On the Solution and Application of Rational Expectations Models with Function-Valued States*

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Abstract

Many variables of interest to economists take the form of time varying distributions or functions. This high-dimensional 'functional' data can be interpreted in the context of economic models with function valued endogenous variables, but deriving the implications of these models requires solving a nonlinear system for a potentially infinite-dimensional function of infinite-dimensional objects. To overcome this difficulty, I provide methods for characterizing and numerically approximating the equilibria of dynamic, stochastic, general equilibrium models with function-valued state variables by linearization in function space and representation using basis functions. These methods permit arbitrary infinite-dimensional variation in the state variables, do not impose exclusion restrictions on the relationship between variables or limit their impact to a finite-dimensional sufficient statistic, and, most importantly, come with demonstrable guarantees of consistency and polynomial time computational complexity. I demonstrate the applicability of the theory by providing an analytical characterization and computing the solution to a dynamic model of trade, migration, and economic geography.

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1 Introduction

In order to understand and evaluate the causes and consequences of economic heterogeneity, it is helpful to have an analytical framework in which the distribution of heterogeneity can change over time and can both affect and be affected by other variables. A perspective in which some of the state variables of an economic model are endogenous random functions allows distributions, as well as objects like demand and supply curves or policy or value functions, to be treated as data. While descriptive models and methods for function-valued time series are undergoing rapid development,[1] interpreting this data requires formulating economic models capable of generating the observed functional data and deriving their implications. For models featuring forward looking decision making and endogenous aggregate variables, this derivation typically requires solving a computationally intractable infinite-dimensional system of nonlinear expectational difference equations. Although heuristic or strongly model dependent methods have been proposed, to date there appears to be no general purpose algorithm which provides a formal guarantee of even an approximate solution to rational expectations models with stochastic function-valued states.

This paper provides such an algorithm. In particular, it demonstrates how the equilibrium conditions for a general class of function-valued rational expectations models, including but not limited to heterogeneous agent dynamic stochastic general equilibrium models, can be linearized directly in function space, with solutions characterized locally by a functional linear process, a tractable empirical model for function-valued time series (Bosq, 2000). Construction of a local solution requires introducing a novel infinite-dimensional extension of the generalized Schur decomposition used to solve finite-dimensional rational expectations models (Klein, 2000) and developing perturbation theory for this object, which may be contributions of independent mathematical interest. The solution can be implemented numerically by a procedure based on finite-dimensional projection approximations which converges to the local solution under mild regularity conditions. I analyze in detail a particular approximation algorithm in this class, a wavelet transform based procedure which yields an approximate solution accurate to within any desired degree in polynomial time.

To demonstrate and evaluate the method, I develop a dynamic spatial model of trade, migration, and economic geography which introduces forward looking migration decisions and spatial shocks into the economic geography model of Krugman (1996). In the model, the spatial distribution of population, wages, and welfare over a continuum of locations is allowed to vary nonparametrically in response to persistent spatially correlated shocks to

[1] See Horváth & Kokoszka (2012); Bosq (2000); Morris (2014); Ferraty & Romain (2011) for surveys of the rapidly expanding field of functional data analysis, which focuses on modeling, estimation, and inference for series of observed or estimated functions.

the desirability of different locations. Due to the spatial structure of trade and production, the spatial distribution of economic activity is a determined by the distribution of population across locations, while the distribution of population is determined by forward looking migration decisions which take into account the expected distribution of economic activity. In this setting, the relationship between these two functions is not easily reduced to low-dimensional summaries or split into "local" and "global" components, but is well characterized by a functional linear model representation. By exploiting an analytical characterization of the solution to (certain parameterizations of) this model, the speed and numerical accuracy of the algorithm are evaluated in practice and shown to be in line with the strong theoretical guarantees.

The core idea behind the solution method is *functional linearization*. By taking the functional derivatives of the equations defining an equilibrium, it is possible to construct a system of equations which can be solved for the functional derivatives at a fixed point in function space of the policy operator, a map from function-valued states to function-valued endogenous variables. In this way, it is possible to recover local information about the solutions, which can then be used to construct a functional Taylor expansion of the policy operator which provides an accurate solution for all functions not too far from the function around which the model is linearized.

Constructing this linear approximation of the policy operators from the functional derivatives of the model equations requires solving a system of quadratic equations in linear operators. In the case of linear or linearized finite-dimensional rational expectations models, the analogous quadratic equation can be solved using matrix decomposition. In particular, Klein (2000) demonstrated that a solution can be found using the generalized Schur (or QZ) decomposition of the matrices of derivatives. In infinite dimensions, an analogous decomposition appears to be absent from the literature, in part because the finite-dimensional version is constructed by induction using eigenvalues, which may fail to exist or have countable cardinality in infinite-dimensional space. Nevertheless, it is possible to construct an analogous decomposition by other methods, described in detail in Appendix A. Under the conditions required for such a decomposition to exist and under further conditions analogous to the well known criterion of Blanchard & Kahn (1980) ensuring that the model has a linear solution, it is possible to solve for the first order expansion of the policy operator.

Calculating this local solution numerically requires representing it in a form that can be evaluated on a computer. A standard procedure for reducing problems in function spaces to finite-dimensional objects is to approximate the functions by projecting the space onto the span of a set of basis functions, such as wavelets, splines, or trigonometric or Chebyshev polynomials, and representing operators on function space in terms of their behavior with

respect to the basis functions. These approaches are referred to as spectral methods and are commonly applied to solve integral and differential equations: see Boyd (2000), Chatelin (2011). If any function we are interested in can be represented reasonably accurately by a finite set of basis functions, the loss from the use of a finite set of functions may be small. The caveat here is that, unlike in classical function approximation problems where the class of functions is known, 'the set of functions we are interested in' is not explicitly assumed, but must be determined by the properties of the model.

The issue that projection methods must overcome is that the class of functions well approximated by finite projection is in fact small in the class of all possible functions which could conceivably arise endogenously as outcomes of an implicitly defined model with function valued variables. To handle this concern, conditions must be imposed on the model which ensure both that the solutions themselves are continuous with respect to projection approximations and that the solutions are operators which have the property that they map functions which are well approximated by basis functions to functions which are well approximated by basis functions. Continuity properties of the generalized Schur decomposition are derived in Appendix B, and a set of restrictions on the model which ensure that basis function approximation is valid is described in Section 5.2.

While the precise statements of the sufficient conditions on a model for projection to be valid are somewhat technical, the conditions themselves are rather mild. Essentially, they rule out certain kinds of maps which take well-behaved smooth functions as input and produce jagged, noisy, or discontinuous functions as output. Many economic models can be represented in forms which satisfy these conditions, and many of those that do not can be modified slightly so that they do, for example by smoothing discontinuous cost functions or adding a small amount of noise to ensure that a distribution remains smooth.

Provided that the regularity conditions hold, implementing the solution is simple and fast. The linearized equilibrium equations can be approximated by projection, either analytically or numerically by quadrature, to produce two pairs of matrices, to which one can apply the finite-dimensional QZ decomposition, solve and combine to form a matrix approximation to the infinite dimensional policy operator. The accuracy of the approximation is then determined by the number of basis functions used and the smoothness of the functions that they are used to approximate. If all the equilibrium conditions are defined using Hölder continuous functions, wavelets provide the smallest and fastest feasible representation. Implementing approximate projection using the Discrete Wavelet Transform, the method converges in a number of operations polynomial in the degree of accuracy of the solution and in numerical experiments gives demonstrably accurate results at high speed. High level conditions are also provided for more general procedures, including for the case when parts of the model

are estimated directly from data.

The dynamics of economic heterogeneity have been considered from a variety of perspectives. Surveys of heterogeneous agent models are available in Krusell & Smith (2006); Heathcote *et al.* (2009); Guvenen (2011); Ljungqvist & Sargent (2004). A canonical framework is the Bewley model (Bewley, 1986), sometimes referred to as the Bewley-Huggett-Aiyagari model after the models and algorithms of Huggett (1993) and Aiyagari (1994). These models produce a time invariant cross-sectional distribution of income and wealth given by the stationary distribution endogenously induced by individual decisions which are themselves determined by the distribution. While not permitting any stochastic variation over time in distributions, the algorithms introduced to solve these nonstochastic models can be used as the first step in the linearization procedure I will provide, to find the point in the space of distributions around which to construct a linearized solution to a model with a stochastic distribution of heterogeneity.

To accommodate the setting where the distribution may evolve stochastically over time, Krusell & Smith (1998) introduce aggregate uncertainty into the Bewley model and provide a procedure to calculate approximate decision rules and generate dynamics of distributions jointly by simulation and representation of the impact of the distribution on decisions through a small set of moments. This method is particularly well suited to the model in that it takes advantage of a feature the authors refer to as 'approximate aggregation.' In the Krusell-Smith model, due to the use of a one-dimensional source of aggregate variation, an economic structure in which the impact of the distribution on the decision problem occurs only indirectly through its impact on prices in a centralized market, and a set of preferences and constraints that yields a decision rule which appears close to linear in individual states over most of the state space, a low-dimensional set of statistics of the wealth distribution suffices to describe its dynamics with a high level of accuracy. As a result, the decision problem can be reduced to a low-dimensional nonlinear decision problem in these statistics with apparently minimal loss of accuracy.

However, many of the features which make the Krusell-Smith method well suited to their model and similar models are far from universal. In particular, the finite-dimensional set of aggregate shocks limits variation in the shape of the functions of interest, and may create difficulty in matching estimates of the functions from cross-sectional data. Unless variation over time in the function lies exactly on the same low-dimensional space along which the model implies the functions move, observed functions may not be consistent with any possible values of the aggregate shocks, and so full information statistical methods will reject the specification completely. The low-dimensional set of aggregate shocks may also prevent consideration of economically important features, whether they affect the shape of

the functions directly, such as shocks to uncertainty, skewness, and higher moments of a distribution, as documented by Guvenen *et al.* (2012) for income distributions, or enter the model via other variables, such as the variety of sources uncertainty included in medium scale DSGE models. While procedures like the Krusell-Smith method can handle some increase in the dimensionality of the space of aggregate shocks by adding more statistics, increasing the dimension of the state space in the intertemporal decision problem can be computationally costly, with naive approaches based on discretization or tensor product function representations scaling exponentially in the number of state variables, and more sophisticated approaches, such as the Smolyak method of Gordon (2011), requiring some degree of difficult to verify regularity in the induced distribution.

Moreover, the restriction of the impact of the distribution to acting only on a finite set of market prices is less tenable in situations where interactions are decentralized, or the distribution enters the decision problem directly. This can be the case, for example, in spatial models, like the one considered in Section 3, where market outcomes differ across locations due to costs of trade or other economic interactions over distance, and in which decisions depend on the entire spatial distribution of economic activity due to both local and long distance interactions.

Many extensions and alternatives to the Krusell-Smith method are available: a *Journal of Economic Dynamics and Control* symposium (Den Haan, 2010) compares a variety of methods. Methods based on linearization or perturbation are not new, and are explored in, among others, Reiter (2009), Chung (2007), Winberry (2014) and Veracierto (2014). Perturbation approaches, which build on the class of linear rational expectations solution methods introduced by Blanchard & Kahn (1980) and extended by Klein (2000) and others, describe variability locally, and are much more amenable to including high-dimensional aggregate shocks than global, fully nonlinear approaches. Reiter (2009) developed the approach of linearizing models around nonlinear functions and distributions, and noted that by doing so, local methods can capture differences of large magnitude in the heterogeneous state between individuals and completely nonlinear responses to those differences while maintaining the tractability of linear methods for aggregate variables.

All of the considered perturbation approaches differ from the one advocated here because they do not consider linearization in function space. Instead, they replace functions by finite-dimensional approximations, either by projection or discretization, and then linearize and construct a solution based on applying algorithms applicable to finite-dimensional expectations models. The difference between linearizing and solving in infinite-dimensional space before taking a finite-dimensional approximation and taking a finite-dimensional approximation before linearizing and solving may seem minor, but the first approach is key to

ensuring that the resulting solution is well defined in terms of the true solution of the model and that the algorithm produces an answer which is provably close to this solution. These approaches also fail to consider that, except under certain regularity conditions, applying a finite-dimensional solution method to the approximated equilibrium conditions need not guarantee that the solution is accurate, even if the approximation of the conditions is, because the solution is not in general continuous with respect to the approximation error.

More generally, none of the methods described, including Krusell-Smith, provide or attempt to provide any formal proof that the approximation converges to a true solution. Although numerical demonstrations may be used to assess features consistent with the accuracy of the methods and so diagnose certain inaccuracies in an approximated solution, they cannot certify that the output of the algorithm is valid. Because it comes with formal guarantees, the functional linearization approach introduced here provides for the first time a benchmark which can be used to characterize a solution to dynamic models with heterogeneous agents which can be assured to be accurate.

Outline

The structure of this paper is as follows. I describe the setting of rational expectations models with function valued states in Section 2. Section 3 describes an illustrative application, a dynamic model of trade, migration and economic geography. Section 4 characterizes and gives necessary conditions for the existence of solution to the linearized model, while Section 5 introduces projection algorithms for calculating this solution and describes conditions for their consistency. Section 6 illustrate the procedure by applying it to solve the geography model from Section 3, and evaluates its performance by comparing to an analytical characterization of the solution. Section 7 concludes. Several appendices contain technical results: Appendix A describes conditions for existence of an infinite-dimensional version of the generalized Schur decomposition, and Appendix B gives conditions under which it is continuous. Appendix C collects all proofs, and Appendix D contains additional figures. An online supplementary appendix provides high level sufficient conditions for the existence of recursive equilibria in function-valued dynamic models.

