right) \in \Upsilon^h(q)$ iff $\left(\tilde{c}^h, \tilde{z}^h, \tilde{b}^h\right)$ solves (H). Υ^h is well-defined and Υ, defined such that $\Upsilon(q^*, \rho) = \displaystyle\int \Upsilon^h(q^*, \rho) d\Phi \quad \forall (q^*, \rho) \in \Delta^*(\omega)$, is convex-valued. The proof of the following lemma is contained after the *proof of theorem 2.*

Lemma 6 Υ^h *is an upper hemicontinuous (uhc) correspondence.*

Recall what Walras' law yielded for the "no-bankruptcy" model as given in the set of equations

$(a.3)$ from the *proof of theorem 1:* (where the middle equations hold for all $t : 0 < t < \bar{T}$):

$$\begin{array}{l}
\left(\int c^h(0)\, d\Phi - \bar{\omega}\right) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,0) \int z_j^h(e,0) d\Phi = 0 \\
\left(\int c^h(t)\, d\Phi - \bar{\omega}\right) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,t) \int z_j^h(e,t) d\Phi - .. \\
... - \int\limits_{h\in\mathcal{H}} \left\{ \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t-1) \right) + \sum_{j\in\mathcal{J}(e)} r_j(e'|e) z_j^h(e,t-1) \right\} d\Phi = 0 \\
\left(\int c^h\left(\bar{T}\right) d\Phi - \bar{\omega}\right) - ... \\
.. - \int\limits_{h\in\mathcal{H}} \left\{ \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, \bar{T}-1) \right) + \sum_{j\in\mathcal{J}(e)} r_j(e'|e) z_j^h(e,\bar{T}-1) \right\} d\Phi = 0
\end{array} . \tag{a.3}$$

For the bankruptcy model, Walras' law now yields the more complicated set of equations (again, the middle equations hold for all $t : 0 < t < \bar{T}$):

$$\begin{array}{l}
\left(\int c^h(0)\, d\Phi - \bar{\omega}\right) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,0) \int z_j^h(e,0) d\Phi = 0 \\
\left(\int c^h(t)\, d\Phi - \bar{\omega}\right) + \sum_e \sum_{j\in\mathcal{J}(e)} \int q_j^h(e,t) z_j^h(e,t) d\Phi - ... \\
.. - \int\limits_{h:\delta^h(t)=1} \left\{ \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} r_j^h(\hat{e},t) z_j^h(\hat{e}, t-1) \right) + \sum_{j\in\mathcal{J}(e)} r_j^h(e'|e,t) z_j^h(e,t-1) \right\} d\Phi + .. \\
... + \int\limits_{h:\delta^h(t)<1} \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} \alpha_j(\hat{e}) r_j(\hat{e}) \left(z_j^h(\hat{e}, t-1)\right)^+ \right) d\Phi + ... \\
... + \int\limits_{h:\delta^h(t)<1} \sum_{j\in\mathcal{J}(e)} \alpha_j(e) r_j(e'|e) \left(z_j^h(e, t-1)\right)^+ d\Phi = 0 \\
\left(\int c^h\left(\bar{T}\right) d\Phi - \bar{\omega}\right) - ... \\
.. - \int\limits_{h\in\mathcal{H}} \left\{ \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, \bar{T}-1) \right) + \sum_{j\in\mathcal{J}(e)} r_j(e'|e) z_j^h(e,\bar{T}-1) \right\} d\Phi = 0
\end{array} . \tag{a.9}$$

The distinction between $\delta^h(t) = 1$ and $\delta^h(t) < 1$ is meant as a means to separate all households (both bankrupt and solvent) who repay the entire amount of what is owed ($\delta^h(t) = 1$) from the remaining group of households ($\delta^h(t) < 1$) that declare bankruptcy ($b^h(t) = 0$) and $B^h(t) > S^h(t)$ (terms defined in $(a.8)$).

Lemma 7 *The aggregate consistency conditions are such that for all time periods $t : 0 < t < \bar{T}$, the equalities given in $(a.9)$ are equivalent to the equalities given in $(a.3)$.*

The proof of lemma 7 is located after the conclusion of the *proof of theorem 2*.

I will now write down the price correspondence

$$\Psi : \int \bar{B}^h d\Phi \rightrightarrows \Delta^*.$$

Given $\left(c^h, z^h, b^h\right)_{h\in\mathcal{H}}$, by definition $(q^*, \rho) \in \Psi\left(\left(c^h, z^h, b^h\right)_{h\in\mathcal{H}}\right)$ iff the dilution variables $\rho = \left(\rho_j(e)\right)_{\forall e, \forall j\in\mathcal{J}(e)}$ are defined as in (2.10) and the asset prices $q^b = \left(q_j^b(e)\right)_{\forall e,\forall j\in\mathcal{J}(e)}$ are defined as in (2.12) and (2.13) and the remaining price vector $(p, q) = \left(p, (q_j(e))_{\forall e,\forall j\in\mathcal{J}(e)}\right)$ satisfies the following maximization problem:

$$(p,q) \in \arg\max \left\{ \begin{array}{l} p(0)\left(\int c^h\left(0\right)d\Phi - \bar{\omega}\right) + \sum_{e}\sum_{j\in\mathcal{J}(e)} q_j(e,0)\int z_j^h(e,0)d\Phi \\ +\sum_{t=1}^{\bar{T}} p(t)\left(\int c^h\left(t\right)d\Phi - \bar{\omega}\right) + \sum_{t=1}^{\bar{T}-1}\sum_{e}\sum_{j\in\mathcal{J}(e)} q_j(e,t)\int z_j^h(e,t)d\Phi \end{array} \right\}. \quad \text{(a.10)}$$

Notice, the prices $(p(t))_{0\leq t\leq \bar{T}}$ are the prices of the numeraire commodity. In equilibrium, they are normalized to 1, but for the price correspondence, they are a variable to be determined. Notice further that the price correspondence is the same as that given in the proof of theorem 1 (only that the aggregate payouts from the price correspondence in $(a.3)$ have been removed as they have value 0).

The correspondence Ψ is well-defined, convex-valued, and uhc. Take any time period t and any asset (e, j). If $\int_{h\in\mathcal{H}} \left(z_j^h(e,t)\right)^- = 0$, then $\int_{h\in\mathcal{H}} \bar{B}^h(q^*, \rho)$ is trivially compact. In such a case, it is possible that undue pessimism about credit payouts (though irrational as discussed in the footnote following (2.10)) causes $q_j(e,t) = 0$ or $q_j^b(e,t) = 0$ for some $b \in \mathcal{B}$.

In all other equilibria, no arbitrage conditions (necessary conditions of equilibria) imply that the asset prices (both $q_j(e,t)$ and $q_j^b(e,t)$) are strictly positive. Therefore, the conditions for lemma 5 are met and $\int_{h\in\mathcal{H}} \bar{B}^h(q^*, \rho)$ is compact.

Define the overall equilibrium correspondence as the Cartesian product $\Upsilon \times \Psi$. The overall correspondence is well-defined, convex-valued, and uhc. It maps from the Cartesian product $\int \bar{B}^h d\Phi \times \Delta^*$ into itself. The set $\int \bar{B}^h d\Phi \times \Delta^*$ is nonempty, convex, and compact. Applying Kakutani's fixed point theorem yields a fixed point of this overall equilibrium correspondence. By definition, the fixed points is such that $\left(c^h, z^h, b^h\right)$ satisfies the household optimization problem (H) $\forall h \in \mathcal{H}$.

Given lemma 7, markets clear. This is because $(a.9)$ is equivalent to $(a.3)$ from the proof of lemma 4 (where the aggregate dividend payouts in $(a.3)$ have value equal to 0). I can directly apply the proof of lemma 4 to obtain the result and complete the *proof of theorem 2*.

Proof of Lemma 5

To show this result, I recognize that the consumptions are bounded (by definition). Then, beginning in time period $\bar{T}$, I will prove that the assets are bounded by backward induction.

Final period: $t = \bar{T}$

By the setup of the truncated equilibrium, no household may declare bankruptcy in time period $\bar{T}$. Thus, from the proof of lemma 1, in the final period $t = \bar{T}$, the payouts with bankruptcy exactly equal those in the general financial model (without bankruptcy). The same argument as in the *proof of lemma 1* yields the result that the assets $\left(\left(z_j^h(e, \bar{T}-1)\right)_{\forall j\in\mathcal{J}(e)}, \left(z_j^h(\hat{e}', \bar{T}-1)\right)_{\forall \hat{e}\neq e, j\in\mathcal{J}(\hat{e})}\right)^T$ are bounded.

Backward induction: $t < \bar{T}$

With $\left(\left(z_j^h(e,t)\right)_{\forall j \in \mathcal{J}(e)}, \left(z_j^h(\hat{e}',t)\right)_{\forall \hat{e} \neq e, j \in \mathcal{J}(\hat{e})} \right)^T$ bounded (this is the inductive hypothesis), the budget constraint at time period $t < \bar{T}$ dictates that the following terms are bounded, both $S^h(t)$ and $B^h(t)$ where

$$\begin{array}{l} S^h(t) = \sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j^h(\hat{e},t) z_j^h(\hat{e},t-1) \right) + \sum_{j \in \mathcal{J}(e)} r_j^h(e'|e,t) z_j^h(e,t-1). \\ \text{and } B^h(t) = -\sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} \alpha_j(\hat{e}) r_j(\hat{e}) \left(z_j^h(\hat{e},t-1)\right)^+ \right) - ... \\ -\sum_{j \in \mathcal{J}(e)} \alpha_j(e) r_j(e'|e) \left(z_j^h(e,t-1)\right)^+ . \end{array} \tag{a.11}$$

By observing $(a.11)$, it is clear how to partition the households at this time period t. I will first consider those households such that $\delta^h(t) < 1$ (bankrupt households by definition) and then consider those households such that $\delta^h(t) = 1$ (either solvent households or bankrupt households such that $\max\{S^h(t), B^h(t)\} = S^h(t)$).