Notation

$\mathcal{H}$, with any subscript, is assumed to be a complete separable Hilbert space. $\mathcal{B}$, with any subscript, is Banach space. The notation $\| \ \|$ is overloaded: if the object a is an element of $\mathcal{B}$, $\|a\|$ is the norm of a in $\mathcal{B}$, and if a is in a Hilbert space $\mathcal{H}$, $\|a\| = \langle a, a \rangle^{\frac{1}{2}}$ is the norm of a in $\mathcal{H}$, where $\langle \ , \ \rangle$ is the associated inner product. If $a \in \mathcal{H}_a$ but it is not clear from context the

space on which a lives, the norm may be denoted $\|a\|_{\mathcal{H}_a}$. $\mathcal{L}(\mathcal{H}_a \to \mathcal{H}_b)$ is the space of bounded linear operators from $\mathcal{H}_a$ to $\mathcal{H}_b$, equipped with the operator norm: for $A \in \mathcal{L}(\mathcal{H}_a \to \mathcal{H}_b)$, $\|A\| = \sup_{\|x\|_{\mathcal{H}_a}=1} \|Ax\|_{\mathcal{H}_b}$. If clarification is required, this norm may be denoted $\|A\|_{op}$. A^* denotes the (Hermitian) adjoint of A: $\forall x \in \mathcal{H}_a, y \in \mathcal{H}_b$, $\langle Ax, y\rangle = \langle x, A^*y\rangle$. A sequence of operators $A_i \in \mathcal{L}(\mathcal{H}_a \to \mathcal{H}_b)$, $i \in \mathbb{N}$ is said to converge in operator norm topology, or 'in norm' to A if $\|A_i - A\| \to 0$. For Γ a Cauchy contour in the extended complex plane $\mathbb{C}_\infty$ (see Conway (1978, Ch. 1 S. 6)) and $f(\lambda) : \mathbb{C}_\infty \to \mathcal{L}(\mathcal{H}_a \to \mathcal{H}_b)$ a function from one complex variable to a linear operator, $\int_\Gamma f(\lambda)d\lambda$ is the path integral of $f(\lambda)$ over the curve Γ, as defined in Gohberg *et al.* (1990, Ch. I). I is the identity operator: if the space $\mathcal{B}_a$ on which it acts needs to be specified, it is written $I_{\mathcal{B}_a}$. For $A \in \mathcal{L}(\mathcal{H}_a \to \mathcal{H}_b)$, $\mathrm{Im}(A)$ is the image of A and $\mathrm{Ker}(A)$ is the kernel of A. For a pair of bounded operators (B, A) each in $\mathcal{L}(\mathcal{H}_a \to \mathcal{H}_b)$, following Gohberg *et al.* (1990), define the spectrum $\sigma(B, A)$ as those $\lambda \in \mathbb{C}$ such that $\lambda A - B$ is not invertible, accompanied by the point ∞ if and only if A does not have bounded inverse, and the resolvent set $\rho(B, A)$ as $\mathbb{C}_\infty \backslash \sigma(B, A)$. An operator pair is said to be Γ-regular if for some nonempty subset $\Gamma \subset \mathbb{C}_\infty$, $\Gamma \subset \rho(B, A)$. Brackets $A[h]$ may optionally be used to denote that h is an argument of linear operator A, parentheses $A(h)$ generally denote that h is an argument of (possibly) nonlinear operator A. For nonlinear functions and operators, $F(a, b)$, F_a and F_b are the partial derivatives with respect to arguments a and b respectively. For a variable x, which may be a function, x' denotes the variable in the next time period, not the derivative. The Fourier transform of a function $f(x)$ is denoted with the scale convention $\hat{f}(\omega) := \mathcal{F}[\mathrm{f(x)}](\omega) := \int \exp(-2\pi\iota\omega x) f(x) dx$.

2 Function-Valued Models and Linearization in Function Space

The class of dynamic economic models which may be placed in a framework amenable to linearization in function space is large. Many economic models define objects of interest, explicitly or implicitly, as functions which solve a set of equations representing conditions such as optimization, market clearing, self-consistency, feasibility, or accounting identities. For example, a consumption function is often represented implicitly as the solution to an Euler equation, or a value function as the fixed point of a Bellman operator. Most trivially, when economic variables take values in Euclidean space, all of the theory developed in this paper will continue to apply. To see how random functions may naturally enter the description of an economic model, let us first consider a simple and illustrative case, before providing a general characterization.

It is common in microeconometric study of the dynamics of income and consumption by consumers or production by firms to model individual behavior by a linear dynamic panel model. A simplified version of this model is given by the assumption that, for each agent i, the variable of interest ζ_{it} follows the autoregressive process $\zeta_{it+1} = \rho\zeta_{it} + \epsilon_{it+1}$, where ϵ_{it+1} is independent of ζ_{it} and across agents and $|\rho| < 1$. While it is conventional to take an interest in the individual persistence parameter ρ, for the purposes of analysis of aggregates and welfare we may also be interested in the cross sectional distribution of the attribute ζ_{it}, which may be represented by pdf $f_t(\zeta)$. Given a measure 1 continuum of agents following this rule, the evolution of this distribution can be determined from its past value and the distribution of the shock ϵ_{it+1}. To model time varying effects such as aggregate shocks, we may let $\epsilon_{it+1} \overset{i.i.d.}{\sim} p_{t+1}(\epsilon)$ across agents, where the density function $p_{t+1}(.)$ may be taken as a function-valued random variable for each t. This models not only mean shifts, as would be captured by time fixed or random effects, but also distributional changes such as the changes in polarization or tail behavior of income risk as documented, for example, in Guvenen *et al.* (2012). Under this assumption, we have a dynamic equation for the evolution of the distribution of ζ, given by the convolution of the past distribution and the shock distribution

$$f_{t+1}(\zeta) = \int p_{t+1}(\zeta - \rho u) f_t(u) du \tag{2.1}$$

which provides a recursive representation for a function-valued economic variable of interest, $f_t(.)$, in terms of current and past values of the state, an operator mapping between them, and an exogenous shock which is also function valued, $p_{t+1}(.)$.

To formalize the linearization procedure for this and other models, and to provide a framework which permits both variables which are predetermined and those determined by forward looking expectations, we provide a notational framework for a general class of models. The notation and structure to be used follows closely that of Schmitt-Grohe & Uribe (2004), who described perturbation procedures for finite-dimensional rational expectations models, with the difference that we now allow state variables to be elements of an infinite-dimensional space. We consider in particular models with a recursive representation described by a set of equilibrium conditions which may be expressed as differentiable operators between separable Banach spaces. A solution to the model defines a recursive law of motion for the endogenous variables in the system in terms of the exogenous variables and past values of endogenous variables. The law is determined implicitly as the solution of a nonlinear expectational difference equation

$$\mathbb{E}F(x, y, x', y', \sigma) = 0_{\mathcal{B}_2} \tag{2.2}$$

where $x \in \mathcal{B}_x$ is a set of predetermined variables, $y \in \mathcal{B}_y$ is a set of endogenous or 'jump' variables, a superscript x', y' indicates the values of these elements in the next time period $t+1$ and the absence thereof indicates values of variables known at time t, $\sigma \in \mathbb{R}$ is a scalar scaling parameter determining the size of fluctuations. The function $F(x, y, x', y', \sigma) : \mathcal{B}_x \times \mathcal{B}_y \times \mathcal{B}_x \times \mathcal{B}_y \times \mathbb{R} \to \mathcal{B}_2$, which we refer to as the *equilibrium operator*, is a map taking the values of the state variables today and tomorrow and the scaling parameter to a space $\mathcal{B}_2$, and $\mathbb{E}$ is the (Bochner) expectation with respect to the law of motion induced by the solution of the model, to be made explicit shortly.[2]

Uncertainty in the model is incorporated solely via exogenous Banach random elements z' on probability space $(\mathcal{B}_z, \Sigma_z, \mu^z)$, which enter into the exogenous law of motion generating a subset of the predetermined variables x_2, with $(x_1, x_2) \in \mathcal{B}_{x_1} \times \mathcal{B}_{x_2} = \mathcal{B}_x$, by the equation $x_2' = h_2(x_2) + \sigma z'$ for $h_2 : \mathcal{B}_{x_2} \to \mathcal{B}_{x_2}$ a given function describing the dependence of future values of x_2 on current values. The shocks z' are normalized to have zero mean $\mathbb{E}[z'] = 0$. As a result, F contains as one subcomponent the formula $x_2' - h_2(x_2)$.

While this form may appear somewhat restrictive, many apparent limitations may be addressed through inclusion of appropriate auxiliary variables and equations. For example, while only variables in two time periods are included, by including lags and leads as separate variables, systems dependent on more time periods may be brought into this recursive form. Likewise, while function-valued uncertainty z' is restricted to enter additively in the model, nonlinear effects of shocks may be included by adding an additional predetermined variable which is a function of the shock: e.g., if z_k enters nonlinearly in F, replacing z_k with x_{2k} and incorporating the equation $x_{2k}' = \mathbb{E}z_k' + \sigma(z_k' - \mathbb{E}z_k')$ can recover the nonlinear effects. Overall, beyond imposing a recursive structure, the form provides a consistent notation but imposes only modest restrictions on the form of the economic model.

A (recursive) solution is given by a set of policy operators which solve the equilibrium equation for any value of the initial predetermined state x and the exogenous shocks z. In each period, y is given by the endogenously determined map $g(x, \sigma)$ from predetermined state x to endogenous state y (or x' to y'), and x' is given by the transition operator $h(x, \sigma) + \sigma\eta z'$ mapping the current predetermined state and shocks to next period's predetermined state, where η denotes the imbedding $\mathcal{B}_{x_2} \to \mathcal{B}_x$, i.e. for $z \in \mathcal{B}_{x_2}$, $\eta[z] = (0, z) \in \mathcal{B}_{x_1} \times \mathcal{B}_{x_2}$, and $h(x, \sigma) = (h_1(x, \sigma), h_2(x_2))$ includes both an endogenously determined transition component h_1 and an exogenous component h_2.

[2]The Bochner integral of a $\mathcal{B}$-valued random variable g on probability space (Ω, Σ, μ) is given by an element $\mathbb{E}g \in \mathcal{B}$ defined for simple functions $g = \sum_{i=1}^n f_i\{\omega \in A_i\}$ for f_i in $\mathcal{B}$, $A_i \in \Sigma$ as $\mathbb{E}g = \sum_{i=1}^n f_i \mu[A_i]$ and for more general random variables g as the strong limit of the Bochner integral of a sequence of simple functions g_n such that $\mu \left\| g - g_n \right\|_{\mathcal{B}} \to 0$. A measurable random element is Bochner integrable if and only if $\mu \|g\|_{\mathcal{B}} < \infty$.

Definition 1. A *recursive solution* is a set of maps $g(x,\sigma):\ \mathcal{B}_x \times \mathbb{R} \to \mathcal{B}_y$, $h_1(x,\sigma):\ \mathcal{B}_x \times \mathbb{R} \to \mathcal{B}_{x_1}$, $h_2(x_2):\ \mathcal{B}_{x_2} \to \mathcal{B}_{x_2}$ such that the *equilibrium conditions* hold:

$$\mathbb{E}F(x, g(x,\sigma), h(x,\sigma) + \sigma\eta z', g(h(x,\sigma) + \sigma\eta z', \sigma), \sigma) = 0_{\mathcal{B}_2} \tag{2.3}$$

for all x, σ, where the expectation $\mathbb{E}$ may now be defined, for each x, σ as the expectation with respect to the pushforward measure of μ^z on $\mathcal{B}_2$ generated by the function $F(x, g(x,\sigma), h(x,\sigma)+\sigma\eta z', g(h(x,\sigma)+\sigma\eta z', \sigma), \sigma):\ (x, z', \sigma) \in \mathcal{B}_x \times \mathcal{B}_{x_2} \times \mathbb{R} \to \mathcal{B}_2$ evaluated at fixed x, σ.

It can be shown that this pointwise in x definition of a solution generates a stochastic process for (x_t, y_t) under mild measurability conditions on the functions chosen.

Condition 1. (i) Let $\{z_t\}_{t=0}^{\infty}$ be an i.i.d. sequence on the infinite product of independent copies of $(\mathcal{B}_z, \Sigma_z, \mu^z)$ and initial value x_0 be defined on $(\mathcal{B}_x, \Sigma_x)$ with distribution μ_0^x, where Σ_x is a sigma field containing Σ_z. (ii) Fix $\sigma \in \mathbb{R}$. Suppose $h(x,\sigma)$ is $(\mathcal{B}_x, \Sigma_x) \to (\mathcal{B}_x, \Sigma_x)$ measurable, $g(x,\sigma)$ is $(\mathcal{B}_x, \Sigma_x) \to (\mathcal{B}_y, \Sigma_y)$ measurable for some Σ_y, and F is measurable with respect to the product sigma field $\Sigma_x \otimes \Sigma_y \otimes \Sigma_x \otimes \Sigma_y$ on $\mathcal{B}_x \times \mathcal{B}_y \times \mathcal{B}_x \times \mathcal{B}_y$

The measurability restrictions on h and g do impose some nontrivial limitations on the class of solutions to be considered by ruling out auxiliary randomness in the policy functions for aggregate variables beyond that included in z. For certain classes of models, randomization may be necessary to ensure existence of a solution, see Miao (2006). If this can be incorporated in z by expanding the state space, this poses no difficulty, but because the model will be solved by approximating near a point with no aggregate variability, the method cannot accommodate models which have no solution without aggregate randomness.

Proposition 1. *The series defined recursively by* $x_0 \sim \mu_0^x$, $x_{2,t+1} = h_2(x_{2,t}) + \sigma z_{t+1}$, $x_{1t+1} = h_1(x_t, \sigma)$, $y_t = g(x_t, \sigma)\ \forall t \geq 0$, *where* h, g *are a recursive solution satisfying Condition 1, is measurable with respect to the infinite product sigma field and* $\mathbb{E}F(x, g(x,\sigma), h(x,\sigma) + \sigma\eta z', g(h(x,\sigma)+\sigma\eta z', \sigma), \sigma)$ *coincides with the conditional expectation of* $F(x_t, g(x_t,\sigma), h(x_t,\sigma)+\sigma\eta z_{t+1}, g(h(x_t,\sigma) + \sigma\eta z_{t+1}, \sigma), \sigma)$ *at time* t *given* $x_t = x$.

Proof. See Appendix. □

In order to ensure computation of a stationary solution, the point around which the model is linearized is a nonstochastic steady state, which allows construction of a solution which is both local and recursive, by ensuring that the point around which the rule is calculated is the same in all periods.

Definition 2. A *nonstochastic steady state* is a set of values $(x^*, y^*) \in \mathcal{B}_x \times \mathcal{B}_y$ such that when $\sigma = 0$ and so function-valued uncertainty disappears, F satisfies

$$F(x^*, y^*, x^*, y^*, 0) = 0$$

Many recursive models will have such a point, at which all aggregate variables are unchanging over time. This is the equilibrium concept used in Bewley-Huggett-Aiyagari models, in which the distribution of heterogeneity is given by an invariant distribution generated by individual decision rules, and its existence can often be guaranteed by fixed point theorem. It may also be calculated consistently by standard methods, such as the iterative algorithms proposed by Huggett (1993) and Aiyagari (1994). In general, determining the nonstochastic steady state of a model involves solving a functional equation, which will differ depending on the details of the model. However, the problem involves determining only a single set of functions rather than an operator valid for any function, and is often quite feasible using standard methods. For example, in models where the decision rule is a function-valued state variable, recursive solutions are often available by dynamic programming, for which there are many feasible approximation algorithms with exponential or similarly fast convergence rates. Calculation of invariant distributions of Markov processes is also often achievable by iterative methods with exponential convergence. More broadly, in the absence of infinite-dimensional uncertainty, the problem usually reduces to a set of integral equations, for which a broad variety of standard numerical integral equation methods may be used.

A linearized solution of the model is given by first order Taylor expansion of $g(.)$ and $h(.)$ with respect to their arguments at the steady state. In order to solve for this, $g(.)$ and $h(.)$ and the operator $F : \mathcal{B}_x \times \mathcal{B}_x \times \mathcal{B}_y \times \mathcal{B}_y \to \mathcal{B}_2$ must be differentiable with respect to their arguments. In Banach space, the appropriate notion of derivative for linearization is (usually) the Fréchet derivative, which is defined analogously to the Fréchet derivative in Euclidean space.[3] If $F(x)$ is operator between Banach spaces $\mathcal{B}_1 \to \mathcal{B}_2$, the Fréchet derivative, if it

[3]For operators defined or differentiable only on subsets (not necessarily subspaces) of infinite-dimensional Banach spaces, such as the positive cone of non-negative measures, it may sometimes be desirable to consider the Hadamard derivative (see Flett (1980)) tangential to a set, which requires a derivative to be defined uniformly only over compact sets and so is weaker than the Fréchet derivative, which requires uniformity over closed balls, which in infinite dimensions are not compact. While the chain rule and a version of the implicit function theorem also apply for this class of derivatives, and so a linear approximation may be defined by the same equations with the Hadamard in place of the Fréchet derivative (and the derivatives exactly coincide on finite-dimensional spaces), the Taylor expansion will in be defined only over the subset on which a Hadamard derivative exists, and have a remainder with size dependent on the direction of the path of the approach, rather than just the norm. The domain restriction may not be a problem in practice, as in most cases the derivatives may be extended by the Hahn-Banach theorem to a larger space in a canonical way, for example, by removing positivity restrictions. If a solution does exist and is Hadamard differentiable, the Hadamard derivatives of the operators of interest will coincide on their domain with the

exists, is the continuous linear operator DF satisfying

$$\lim_{\|h\|_1 \to 0} \frac{\|F(x+h) - F(x) - DF[h]\|_2}{\|h\|_1} = 0. \tag{2.4}$$

In practice, calculation of Fréchet derivatives of Banach space-valued operators is not difficult: they obey many of the standard rules of Euclidean-valued derivatives including linearity, additivity, and the product rule, and many standard operators have known derivatives: see e.g. Kesavan (2004). Most importantly, the Fréchet derivative follows a version of the chain rule: for two Fréchet differentiable operators F, G, $D(F \circ G)[h] = DF[DG[h]]$. Fréchet differentiability is strictly stronger than directional, or Gateaux differentiability, which requires only the existence of a limit in the direction of a fixed element h. As the Gateaux derivatives of F in any direction $h \in \mathcal{B}$ may be calculated as the scalar derivative $\frac{d}{d\tau}F(x + \tau h)$ at $\tau = 0$ and must coincide with the Fréchet derivative when the latter exists, the form of the Fréchet derivative is easily determined. The Fréchet derivative preserves linear operators, so integration, differentiation, multiplication by a function, and any composition thereof have derivatives equal to themselves. A special class of operators which arises frequently in economic models is the composition of one function with another, referred to as a Nemytskii operator. Under appropriate boundedness, differentiability, and integrability conditions on $f(s_1, s_2)$, the composition $f(g(x), s_2)$, viewed as a map from the function $g(.)$ of x to the function $f(g(x), s_2)$ of x, s_2 is a Fréchet differentiable function of $g(.)$ at the point $g^*(.)$ with derivative $f_{s_1}(g^*(x), s_2) \cdot [h(.)]$: that is, the derivative is given by multiplication of the direction in which g changes by the partial derivative of f with respect to the element with which it is composed (Kesavan, 2004). In Banach space, Taylor's theorem for Fréchet derivatives gives a linear approximation of a differentiable operator $F(x) : \mathcal{B}_1 \to \mathcal{B}_2$ as $F(x + h) = F(x) + DF[h] + o(\|h\|_1)$.

It is important to contrast a Taylor expansion in function space with a local solution for finite-dimensional models. The point at which the linearization is constructed is the stationary state of the model in the situation where the variance of function-valued shocks is taken to 0. This is not the same as shutting down all variability in the model. In most heterogeneous agent models, individuals face a distribution of idiosyncratic uncertainty which may be arbitrarily dispersed and induces a nondegenerate stationary distribution of heterogeneity, in which the state of each individual evolves stochastically over time. In these models, the steady state function is the stationary distribution of heterogeneity, in the absence of aggregate shocks. For example, the unemployment rate can be constant

extended operators on the total space. When this is the case, the same first order approximation may be constructed and approximated by the algorithm provided, but will be valid only for directions in which Hadamard differentiability holds.

over time while each individual faces employment risk, with the number of people entering and leaving unemployment equal. Similarly, linearization of the policy operator does not imply all decision rules are linear: decisions with respect to individual characteristics may be arbitrarily nonlinear. Instead, the relationship between function-valued state variables is expressed in terms of linear operators. Loosely, the value of one nonlinear function at each point can be thought of as approximated by a linear function of the values of each other function at each point.[4] As a result, a first order functional Taylor expansion can describe rather complicated patterns of behavior.

To illustrate the process of linearization, consider the law of motion for the distribution provided by the panel model in (2.1): a linear approximation will describe the law of motion in the case of 'small' changes in the distribution. To consider the model in the case of small i.i.d. over time aggregate shocks to the cross sectional distribution $p_{t+1}(\epsilon)$ of idiosyncratic shocks, write the law deviations from the mean as $p_{t+1}(\epsilon) - p^*(\epsilon) = \sigma z_{t+1}(\epsilon)$, for $z_{t+1}(\epsilon)$ an i.i.d. over time Bochner mean 0 random function so that at $\sigma = 0$ the distribution of ϵ is constant over time at a fixed distribution $p^*(\epsilon)$. In the above notation $x_2' = p_{t+1}(\epsilon) - p^*(\epsilon)$, $z' = z_{t+1}(\epsilon)$ and $h_2(x_2) = 0$ because we have assumed that the exogenous aggregate shocks are not persistent.[5] To complete the description of the model, we may take as the endogenous predetermined variables $x_1 = f_t$, $x_1' = f_{t+1}$, and

$$F(x_1, x_2, x_1', x_2') = \begin{bmatrix} f_{t+1}(\zeta) - \int p_{t+1}(\zeta - \rho u) f_t(u) du \\ p_{t+1}(\epsilon) - p^*(\epsilon) \end{bmatrix}$$

as the equilibrium operator defining the model. In this case, all variables are predetermined or exogenous, so there is no y variable. A linear approximation with respect to f and p is given by taking the functional derivative of F with respect to p_t,f_t, p_{t+1}, f_{t+1} around a nonstochastic steady state f^*, p^* satisfying $p(\epsilon) = p^*(\epsilon)$, $f^*(\zeta) = \int p^*(\zeta - \rho u) f^*(u) du$, which exists whenever $|\rho| < 1$ under mild conditions on the density p^* of the error term: see Christensen (2014). Applying the chain rule and the product rule, a Taylor expansion of the

[4]This description is accurate for discrete functions: for general functions, the proper statement is that each linear functional of the output function is equal to a different linear functional of the input function.

[5]Allowing $h_2(x_2)$ in the model to be nonzero would represent persistence in the aggregate shock to the distribution of error terms. After linearization, the cross sectional distribution of ζ, the observable individual characteristic, would then be approximated by a functional ARMA(1,1) process, instead of a functional AR(1).

law of motion for f_{t+1} in p_{t+1} and f_t is given by

$$f_{t+1}(\zeta) = \int p^*(\zeta - \rho u) f^*(u) du + \int p^*(\zeta - \rho u)[f_t(u) - f^*(u)] du + \int [p_{t+1}(\zeta - \rho u) - p^*(\zeta - \rho u)] f^*(u) du + o(\|(f_t - f^*, p_{t+1} - p^*)\|).$$

Substituting in the (already linear) law of motion $p_{t+1} = p^* + \sigma z_{t+1}$ and the steady state relation, obtain

$$f_{t+1}(\zeta) - f^*(\zeta) = \int p^*(\zeta - \rho u)[f_t(u) - f^*(u)] du + \sigma \int z_{t+1}(\zeta - \rho u) f^*(u) du + o(\|(f_t - f^*, p_{t+1} - p^*)\|)$$

which expresses the deviation from the steady state in time $t+1$ as given by a linear operator applied to the deviation from steady state in time t plus, by linearity of the expectation and of the integral operator applied to $z_{t+1}(.)$, a mean 0 exogenous Banach random element. That is, it may be written as

$$f' - f^* \approx B[f - f^*] + \sigma \varepsilon'$$

for some linear operator B and some mean zero noise ε', a linear functional autoregression as in Bosq (2000), so long as both the noise and the deviation from a steady state are small. As similar procedures may be applied to more general dynamic panel data models, one sees that a functional linear process may provide a local approximation to the law of motion for distributions of cross-sectional aggregates for a wide range of commonly used empirical models of individual and aggregate behavior.[6]

In what follows, I will show how to take a linearization of a much broader class of models, and use it to solve for the dynamics and responses of the state variables of the models to endogenous and exogenous changes.

3 Example: Trade, Migration, and Economic Geography

The following is a dynamic model of economic geography based on the spatial model of Krugman (1996). In particular, it borrows whole cloth the static spatial equilibrium of that model, which determines wages, output, production, and prices at a continuum of locations given a distribution of population, and replaces the dynamic structure, which was

[6]For models as simple as in 2.1, it is possible to characterize the behavior without approximations: the random linear operator model generalizes the random coefficients model to infinite dimensions, and has been analyzed in Skorohod (1984).

given by an ad hoc behavioral rule chosen purely for tractability, with an intertemporally optimizing dynamic stochastic model of location choice. The economic phenomena one hopes to understand by making such a substitution are the dynamics of regional economies in response to aggregate shocks which may be global or asymmetric across regions. To capture the temporal structure of the dynamics, the model imposes adjustment costs on the movement of population, so that adjustment to regional shocks must take place in the short run by movements of prices and quantities and only gradually by shifts in population. This feature, along with explicit modeling of preferences, also has the advantage of allowing evaluation of the welfare implications of regional shocks. This is a feature which is relatively uncommon in equilibrium models with spatial structure, which tend to assume immediate adjustment of populations to equalize welfare across regions, complete markets in insurance for regional uncertainty, or complete labor immobility. An exception is provided by Caliendo *et al.* (2015), which is descriptively quite rich, and from which the model incorporates the structure of the intertemporal migration problem, as introduced originally by Artuç *et al.* (2010).

The proposed model differs from Caliendo *et al.* (2015) in market structure and in allowing a continuum of locations. As will be shown, the use of a continuum provides several advantages. Beyond the greater generality and ability to fit to arbitrary resolution, the continuum allows substantially greater tractability and analytical insight under certain parameterizations of the model. Most importantly, in the special case, also considered by Krugman (1996), of a completely spatially homogeneous geography with a continuum of locations where no location differs ex ante from any other, the linearized solution to the model can be described analytically. This makes the model a useful test case for numerical algorithms that attempt to approximate this solution numerically.

We begin with the intertemporal decision problem, which can be analyzed independently of the static equilibrium structure. Notation follows Krugman (1996). Individuals working in the tradeables sector at location x in geography G, a set of locations with a distance metric which for now we take to be a subset of Euclidean space, receive in each period t a real wage $\omega_t(x)$ and a value of regional amenities $\nu_t(x)$, both taken as given by the worker. A worker in location x at time t may decide to move to location x' in period $t+1$ at a cost $c(x'-x)$ which is a convex function of distance traveled (this could be relaxed to be non-translation invariant, but this simplifies some results). Workers are risk neutral with time-separable additive utility and discount the future at rate β. In each period they also receive independent and identically distributed across time and worker shocks $\epsilon_t(x')$ to their utility for each potential choice of location x', distributed according to a Gumbel process (Maddison *et al.* , 2014), whose finite-dimensional marginal distributions are independent

Type I extreme value random variables. The Bellman equation for the decision problem is therefore given by

$$\tilde{V}_t(x) = \max_{x'}\{\omega_t(x) + \nu_t(x) + c(x' - x) + \epsilon_t(x') + \beta E_t \tilde{V}_{t+1}(x')\}$$

For reasons of tractability, it is easier to work with the conditional expectation of this equation: denoting $V_t(x) := E_t\tilde{V}_{t+1}(x)$, we obtain

$$V_t(x) = E_t \max_{x'}\{\omega_{t+1}(x) + \nu_{t+1}(x) + c(x' - x) + \epsilon_{t+1}(x') + \beta V_{t+1}(x')\}$$

As a result, the location decision satisfies a continuous analogue of a multinomial logit decision rule: the conditional density of choices at location x' given current location x is given by

$$p(x'|x, V) = \exp(c(x' - x) + \beta V(x'))/\int \exp(c(x' - x) + \beta V(x'))dx'.$$

The use of extreme value shocks to generate a logit formulation for the policy function is similar to that used in Caliendo *et al.* (2015), with the difference that here the decision rule is defined over a continuum. To simplify notation, we will write the partition function of this conditional density as

$$f(x, V) := \int \exp(c(x' - x) + \beta V(x'))dx'.$$

Using the closed form characterization for the expectation of the maximum of a Gumbel process, it is possible to write the expectation over the maximum in terms of the partition function, allowing the Bellman equation to be simplified to

$$V_t(x) = E_t\omega_{t+1}(x) + \nu_{t+1}(x) + \log f(x, V_{t+1}) + \gamma \tag{3.1}$$

where γ is the Euler-Mascheroni constant (≈ 0.577). Due to this explicit form, no numerical optimization is needed to compute the value function. Since it can be shown that Blackwell's conditions hold, the Bellman operator is a contraction and the steady state value can be found by iteration of the contraction mapping.