Part I: Households $h : \delta^h(t) < 1$

Given the realized sequence $\left(e^h\right)^{t-1} = \left(e^h(0), ..., e^h(t-1)\right)$, household h must choose a portfolio of assets

$$\left(\left(z_j^h(e,t-1)\right)_{\forall j \in \mathcal{J}(e)}, \left(z_j^h(\hat{e}',t-1)\right)_{\forall \hat{e} \neq e, j \in \mathcal{J}(\hat{e})} \right)^T$$

to hedge against its idiosyncratic risk. The idiosyncratic risk stems from the fact that the income state $e^h(t)$ can be any $e' : \pi(e'|e^h(t-1)) > 0$. By the definition of the bankruptcy problem, household h cannot declare bankruptcy at time period t if $\omega^h(t) > \omega_{med}$ or if $b^h(t-1) = 0$. For the part under consideration, $\delta^h(t) < 1$ implies that for some income state realization $e^h(t)$, the household h prefers declaring bankruptcy $b^h(t) = 0$ (and is legally allowed to do so).

Thus, for the realized sequence $\left(e^h\right)^t = \left(\left(e^h\right)^{t-1}, e^h(t) \right)$ at which $\delta^h(t) < 1$, the inequality $B^h(t) > S^h(t)$ holds. Suppose that there exists a sequence $\left(z_j^\nu(e,t-1)\right)$ s.t. $z_j^\nu(e,t-1) \to -\infty$ as $\nu \to \infty$ for any asset (e,j). Under assumption $(A.5)$, which says that $\alpha >> 0$, and assumption $(A.3)$, which says that $r_j(e'|e) > 0$ and $r_j(\hat{e}) > 0$, the term $B^h(t)$ becomes unbounded. This contradiction proves that the assets $\left(\left(z_j^h(e,t-1)\right)_{\forall j \in \mathcal{J}(e)}, \left(z_j^h(\hat{e}',t-1)\right)_{\forall \hat{e} \neq e, j \in \mathcal{J}(\hat{e})} \right)^T$ are bounded above.

With the assets bounded above, suppose that for some asset (e,j), there exists a sequence $\left(z_j^\nu(e,t-1)\right)$ s.t. $z_j^\nu(e,t-1) \to -\infty$ as $\nu \to \infty$. By assumption $(A.4)$, with strictly positive probability the income state at period t will be such that $\omega^h(t) > \omega_{med}$ and the household cannot declare bankruptcy (by law). Thus, the household has financial wealth equal to

$$S^h(t) = \sum_{\hat{e} \neq e} \left(\sum_{j \in \mathcal{J}(\hat{e})} r_j^h(\hat{e},t) z_j^h(\hat{e},t-1) \right) + \sum_{j \in \mathcal{J}(e)} r_j^h(e'|e,t) z_j^h(e,t-1).$$

With $r_j^h(e'|e,t) = r_j(e'|e) > 0$ if $z_j^h(e,t-1) < 0$ and $r_j^h(\hat{e},t) = r_j(\hat{e}) > 0$ if $z_j^h(\hat{e},t-1) < 0$ from assumption $(A.3)$, the unbounded sequence of assets $z_j^\nu(e,t) \to -\infty$ leads to the unbounded term $S^h(t)$, a contradiction.

Part II: Households $h : \delta^h(t) = 1$

Given the realized sequence $\left(e^h\right)^{t-1} = \left(e^h(0), ..., e^h(t-1)\right)$, household h must choose a portfolio of assets

$$\left(\left(z_j^h(e,t-1)\right)_{\forall j \in \mathcal{J}(e)}, \left(z_j^h(\hat{e}',t-1)\right)_{\forall \hat{e} \neq e, j \in \mathcal{J}(\hat{e})} \right)^T$$

to hedge against its idiosyncratic risk. No matter which income state $e^h(t)$ appears, the term $(a.12)$ must remain bounded. For the households $h : \delta^h(t) = 1$, no matter which income state $e^h(t)$ is realized, the household will always prefer $\delta^h(t) = 1$ (either remain solvent or declare bankruptcy with $\max\{S^h(t), B^h(t)\} = S^h(t)$).

From part I, I already know that

$$\left(\left(z_j^h(e,t-1)\right)_{\forall j \in \mathcal{J}(e)}, \left(z_j^h(\hat{e}',t-1)\right)_{\forall \hat{e} \neq e, j \in \mathcal{J}(\hat{e})}\right)^T$$

are bounded for any and all households that such that $\delta^h(t) < 1$ for some realization $e^h(t)$.

Impose an artificial bound on the assets of some household $h : \delta^h(t) = 1$, $z_j^h(e,t-1) \geq -K_j(e)$ $\forall t$, $\forall e$, and $\forall j \in \mathcal{J}(e)$. Define the vector $K = (K_j(e))_{\forall e, j \in \mathcal{J}(e)} \in \mathbb{R}_+^J$. Then $\int_{h \in \mathcal{H}} \bar{B}^h(q^*, \rho)$ is compact, markets clear, and the aggregate consistency (AC) conditions hold. Consider what happens as $K \to \infty$. If the constraints cease to bind, then the assets are bounded. I will assume that some of the constraints continue to bind as $K \to \infty$ (that is, $\int_{h \in \mathcal{H}} \bar{B}^h(q^*, \rho)$ becomes unbounded) and show that this leads to a contradiction.

If some of the constraints continue to bind, it must be the case that the markets are open, $\int_{h \in \mathcal{H}} \left(z_j^h(e,t-1)\right)^- d\Phi < 0$. From the no arbitrage conditions as well as equations (2.11) – (2.13), the asset prices satisfy $q_j(e,t-1) > 0$ and $q_j^b(e,t-1) > 0$ $\forall b \in \{1,2\}$ and the dilutions satisfy $\rho_j(e,t) \leq 1$, and $\rho_j^b(e,t) \leq 1$ $\forall b \in \{1,2\}$ over all assets e and $j \in \mathcal{J}(e)$. Recall, in particular, equation (2.11) :

$$\rho_j(e,t) = 1 + \frac{\int_{h \in \mathcal{H}_t'} (1 - \delta^h(t)) \left(z_j^h(e,t-1)\right)^- d\Phi}{\int_{h \in \mathcal{H}} \left(z_j^h(e,t-1)\right)^+ d\Phi}.$$

As the assets $z_j^h(e,t-1)$ for any potential households $h : \delta^h(t) < 1$ are bounded, then $\rho_j(e,t) \to 1$ $\forall e, \forall j \in \mathcal{J}(e)$ as $\int_{h \in \mathcal{H}} \bar{B}^h(q^*, \rho)$ becomes unbounded. Thus $r_j^h(\hat{e}) \to r_j(\hat{e})$ and $r_j^h(e'|e) \to r_j(e'|e)$ as $\int_{h \in \mathcal{H}} \bar{B}^h(q^*, \rho)$ becomes unbounded. Repeating the same argument for the repayment rates $\rho_j^b(e,t)$ specific to households $h : b^h(t) = b$, then $\rho_j^b(e,t) \to 1$ $\forall b \in \{1,2\}$, $\forall e$, and $\forall j \in \mathcal{J}(e)$ as $\int_{h \in \mathcal{H}} \bar{B}^h(q^*, \rho)$ becomes unbounded. From (2.13) :

$$q_j^b(e,t-1) = q_j(e,t-1) \cdot E_{t-1}\left(\frac{\rho_j^b(e,t)}{\rho_j(e,t)}\right)$$

all households face the same asset prices in the limit $\left(q_j^h(e,t-1)\right)_{\forall e, j \in \mathcal{J}(e)} \to (q_j(e,t-1))_{\forall e, j \in \mathcal{J}(e)}$. The asset prices at the limit are the market-clearing prices for the general financial model (without bankruptcy). Therefore, at the limit, the bankruptcy equilibrium prices are identical to those of the general financial equilibrium and the term $(a.11)$ is equivalent to $(a.7)$.

Making use of what was previously shown in the *proof of lemma 1* for the general financial model (without bankruptcy), the assets

$$\left(\left(z_j^h(e,t)\right)_{\forall j \in \mathcal{J}(e)}, \left(z_j^h(\hat{e}',t)\right)_{\forall \hat{e} \neq e, j \in \mathcal{J}(\hat{e})}\right)^T$$

are bounded (in the limit as $\int_{h\in\mathcal{H}} \bar{B}^h(q^*,\rho)$ becomes unbounded). This contradicts that the constraints $z_j^h(e,t-1) \geq -K_j(e)\ \ \forall t,\ \forall e$, and $\forall j \in \mathcal{J}(e)$ will continue to bind as $K \to \infty$.

Proof of Lemma 6

I will define the budget correspondence

$$\bar{B}^h : \Delta^*(\omega) \rightrightarrows \bar{B}^h(q^*,\rho)$$

such that given $(q^*,\rho) \in \Delta^*(\omega)$, the values of the correspondence $\bar{B}^h$ are the entire budget set $\bar{B}^h(q^*,\rho)$. This correspondence is trivially uhc. The following proof will show that $\bar{B}^h$ is also lhc. Using the maximum principle (with a continuous utility function in the household optimization problem), Υ^h is a uhc correspondence.

Claim 2 *$\bar{B}^h$ is an lhc correspondence.*

Proof. Consider a sequence $\left(q^\nu, \left(q^{b\nu}\right)_{b\in\mathcal{B}}, \rho^\nu\right) \to \left(q, \left(q^b\right)_{b\in\mathcal{B}}, \rho\right)$ with $\left(c^h, z^h, b^h\right) \in \bar{B}^h\left(q, \left(q^b\right)_{b\in\mathcal{B}}, \rho\right)$ for some h (I will drop the superscript now). I will find some scaling factor $(\theta^\nu(t))_{0\leq t\leq \bar{T}}$ such that the scaled consumption and scaled assets:

$$\begin{aligned} c^\nu(t) &= \left(\min_{0\leq t\leq \bar{T}} \theta^\nu(t)\right) c(t)\ \ \forall t \\ z_j^\nu(e,t) &= \left(\min_{0\leq t\leq \bar{T}} \theta^\nu(t)\right) z_j(e,t)\ \ \forall t, \forall e,\ \text{and}\ \forall j \in \mathcal{J}(e) \\ b^\nu(t) &= b(t), \end{aligned}$$

$\exists N$ s.t. $\forall \nu \geq N$, $(c^\nu, z^\nu, b^\nu) \in \bar{B}^h\left(q^\nu, \left(q^{b\nu}\right)_{b\in\mathcal{B}}, \rho^\nu\right)$ and $(c^\nu, z^\nu, b^\nu) \to (c,z,b)$. For simplicity, define $\theta^\nu = \left(\min_{0\leq t\leq \bar{T}} \theta^\nu(t)\right)$.