The above constitutes the forward looking component of the model. To determine the implications of the chosen policy for dynamics of the equilibrium, assume that *at each location* there is a continuum of workers, who each receive independent and identically distributed preference shocks, and that the total mass of workers has measure 1 and is distributed across

locations at time t with density at location x given by $\lambda_t(x)$. Since the conditional density over locations given an initial state x is given by $p(x'|x, V)$, the time evolution of the density of workers across regions is given by the (adjoint) Markov transition operator

$$\lambda_{t+1}(x') = \int_G p(x'|x, V_t)\lambda_t(x)dx \tag{3.2}$$

taking the current distribution of population $\lambda_t(x)$ to the next period distribution $\lambda_{t+1}(x)$.

Together, λ_t and $\tilde{V}_t$ constitute the endogenous function-valued state variables of the model. To complete the model, one computes a static spatial equilibrium which generates a value of real wages at each location $\omega_t(x)$ given a distribution of population across places. A number of assumptions on market structure, trade, and geographical spillovers are possible here, with many models of trade and geography taking similar functional forms as discussed by Allen & Arkolakis (2014). A simple benchmark choice is the model of increasing returns, monopolistic competition, and iceberg trade costs of Krugman (1996), whose static structure can be borrowed without change. Specifically, we copy exactly the block of equations (A.24)-(A.27) of that model to determine wages given population. See Krugman (1996) for derivation and more detailed explanation.

Variables included in these equations are $Y(x)$, output at location x, $T(x)$, the price index at location x, and $w(x)$, the nominal wage in terms of the nontraded good. Parameters used are σ, the elasticity of substitution of the CES aggregator across varieties, μ, the Cobb-Douglas expenditure share on traded goods, and τ, the scale factor in the proportional iceberg trade costs $1 - \mathrm{e}^{-\tau|x-y|}$ of shipping a good from point x to point y. Given a predetermined distribution of population $\lambda_t(x)$, a static equilibrium of the model is given by functions $\{Y_t(x), T_t(x), w_t(x), \omega_t(x)\}$ satisfying the system of nonlinear integral equations

$$\begin{aligned}
Y_t(x) &= 1 - \mu + \mu\lambda_t(x)w_t(x) && (3.3)\\
T_t(x) &= \left[\frac{\tau(\sigma-1)}{2}\int_G \lambda_t(z)w_t(z)^{1-\sigma}e^{\tau(1-\sigma)|x-z|}dz\right]^{\frac{1}{1-\sigma}} && (3.4)\\
w_t(x) &= \left[\frac{\tau(\sigma-1)}{2}\int_G Y_t(z)T_t(z)^{\sigma-1}e^{-\tau(\sigma-1)|x-z|}dz\right]^{\frac{1}{\sigma}} && (3.5)\\
\omega_t(x) &= w_t(x)T_t(x)^{-\mu} && (3.6)
\end{aligned}$$

This system of equations is not analytically tractable, and has no explicit solution for ω_t in terms of λ_t. However, a solution in general exists and under certain conditions on parameters one may be able to calculate an implicit solution.

The dynamic specification of the model is completed by the inclusion of aggregate un-

certainty. As described above, while agents take into consideration the expectation of real wages, in the absence of additional inputs, these evolve deterministically. A number of potential sources of aggregate uncertainty can arise which affect the evolution of population across regions. For the purpose of the decision problem over locations, however, any source of uncertainty which affects the static equilibrium of the model exerts its effect only through its impact on the real living standards at different locations, $\omega_t(x) + \nu_t(x)$. Amenity value is exogenous in this specification of the model, and shocks to amenities across locations can reflect natural mechanisms like patterns of weather or natural disasters, or outcomes of (exogenous) local policies. Disturbances to variables determined within the static equilibrium of the model, such as changes in productivity (which may vary by location) in traded or nontraded sectors, changes in trade costs, or relative preferences for different varieties of good, will all show up in real wages. Further, because these are determined as the outcome of a purely static process, any persistence in these deviations (aside from that transmitted through the dynamics of population, described above), must come from outside the model. As a result, for the purposes of deriving the dynamics of economic activity and population, it is equivalent to model all shocks as changes to the exogenous value of amenities $\nu_t(x)$ at time t, and to provide exogenously specified dynamics for these shocks.

While many forms are possible, because the model will end up being linearized, it is sufficient to consider a linear specification for the dynamics of $\nu_t(x)$. For simplicity of illustration and, later, computation, I consider a first order functional autoregression specification with translation invariant transition operator, thereby restricting to shocks which do not diffuse differently from ex-ante identical locations:

$$\nu_{t+1}(x) = \int_G \Gamma(x - z)\nu_t(z)dz + \varepsilon_{t+1}(x) \tag{3.7}$$

In the above, $\Gamma(.)$ is some bounded, smooth, square-integrable function parameterizing the degree of spatial diffusion of shocks, and $\varepsilon_t(x)$ is an i.i.d. function-valued Banach random element with covariance operator Σ. Note that the additive formulation of the shock ν_t is without loss of generality even when interpreted as shocks to the trade component of the model, as subsequent to linearization, up to appropriate reparameterization of Γ and Σ, all specifications lead to a representation in the linearized Bellman equation as an additive shock to $\omega_t(x)$. While this is without loss of generality for the purpose of determining dynamic properties of the model, specification of the particular form in which shocks enter could be used to aid identification of the effects of these particular shocks, by specifying Γ and Σ as results of the composite effects of multiple shocks. However, since all static variables are determined jointly and contemporaneously, identification requires the aid of functional

form assumptions on the covariance of shocks or the validity of external instruments, and so no conclusions can be drawn without additional assumptions. The additive structure does have implications for identification of the model in the case where amenities are not directly observed, as the impact of shocks to real wages and amenities must be disentangled to identify the effects of each, but given the dynamics of the sum, the division has no effect on the dynamics of the other model variables, at least to first order. Similarly, as in Caliendo *et al.* (2015), the idiosyncratic valuation shocks which induce diffusion of population may be divided between real income and amenities, affecting interpretation but not the dynamics of the model.

While one may consider parameterizations under which it is necessary to solve numerically for many components of linearized model, for this model, it is possible to construct a particularly tractable special case in which the steady state and projections of derivatives can be computed exactly. If we assume that the geography is spatially homogeneous, such as the case of a circle, a sphere, an infinite line or plane, or higher-dimensional analogues of the preceding, the steady state of the system has a closed form solution. In particular, set $\nu_t(x)$ to 0 in all periods and conjecture that the initial distribution of population is uniform over the real line, in the sense that population measure over any interval is given by Lebesgue measure over the interval. Then it can be seen that a solution of the static equilibrium component of the model is given by $\{Y_t(x), T_t(x), w_t(x), \omega_t(x)\}$ which are all constant over x and equal to 1. Plugging this into the Bellman equation under the assumption that $\omega_t = 1$ is constant over time, shows that, because $c(x' - x)$ is translation invariant, $V(x) = \bar{V}$ constant is the unique solution of the Bellman equation. Placing this in $p(x'|x, V)$, we obtain that $p(x'|x, \bar{V}) \propto \exp(c(x' - x))$ and so is also translation invariant, and in particular if $c(x' - x) = \log g(x' - x)$ for any nonnegative function $g(.)$, the transition equation for λ_t is given by a convolution with a density proportional to $g(.)$. For example, if $c(x' - x) = -\frac{1}{2c}(x' - x)^2$, quadratic adjustment costs, equation (3.2) is given by convolution with a Gaussian with standard deviation c, and if $c(x' - x) = -\frac{1}{c}|x' - x|$, equation (3.2) is given by convolution with a Laplace distribution with dispersion parameter c. Because convolution is spatially invariant, the unique steady state of this transition equation on a translation-invariant domain is the uniform distribution, thus verifying the initial conjecture. For convenience, note that in steady state the partition function $f(x, \bar{V})$ is a constant, $\bar{f}$.

Given the existence of a steady state, the dynamics of the model local to this point can be expressed by taking functional derivatives of the operators. To express this model in format appropriate for solution by a functional linear rational expectations algorithm, express the model recursively in terms of jump variable $V(x)$ and predetermined variables $\lambda(x)$ and $\nu(x)$ and their next period values V', λ', ν', solving out the static variables, which

may be expressed at each time as a deterministic function of these three states, which are completely sufficient to solve for the dynamics of the model. We consider perturbations of V and ν as elements of $L^2(\mathbb{R})$ and perturbations of λ, a probability distribution, as an element of $L^2_0(\mathbb{R})$, the space of square integrable functions on $\mathbb{R}$ integrating to 0, ensuring that densities integrate to 1.

First, the transition equation is linear with respect to λ with derivative given by a convolution of the argument with density proportional to $\exp(c(x'-x))$, an operator which we can denote as $P[.] := \int \frac{1}{\bar{f}} \exp(c(x'-x) + \beta \bar{V})[.]dx'$. This can be interpreted as convolution with Gibbs distribution with potential given by the cost of moving: in the absence of disturbances to the value of a different locations, given a current population at each location, next period population spreads out by an amount proportional to the cost of distance. The Bellman equation is linear in V with derivative equal to the identity and has functional derivative with respect to V' given by $\frac{\beta}{\bar{f}} \int \exp(c(x'-x) + \beta \bar{V})[.]dx' = \beta P[.]$. The transition equation has derivative equal to the identity with respect to λ' and has derivative with respect to V given by $\beta \int_G \frac{1}{\bar{f}} \exp(c(x'-x) + \beta \bar{V})[.] - \frac{1}{\bar{f}} \exp(c(x'-x) + \beta \bar{V}) \frac{\beta}{\bar{f}} \int \exp(c(z'-x) + \beta \bar{V})[.]dz'dx$, which equals $\beta P - \beta PP$. The transition equation for ν is linear in ν and ν', with derivative with respect to ν given by $\Gamma[.] := \int \Gamma(x,z)[.]dz$ and ν' by the identity. Finally, although no closed form expression exists for $\omega(x)$ in terms of $\lambda(x)$, its functional derivative $\frac{d\omega}{d\lambda}$ with respect to $\lambda(x)$, which is all that is needed, can be determined by implicit differentiation of equations (3.3),(3.4), (3.5), and (3.6): the exact formula is derived in Appendix C.2 as equation (C.1).

Together these calculations fully characterize the derivatives of the model's equilibrium conditions with respect to the state variables. Arranging these derivatives into blocks with elements given by linear operators, the linearization of the equilibrium conditions of this model can be expressed in a form suitable for application of our solution methods, as a pair of linear operators representing the derivatives of the equilibrium conditions of the model with respect to today's state variables (λ, ν, V) and tomorrow's state variables (λ', ν', V').

$$\left(\begin{bmatrix} 0 & 0 & I \\ P & 0 & \beta P - \beta PP \\ 0 & \Gamma & 0 \end{bmatrix}, \begin{bmatrix} \frac{d\omega}{d\lambda} & I & \beta P \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \right) \tag{3.8}$$

In this pair of operators, the columns correspond to function-valued state variables, while the rows correspond to the linearized equations defining the equilibrium. In order, these are the Bellman equation, the transition law for the population distribution, and the law of motion for the function-valued shock to the distribution of amenities.

In what follows, I will show how to take the derivatives of a model expressed in this form

and use them to solve for the linearized dynamics and responses of the state variables of the model to endogenous and exogenous changes.

4 Characterization of Equilibrium Solution

While simple models with a high degree of structure may have linear approximations to solutions which can be found easily by heuristic methods, for larger or more complicated systems, and especially those where components are mutually determined, a more systematic approach is required. Fortunately, the steps involved in finding a linear solution may be described explicitly and so reduced to an algorithm which automates construction. The idea behind the method is to use a decomposition of the equilibrium conditions into components which may be solved separately and recursively, by taking a component whose evolution may be expressed as a function of past variables and a component which is solved by iterating forward expectations of future variables. While in some models, the components which are solved by looking backwards and the components which are solved by looking forwards may be identified with separate variables in the system, this is not true in general. Instead, this separation must be determined endogenously in such a way that initial and end-point conditions of the system are satisfied. What this often consists of is the requirement that some choice variables or other endogenous variables must be chosen to affect the expected evolution of other variables so that they satisfy an endpoint condition.

This is the source of the logic behind the cross-equation restrictions implied by many classical rational expectations models: the path of the endogenous variables must be determined jointly, and so expectations regarding one variable possibly far in the future may cause another variable to move far in advance. For example, in the Krugman (1979) model of balance-of-payments crises (and more recent models of the same), foreign exchange investor behavior is tied down by expected future optimality conditions at the point when the sovereign runs out of reserves. In the Dornbusch (1976) overshooting model, the nominal exchange rate follows a nonmonotonic path to ensure consistency with both short run price rigidity and a long run purchasing power parity anchor. In infinite-horizon settings, the role of an end-point condition which coordinates expectations is played by analogous long-run optimality or consistency conditions. For example, in the fiscal theory of the price level, the price adjusts to ensure consistency of long run expectations of the government budget deficit. Most commonly (at least in real models: see Cochrane (2011) for a discussion of complications in nominal models), long-run behavior is determined by a condition, such as transversality, which is satisfied when variables follow a dynamic path which is stationarity. While many types of long run restrictions are possible in models with function-valued

state variables, we will provide an algorithm for this most common case, in which the specified model takes recursive form over an infinite horizon and endpoint conditions require (or permit) a stationary solution. Although some modification is possible, including requiring asymptotic convergence (or slow divergence) at a particular rate possibly above or below 1, due to the infinite-dimensional nature of the parameter space, arbitrary endpoint conditions introduce substantial complications and so these will not be discussed further.

The requirement that it is possible to separate into solvable components also imposes one more technical limitation: to ensure orthogonality of projections, in what follows, we specialize from the setting of arbitrary separable Banach spaces to require all variables to live on separable Hilbert spaces: $\mathcal{H}_1 = \mathcal{H}_x \times \mathcal{H}_y$ and $\mathcal{H}_2$ replace $\mathcal{B}_1 = \mathcal{B}_x \times \mathcal{B}_y$ and $\mathcal{B}_2$, respectively. This will also be helpful in the numerical implementation. For models defined on spaces which can be densely embedded into a Hilbert space, it is often possible to extend the derivatives to the full Hilbert space by completion. However, norm convergence results must then be taken with respect to the Hilbert space norm.

For an economic model with recursive solution which is differentiable and generates a stationary stochastic process, we describe conditions that the functional derivatives of the solution operators $g(.)$ and $h(.)$ must satisfy, which will allow these derivatives to be calculated numerically.

Let the equilibrium conditions for the model of interest be given by 2.3 on page 11

$$G(x, \sigma) := \mathbb{E}F(x, g(x, \sigma), h(x, \sigma) + \sigma\eta z', g(h(x, \sigma) + \sigma\eta z', \sigma), \sigma) = 0$$

for all x, σ and assume $G(x, \sigma)$ is Fréchet differentiable with respect to x, σ.

Take the derivative with respect to x (evaluated at $(x^*, x^*, y^*, y^*, 0)$) to obtain

$$F_x + F_{x'}h_x + F_y g_x + F_{y'}g_x h_x = 0$$

In matrix form

$$\begin{bmatrix} F_{x'} & F_{y'} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & g_x \end{bmatrix} \begin{bmatrix} h_x \\ h_x \end{bmatrix} = -\begin{bmatrix} F_x & F_y \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix} \tag{4.1}$$

Define $A = \begin{bmatrix} F_{x'} & F_{y'} \end{bmatrix}$, $B = -\begin{bmatrix} F_x & F_y \end{bmatrix}$ mapping $\mathcal{H}_1 := \mathcal{H}_x \times \mathcal{H}_y \to \mathcal{H}_2$.

We seek to partially characterize the policy operators $h(x, \sigma)$ and $g(x, \sigma)$ by solving for their first derivatives with respect to the 'predetermined' state variable x, h_x and g_x. Written

as

$$A \begin{bmatrix} I & 0 \\ 0 & g_x \end{bmatrix} \begin{bmatrix} h_x \\ h_x \end{bmatrix} = B \begin{bmatrix} I \\ g_x \end{bmatrix}$$

this can be seen as an equation in terms of a pair of linear operators (B, A) which may be solved in terms of a joint decomposition of the pair. In general, multiple solutions to this system are possible: however, additional considerations provide some constraint as to the nature of acceptable solutions. In particular, conditions such as transversality conditions in optimization and No Ponzi Game conditions often rule out equilibria in which (some) state variables explode. More generally, a local solution method is attractive largely to the extent that the system remains with high probability in a neighborhood of the state around which the linearization applies.

For these reasons, we seek a solution to these equilibrium conditions which also induces stable, or stationary, dynamics. For finite-dimensional deterministic dynamical systems, sufficient conditions for the local stability around the steady state may be characterized by the eigenvalues of the linearized transition rule: in discrete time, eigenvalues less than one in modulus imply stability, in continuous time, eigenvalues must have real part less than 0. For infinite-dimensional dynamical systems, analogous conditions apply (see Gohberg *et al.* (1990, Ch. IV.3)). For rational expectations models characterized in terms of expectations, dynamics of state variables may be characterized not only by past values, but also by expectations of future values, and, in particular, certain variables may be allowed to 'jump,' which is to say that in response to a stochastic change in the current state, some variables may change discontinuously in order to satisfy the equilibrium conditions. As a result, stability conditions for this class of models differ from those for deterministic dynamical systems. Most notably, they may exhibit 'saddle-path stability,' in which the system evolves toward the steady state only along a lower-dimensional manifold and so only a (possibly null) subset of eigenvalues satisfy the stability conditions. A stable solution exists in such a case if the jump variables may adjust to ensure that the system stays on this stable manifold.

In the finite-dimensional case, stable solutions to this matrix pair equation may be characterized in terms of the Jordan decomposition of the pair, as in the seminal work of Blanchard & Kahn (1980), or in the case where singularity may be possible or numerical stability is desired, in terms of the generalized Schur decomposition as in Klein (2000). In the infinite-dimensional case, one may, under certain regularity conditions, apply analogues of these decompositions. To provide robustness to singularity and ensure numerical stability, this paper applies an analogue of the generalized Schur decomposition. As such a decomposition appears to be absent from the literature, Appendix A provides a detailed characterization and a proof of existence under a mild set of regularity conditions. The key idea of the proof is

to use the generalized resolvent operator to construct potentially non-orthogonal subspaces on which the operator pair acts corresponding to elements of the spectrum outside and inside the unit circle, and then show that orthogonalizing the subspaces to ensure unitarity of the transform preserves the spectrum.

For the purpose of characterizing the equilibrium conditions of the model, the following description suffices. If (B, A) satisfy the regularity conditions of Lemma(4) in Appendix A, among which are that (B, A) are bounded operators and that (B, A) are Γ-regular on the unit circle: $\gamma A - B$ has bounded inverse for any complex γ satisfying $|\gamma| = 1$, i.e. the unit circle is in the resolvent set, there exists a decomposition

$$(B, A) = (Q^* T U, Q^* S U)$$

in which U and Q are unitary operators and S and T may be decomposed as

$$(T, S) = \left(\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \right)$$

conformable with the decomposition $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$ and $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ such that the images of U_1^* and U_2^* respectively decompose $\mathcal{H}_1$ into two orthogonal subspaces $\mathcal{H}_{11}$ and $\mathcal{H}_{12}$ and the spectrum of (T_{11}, S_{11}) lies inside the unit circle, so S_{11} has bounded inverse. We may further decompose $U_1,\ U_2$ by considering their actions on $\mathcal{H}_y$ and $\mathcal{H}_x$. Write $U_{11} := U_1 \varphi^X$, $U_{12} := U_1 \varphi^Y$, $U_{21} := U_2 \varphi^X$, $U_{22} := U_2 \varphi^Y$ where $\varphi^X : \mathcal{H}_x \to \mathcal{H}_x \times \{0\} \subseteq \mathcal{H}_1$ and $\varphi^Y : \mathcal{H}_y \to \{0\} \times \mathcal{H}_y \subseteq \mathcal{H}_1$ are imbeddings.

Remark. The assumption of boundedness of the operator pair is not fundamental. Rather, it reflects the choice of space on which the operators are defined. See Kurbatova (2009) for a way in which to define the domain on which the pair acts so that boundedness holds and the above decomposition may be constructed for operator pairs unbounded with respect to the original choice of space $\mathcal{H}_X$ by restriction to a subspace. The use of potentially unbounded operators may be useful if equilibrium conditions are defined in terms of differential operators, as is common in continuous time versions of the models studied in this paper, as in Achdou *et al.* (2013). In discrete time, the conditions of interest are generally defined in terms of integral equations and so boundedness usually holds without restrictions.

In contrast, Γ-regularity imposes nontrivial restrictions. By requiring existence of a bounded operator with bounded inverse between the two spaces, it requires that $\mathcal{H}_X$ and $\mathcal{H}_Y$ be isomorphic, reflecting the traditional condition that to have a unique set of solutions, it is necessary that there be as many equations as unknowns. In addition to ruling out unit

roots, invertibility on the unit circle also rules out long memory behavior. To see this, note that because the resolvent set of an operator pair is open, invertibility must also hold in an open neighborhood of the unit circle, and so it cannot be the case that the spectrum has a limit point in the unit circle, as occurs in certain processes with long memory.

In terms of classes of operators which this assumption excludes, it rules out the presence of a continuous spectrum in the neighborhood of the unit circle. A prominent example of an operator pair with a continuous spectrum is an identity paired with a multiplication operator (which can arise as the functional derivative of a composition operator), i.e. (F, I) with $F[g(x)] = f(x) \cdot g(x)$, which has continuous spectrum taking all values attained by $f(x)$. If $|f(x)|$ has a limit point equal to 1, this operator pair is not Γ-regular. In this case a spectral decomposition can be constructed analytically, and a solution will exist with long memory or unit root behavior (depending on the behavior of $f(x)$ as it approaches 1), but for general models which fail to be Γ-regular with no closed form spectral decomposition, numerical approximations of the decomposition based on projection methods will be highly unstable.

The generalized Schur decomposition allows us to rewrite our decomposition as

$$\begin{aligned} Q^* \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & g_x \end{bmatrix} \begin{bmatrix} h_x \\ h_x \end{bmatrix} \\ = Q^* \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix} \end{aligned} \tag{4.2}$$

Unitarity of Q allows it to cancel on both sides, leaving, after simplification,

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} (U_{11} + U_{12}g_x)h_x \\ (U_{21} + U_{22}g_x)h_x \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} U_{11} + U_{12}g_x \\ U_{21} + U_{22}g_x \end{bmatrix}.$$

To find a stable solution, first solve for g_x, which determines the jump variables in terms of the predetermined variables, and then use this to find the value of h_x. To ensure that the second line holds trivially, it is sufficient to find $g_x : \mathcal{H}_x \to \mathcal{H}_y$ such that

$$U_{21} + U_{22}g_x = 0 \tag{4.3}$$

always. In principle, there may be many solutions, one solution, or no solution to this problem. In the case that $U_{22}U_{22}^*$ has bounded inverse on the space $\mathrm{Im}(U_2)$, at least one solution exists, given by what is referred to in numerical analysis as the 'minimum norm

solution' (Golub & van Loan, 1996, Ch. 4) to the linear equation (4.3),

$$g_x = -U_{22}^*(U_{22}U_{22}^*)^{-1}U_{21}. \tag{4.4}$$

It is worth noting how the condition that $U_{22}U_{22}^*$ has bounded inverse relates to the eigenvalue criteria in Blanchard & Kahn (1980) and subsequent rational expectations solution procedures. The existence of a bounded inverse implies $U_{22}U_{22}^*$ is bijective, and so U_{22} is surjective onto $\mathrm{Im}(U_2)$, which is mapped isometrically to $\mathcal{H}_{12}$ by the continuous and invertible linear transformation U_2^*. Therefore, there exists a linear surjection from $\mathcal{H}_y \to \mathcal{H}_{12}$. In finite dimensions, this requires that the dimension of the space of 'jump variables' y is at least as large as the dimension of the eigenspace corresponding to the 'unstable' generalized eigenvalues. Note however that in infinite dimensions, both of these spaces are infinite dimensional and the spectrum is generally uncountable, so this criterion cannot be expressed in terms of a relationship between the 'number of eigenvalues greater than one' and the 'number of jump variables'.

There is also an analogous condition characterizing uniqueness of the solution. Consider the case in which U_{22} has nontrivial null space. Then if g_x is a solution and $\tilde{g}$ is an operator whose range is a subset of Ker U_{22}, $g_x + \tilde{g}$ is also satisfies $U_{21} + U_{22}(g_x + \tilde{g}) = 0$. Thus, a solution is unique only if U_{22} has trivial null space. Formally, a solution is unique if and only if U_{22} is $\mathcal{H}_y$-complete: $\forall y \in \mathcal{H}_y$, $U_{22}y = 0$ implies $y = 0$. If U_{22} is complete and surjective, then it is bijective, and so, by the bounded inverse theorem has a bounded inverse and so

$$g_x = -{U_{22}}^{-1}U_{21} \tag{4.5}$$

is the unique solution.[7] In finite dimensions, a necessary condition for a linear operator to have trivial null space is that the domain and range spaces are of the same dimension. This therefore corresponds to the case in which the number of jump variables and unstable eigenvalues is exactly equal. Note that while there is a burgeoning literature on the characterization and implications of completeness in econometric models (see, e.g., Andrews (2011)), this is generally in the context of operators which are not surjective and do not have bounded inverse.

If U_{22} is surjective but not complete, the system is said to be underdetermined, and there may be many solutions, of which $-U_{22}^*(U_{22}U_{22}^*)^{-1}U_{21}$, the minimum norm solution is one. In this case, any solution to the system must be equal to $-U_{22}^*(U_{22}U_{22}^*)^{-1}U_{21}$ plus an operator $\tilde{g}$ whose range is in the kernel of U_{22}, which is the complement of the range of U_{22}^*,

[7] If you don't like the axiom of choice, this will still be the unique solution but may or may not be a bounded operator.

and so for all $x \in \mathcal{H}_x$ $\| - U_{22}^*(U_{22}U_{22}^*)^{-1}U_{21}x + \tilde{g}x\| = \| - U_{22}^*(U_{22}U_{22}^*)^{-1}U_{21}x\| + \|\tilde{g}x\| \geq \| - U_{22}^*(U_{22}U_{22}^*)^{-1}U_{21}x\|$ hence the description 'minimum norm'. This corresponds to the case in finite dimensions in which there are more jump variables than unstable eigenvalues. In this case, one may calculate a canonical solution with minimum norm, but there are also a continuum of other solutions in which arbitrary terms may be added in the eigenspaces corresponding to the jump variables so long as these terms are sent to zero by the expectation operator. This situation corresponds to the partial identification result when completeness fails in nonparametric instrumental variables estimation described in Santos (2012). While indeterminacy in the finite-dimensional case has received extensive study, for brevity and to avoid technical complications, I will consider only cases in which the solution is unique, in which case U_{22}^{-1} is bounded and well defined.

Given a solution for g_x, the evolution equation for the predetermined variables may be expressed in terms of this solution. Imposing 4.3, the equilibrium conditions hold if $S_{11}(U_{11} + U_{12}g_x)h_x = T_{11}(U_{11} + U_{12}g_x)$. Since S_{11} has bounded inverse by construction, this gives

$$h_x = (U_{11} + U_{12}g_x)^{-1}S_{11}^{-1}T_{11}(U_{11} + U_{12}g_x) \tag{4.6}$$

is a solution so long as $U_{11} + U_{12}g_x$ has bounded inverse. Moreover, this operator is similar to $S_{11}^{-1}T_{11}$ and so has identical spectrum. In particular, by the construction of S_{11} and T_{11} the spectrum of this operator is inside the complex unit circle. So, by Gohberg *et al.* (1990, Thm IV.3.1), the difference equation $x_{t+1} = h_x x_t,\ x_0 = x \in \mathcal{H}_x$ has a unique solution for any given x, given by $x_t = (h_x)^t x$, which converges to 0. Thus, we say that h_x is a stable solution. Moreover, under these conditions, Bosq (2000, Thm 3.1) implies that the Hilbert AR(1) functional linear process given by $x_{t+1} = h_X x_t + \xi_t$, where ξ_t is a $\mathcal{H}_x$ random element uncorrelated over t has a unique covariance stationary solution, and so we are justified in referring to h_x as a stationary solution.