The budget set $\bar{B}^h\left(q, \left(q^b\right)_{b\in\mathcal{B}}, \rho\right)$ has the so-called scaling propety (so called by Dubey et. al. (2005)) meaning that it is fairly straightforward to define the sequence of scaling fractions $\theta^\nu(t) \in [0,1]$ for $0 \leq t \leq \bar{T}$. This is done by induction.

<u>Initialization:</u> $t = 0$
Since the bankruptcy variables do not play a role in the budget constraint at $t = 0$, this proof follows exactly as in the *proof of lemma 3.*

<u>Induction:</u> $0 < t \leq \bar{T}$
Pick any time period $t : 0 < t \leq \bar{T}$ and suppose that the budget constraints are satisfied for all prior time periods (induction hypothesis). Define the asset payouts for this time period as $ap(t) = S^h(t)$ (the situation when $\delta^h(t) = 1$) or $ap(t) = B^h(t)$ (the situation when $\delta^h(t) = 1$) where

$$S^h(t) = \sum_{\hat{e}\neq e}\left(\sum_{j\in\mathcal{J}(\hat{e})} r_j^h(\hat{e},t) z_j^h(\hat{e},t-1)\right) + \sum_{j\in\mathcal{J}(e)} r_j^h(e'|e,t) z_j^h(e,t-1).$$

and $B^h(t) = -\sum_{\hat{e}\neq e}\left(\sum_{j\in\mathcal{J}(\hat{e})} \alpha_j(\hat{e}) r_j(\hat{e}) \left(z_j^h(\hat{e},t-1)\right)^+\right) - ...$
$-\sum_{j\in\mathcal{J}(e)} \alpha_j(e) r_j(e'|e) \left(z_j^h(e,t-1)\right)^+$.

If $\omega^h(t) - c(t) - \sum_e \sum_{j\in\mathcal{J}(e)} q_j^h(e,t)z_j(e,t) + ap(t) > 0$, then $\exists N$ s.t.

$$\omega^h(t) - c(t) - \sum_e \sum_{j\in\mathcal{J}(e)} \left(q_j^h\right)^\nu (e,t)z_j(e,t) + ap(t) > 0$$

holds $\forall\nu \geq N$. Define $\theta^\nu(t) = 1$ for this case.

Otherwise, $\omega^h(t) - c(t) - \sum_e \sum_{j\in\mathcal{J}(e)} q_j^h(e,t)z_j(e,t) + ap(t) = 0$ and thus $c(t) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j^h(e,t)z_j(e,t) - ap(t) > 0$ using the assumption that $\omega^h(t) > 0$.

Define $\theta^\nu(t) \in [0,1]\ \ \forall\nu$ as:

$$\theta^\nu(t) = \begin{array}{ll} \frac{\omega^h(t)}{c(t)+\sum_e \sum_{j\in\mathcal{J}(e)}\left(q_j^h\right)^\nu (e,t)z_j(e,t)-ap(t)} & \text{if } num < den \text{ and } den \neq 0 \\ 1 & \text{otherwise} \end{array}$$

where $num = \omega^h(t)$ and $den = c(t) + \sum_e \sum_{j\in\mathcal{J}(e)} \left(q_j^h\right)^\nu (e,t)z_j(e,t) - ap(t)$. With $num \to den$ and knowing that for some N_t, $\forall\nu \geq N_t$, $den > 0$, then $\theta^\nu(t) \to 1$.

Notice that both $S^h(t)$ and $B^h(t)$ are linear expressions of $\left(z_j^h(e,t-1)\right)_{\forall e, j\in\mathcal{J}(e)}$. As a result, scaling the vector of assets by a fraction θ^ν is equivalent to scaling the entire term $S^h(t)$ or $B^h(t)$ by the same fraction θ^ν. By holding the bankruptcy variable fixed, $b^\nu(t) = b(t)$, then the household will have either the asset payouts $ap(t) = S^h(t)$ or the asset payouts $ap(t) = B^h(t)\ \ \forall\nu \geq 0$. The following equations verify that the scaled $\left(c^\nu(t), \left(z_j^\nu(e,t-1), z_j^\nu(e,t)\right)_{\forall e, j\in\mathcal{J}(e)}, b^\nu(t) = b(t)\right)$ satisfy the budget constraint $\forall\nu \geq N_t$ at time period t :

$$\begin{array}{l} c^\nu(t) + \sum_e \sum_{j\in\mathcal{J}(e)} \left(q_j^h\right)^\nu (e,t)z_j^\nu(e,t) - ap(t) = \\ \theta^\nu c(t) + \theta^\nu \sum_e \sum_{j\in\mathcal{J}(e)} \left(q_j^h\right)^\nu (e,t)z_j(e,t) - \theta^\nu ap(t) \leq \\ \theta^\nu(t)\left[c(t) + \sum_e \sum_{j\in\mathcal{J}(e)} \left(q_j^h\right)^\nu (e,t)z_j(e,t) - ap(t)\right] \leq \omega^h(t). \end{array}$$

Thus, $\exists N = \max\{N_0, ..., N_{\overline{T}}\}$ s.t. $\forall\nu \geq N$, $(c^\nu, z^\nu, b^\nu) \in \bar{B}^h\left(q^\nu, \left(q^{b\nu}\right)_{b\in\mathcal{B}}, \rho^\nu\right)$ and also $(c^\nu, z^\nu, b^\nu) \to (c, z, b)$. This completes the proof. ■

Proof of Lemma 7

Take any time period $t : 0 < t < \bar{T}$. First note that the following term from equations $(a.3)$ can be removed by aggregating asset payouts:

$$\int\limits_{h\in\mathcal{H}} \left\{ \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e}) z_j^h(\hat{e}, t-1) \right) + \sum_{j\in\mathcal{J}(e)} r_j(e'|e) z_j^h(e, t-1) \right\} d\Phi = 0.$$

This leaves the equation of $(a.3)$ for this given time period t as:

$$\left(\int c^h\,(t)\, d\Phi - \bar{\omega} \right) + \sum_e \sum_{j\in\mathcal{J}(e)} q_j(e,t) \int z_j^h(e,t) d\Phi = 0. \tag{a.12}$$

Add and subtract the following term

$$\int\limits_{h:b^h(t)=0} \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} \rho_j(\hat{e},t) r_j(\hat{e}) \left(z_j^h(\hat{e}, t-1)\right)^+ \right) d\Phi + ..$$

$$.. + \int\limits_{h:b^h(t)=0} \sum_{j\in\mathcal{J}(e)} \rho_j(e,t) r_j(e'|e) \left(z_j^h(e, t-1)\right)^+ d\Phi.$$

from the set of equations $(a.9)$ yields the simplified set of equations:

$$\begin{aligned}
&\left(\int c^h(t)\, d\Phi - \bar{\omega}\right) + \sum_e \sum_{j\in\mathcal{J}(e)} \int q_j^h(e,t) z_j^h(e,t) d\Phi - ... \\
&... - \int\limits_{h\in\mathcal{H}} \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} \rho_j(\hat{e},t) r_j(\hat{e}) \left(z_j^h(\hat{e},t-1)\right)^+ \right) d\Phi - ... \\
&... - \int\limits_{h\in\mathcal{H}} \sum_{j\in\mathcal{J}(e)} \rho_j(e,t) r_j(e'|e) \left(z_j^h(e,t-1)\right)^+ d\Phi - .. \\
&... - \int\limits_{h\in\mathcal{H}} \delta^h(t) \sum_{\hat{e}\neq e} \left(\sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e},t) \left(z_j^h(\hat{e},t-1)\right)^- \right) d\Phi - .. \\
&... - \int\limits_{h\in\mathcal{H}} \delta^h(t) \sum_{j\in\mathcal{J}(e)} r_j(e'|e) \left(z_j^h(e,t-1)\right)^- d\Phi = 0.
\end{aligned} \tag{a.13}$$

after using the definition of $\delta^h(t) = -\frac{num}{den}$ for $h : \delta^h(t) < 1$. Recall that

$$\begin{aligned}
num &= \sum_{\hat{e}\neq e} \sum_{j\in\mathcal{J}(\hat{e})} \left(r_j^h(\hat{e},t) + \alpha_j(\hat{e})\right) \left(z_j^h(\hat{e},t_{-1})\right)^+ + \sum_{j\in\mathcal{J}(e)} \left(r_j^h(e'|e,t) + \alpha_j(e)\right) \left(z_j^h(e,t_{-1})\right)^+. \\
den &= \sum_{\hat{e}\neq e} \sum_{j\in\mathcal{J}(\hat{e})} r_j(\hat{e}) \left(z_j^h(\hat{e},t-1)\right)^- + \sum_{j\in\mathcal{J}(e)} r_j(e'|e) \left(z_j^h(e,t-1)\right)^-.
\end{aligned}$$

By definition, $\pi(e'|e) = \dfrac{\int_{h:(e^h(t),e^h(t+1))=(e,e')} d\Phi}{\int_{h:e^h(t)=e} d\Phi}$ and since the integration is a linear operation, the equation $(a.13)$ is equivalent to:

$$\begin{aligned}
&\left(\int c^h(t)\, d\Phi - \bar{\omega}\right) + \sum_e \sum_{j\in\mathcal{J}(e)} \int q_j^h(e,t) z_j^h(e,t) d\Phi - ... \\
&... - \sum_e \sum_{j\in\mathcal{J}(e)} \rho_j(e,t) r_j(e) \int\limits_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^+ d\Phi - .. \\
&... - \sum_e \sum_{j\in\mathcal{J}(e)} r_j(e) \int\limits_{h\in\mathcal{H}} \delta^h(t) \left(z_j^h(e,t-1)\right)^- d\Phi = 0.
\end{aligned} \tag{a.14}$$

after using the definition $r_j(e) = \sum_{e'} \pi(e'|e)\, r_j(e'|e)$. Recall the aggregate consistency (AC) condition for asset e and $j \in \mathcal{J}(e)$ as defined in (2.10) :

$$\rho_j(e,t) \int_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^+ d\Phi = -\int_{h\in\mathcal{H}} \delta^h(t) \left(z_j^h(e,t-1)\right)^- d\Phi.$$

Thus, multiplying each (AC) condition by $r_j(e)$ and summing up over all assets (e,j) yields:

$$\begin{aligned}
&\sum_e \sum_{j\in\mathcal{J}(e)} \rho_j(e,t) r_j(e) \int\limits_{h\in\mathcal{H}} \left(z_j^h(e,t-1)\right)^+ d\Phi + .. \\
&.. + \sum_e \sum_{j\in\mathcal{J}(e)} r_j(e) \int\limits_{h\in\mathcal{H}} \delta^h(t) \left(z_j^h(e,t-1)\right)^- d\Phi = 0.
\end{aligned}$$

After using this equation to simplify $(a.14)$, what remains is

$$\left(\int c^h(t)\,d\Phi-\bar{\omega}\right)+\sum_e\sum_{j\in\mathcal{J}(e)}\int q_j^h(e,t)z_j^h(e,t)d\Phi=0.$$

The equation (2.15) :

$$\int_{h\in\mathcal{H}}q_j^h(e,t)z_j^h(e,t)d\Phi=q_j(e,t)\int_{h\in\mathcal{H}}z_j^h(e,t)d\Phi\quad\forall e,\forall j\in\mathcal{J}(e)$$

allows me to simplify $(a.14)$ even further:

$$\left(\int c^h(t)\,d\Phi-\bar{\omega}\right)+\sum_e\sum_{j\in\mathcal{J}(e)}q_j(e,t)\int z_j^h(e,t)d\Phi=0.\tag{a.15}$$

As the simplified set of equations $(a.15)$ (recall this holds $\forall t: 0<t<\overline{T}$) derived from $(a.9)$ is equivalent to set of equations $(a.13)$ derived from $(a.3)$, then the set of equations $(a.9)$ are equivalent to the set $(a.3)$. This completes the proof.

Appendix B

Proof of Theorem 5

The proof breaks down into two basic analyses: a partial equilibrium analysis and a general equilibrium analysis.

First, I will consider the partial equilibrium analysis. Let's analyze the bankruptcy decision of two households, h and k. Both households bring financial wealth $w(t)$ into the period and have bankrutpcy indicator $b^h(t) = b^k(t) = 1$. Household h has income state 1, $e^h(t) = 1$ and household k has income state 2, $e^k(t) = 2$. Recall that the optimization for both households is given as:

$$\begin{array}{cc} \max\limits_{c,z,w',b'} & u(c) + \beta \sum_{e'} \pi\left(e'|e\right) V\left(e', w', b'; \Phi'\right) \\ subj.\ to & c\left(e,w,b;\Phi\right) + q^h\left(\Phi\right) \circ z\left(e,w,b;\Phi\right) = \omega_e + w \end{array} \tag{b.1}$$

where $w'\left(e,w,b;\Phi\right) = 0$ if $b'\left(e,w,b;\Phi\right) = 0$ and $w'\left(e,w,b;\Phi\right) = r^h(\Phi) z^h\left(e,w,b;\Phi\right)$ otherwise. Define the optimal consumption and asset choice of household h as $\left(c^h(t), w_1^h(t+1), w_3^h(t+1)\right)$ if household h is planning to declare bankruptcy given realization $e^h(t+1) = 1$ and as $\left(\widetilde{c^h(t)}, \widetilde{w_1^h(t+1)}, \widetilde{w_3^h(t+1)}\right)$ if h is planning on solvency. Likewise, define the optimal consumption and asset choice of household k as $\left(c^k(t), w_2^k(t+1), w_3^k(t+1)\right)$ for bankruptcy and $\left(\widetilde{c^k(t)}, \widetilde{w_2^k(t+1)}, \widetilde{w_3^k(t+1)}\right)$ for solvency.

I will now define a wealth parameter that will serve as the lower bound on wealth $w(t)$. Let $\underline{w}$ be such that for any $w^k(t) \leq \underline{w}$, household k will still choose to declare bankruptcy even if $\gamma_2 = \epsilon$ for $\epsilon > 0$ small. The proof of the following lemma is located at the completion of the *proof of theorem 5*.

Lemma 8 *For any wealth $w(t) > \underline{w}$, the following strict inequalities hold:*

- $u\left(c^h(t)\right) + \beta V\left(1, w_1^h(t+1), 0\right) > u\left(c^k(t)\right) + \beta V\left(2, w_2^k(t+1), 0\right).$
- $u\left(\widetilde{c^k(t)}\right) + \beta V\left(2, \widetilde{w_2^k(t+1)}, 2\right) > u\left(\widetilde{c^h(t)}\right) + \beta V\left(1, \widetilde{w_1^h(t+1)}, 2\right).$
- $u\left(c^h(t)\right) + \beta V\left(3, w_3^h(t+1), 2\right) < u\left(c^k(t)\right) + \beta V\left(3, w_3^k(t+1), 2\right) < ...$

  $...u\left(\widetilde{c^k(t)}\right) + \beta V\left(3, \widetilde{w_3^k(t+1)}, 2\right) < u\left(\widetilde{c^h(t)}\right) + \beta V\left(3, \widetilde{w_3^h(t+1)}, 2\right).$

The following fact is important. If $\gamma_1 = 1$, then trivially

$$u\left(c^h(t)\right) + \beta V\left(1, w_1^h(t+1), 0\right) > u\left(\widetilde{c^h(t)}\right) + \beta V\left(1, \widetilde{w_1^h(t+1)}, 2\right).$$

By continuity, this strict inequality holds in some open set containing $\gamma_1 < 1$.

From lemma 8 and the above fact, $\exists \underline{\gamma_1} \in (0,1)$ s.t. $\left(c^h(t), w_1^h(t+1), w_3^h(t+1)\right)$ is the optimal solution to $(b.1)$ for household h $\forall \gamma_1 \geq \underline{\gamma_1}$.[14] Likewise, $\exists \overline{\gamma_2} \in (0,1)$ s.t. the consumption and wealth

[14] In other words, $u\left(c^h(t)\right) + \beta\gamma_1 V\left(e_1, w_1^h(t+1), 0\right) + \beta(1-\gamma_1) V\left(e_3, w_3^h(t+1), 2\right) > u\left(\widetilde{c^h(t)}\right) + \beta\gamma_1 V\left(e_1, \widetilde{w_1^h(t+1)}, 2\right) + \beta(1-\gamma_1) V\left(e_3, \widetilde{w_3^h(t+1)}, 2\right).$

choice $\left(\widetilde{c^k(t)}, \widetilde{w_2^k(t+1)}, \widetilde{w_3^k(t+1)}\right)$ is the optimal solution to $(b.1)$ for household k $\forall \gamma_2 \leq \overline{\gamma_2}$.[15] The proof of the following lemma is located at the completion of the *proof of theorem 5*.

Lemma 9 $\underline{\gamma_1} = \overline{\gamma_2}$.

Define $\bar{\gamma} = \underline{\gamma_1} = \overline{\gamma_2}$. I can now summarize the bankruptcy decisions of households h and k for any wealth $w(t) > \underline{w}$. If $\gamma_1 \geq \bar{\gamma} \geq \gamma_2$, then separation occurs. That is, household h plans to declare bankruptcy (and will declare for the state realization that allows it) and household k does not plan to declare bankruptcy.

Second, I will consider the general equilibrium analysis. The value of $\bar{\gamma}(w)$ found in the partial equilibrium analysis is clearly a function of wealth w. More specifically, it is a strictly increasing function of w (if a household brings less wealth w into the current period, they are more likely to declare bankruptcy). Define $w(\gamma_1) = w\left(t : e(t-1) = e(t) = 1, w(t-1) = 0, b(t) = 1\right)$ as the wealth brought into period t by a household with a constant realization of states $e(t-1) = e(t) = 1$ who declares bankruptcy in time period $t-1$ and plans to declare in time period $t+1$ (if the state $e(t+1) = 1$ is realized). The separation will then actually be written as $\gamma_1 > \bar{\gamma}(w(\gamma_1)) > \gamma_2$, which is an assumption of theorem 5.

In order to show that the expected repayment rates $\left(\rho^1(t), \rho^2(t)\right)$ satisfy $\rho^1(t) < \rho^2(t)$ $\forall t$, I will find an upper bound $\rho^1(t) \leq \overline{\rho}^1$ $\forall t$ and a lower bound $\rho^2(t) \geq \underline{\rho}^2$ $\forall t$ such that $\overline{\rho}^1 < \underline{\rho}^2$. The proofs of the following lemmas are located at the completion of the *proof of theorem 5*.

Lemma 10 $\overline{\rho}^1 = 1 - \frac{(1-\gamma_2)(\gamma_1)^4}{(1-\gamma_2)(\gamma_1)^3 + (1-\gamma_1)(\gamma_2)^3}$.

Lemma 11 $\underline{\rho}^2 = 1 - \frac{(\gamma_2)^3 + \gamma_1 \frac{2(1-\gamma_2)}{(\gamma_2)^2 + 2(1-\gamma_2)}}{(\gamma_2)^2 + \frac{1}{2}\gamma_2 + \frac{1}{3} + \frac{2(1-\gamma_2)}{(\gamma_2)^2 + 2(1-\gamma_2)}}$.

The *proof of theorem 5* is complete upon using the final assumption of theorem 5 to obtain $\overline{\rho}^1 < \underline{\rho}^2$.