For purposes of numerical approximation, we would like to write h_x in a more convenient form.

Lemma 1. *Let $(U_{22}U_{22}^*)^{-1}$ be bounded and let g_x solve (4.3). Then*

$$\begin{aligned} h_x &= (U_{11} + U_{12}g_x)^{-1}S_{11}^{-1}T_{11}(U_{11} + U_{12}g_x) \\ &= (\varphi^{X*}\varphi^X + g_x^*g_x)^{-1}(g_x^*U_{12}^* + U_{11}^*)S_{11}^{-1}T_{11}(U_{11} + U_{12}g_x) \end{aligned} \tag{4.7}$$

Proof. See Appendix C. □

As a corollary, note that $\varphi^{X*}\varphi^X + g_x^*g_x$ is a quadratic form satisfying $\inf_{\|x\|_{\mathcal{H}_x}=1}\|(\varphi^{X*}\varphi^X + g_x^*g_x)x\| \geq \inf_{\|x\|_{\mathcal{H}_x}=1}\|\varphi^{X*}\varphi^X x\| = 1$ and so this inverse always exists and is bounded. Therefore,

if $(U_{22}U_{22}^*)^{-1}$ is bounded, a stable solution exists for h_x, and if U_{22}^{-1} is bounded, this solution is unique.

Note that existence of a solution to the operator equation (4.1) is a necessary condition for the existence of a differentiable solution consistent with the equilibrium conditions of the model but is not sufficient. For an overview of high level conditions that might be used to ensure existence of a solution, see the online supplementary materials.

5 Algorithm

Having a formula for the functional derivatives of the policy operators in terms of the functional derivatives of the equilibrium conditions is not sufficient to implement the formula unless the components of that formula, defined in terms of the generalized Schur decomposition can be found. While there are some cases where this can be done analytically, these require a high degree of structure to be imposed, often requiring, for example, the model to take a partial equilibrium structure where aggregate variables are taken as exogenous, or the opposite, require individual decisions not to depend on the aggregate state. Beyond these and some other idiosyncratic cases, a numerical procedure is needed to construct the solution. This can be done using projection of the equilibrium conditions onto a finite set of basis functions, so long as the model takes a structure where the approximation error this introduces can be controlled. Conditions under which this holds can often be verified easily, and in particular hold for the economic geography model described in Section 5.

5.1 Cases in which a known decomposition exists

To compute the functional derivatives of the equilibrium policy operators of a rational expectations model with Hilbert-valued states, it is generally necessary to separate the state space into forward looking and backward-looking, or 'unstable' and 'stable' components. In some special cases, these components correspond to known or analytically identifiable state variables. This generally requires that certain derivatives equal 0: a type of exclusion restriction which ensures that backward-looking variables are not influenced by forward looking variables or vice versa. Exclusion restrictions of these sort are prevalent in partial equilibrium models, in which a forward looking decision may be made given a persistent and purely exogenous state variable. For example, if the feedback between population and economic activity were to be removed from the geography model described in Section 3, the migration decision problem given an exogenous distribution of wages would fall into this class. Similar exclusion restrictions may also arise in cases where the equilibrium environment and decision

problem are carefully tailored so that a persistent backward looking state has no impact on forward looking decisions which do affect the state. A special case of this structure is when decision making is purely myopic, either due to a carefully tailored incentive structure or due to behavioral constraints on the decision makers. Models with these kinds of exclusion restrictions may be described as *triangular*. While the restrictions required to ensure that such a condition holds are often stringent, the computational and analytical tractability that they allow makes them an important special case.

Let us consider two kinds of triangular models, roughly corresponding to the cases described above where forward-looking decisions are not influenced by a persistent state and where a persistent state is not influenced by a forward looking decision. I will call such cases *upper triangular* and *lower triangular*, respectively, for reasons that will become apparent. In the upper triangular case, the partial derivative of the equilibrium conditions with respect to the predetermined state variables x is 0 in the equations describing the forward looking decision. This can occur in models with myopic decision making, either due to behavioral constraints or due to a structure of preferences, production or technology designed to produce the knife-edge condition that the optimal decision is independent of the state of the world. In the notation from above, in such cases, the derivatives may be decomposed as

$$(B, A) = [-F_x \ -F_y, F_{x'} \ F_{y'}] = \begin{bmatrix} T_{11} & T_{12} & , & S_{11} & S_{12} \\ 0 & T_{22} & & 0 & S_{22} \end{bmatrix}$$

without any (additional) unitary transformation, so $Q = U = I$, and the forward and backward looking state variables may be identified with y and x, respectively. Applying previous results on the derivatives of the policy functions, obtain $g_X = 0$ and $h_X = S_{11}^{-1} T_{11}$. This says that, consistent with the intuition, the forward looking state has 0 derivative with respect to the persistent backward-looking one, and the backward looking state evolves autonomously. This is a locally stable solution if the spectrum of $S_{11}^{-1} T_{11}$ lies within the unit circle.

The lower triangular case occurs when the derivative of the equilibrium conditions with respect to the jump state variable y is 0 in the equations describing the backward looking persistent state variable. This can occur in partial equilibrium or small open economy type settings, in which aggregate states or distributions are determined completely exogenously. In this case, the derivatives are decomposed as

$$(B, A) = [-F_x \ -F_y, F_{x'} \ F_{y'}] = \begin{bmatrix} T_{11} & 0 & , & S_{11} & 0 \\ T_{21} & T_{22} & & S_{21} & S_{22} \end{bmatrix}$$

without any unitary transform, so again y and x may be identified with forward and

backward-looking state variables. In this case, a slightly different calculation may be applied to obtain the derivatives of the policy operators. Applying the condition

$$\begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} h_x \\ g_x h_x \end{bmatrix} = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix}$$

obtain the solution $h_x = S_{11}^{-1} T_{11}$, which is to say that x evolves according to a law of motion which does not depend on the forward looking jump state y. The derivative g_X of the policy function giving y in terms of x, satisfies the equation

$$T_{22} g_x = S_{21} h_x - T_{21} + S_{22} g_x h_x$$

which may either be expressed recursively as an infinite sum, or by treating g_x as an element of the Banach space of bounded linear operators, in terms the inverse of the linear operator $T_{22}[.] - S_{22}[.] S_{11}^{-1} T_{11}$, if it exists, as $g_x = (T_{22}[.] - S_{22}[.] S_{11}^{-1} T_{11})^{-1} (S_{21} S_{11}^{-1} T_{11} - T_{21})$. [8] While neither of these formulas is particularly straightforward to apply, often inverses may be computable in closed form, allowing simple evaluation of the effect of a state variable of interest on an intertemporal decision problem.

5.2 Numerical Evaluation by Projection

In practice, one often finds that the restrictions required to ensure that a system takes triangular form are not economically sensible. Especially in general equilibrium problems, forward looking decisions both influence and are influenced by the evolution of persistent states. In such cases, it becomes necessary to apply a method which can separate the forward and backward looking subspaces under general conditions. Unfortunately, closed form solutions are rarely available for the generalized Schur decomposition of systems of operator equations, and one must instead turn to numerics. For an algorithm to be useful, it must take data which are computable from representations of the derivatives of the equilibrium conditions and output an approximate decomposition. This suggests application of methods based on sampling, where the derivative operators are approximated by finite-dimensional objects to which a decomposition may be applied numerically.

A particularly simple way to perform this approximation is to approximate any infinite-dimensional Hilbert space by an increasing sequence of subspaces, possibly spanned by a

[8] If we may use the familiar notation $\mathrm{vec}(g_x)$ to denote the map from the space of operators to an isomorphic Banach space, viewed as a vector space, we may write this formula suggestively as $\mathrm{vec}(g_x) = (I \otimes T_{22} - (S_{11}^{-1} T_{11})^* \otimes S_{22})^{-1} \mathrm{vec}(S_{21} S_{11}^{-1} T_{11} - T_{21})$, which gives the solution in terms of the finite-dimensional vec operator and the Kronecker product when $\mathcal{H}_x$ and $\mathcal{H}_y$ are finite dimensional.

standard set of basis functions. On such spaces, the derivative operators of interest are finite-dimensional matrices. As the number of basis functions grows, representation of any function in the space becomes increasingly accurate, and one may hope that at a sufficient level of detail, the finite-dimensional system accurately approximates the infinite-dimensional one. If this is the case, it may be possible to simply apply solution algorithms for finite-dimensional linear rational expectations algorithms to produce finite-dimensional approximations of the policy functions.

While intuitively appealing, there is an important step missing in the above logic. In order for the finite-dimensional solutions to be accurate, at least asymptotically, it is necessary that when the input of the finite-dimensional rational expectation algorithm is sufficiently close to the truth, that the output also be close: the solution must be continuous with respect to some topology. It turns out that the difficult step to show here is the continuity of the generalized Schur decomposition. While the generalized Schur decomposition is known to be stable in finite dimensions (Golub & Van Loan, 1996), only limited results are available for the infinite-dimensional case. In infinite dimensions, different topologies are not equivalent, and so one must choose a topology with respect to which the approximation converges. The results of Stewart (1973) ensuring continuity of the generalized Schur subspaces and the Rayleigh components ((S_{ij}, T_{ij}) in previous notation) apply in infinite-dimensional spaces, demonstrating, under certain conditions, the continuity of these objects with respect to the Hilbert-Schmidt norm. In infinite dimensions, the use of this norm imposes unduly strong summability conditions on most operators of interest, to the point that many operators used in practice, such as the identity, do not have finite norm. As a result, in the appendix, I have demonstrated a generalization of this result to the operator norm. If a sampling procedure converges to the true derivatives in operator norm, the generalized Schur decomposition will also converge in the same norm.

While reassuring, continuity in operator norm is in fact of limited applicability without some important auxiliary hypotheses. In particular, it is known that a finite-dimensional matrix may approximate an infinite-dimensional operator in operator norm only if that operator is compact. This presents something of a difficulty, as essentially no economic models with function-valued states have derivatives which are compact operators. However, there exists a limited but far from trivial subclass of models in which it is nevertheless possible to construct the generalized Schur decomposition of a set of operators which consistently approximates the true equilibrium derivatives in operator norm, and so to which the continuity result applies. I refer to models which satisfy this condition as *asymptotically diagonal*.

Definition 3. The operator pair (B, A) is *asymptotically diagonal* if there exists a known linear isometry such that $\mathcal{H}_1$ is isometrically isomorphic to $\mathcal{H}_2$, and the representation of

the operator pair with respect to this isometry (which will also be denoted (B, A)) satisfies the decomposition $(B, A) = (B_I, A_I) + (B_C, A_C)$ such that B_C and A_C are compact and there exist known finite partitions of $\mathcal{H}_x, \mathcal{H}_y \subset \mathcal{H}_1 \cong \mathcal{H}_2$ into orthogonal subspaces $\{\mathcal{H}_j\}_{j=1}^J$ conformable with the partition into $\mathcal{H}_x$ and $\mathcal{H}_y$, usually corresponding to variables making up X and Y, such that for each pair $(i, j) \in \{1 \dots J\}^2$, $A_{I_{ij}} := \text{Proj}_{\mathcal{H}_i} A_I \text{Proj}_{\mathcal{H}_j}$ and $B_{I_{ij}} := \text{Proj}_{\mathcal{H}_i} B_I \text{Proj}_{\mathcal{H}_j}$ satisfy $A_{I_{ij}}$ and $B_{I_{ij}}$ are each either equal either to the zero operator or to a scalar multiple of the identity I_{ij}, where I_{ij} is defined for $i = j$ as the identity operator on $\mathcal{H}_i$ and for $i \neq j$ is defined as the identity from $\mathcal{H}_j$ to $\mathcal{H}_i$ if $\mathcal{H}_i \cong \mathcal{H}_j$.

Informally, this statement says that asymptotically diagonal systems can be broken up into a compact part and a part for which all subcomponents are equal to the identity. The typical form for an asymptotically diagonal operator pair is a set of square block operators acting on a space of J functions, where each block contains an identity operator, a compact operator, or a sum of a compact operator and an identity operator. For example if $J = 2$, (B, A) may take the form

$$\left(\begin{bmatrix} c_1 I_{11} + C_1 & c_2 I_{12} + C_2 \\ c_3 I_{21} + C_3 & c_4 I_{22} + C_4 \end{bmatrix}, \begin{bmatrix} c_5 I_{11} + C_5 & c_6 I_{12} + C_6 \\ c_7 I_{21} + C_7 & c_8 I_{22} + C_8 \end{bmatrix} \right) \tag{5.1}$$

where c_1 through c_8 are real scalars (possibly 0) and C_1 through C_8 are compact operators, for example integral operators of the form $\int K(x, z)[f(z)]dz$ for some bounded smooth function $K(x, z) : [0, 1)^2 \to \mathbb{R}^1$ in the case where $\mathcal{H}_j$ is $L^2[0, 1)$. Here (B_C, A_C) collects the C components, and (B_I, A_I) collects the cI components.

Asymptotic diagonality ensures that the model has a tractable form 'up to a compact perturbation.' In particular, it can be seen that (B_I, A_I) is block diagonal with respect to any orthonormal basis of $\mathcal{H}_1$ conformable with the partition into subspaces $\{\mathcal{H}_j\}_{j=1}^J$ with blocks which are J-dimensional square pencils which are, importantly, all identical. For example, for a pair in the form of (5.1), for any orthonormal basis $\{\phi_{i1}\}_{i=1}^\infty$ of $\mathcal{H}_{j=1}$, which must have a corresponding basis $\{\phi_{i2}\}_{i=1}^\infty$ for $\mathcal{H}_{j=2}$ if $\mathcal{H}_{j=1} \cong \mathcal{H}_{j=2}$, (if not, the off-diagonal components c_2, c_3, c_6, and c_7 must all be 0, as no identity can be defined), the action of (B_I, A_I) on the coefficients corresponding to functions (ϕ_{i1}, ϕ_{i2}) is given by the pair of 2×2 matrices

$$\left(\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}, \begin{bmatrix} c_5 & c_6 \\ c_7 & c_8 \end{bmatrix} \right)$$

for *any* $i = 1 \dots \infty$. This provides a representation of (B_I, A_I) as block diagonal with respect to the the orthonormal basis $\{\{e_{ij}\}_{i=1}^\infty\}_{j=1}^2$ of $\mathcal{H} = \mathcal{H}_{j=1} \times \mathcal{H}_{j=2}$ with $e_{i1} = (\phi_{i1}, 0)$, $e_{i2} = (0, \phi_{i2})$, with blocks corresponding to pairs identified by common index i. As a result,

to construct the generalized Schur decomposition of (B_I, A_I), it suffices to calculate a single J-dimensional decomposition of the matrix pencil representing any particular block and to concatenate the identical and orthogonal blocks.