Proof of Lemma 8

The proof breaks down to showing that each of the following five strict inequalities holds:

$$u\left(c^h(t)\right) + \beta V\left(1, w_1^h(t+1), 0\right) > u\left(c^k(t)\right) + \beta V\left(2, w_2^k(t+1), 0\right). \tag{b.2}$$

$$u\left(\widetilde{c^k(t)}\right) + \beta V\left(2, \widetilde{w_2^k(t+1)}, 2\right) > u\left(\widetilde{c^h(t)}\right) + \beta V\left(1, \widetilde{w_1^h(t+1)}, 2\right). \tag{b.3}$$

$$u\left(c^h(t)\right) + \beta V\left(3, w_3^h(t+1), 2\right) < u\left(c^k(t)\right) + \beta V\left(3, w_3^k(t+1), 2\right). \tag{b.4}$$

$$u\left(c^k(t)\right) + \beta V\left(3, w_3^k(t+1), 2\right) < u\left(\widetilde{c^k(t)}\right) + \beta V\left(3, \widetilde{w_3^k(t+1)}, 2\right). \tag{b.5}$$

$$u\left(\widetilde{c^k(t)}\right) + \beta V\left(3, \widetilde{w_3^k(t+1)}, 2\right) < u\left(\widetilde{c^h(t)}\right) + \beta V\left(3, \widetilde{w_3^h(t+1)}, 2\right). \tag{b.6}$$

Throughout this proof, $\epsilon > 0$ is an arbitrarily small real number.

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[15] That is, $u\left(\widetilde{c^k(t)}\right) + \beta\gamma_2 V\left(e_2, \widetilde{w_2^k(t+1)}, 2\right) + \beta(1-\gamma_2) V\left(e_3, \widetilde{w_3^k(t+1)}, 2\right) > u\left(c^k(t)\right) + \beta\gamma_2 V\left(e_2, w_2^k(t+1), 0\right) + \beta(1-\gamma_2) V\left(e_3, w_3^k(t+1), 2\right).$

Proof. Inequality (b.5)

(i) Since $w(t) \geq \underline{w}$, household k will not declare bankruptcy if $\gamma_2 = \epsilon$. Thus, since the expression $u(\cdot) + \beta V(3, \cdot)$ is the certainty value for $\gamma_2 = 0$, the optimal consumption and asset choices when $\gamma_2 = \epsilon$ (defined as $c^k(\widetilde{t; \gamma_2} = \epsilon), w_3^k(\widetilde{t+1; \gamma_2} = \epsilon)$) will satisfy:

$$u\left(c^k(\widetilde{t; \gamma_2 = \epsilon})\right) + \beta V\left(3, w_3^k(\widetilde{t+1; \gamma_2} = \epsilon), 2\right) >$$

$$u\left(c^k(t)\right) + \beta V\left(3, w_3^k(t+1), 2\right)$$

where $\left(c^k(t), w_3^k(t+1)\right)$ are the optimal consumption and wealth choice for the true value of γ_2.

(ii) Now consider $\gamma_2 = 1 - \epsilon$. By definition (again since $u(\cdot) + \beta V(2, \cdot)$ is the certainty value for $\gamma_2 = 1$),

$$u\left(c^k(t; \widetilde{\gamma_2 = 1} - \epsilon)\right) + \beta V\left(2, w_2^k(t + \widetilde{1; \gamma_2} = 1 - \epsilon), 2\right) > \tag{b.7}$$

$$u\left(c^k(t)\right) + \beta V\left(2, w_2^k(t+1), 2\right)$$

where $\left(c^k(t), w_2^k(t+1)\right)$ are the optimal consumption and wealth choice for the true value of γ_2. I obtain:

- $V\left(3, w_3^k(t + \widetilde{1; \gamma_2} = 1 - \epsilon), 2\right) \approx V\left(2, w_2^k(t + \widetilde{1; \gamma_2} = 1 - \epsilon), 2\right) + u(\omega_3) - u(\omega_2)$.
- $V\left(3, w_3^k(t+1), 2\right) \leq V\left(2, w_2^k(t+1), 2\right) + u(\omega_3) - u(\omega_2)$ where $\left(w_2^k(t+1), w_3^k(t+1)\right)$ are the optimal wealth choices for the true value of γ_2.

The first bullet point holds since the shock is unanticipated when $\gamma_2 = 1 - \epsilon$. Both bullet points rely on the fact that $w_2^k(t+1) = w_3^k(t+1)$ when $b_2^k(t+1) = 2$. Therefore, using the bullet points together with inequality $(b.7)$ yields:

$$u\left(c^k(t; \widetilde{\gamma_2 = 1} - \epsilon)\right) + \beta V\left(3, w_3^k(t + \widetilde{1; \gamma_2} = 1 - \epsilon), 2\right) >$$

$$u\left(c^k(t)\right) + \beta V\left(3, w_3^k(t+1), 2\right)$$

where $\left(c^k(t), w_3^k(t+1)\right)$ are the optimal consumption and wealth choice for the true value of γ_2.

For future reference, the following will be referred to as the "convexity argument". The true value of γ_2 lies between the two extremes $(\epsilon, 1-\epsilon)$. Let $z^k(\gamma_2) = \theta z_\epsilon + (1-\theta) z_{1-\epsilon}$ denote the optimal asset choice at the true value of γ_2. Also, $\left(\widetilde{c^k(t)}, \widetilde{w_3^k(t+1)}\right)$ will denote the optimal consumption and wealth at the true value of γ_2. Then

$$\widetilde{c^k(t)} = \theta c^k(\widetilde{t; \gamma_2} = \epsilon) + (1-\theta) c^k(t; \widetilde{\gamma_2 = 1} - \epsilon)$$

since from the budget constraint:

$$c = \omega_e + w - q^h\left(\theta z_\epsilon + (1-\theta) z_{1-\epsilon}\right) = \theta c_\epsilon + (1-\theta) c_{1-\epsilon}.$$

Likewise for the wealth term:

$$\widetilde{w_3^k(t+1)} = \theta w_3^k(\widetilde{t+1}; \gamma_2 = \epsilon) + (1-\theta) w_3^k(t + \widetilde{1; \gamma_2} = 1 - \epsilon).$$

Using the concavity of $u(\cdot)$ and $V(\cdot)$:

$$u\left(\widetilde{c^k(t)}\right)+\beta V\left(3,\widetilde{w_3^k(t+1)},2\right)\geq$$

$$\theta\left[u\left(\widetilde{c^k(t;\gamma_2=\epsilon)}\right)+\beta V\left(3,\widetilde{w_3^k(t+1;\gamma_2=\epsilon)},2\right)\right]+$$

$$(1-\theta)\left[u\left(\widetilde{c^k(t;\gamma_2=1-\epsilon)}\right)+\beta V\left(3,\widetilde{w_3^k(t+1;\gamma_2=1-\epsilon)},2\right)\right]>$$

$$u\left(c^k(t)\right)+\beta V\left(3,w_3^k(t+1),2\right)$$

where $\left(c^k(t),w_3^k(t+1)\right)$ are the optimal consumption and wealth choice for the true value of γ_2. This finishes the proof of inequality $(b.5)$. ■

Proof. Inequality (b.6)

(i) First consider $\gamma_1=\epsilon$. Since the expression $u(\cdot)+\beta V(3,\cdot)$ is the certainty value for $\gamma_1=0$:

$$u\left(\widetilde{c^h(t;\gamma_1=\epsilon)}\right)+\beta V\left(3,\widetilde{w_3^h(t+1;\gamma_1=\epsilon)},2\right)>$$

$$u\left(\widetilde{c^k(t)}\right)+\beta V\left(3,\widetilde{w_3^k(t+1)},2\right)$$

where $\left(\widetilde{c^k(t)},\widetilde{w_3^k(t+1)}\right)$ are the optimal consumption and wealth choices for household k for the true value of γ_2.

(ii) Next consider $\gamma_1=1-\epsilon$. By definition (again since $u\left(\cdot\right)+\beta V\left(1,\cdot\right)$ is the certainty value for $\gamma_1=1$),

$$u\left(\widetilde{c^h(t;\gamma_1=1-\epsilon)}\right)+\beta V\left(1,\widetilde{w_1^h(t+1;\gamma_1=1-\epsilon)},2\right)> \tag{b.9}$$

$$u\left(\widetilde{c^k(t)}\right)+\beta V\left(1,\widetilde{w_1^k(t+1)},2\right)$$

for all consumption and wealth choices $\left(\widetilde{c^k(t)},\widetilde{w_1^k(t+1)}\right)$. I obtain the following two conditions:

- $V\left(3,\widetilde{w_3^h(t+1;\gamma_1=1-\epsilon)},2\right)\approx V\left(1,\widetilde{w_1^h(t+1;\gamma_1=1-\epsilon)},2\right)+u(\omega_3)-u(\omega_1)$.
- $V\left(3,\widetilde{w_3^k(t+1)},2\right)\leq V\left(1,\widetilde{w_1^k(t+1)},2\right)+u(\omega_3)-u(\omega_1)$ where $\left(\widetilde{w_1^k(t+1)},\widetilde{w_3^k(t+1)}\right)$ are the optimal wealth choices for the true value of γ_2.

The first bullet point follows since the shock is unanticipated when $\gamma_1=1-\epsilon$. Therefore, using the bullet points together with inequality $(b.9)$ yields:

$$u\left(\widetilde{c^h(t;\gamma_1=1-\epsilon)}\right)+\beta V\left(3,\widetilde{w_3^h(t+1;\gamma_1=1-\epsilon)},2\right)>$$

$$u\left(\widetilde{c^k(t)}\right)+\beta V\left(3,\widetilde{w_3^k(t+1)},2\right)$$

where $\left(\widetilde{c^k(t)}, \widetilde{w_3^k(t+1)}\right)$ are the optimal consumption and wealth choices for household k for the true value of γ_2.

Applying the convexity argument finishes the proof of inequality $(b.6)$. ■

Proof. Inequality (b.3)

I want to show that

$$u\left(\widetilde{c^k(t)}\right) + \beta V\left(2, \widetilde{w_2^k(t+1)}, 2\right) > u\left(\widetilde{c^h(t)}\right) + \beta V\left(1, \widetilde{w_1^h(t+1)}, 2\right).$$

Suppose not, then combined with inequality $(b.6)$, I obtain:

$$u\left(\widetilde{c^h(t)}\right) + \beta\gamma_2 V\left(1, \widetilde{w_1^h(t+1)}, 2\right) + \beta(1-\gamma_2)V\left(3, \widetilde{w_3^h(t+1)}, 2\right) > \tag{b.10}$$

$$u\left(\widetilde{c^k(t)}\right) + \beta\gamma_2 V\left(2, \widetilde{w_2^k(t+1)}, 2\right) + \beta(1-\gamma_2)V\left(3, \widetilde{w_3^k(t+1)}, 2\right).$$

In the strict inequality $(b.10)$, both households h and k are making optimal decisions expecting solvency and $V(2, w, 2) > V(1, w, 2)$ for any values of wealth w.[16] The vector $\left(\widetilde{c^h(t)}, \widetilde{w_1^h(t+1)}, \widetilde{w_3^h(t+1)}\right)$ is feasible, so the inequality $(b.10)$ contradicts that $\left(\widetilde{c^k(t)}, \widetilde{w_2^k(t+1)}, \widetilde{w_3^k(t+1)}\right)$ are optimizing decisions for household k. This contradiction completes the proof of inequality $(b.3)$. ■

To prove inequality $(b.2)$, I will make an additional assumption that will make the inequality hold trivially. Then I will show that this assumption is vacuous. Define the new function $\Delta(\gamma_2) \geq 0$ to be such that

$$u\left(c^h(t)\right) + \beta V\left(1, w_1^h(t+1), 0\right) \geq u\left(c^k(t)\right) + \beta V\left(2, w_2^k(t+1), 0\right) \tag{b.11}$$

where $\left(c^k(t), w_2^k(t+1)\right)$ are the optimal asset and wealth choice for household k at the true γ_2, $\left(c^h(t), w_1^h(t+1)\right)$ are the optimal asset and wealth choice for household h at the true γ_1, and $\gamma_1 \geq \gamma_2 + \Delta(\gamma_2)$. The strict inequality version is obtained as well. If $\gamma_1 > \gamma_2 + \Delta(\gamma_2)$, then inequality $(b.11)$ becomes a strict inequality (and equivalent to the desired $(b.2)$).

Proof. $\Delta(\gamma_2) = 0 \ \ \forall \gamma_2$.

This proof will be conducted in two steps: (i) $\Delta(\gamma_2)$ is a nondecreasing function of γ_2 and (ii) $\Delta(\gamma_2) \to 0$ for $\gamma_2 \to 1$.

(i) Define two persistences for household k, γ_2 and $\hat{\gamma}_2$ with $\gamma_2 > \hat{\gamma}_2$. Then define γ_1 and $\hat{\gamma}_1$ for household h as $\gamma_1 = \gamma_2 + \Delta(\gamma_2)$ and $\hat{\gamma}_1 = \hat{\gamma}_2 + \Delta(\hat{\gamma}_2)$.

With $\gamma_2 > \hat{\gamma}_2$, then

$$u\left(c^k(t;\gamma_2)\right) + \beta V\left(2, w_2^k(t+1;\gamma_2), 0\right) > u\left(c^k(t;\hat{\gamma}_2)\right) + \beta V\left(2, w_2^k(t+1;\hat{\gamma}_2), 0\right). \tag{b.12}$$

[Suppose not, then $\left(c^k(t;\gamma_2), w_2^k(t+1;\gamma_2)\right)$ is not an optimal solution to the household problem (3.1)].

By the definition of $\Delta(\gamma_2)$:

[16] The risk-sharing is always better for the less persistant income state $e = 2$.

- $u\left(c^h(t;\gamma_1)\right) + \beta V\left(1, w_1^h(t+1;\gamma_1), 0\right) = u\left(c^k(t;\gamma_2)\right) + \beta V\left(2, w_2^k(t+1;\gamma_2), 0\right).$
- $u\left(c^h(t;\hat{\gamma}_1)\right) + \beta V\left(1, w_1^h(t+1;\hat{\gamma}_1), 0\right) = u\left(c^k(t;\hat{\gamma}_2)\right) + \beta V\left(2, w_2^k(t+1;\hat{\gamma}_2), 0\right).$

For any wealth, $V(2,w,0) > V(1,w,0)$ (as discussed above). Thus, the first bullet point implies $u\left(c^h(t;\gamma_1)\right) > u\left(c^k(t;\gamma_2)\right)$ and the second implies $u\left(c^h(t;\hat{\gamma}_1)\right) > u\left(c^k(t;\hat{\gamma}_2)\right)$. Since utility is strictly increasing, then

$$\begin{aligned} c^h(t;\gamma_1) &> c^k(t;\gamma_2) \\ c^h(t;\hat{\gamma}_1) &> c^k(t;\hat{\gamma}_2). \end{aligned} \tag{b.13}$$

From the equalities in the bullet points above, the inequality $(b.12)$ implies:

$$u\left(c^h(t;\gamma_1)\right) + \beta V\left(1, w_1^h(t+1;\gamma_1), 0\right) > u\left(c^h(t;\hat{\gamma}_1)\right) + \beta V\left(1, w_1^h(t+1;\hat{\gamma}_1), 0\right).$$

With a single asset, the decisions by households h and k to declare bankruptcy (given state realizations $e=1$ or $e=2$, respectively) imply $w_1^h(t+1) = w_2^k(t+1) = 0$. Plugging in $w_1^h(t+1) = w_2^k(t+1) = 0$, the collected equations:

$$\begin{aligned} u\left(c^h(t;\gamma_1)\right) - u\left(c^k(t;\gamma_2)\right) &= \beta V(2,0,0) - \beta V(1,0,0) \\ u\left(c^h(t;\hat{\gamma}_1)\right) - u\left(c^k(t;\hat{\gamma}_2)\right) &= \beta V(2,0,0) - \beta V(1,0,0) \\ u\left(c^h(t;\gamma_1)\right) &> u\left(c^h(t;\hat{\gamma}_1)\right) \\ u\left(c^k(t;\gamma_2)\right) &> u\left(c^k(t;\hat{\gamma}_2)\right) \end{aligned}$$

together with the strict concavity of $u(\cdot)$ imply

$$c^h(t;\gamma_1) - c^k(t;\gamma_2) > c^h(t;\hat{\gamma}_1) - c^k(t;\hat{\gamma}_2). \tag{b.14}$$

The consumption $c^h(\cdot)$ is concave in γ_1. This is seen from the implicit function theorem applied to the first order condition with respect to z :

$$-qDu\left(c^h(t;\gamma_1)\right) + \beta(1-\gamma_1) D_w V\left(3, w_3^h(t+1;\gamma_1), 2\right) = 0.$$

Likewise, $c^k(\cdot)$ is concave in γ_2. Equation $(b.14)$ can be rearranged to reveal:

$$c^h(t;\gamma_1) - c^h(t;\hat{\gamma}_1) > c^k(t;\gamma_2) - c^k(t;\hat{\gamma}_2).$$

From the inequalities in $(b.13)$: $c^h(t;\gamma_1) > c^k(t;\gamma_2)$ and $c^h(t;\hat{\gamma}_1) > c^k(t;\hat{\gamma}_2)$, the concavity of consumption implies:

$$\gamma_1 - \hat{\gamma}_1 > \gamma_2 - \hat{\gamma}_2.$$

Thus, $\Delta(\gamma_2) = \gamma_1 - \gamma_2 > \hat{\gamma}_1 - \hat{\gamma}_2 = \Delta(\hat{\gamma}_2)$, so the function $\Delta(\gamma_2)$ is nondecreasing.

(ii) As $\gamma_2 \to 1$, then obviously $\gamma_1 \to 1$. The expressions $u(\cdot) + \beta V(1,\cdot)$ and $u(\cdot) + \beta V(2,\cdot)$ are the certainty values for $\gamma_1 = 1$ and $\gamma_2 = 1$, respectively. Further, $V(1,\cdot) \to V(2,\cdot)$ as $\gamma_2 \to \gamma_1$. Thus, inequality $(b.11)$ is obtained for any $\epsilon > 0$ with $\gamma_1 = \gamma_2 + \epsilon$. This shows that $\Delta(\gamma_2) \to 0$ for $\gamma_2 \to 1$. ■

Proof. Inequality (b.4)

I want to show that

$$u\left(c^h(t)\right) + \beta V\left(3, w_3^h(t+1), 2\right) < u\left(c^k(t)\right) + \beta V\left(3, w_3^k(t+1), 2\right).$$

Suppose not, then combined with inequality $(b.2)$, I obtain:

$$u\left(c^h(t)\right) + \beta\gamma_2 V\left(1, w_1^h(t+1), 0\right) + \beta(1-\gamma_2)V\left(3, w_3^h(t+1), 2\right) > \tag{b.15}$$
$$u\left(c^k(t)\right) + \beta\gamma_2 V\left(2, w_2^k(t+1), 0\right) + \beta(1-\gamma_2)V\left(3, w_3^k(t+1), 2\right).$$

In the strict inequality $(b.15)$, both households h and k are making optimal decisions expecting to declare bankruptcy and $V(2, w, 2) > V(1, w, 2)$ for any values of wealth w (as discussed above). The vector $\left(c^h(t), w_1^h(t+1), w_3^h(t+1)\right)$ is feasible, so the inequality $(b.15)$ contradicts that $\left(c^k(t), w_2^k(t+1), w_3^k(t+1)\right)$ are optimizing decisions for household k. This contradiction completes the proof of inequality $(b.4)$. ■

Proof of Lemma 9

This proof will be finished when I prove that both (i) $\underline{\gamma_1} < \overline{\gamma_2}$ and (ii) $\underline{\gamma_1} > \overline{\gamma_2}$ lead to contradictions.

i. Suppose that $\underline{\gamma_1} < \overline{\gamma_2}$. Then, there exists some value $\gamma \in (\underline{\gamma_1}, \overline{\gamma_2})$ such that when $\gamma_1 = \gamma_2 = \gamma :$

$$u\left(\widetilde{c^k(t)}\right) + \beta\gamma_2 V\left(2, \widetilde{w_2^k(t+1)}, 2\right) + \beta(1-\gamma_2)V\left(3, \widetilde{w_3^k(t+1)}, 2\right) > \tag{b.16}$$
$$u\left(c^k(t)\right) + \beta\gamma_2 V\left(2, w_2^k(t+1), 0\right) + \beta(1-\gamma_2)V\left(3, w_3^k(t+1), 2\right)$$

and

$$u\left(c^h(t)\right) + \beta\gamma_1 V\left(1, w_1^h(t+1), 0\right) + \beta(1-\gamma_1)V\left(3, w_3^h(t+1), 2\right) > \tag{b.17}$$
$$u\left(\widetilde{c^h(t)}\right) + \beta\gamma_1 V\left(1, \widetilde{w_1^h(t+1)}, 2\right) + \beta(1-\gamma_1)V\left(3, \widetilde{w_3^h(t+1)}, 2\right).$$