Generally speaking, the isometry condition will be fulfilled by any model which uniquely determines an equilibrium, as it generalizes the familiar requirement that a model have an identical number of equations and unknowns, so the space into which the equilibrium conditions map will generally have a canonical isomorphism to the space of unknown states. This holds similarly for the J subspaces, which usually correspond to interpretable variables in the context of the model, with isomorphisms between spaces of variables likewise defined canonically. For example, in the example model of economic geography, the distribution of wages and the distribution of amenities may be defined as functions on the same space defined in the same units (such as dollars, or utils).

The use of a restriction of this kind is that identity components are common components of the derivative operators of many models, because many conditions take the form of defining a variable or assigning it a value, but are not compact, and so cannot be approximated directly by finite-dimensional approximations. The remainder of (B_I, A_I) after projection onto any subspace does not go to 0, but because it takes a tractable diagonal form, it is known. In contrast, for the compact component, the remainder when projecting onto an increasing sequence of subspaces does go to 0 and so is asymptotically negligible. By combining these two components, it is possible to use a finite-dimensional projection to approximate the operator pencil on a finite-dimensional subspace and leave a remainder on the orthogonal complement space a which is known up to an asymptotically negligible perturbation. In this way, one can use a finite set of computations to compute a generalized Schur decomposition corresponding to an operator pencil which is close in norm to the true infinite-dimensional pencil, and so by the continuity in norm of the decomposition, yields a decomposition which is close to the true one.

The disadvantage of such a restriction is that requiring all components of the derivatives to either be compact or to be composed of identity operators restricts the functional forms of allowable models, potentially in ways which rule out economically meaningful effects. For example, in a model with a distribution of characteristics which evolve independently across individuals driven by a Markov process, the distribution is a state variable and its evolution is described by an adjoint Markov operator. When the conditional density of the process given any initial state is sufficiently smooth, the operator mapping the density today to the density tomorrow will be compact and the density tomorrow enters via an identity. However, when there is a point mass in the conditional density, the transition operator need not be compact. Point masses can describe inertia, such as that induced by fixed costs (Stokey,

2008) or indexation, or mass movement along a discontinuous path. Similarly, decision problems where the object chosen is a function, ubiquitous in economics in the form of best-response policies, can yield first order conditions in which the function which is a choice variable enters into a nonlinear utility function, resulting in a functional derivative which is a multiplication operator, which may be noncompact. In some cases, it may be possible to transform the condition into one where the noncompact operator is an identity by applying its inverse to the equation, but this can eliminate only one non-identity operator from the equation. In the case where this decision problem over functions faces a state variable which is also a function, as in games or contracting problems, or government choosing nonlinear policies over a continuum of agents, goods, or locations, there may be multiple nonlinear operators in the decision problem which may prevent reducing to an asymptotically diagonal form.

When possible, applying an invertible transformation to both sides of an equilibrium equation can ensure that the asymptotically diagonal form holds without making changes to the model itself. In other cases, it may be possible to construct a modified model which is close to the original but which satisfies the condition that its derivatives are asymptotically diagonal. For example, if compactness fails due to a law of motion with discrete jumps to a fixed value, creating a discontinuity in the distribution at that value, the discontinuity may be removed if the discrete jump is accompanied by a small amount of continuously distributed noise, thereby smoothing out the conditional distribution, though the shape of the resulting distribution may be very close if the noise is small enough. Similarly, discrete actions induced by hard constraints can be made to vary continuously by replacing hard constraints with smooth but sharply growing penalties which induce similar but smooth behavior. These sorts of smoothing transformations are commonly used to employ numerical methods which rely on smoothness (see Den Haan (2010) for commentary), though it should be noted that these changes in the model may not be innocuous. While the resulting behavior at the individual level may be extremely similar, the resulting operator describing the law of motion for the distribution across individuals, which is now compact, as desired, must be far away from the true operator for some input functions. As a result, this approach does not guarantee that the resulting aggregate behavior will be close. Instead, it provides a solution to a different model, with similar individual level behavior. However, if the additional noise or smoothing of the constraint is empirically justified, this is not necessarily a concern. For example, the extreme value heterogeneity in location preference in the model of migration decisions not only ensures a smooth law of motion for the population distribution, it also reflects the believable feature that there is idiosyncratic preference heterogeneity which ensures that individuals do not all move to the same place.

When the derivatives of the equilibrium conditions of a model are asymptotically diagonal (and a unique stable equilibrium exists local to the steady state), computation of the first order approximation of the policy operators is both straightforward and computationally fast, in the sense that a consistent approximation can be computed to any desired precision in time polynomial in the number of basis functions used in the approximation. The procedure consists of projecting the equilibrium derivative operators onto a finite-dimensional orthogonal subspace, computing the policy operators on that subspace by applying directly a standard first order rational expectations solution algorithm for finite-dimensional models, and computing the policy operator on the orthogonal complement of that subspace analytically using (B_I, A_I). The operator norm precision of the resulting approximation is then asymptotically of no higher order than the operator norm error in the projection approximation of (B_C, A_C). While compactness alone ensures only that this projection error goes to 0 as the number of basis functions increases, when the compact component takes the form of integral operators $\int K(x,z)[f(z)]dz$, mild smoothness conditions (or other limited complexity conditions) on the kernel, can be used to ensure a rate of convergence. Moreover, in the case where projections cannot be calculated analytically, for example because the kernel function can only be accessed by point evaluation and so integrals must be computed approximately by quadrature, similar smoothness conditions ensure that the additional error induced is controllable.

As a very wide variety of schemes for approximating such an operator may be applied, I first provide a general purpose bound in terms of operator norm error, then offer an example of a set of conditions and an approximation scheme which ensure that this bound goes to 0. Specifically, under a Hölder continuity and compact support condition on the kernel, I demonstrate convergence rates for a Coiflet representation and 1-point quadrature scheme derived from Beylkin *et al.* (1991) which justify the high speed and accuracy that this method has exhibited in numerical experimentation relative to a wide variety of other basis function choices and quadrature schemes. Results in Beylkin *et al.* (1991) may easily be used to extend to the case of singular kernels and other wavelet classes. One particular alternative which may be attractive in some cases is to use an operator calibrated to data. In this case, plugging in any matrix-valued operator norm consistent estimator (as in Guillas (2001) for functional autoregressions, or Park & Qian (2012) or Benatia *et al.* (2015) for functional regression) produces a consistent estimator of the policy functions.

Formally, sufficient conditions for consistent approximation are given by

Condition 2. (i) (B, A) $\mathcal{H}_1 \to \mathcal{H}_2$ is an asymptotically diagonal pair of bounded operators, Γ-regular with respect to closed Cauchy curve Γ (i.e., per Definition (4), $\lambda A - B$ is invertible for all λ in a closed curve $\Gamma \subset \mathbb{C}_\infty$ separating the extended complex plane into an interior

and exterior subsets), with generalized Schur decomposition with respect to Γ given by

$$(B, A) = [Q_1^*, Q_2^*] \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

(ii) $\text{dif}(\begin{smallmatrix} T_{11} & S_{11} \\ T_{22} & S_{22} \end{smallmatrix}) > 0$, where the dif operator is defined in B.4 on page 79 in Appendix B as a measure of continuity of the generalized Schur decomposition with respect to perturbations
(iii) $U_{22} = U_2 \varphi^X$ is invertible

These conditions on the derivatives of the model are not entirely general but apply to fairly broad classes of models. As mentioned before, asymptotic diagonality rules out certain classes of models which display excessive 'frequency mixing'. The general property of operators which this rules out is a transfer of energy between frequencies which fails to dissipate as frequency increases to infinity. In these cases, input functions with a high degree of regularity are passed to outputs which may be irregular, impeding the ability to represent the system uniformly in time with respect to classes of regular functions which can be well approximated by standard function approximation. This transfer of energy to higher and higher frequencies is commonly described in models of physical systems as an aspect of (weak) turbulence, and generally requires numerical methods different from those described here. Γ-regularity ensures that forward and backward looking components of the system can be distinguished. In the typical case where Γ is the complex unit circle, it rules out unit roots and continuous spectra around unity, and so imposes some restriction on the time series properties of the systems which can be analyzed by this method.

Condition (ii) on the dif operator of the pair similarly imposes that the forward and backward looking components are well-separated, ensuring their continuity with respect to small perturbations in the operators: see Appendix B for an exact definition and further discussion. Heuristically speaking, the dif constant is a measure of the separation between the forward and backward subspaces which depends on the spectral gap between the subspaces and the degree of nonnormality (or deviation from a diagonalizable pair) of the operator pair. In the case where the operator pair (B, A) is diagonalizable, it is equal to the minimum distance between the spectra of (T_{11}, S_{11}) and (T_{22}, S_{22}) and so positivity is implied by Γ-regularity. Γ-regularity is also sufficient in the case that (B, A) is finite dimensional (see Stewart & Sun (1990) Thm VI.1.11) or in the case in which either B or A is invertible, in which case it follows from the Sylvester-Rosenblum theorem for operators (Bhatia & Rosenthal, 1997), though the exact size will depend on the degree of nonnormality.[9]

[9]I conjecture that there may exist an analogue of the Sylvester-Rosenblum theorem for the operator

Condition (iii) is necessary for existence and uniqueness of derivatives of a policy function which are consistent with the equilibrium conditions: it ensures that there is a correct model to be approximated.

To provide a consistent approximation, it is necessary to choose a sequence of finite-dimensional orthogonal subspaces which converge to $\mathcal{H}$. Generally these will be defined as the closed linear span of an increasing sequence of functions in a set of complete orthonormal bases of $\{\mathcal{H}_j\}_{j=1}^J$, though orthonormality is mainly a computational and notational convenience. As one often does not have access to an exact projection, it is sufficient to request a consistent approximation to one instead. For consistency, we require approximations satisfying the following properties

Condition 3. (i) Let $\{\pi_j^{K_j}\}_{j=1}^J$ be J orthogonal projections onto K_j-dimensional orthogonal subspaces of $\{\mathcal{H}_j\}_{j=1}^J$ respectively such that Im $\pi_i^{K_i} \cong$ Im $\pi_j^{K_j}$ if $\mathcal{H}_i \cong \mathcal{H}_j$ (i.e., π_j^K and $\pi_i^{K_i}$ map to subspaces which are identified of elements of the partition which are themselves identified), and let $\pi^K = \sum_{j=1}^J \pi_j^{K_j}$ project onto the $K = \sum_{j=1}^J K_j$ -dimensional union of these subspaces. Define $(B^K, A^K) := \pi^K(B, A)\pi^K$, and $(B_C^K, A_C^K) := \pi^K(B_C, A_C)\pi^K$. Let

$$\max\{\left\|B_C^K - B_C\right\|_{op}, \left\|A_C^K - A_C\right\|_{op}\} \leq \eta_K$$

for some sequence η_K decreasing to 0 as $K \to \infty$.

(ii) Let $(\tilde{B}^K, \tilde{A}^K)$ be a sequence of matrix approximations of (B^K, A^K) on a Euclidean space isomorphic to Im π^K satisfying

$$\max\{\left\|\tilde{B}^K - B^K\right\|_{op}, \left\|\tilde{A}^K - A^K\right\|_{op}\} \leq \zeta_K$$

for some sequence ζ_K decreasing to 0 as $K \to \infty$.

In practice, as the J subspaces of $\mathcal{H}$ represent distinct functions used as state variables (for example, a value function and a distribution over agents), these approximations are given by first choosing an appropriate complete series basis for each function of interest and representing each function with respect to an increasing number of terms in that series. The numerical representation of the operators with respect to the series $(\tilde{B}^K, \tilde{A}^K)$ is then calculated by interpolation, quadrature, exact sampling in some special cases, or estimation. Note that we require only consistency of the projections over (B_C, A_C). Both on and off the projected space (B_I, A_I) has exact representation as a set of scalar multiples of identity matrices on Im π^K and as identity operators on the orthogonal complement of that space.

defining the dif constant, in which case condition (ii) is entirely redundant. However, the method of proof does not straightforwardly generalize.

Given a choice of spaces onto which to project and a consistent approximation of the projected operators, we can define our approximate operators by calculating policy operators $\tilde{g}_x^K$, $\tilde{h}_x^K$ and $g_x^{K\perp}, h_x^{K\perp}$ from (B, A) separately on Im π^K and Ker π^K, respectively and composing them. We may define these operators as follows.

Denote the generalized Schur decomposition with respect to Γ of the finite-dimensional matrix representation of $(\tilde{B}^K, \tilde{A}^K)$ as

$$[\tilde{Q}_1^{*K}, \tilde{Q}_2^{*K}] \begin{bmatrix} \tilde{T}_{11}^K & \tilde{T}_{12}^K \\ 0 & \tilde{T}_{22}^K \end{bmatrix}, \begin{bmatrix} \tilde{S}_{11}^K & \tilde{S}_{12}^K \\ 0 & \tilde{S}_{22}^K \end{bmatrix} \begin{bmatrix} \tilde{U}_{11}^K & \tilde{U}_{12}^K \\ \tilde{U}_{21}^K & \tilde{U}_{22}^K \end{bmatrix}.$$

Note that because this is a finite-dimensional matrix pair, this may be calculated in $O(K^3)$ time by the QZ algorithm: see Golub & Van Loan (1996). Applying the formulas for the policy operators to this restricted space, define $\tilde{g}_x^K = -(\tilde{U}_{22}^K)^{-1}\tilde{U}_{21}^K$, $\tilde{h}_x^K = (\tilde{U}_{11}^K + \tilde{U}_{12}^K\tilde{g}_X^K)^{-1}(\tilde{S}_{11}^K)^{-1}\tilde{T}_{11}^K(\tilde{U}_{11}^K + \tilde{U}_{12}^K\tilde{g}_X^K)$. These define an approximation of g_x and h_x respectively on the space Im π^K.