Since the values $\gamma_1 = \gamma_2$, the household choices for h and k are the same. Thus $\left(c^h(t), w_1^h(t+1), w_3^h(t+1)\right) = \left(c^k(t), w_2^k(t+1), w_3^k(t+1)\right)$ and same for the ˜ choices when the households are planning to remain solvent. Thus, the equations $(b.16)$ and $(b.17)$ above read as $A > B$ and $B > A$ for the appropriately defined terms A and B. This is a contradiction.

ii. Suppose that $\underline{\gamma_1} > \overline{\gamma_2}$. Then, there exists some value $\gamma \in (\overline{\gamma_2}, \underline{\gamma_1})$ such that when $\gamma_1 = \gamma_2 = \gamma :$

$$u\left(\widetilde{c^k(t)}\right) + \beta\gamma_2 V\left(2, \widetilde{w_2^k(t+1)}, 2\right) + \beta(1-\gamma_2)V\left(3, \widetilde{w_3^k(t+1)}, 2\right) < \tag{b.18}$$
$$u\left(c^k(t)\right) + \beta\gamma_2 V\left(2, w_2^k(t+1), 0\right) + \beta(1-\gamma_2)V\left(3, w_3^k(t+1), 2\right)$$

and

$$u\left(c^h(t)\right) + \beta\gamma_1 V\left(1, w_1^h(t+1), 0\right) + \beta(1-\gamma_1)V\left(3, w_3^h(t+1), 2\right) < \tag{b.19}$$
$$u\left(\widetilde{c^h(t)}\right) + \beta\gamma_1 V\left(1, \widetilde{w_1^h(t+1)}, 2\right) + \beta(1-\gamma_1)V\left(3, \widetilde{w_3^h(t+1)}, 2\right).$$

As above, since the values $\gamma_1 = \gamma_2$, the household choices for h and k are the same and so the equations $(b.18)$ and $(b.19)$ read as $A < B$ and $B < A$ for the appropriately defined terms A and B. This is a contradiction.

Proof of Lemma 10

Let $(\chi_1(t), \chi_2(t))$ be variables such that $\chi_i(t)$ is the fraction of total households that are bankrupt at time t with income state i (that is, $b(t) = 0$ and $e(t) = i$). This fraction is bounded above and below by $\left(\underline{\chi}_1, \overline{\chi}_1, \underline{\chi}_2, \overline{\chi}_2\right)$ such that $\underline{\chi}_1 \leq \chi_1(t) \leq \overline{\chi}_1$ and $\underline{\chi}_2 \leq \chi_2(t) \leq \overline{\chi}_2$ $\forall t$.

The households with asset sales in pool $b = 1$ are those households with income states $e = 1$ and $e = 2$. As the asset sales for those households planning to declare bankruptcy from this pool (those with states $e = 1$) are larger than those for households not planning to declare, an upper bound for the expected repayment rates $\rho^1(t)$ is defined as:

$$\overline{\rho}^1 = 1 - \frac{\underline{\chi}_1 (\gamma_1)^2}{\underline{\chi}_1 \gamma_1 + \overline{\chi}_2 \gamma_2}. \tag{b.20}$$

The numerator is the smallest fraction of households that meet the following requirements: (i) bankrupt at time $t-1$ with $e(t-1) = 1$, (ii) receive the realizations $e(t) = e(t+1) = 1$, and (iii) declare bankruptcy at the first opportunity (in time period $t+1$, from pool $b(t) = 1$). The denominator contains all the households with asset sales in pool $b(t) = 1$ and allows for the largest fraction of households that remain solvent (households with state $e(t-1) = 2$).

Recall that the measure across income states is given by $(\mu_1, \mu_2, \mu_3) \in \Delta^2$. The values for $\left(\underline{\chi}_1, \overline{\chi}_2\right)$ are given by:

$$\begin{aligned} \underline{\chi}_1 &= \frac{1}{2}\mu_1 (\gamma_1)^2 . \\ \overline{\chi}_2 &= \frac{1}{2}\mu_2 (\gamma_2)^2 . \end{aligned} \tag{b.21}$$

With the transition matrix Π given in (3.1), the measure (μ_1, μ_2, μ_3) can be calculated in terms of the persistence parameters $(\gamma_1, \gamma_2, \gamma)$:

$$\begin{aligned} \mu_1 (1-\gamma_1) &= \mu_2 (1-\gamma_2) = \mu_3 \left(\frac{1-\gamma}{2}\right). \\ \mu_1 &= \frac{1-\gamma_2}{\frac{(1-\gamma_1)(1-\gamma_2)}{\left(\frac{1-\gamma}{2}\right)} + (1-\gamma_1) + (1-\gamma_2)}. \\ \mu_2 &= \frac{1-\gamma_1}{\frac{(1-\gamma_1)(1-\gamma_2)}{\left(\frac{1-\gamma}{2}\right)} + (1-\gamma_1) + (1-\gamma_2)}. \end{aligned} \tag{b.22}$$

Upon evaluating the equality $(b.20)$ after replacing the variables with the expressions in $(b.21)$ and $(b.22)$, the result is obtained:

$$\overline{\rho}^1 = 1 - \frac{(1-\gamma_2)(\gamma_1)^4}{(1-\gamma_2)(\gamma_1)^3 + (1-\gamma_1)(\gamma_2)^3}.$$

Proof of Lemma 11

Recall the definitions of $(\chi_1(t), \chi_2(t))$ from the previous proof. Suppose that household k brings in wealth $w^k(t-2) \geq 0$ into time $t-2$ with $e(t-2) = 2$. This may be either a result of having just declared bankruptcy or having previously received the income state $e(t-3)=3$. Upon receiving the state realizations $e(t-1)=e(t)=2$, the optimal asset choices are ordered as follows:

$$\begin{aligned} z^k(t) &> 3 \cdot z^k(t-2). \\ z^k(t) &> 2 \cdot z^k(t-1). \end{aligned}$$

Suppose that household h brings in wealth $w^h(t) \geq 0$ into time t with $e(t)=1$. Then $z^h(t) > z^k(t)$.

To find the expected repayment rate, I must consider the fractions of households with certain asset sales and the size of those asset sales. With the inequalities above in terms of $z^k(t)$, I can divide out the asset sale size $z^k(t)$ and obtain an expression for $\underline{\rho}^2$ in terms of fractions of households. The households that may possibly declare bankruptcy out of pool $b(t)=2$ are those with three consecutive realizations of state $e=2$ and those just transitioning from $e=3$ to $e=1$. A lower bound $\underline{\rho}^2$ is defined as:

$$\underline{\rho}^2 = 1 - \frac{\mu_3\left(\frac{1-\gamma}{2}\right)\gamma_1 + \overline{\chi}_2\left(\gamma_2\right)^3 + \mu_3\left(\frac{1-\gamma}{2}\right)\left(\gamma_2\right)^3}{\mu_3\left(\frac{1-\gamma}{2}\right) + \sum_{n=1}^{3}\frac{1}{n}\left(\overline{\chi}_2 + \mu_3\left(\frac{1-\gamma}{2}\right)\right)\left(\gamma_2\right)^{3-n}}. \tag{b.23}$$

Using the expressions $(b.21)$ and $(b.22)$, the equality $(b.23)$ yields the result:

$$\underline{\rho}^2 = 1 - \frac{\left(\gamma_2\right)^3 + \gamma_1\frac{2(1-\gamma_2)}{(\gamma_2)^2+2(1-\gamma_2)}}{\left(\gamma_2\right)^2 + \frac{1}{2}\gamma_2 + \frac{1}{3} + \frac{2(1-\gamma_2)}{(\gamma_2)^2+2(1-\gamma_2)}}.$$

Proof of Theorem 6

The partial equilibrium analysis is identical to that considered in the *proof of theorem 5.* Define

$$w(\gamma_i) = w\left(t : e(t-i) = .. = e(t) = i, w(t-i) = 0, b(t-i) = 0, .., b(t) = i\right)$$

as the wealth brought into period t by a household with a constant realization of states $e(t-i) = .. = e(t) = i$ who declares bankruptcy in time period $t-i$ and plans to declare in time period $t+1$ (if the state $e(t+1)=i$ is realized). The separation will then actually be written as $\gamma_1 > \bar{\gamma}(w(\gamma_1)) > \gamma_2 > \bar{\gamma}(w(\gamma_2)) > ... > \gamma_I$, which is an assumption of theorem 6.

In order to show that the expected repayment rates $\left(\rho^i(t)\right)_{i=1,..,I}$ satisfy $\rho^i(t) < \rho^j(t) \;\; \forall i<j$ and $\forall t$, I will first find upper bounds $\overline{\rho}^i$ such that $\rho^i(t) \leq \overline{\rho}^i \;\; \forall i < I$ and $\forall t$. The proof of the following lemma is located at the completion of the *proof of theorem 6.*

Lemma 12 $\overline{\rho}^i < \overline{\rho}^j \;\; \forall i<j<I$ *and* $\frac{\rho^i(t)}{\overline{\rho}^i} \leq \frac{\rho^j(t)}{\overline{\rho}^j} \;\; \forall i<j<I$ *and* $\forall t$.

Taken together, the statement of lemma 12 implies that $\rho^i(t) < \rho^j(t) \;\; \forall i<j<I$ and $\forall t$. I have left only to show that the expected repayment rate of the pool $b(t)=I-1$ is strictly less than the repayment rate of the pool $b(t)=I$. This latter pool is the pool of all households that have either never declared bankruptcy or have waited long enough following a bankruptcy to have the bankruptcy flag removed. The proof of the following lemma is located at the completion of the *proof of theorem 6.*

Lemma 13 $\overline{\rho}^{I-1} < \underline{\rho}^I \leq \rho^I(t) \;\; \forall t$.

Proof of Lemma 12

Let $(\chi_1(t), \chi_2(t), ..., \chi_I(t))$ be variables such that $\chi_i(t)$ is the fraction of total households that are bankrupt at time t with income state i (that is, $b(t) = 0$ and $e(t) = i$). This fraction is bounded above and below by $\left(\underline{\chi}_1, \overline{\chi}_1, ..., \underline{\chi}_I, \overline{\chi}_I\right)$ such that $\underline{\chi}_i \leq \chi_i(t) \leq \overline{\chi}_i \quad \forall i \in \{1, ..., I\}$ and $\forall t$.

Recall that the measure across income states is given by $(\mu_1, ..., \mu_I, \mu_E) \in \Delta^I$.The definitions of $\left(\underline{\chi}_i, \overline{\chi}_i\right)$ are given as:

$$\begin{aligned}
\underline{\chi}_i &= \frac{1}{i+1}\mu_i (\gamma_i)^{i+1} \quad \forall i \in \{1, ..., I-1\}. \\
\overline{\chi}_i &= \frac{1}{i+1}\mu_i (\gamma_i)^{i+1} + \frac{1}{i+1}\mu_E \left(\frac{1-\gamma}{I}\right)(\gamma_i)^i \quad \forall i \in \{1, ..., I-1\}. \\
\overline{\chi}_I &= \frac{1}{I}\mu_I (\gamma_I)^I .
\end{aligned} \tag{b.24}$$

If I is large or if γ_I is close to 1, then $\underline{\chi}_i \approx \overline{\chi}_i$. I will use this approximation for the remainder of the argument (for notational simplicity).

As the asset sales for those households planning to declare bankruptcy from pool $b(\cdot) = i$ (those with states $e(\cdot) = i$) are larger than those for households not planning to declare, an upper bound for the expected repayment rates $\rho^i(t)$ is defined as:

$$\overline{\rho}^i = 1 - \frac{\underline{\chi}_i (\gamma_i)^{i+1}}{\underline{\chi}_i (\gamma_i)^i + \sum_{n=i+1}^{I} \overline{\chi}_n (\gamma_n)^i} \quad \forall i < I. \tag{b.25}$$

This is an extension of equation $(b.20)$.

The remainder of the proof is divided into two steps: (i) proving that $\overline{\rho}^i < \overline{\rho}^j \quad \forall i < j < I$ and (ii) proving that $\frac{\rho^i(t)}{\overline{\rho}^i} \leq \frac{\rho^j(t)}{\overline{\rho}^j} \quad \forall i < j < I$ and $\forall t$.

1. For I large, then $\underline{\chi}_i \approx \overline{\chi}_i$ and I will denote this fraction as simply χ_i. Consider $\left(\overline{\rho}^i, \overline{\rho}^j\right)$ for any $i < j$. As I is large, then the denominator in both $\overline{\rho}^i$ and $\overline{\rho}^j$ contain approximately the same number of terms.[17] The expressions $\left(\overline{\rho}^i, \overline{\rho}^j\right)$ can be rewritten as:

$$\begin{aligned}
\overline{\rho}^i &= 1 - \frac{\gamma_i}{1 + \sum_{n=i+1}^{I} \frac{\chi_n(\gamma_n)^i}{\chi_i(\gamma_i)^i}}. \\
\overline{\rho}^j &= 1 - \frac{\gamma_j}{1 + \sum_{n=j+1}^{I} \frac{\chi_n(\gamma_n)^j}{\chi_j(\gamma_j)^j}}.
\end{aligned}$$

   Due to the following facts:

   - $\gamma_i > \gamma_j$ for $i < j$ and
   - $\frac{\chi_n(\gamma_n)^i}{\chi_i(\gamma_i)^i} < \frac{\chi_n(\gamma_n)^j}{\chi_j(\gamma_j)^j}$ for any fixed n where $i < j < n \leq I$,

   then $\overline{\rho}^i < \overline{\rho}^j$.

[17] More specifically, one of the denominators will have strictly more terms than the other, but the difference caused by these additional terms is arbitrarily small for I large.

2. To determine the upper bounds $\overline{\rho}^i$, I am using the result that the asset sales for those planning to declare bankruptcy are larger than the asset sales of those planning to remain solvent. In essence, I am taking the complete definition of the expected repayment rate $\rho^i(t+1)$, as given in equation (2.12), and dividing through by the asset choice of household h where $e^h(t-i) = .. = e^h(t) = I$ and $b^h(t-i) = 0, ..., b^h(t) = i$. For simplicity, define the asset choice

$$z(e = j, b = i) = z\left(t : e(t-i) = j, .., e(t) = j, b(t-i) = 0, .., b(t) = i\right)$$

as the asset choice of a household with $b(t-i) = 0, ..., b(t) = i$ who continues to receive the state realizations $e = j$. Let's compare the upper bound $\overline{\rho}^i$ to the expected repayment rate:

$$\begin{aligned}
\overline{\rho}^i &= 1 - \frac{\underline{\chi}_i \left(\gamma_i\right)^{i+1}}{\underline{\chi}_i \left(\gamma_i\right)^i + \sum_{n=i+1}^{I} \overline{\chi}_n \left(\gamma_n\right)^i}. \\
\rho^i(t+1) &= 1 - \frac{\chi_i \left(\gamma_i\right)^{i+1} \cdot z(e = i, b = i)}{\chi_i \left(\gamma_i\right)^i \cdot z(e = i, b = i) + \sum_{n=i+1}^{I} \chi_n \left(\gamma_n\right)^i \cdot z(e = n, b = i)}.
\end{aligned}$$

If $i < j$, then the difference between $z(e = i, b = i)$ and $z(e = I, b = i)$ is larger than the difference between $z(e = j, b = j)$ and $z(e = I, b = j)$. This proves that the fraction $\frac{\rho^j(t)}{\overline{\rho}^j}$ is larger than the fraction $\frac{\rho^i(t)}{\overline{\rho}^i}$ $\forall t$. Considering all possible $i < j$, the desired result obtains:

$$\frac{\rho^i(t)}{\overline{\rho}^i} \leq \frac{\rho^j(t)}{\overline{\rho}^j} \ \forall i < j < I \text{ and } \forall t.$$

Proof of Lemma 13

Combine the equations $(b.24)$ and $(b.25)$ for $i = I - 1$:

$$\overline{\rho}^{I-1} = 1 - \frac{\mu_{I-1} \left(\gamma_{I-1}\right)^{2I}}{\mu_{I-1} \left(\gamma_{I-1}\right)^{2I-1} + \mu_I \left(\gamma_I\right)^{2I-1}}. \tag{b.26}$$

The measures $\left(\mu_{I-1}, \mu_I\right)$ can be calculated given the values of the persistence parameters $\left(\gamma_1, ..., \gamma_I, \gamma\right)$. This calculation leads to the following relation:

$$\frac{\mu_{I-1}}{\mu_I} = \frac{1 - \gamma_I}{1 - \gamma_{I-1}}.$$

Therefore, equation $(b.26)$ is given as:

$$\overline{\rho}^{I-1} = 1 - \frac{\left(1 - \gamma_I\right) \left(\gamma_{I-1}\right)^{2I}}{\left(1 - \gamma_I\right) \left(\gamma_{I-1}\right)^{2I-1} + \left(1 - \gamma_{I-1}\right) \left(\gamma_I\right)^{2I-1}}. \tag{b.27}$$

I have left to define the lower bound $\underline{\rho}^I$ such that $\underline{\rho}^I \leq \rho^I(t)$ $\forall t$ and then prove that $\overline{\rho}^{I-1} < \underline{\rho}^I$. Suppose that household k brings in wealth $w^k(t-\tau) \geq 0$ into time $t - \tau$ with $e(t-\tau) = I$. This may be either a result of having just declared bankruptcy or having previously received the income state $e(t-\tau-1)=E$. Upon receiving the state realizations $e(t-\tau)=..=e(t)=I$, the optimal asset choices are ordered as follows:

$$z^k(t) > (\tau+1)\cdot z^k(t-\tau).$$

I can repeat for any household with a sequence of state realizations $e(t-\tau)=..=e(t)=i<I$. Then the expected repayment rate will have a lower bound $\underline{\rho}^I$ that is a function only of the asset $z^k(t)$ where household k is such that $e(t-\tau)=..=e(t)=I$. Dividing out the asset choices leaves the lower bound $\underline{\rho}^I$ as a function only of fractions of households.

The following lower bound can be considered an extension of equation $(b.23)$:

$$\underline{\rho}^I = 1 - \frac{\overline{\chi}_I\left(\gamma_I\right)^{I+1} + \mu_E\left(\frac{1-\gamma}{I}\right)\left(\gamma_1 + \sum_{i=2}^{I}\left(\gamma_i\right)^{i+1}\right)}{\overline{\chi}_I\left(\gamma_I\right)^{I} + \mu_E\left(\frac{1-\gamma}{I}\right)\left(\gamma_1 + \sum_{i=2}^{I}\sum_{j=0}^{i}\frac{1}{j+1}\left(\gamma_i\right)^{i-j}\right)}. \tag{b.28}$$

Using $(b.24)$ and the fact that the stationary transition matrix Π forces $\mu_I\left(1-\gamma_I\right) = \mu_E\left(\frac{1-\gamma}{I}\right)$ (an extension of $(b.22)$), the lower bound $\underline{\rho}^I$ can be written as:

$$\underline{\rho}^I = 1 - \frac{\frac{\left(\gamma_I\right)^{2I+1}}{I} + \left(1-\gamma_I\right)\left(\gamma_1 + \sum_{i=2}^{I}\left(\gamma_i\right)^{i+1}\right)}{\frac{\left(\gamma_I\right)^{2I}}{I} + \left(1-\gamma_I\right)\left(\gamma_1 + \sum_{i=2}^{I}\sum_{j=0}^{i}\frac{1}{j+1}\left(\gamma_i\right)^{i-j}\right)}. \tag{b.29}$$

The proof is now complete since $\overline{\rho}^{I-1}$ (from equation $(b.27)$) is strictly less than $\underline{\rho}^I$ (from equation $(b.29)$) upon using the final assumption of theorem 6.

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