As the restriction of the policy function to this space need not, in general, consistently approximate the policy functions on $\mathcal{H}$ as a whole, we supplement by an approximation on the orthogonal complement space, $g_x^{K\perp}, h_x^{K\perp}$. To do this, we approximate by considering only (B_I, A_I) on this space: this is a reasonable approximation because for K large enough, the contribution of (B_C, A_C) on the remainder becomes negligible. Consider a set of complete orthonormal bases of $\mathcal{H}_j$, $\{e_{ij}\}_{j=1}^J$, $i = 1 \ldots \infty$, where e_{sj} and e_{tk} are identified if $s = t$ and $\mathcal{H}_j \cong \mathcal{H}_k$. Then, by construction, for all i, (B_I, A_I) maps the closure of their span $\overline{\text{Span}}\{e_{ij}\}_{j=1}^J$ to itself and moreover, the representation of this map is identical for all i. Informally, (B_I, A_I) is ($\mathcal{H}$-equivalent to by Parseval's identity) a block diagonal matrix pair over this complete orthonormal basis with identical $J \times J$ blocks. Further, because an identity matrix has identity representation with respect to any choice of basis, we may choose a basis such that $\{e_{ij}\}_{j=1}^J$, $i = K+1 \ldots \infty$ are a complete orthonormal basis of Ker π^K (these may or may not be the remaining elements of an orthonormal basis the projection onto the span of which defines π^K, though this representation is convenient). By the orthogonality of the blocks, it is sufficient to define the policy function separately on each block. This can be done by applying the solution formula to any J-dimensional block i, regarded as a pair of $J \times J$ matrices, (B_I^i, A_I^i). These have generalized Schur decomposition

$$(B_I^i, A_I^i) = [Q_1^{i*}, Q_2^{i*}] \begin{bmatrix} T_{11}^i & T_{12}^i \\ 0 & T_{22}^i \end{bmatrix}, \begin{bmatrix} S_{11}^i & S_{12}^i \\ 0 & S_{22}^i \end{bmatrix} \begin{bmatrix} U_{11}^i & U_{12}^i \\ U_{21}^i & U_{22}^i \end{bmatrix}$$

on each block i, where U_{11}^i and U_{21}^i acts on the J_1 elements contained in $\mathcal{H}_x$ and U_{12}^i and U_{22}^i

act on the $J - J_1$ elements contained in $\mathcal{H}_y$. The corresponding block of the policy operators are given by $g_x^i = -(U_{22}^i)^{-1}U_{21}^i$, $h_x^i = (U_{11}^i + U_{12}^i g_x^i)^{-1}(S_{11}^i)^{-1}T_{11}^i(U_{11}^i + U_{12}^i g_x^i)$. To define our approximation on the orthogonal complement of Im π^K, we simply concatenate the blocks, giving sequential representations

$$g_x^{K\perp} = \sum_{i=K+1}^{\infty} \sum_{j=J_1+1}^{J} \sum_{k=1}^{J_1} (g_x^i)_{(j-J_1)k} \langle e_{ik}, [.] \rangle e_{ij}$$

and

$$h_x^{K\perp} = \sum_{i=K+1}^{\infty} \sum_{j=1}^{J_1} \sum_{k=1}^{J_1} (h_x^i)_{jk} \langle e_{ik}, [.] \rangle e_{ij}.$$

Note that since each block is identical, calculation of the policy function needs be performed for only one representative block, with running time dominated by the QZ algorithm, of order $O(J^3)$ typically negligible.

The approximation to the policy operators on $\mathcal{H}$ are given by $g_K := \tilde{g}_x^K + g_x^{K\perp}$ and $h_K := \tilde{h}_x^K + h_x^{K\perp}$. A summary of the steps leading to their construction is provided as Algorithm 1. Under the conditions given, these aproximations are consistent in operator norm. We have the result

Theorem 1. *Let (B, A) and their approximations $(\tilde{B}^K, \tilde{A}^K)$ satisfy Conditions (2) and (3). Then $\|g_K - g_x\|_{op} \to 0$ and $\|h_K - h_x\|_{op} \to 0$ as $K \to \infty$. In particular, there exists some $\bar{K}$ and some constant C such that for $K > \bar{K}$, $\|g_K - g_x\|_{op} \leq C(\zeta_K + \eta_K)$ and $\|h_K - h_x\|_{op} \leq C(\zeta_K + \eta_K)$.*

Proof. See Appendix. □

The idea behind the consistency argument is to show that the generalized Schur decomposition of the combined approximation on and off Im π^K converges in operator norm and then apply perturbation theorems ensuring continuity in operator norm for the Schur projectors and Rayleigh components. Then by applying orthogonality, show that the policy functions corresponding to the generalized Schur decompositions on and off Im π^K are equivalent to the policy functions corresponding to the Schur decomposition of the approximate operator as a whole. The exact constant C and $\bar{K}$ are both decreasing functions of the dif constant of (B, A). While the rate of convergence is unaffected, for highly non-normal operators or those with a small gap between the spectrum of the forward and backward looking components, the constant on the rate may be large.

Algorithm 1 Construction of g_K, h_K

Inputs: An equilibrium operator $F(x, y, x', y', \sigma)$ satisfying Condition 2, Im π^K a K-dimensional subspace satisfying Condition 3, and $\{\{e_{ij}\}_{j=1}^{J}\}_{i=K+1}^{\infty}$ a conformable orthonormal basis for the orthogonal complement of Im π^K

Output: g_K, h_K approximate functional derivatives of recursive solution with respect to x

1. Compute steady state (x^*, y^*) s.t. $F(x^*, y^*, x^*, y^*, 0) = 0$
2. $(B, A) \leftarrow \left(-\begin{bmatrix} F_x & F_y \end{bmatrix}, \begin{bmatrix} F_{x'} & F_{y'} \end{bmatrix}\right)$ Calculate functional derivatives at steady state
3. Decompose (B, A) into $(B_I, A_I) + (B_C, A_C)$ compact and identity components as per Definition 3
4. Construct $(\tilde{B}^K, \tilde{A}^K)$, a K-dimensional approximate projection of (B, A) onto Im π^K, satisfying Condition 3, using Algorithm 2 or other method
5. Build components of policy operator on Im π^K and Ker π^K
   (a) Build policy operators on Im π^K using approximate projections
      i. $[\tilde{Q}_1^{*K}, \tilde{Q}_2^{*K}] \begin{bmatrix} \tilde{T}_{11}^K & \tilde{T}_{12}^K \\ 0 & \tilde{T}_{22}^K \end{bmatrix}, \begin{bmatrix} \tilde{S}_{11}^K & \tilde{S}_{12}^K \\ 0 & \tilde{S}_{22}^K \end{bmatrix} \begin{bmatrix} \tilde{U}_{11}^K & \tilde{U}_{12}^K \\ \tilde{U}_{21}^K & \tilde{U}_{22}^K \end{bmatrix} \leftarrow QZ(\tilde{B}^K, \tilde{A}^K)$ Apply QZ algorithm to obtain generalized Schur decomposition of $(\tilde{B}^K, \tilde{A}^K)$
      ii. $\tilde{g}_x^K \leftarrow -(\tilde{U}_{22}^K)^{-1}\tilde{U}_{21}^K$, $\tilde{h}_x^K \leftarrow (\tilde{U}_{11}^K + \tilde{U}_{12}^K\tilde{g}_X^K)^{-1}(\tilde{S}_{11}^K)^{-1}\tilde{T}_{11}^K(\tilde{U}_{11}^K + \tilde{U}_{12}^K\tilde{g}_X^K)$
   (b) Build policy operators on Ker π^K by analytical decomposition of (B_I, A_I)
      i. $[(B_I^i, A_I^i)]_{jk} \leftarrow \langle B_I e_{ij}, e_{ik}\rangle, \langle A_I e_{ij}, e_{ik}\rangle \ \forall j, k = 1 \dots J$ Construct (B_I^i, A_I^i) (identical for all i) using $\{e_{ij}\}_{j=1}^{J}$ for some i
      ii. $[Q_1^{i*}, Q_2^{i*}] \begin{bmatrix} T_{11}^i & T_{12}^i \\ 0 & T_{22}^i \end{bmatrix}, \begin{bmatrix} S_{11}^i & S_{12}^i \\ 0 & S_{22}^i \end{bmatrix} \begin{bmatrix} U_{11}^i & U_{12}^i \\ U_{21}^i & U_{22}^i \end{bmatrix} \leftarrow QZ(B_I^i, A_I^i)$ Apply QZ algorithm to (B_I^i, A_I^i)
      iii. $g_x^i \leftarrow -(U_{22}^i)^{-1}U_{21}^i$, $h_x^i \leftarrow (U_{11}^i + U_{12}^i g_x^i)^{-1}(S_{11}^i)^{-1}T_{11}^i(U_{11}^i + U_{12}^i g_x^i)$ Build policy functions over $\overline{\text{Span}}\{e_{ij}\}_{j=1}^{J}$
      iv. Add identical components for all $i = K + 1 \dots \infty$

$$g_x^{K\perp} = \sum_{i=K+1}^{\infty} \sum_{j=J_1+1}^{J} \sum_{k=1}^{J_1} (g_x^i)_{(j-J_1)k} \langle e_{ik}, [.]\rangle e_{ij}$$

$$h_x^{K\perp} = \sum_{i=K+1}^{\infty} \sum_{j=1}^{J_1} \sum_{k=1}^{J_1} (h_x^i)_{jk} \langle e_{ik}, [.]\rangle e_{ij}$$

   for J_1 blocks in $\mathcal{H}_x$, $J - J_1$ in $\mathcal{H}_y$.
6. $g_K \leftarrow \tilde{g}_x^K + g_x^{K\perp}$, $h_K \leftarrow \tilde{h}_x^K + h_x^{K\perp}$ Add components

5.3 Implementation: Wavelet Transform

Overall, the computational effort needed to obtain ϵ-close approximations is driven by the rates η_K and ζ_K. If efficient (or exact) evaluation schemes are used, the projection error η_K tends to dominate: this may not be the case if the value of the projection coefficients is determined by estimation, in which case the accuracy of ζ_K is limited by the quantity of data available. To more precisely quantify the size of these errors, we provide an example of a set of conditions on (B, A), the approximating subspace Im π^K, and the evaluation method for the projections which provides precise rates. In more general situations, where these conditions don't hold or other approximation methods are desired, similar approaches can guarantee the high level conditions of the preceding theorem.

In particular, I cover the example of Coiflet sampling and representation as in Beylkin *et al.* (1991) for Fredholm integral operators with Hölder-continuous kernel. Fredholm integral operators are a canonical example of operators which are given by a compact component and potentially a component given by an identity, and appear frequently in examples. Wavelet sampling methods provide a particularly fast and accurate method for approximating these operators even when the kernel can only be accessed by pointwise evaluation, perhaps because it is a complicated function which has itself been numerically approximated, such as a function of a steady state calculated numerically by fixed point iteration.

Condition 4. (i) Let $\{\mathcal{H}_j\}_{j=1}^J$ be given by the spaces of square integrable periodic functions of dimension d_j with domain normalizable to $[0,1)^{d_j}$, $\mathcal{H}_j = L^2_{per}[0,1)^{d_j}$. Let (B_C, A_C) consist on each block (i,j) of $r = B$ or $r = A$ of integral operators mapping $f(y) \in \mathcal{H}_j$ to $f(x) \in \mathcal{H}_i$ $f(x) = \int_{[0,1)^{d_j}} K_{r,ij}(x,y)[f(y)]dy$ such that for all r, i, j $\sup_{x,y \in [0,1)^{d_i} \times [0,1)^{d_j}} |K_{r,ij}(x,y)| < \infty$ and $K_{r,ij}(x,y) \in \Lambda^{\alpha_{r,ij}}([0,1)^{d_i} \times [0,1)^{d_j})$, the space of $\alpha_{r,ij}$-Hölder continuous periodic functions on $[0,1)^{d_i} \times [0,1)^{d_j}$ for some $\alpha_{r,ij} > 0$.[10]

(ii) Let Im π^{K_j} be the subspace spanned for each j by a tensor product of d_j one-dimensional orthonormal Coiflet wavelet multiresolution analyses with mother wavelet ψ and scaling function ϕ with each bounded, having support which is a compact interval, and a number of vanishing moments greater than or equal to $\min_{r,i,j} \alpha_{r,ij}$. Let the matrix representation of $(\tilde{B}^K, \tilde{A}^K)$ on this space be given by $\pi^K(B_I, A_I)\pi^K$ plus a matrix where the i,j block is given by the discrete wavelet transforms over rows then columns of the $K_i \times K_j$ matrix whose (s,t) entry is $\frac{1}{\sqrt{K_i K_j}} K_{r,ij}(x_s, y_t)$, where $\{x_s\}_{s=1}^{K_i}$ and $\{y_t\}_{t=1}^{K_j}$ are dyadic grids

[10] A function f(x) is Hölder continuous on domain I of order $\alpha \in (0,1]$ if $\sup_{x,y \in I} |f(x) - f(x)| \leq K|x-y|^\alpha$ and is Hölder continuous of non-integer order $\alpha > 1$ if it is $\lfloor \alpha \rfloor$ times continuously differentiable with $\lfloor \alpha \rfloor^{th}$ derivatives Hölder continuous of order $\alpha - \lfloor \alpha \rfloor$. .

over $[0,1)^{d_i}$ and $[0,1)^{d_j}$ respectively.

Remark. On (i): These assumptions can be slightly relaxed through different choices of wavelet basis. Periodicity is convenient for proofs because it does not require any special treatment of boundaries: it also fits the example model presented. Depending on the problem, this may be relaxed by one of a number of boundary extension methods: see Mallat (2008). Compact support can be replaced by a tail condition by sampling an increasing spatial domain. Boundedness of the kernel can likewise be dispensed with provided the operator remains compact and some knowledge of the singularity is available: Beylkin *et al.* (1991) provides methods and convergence results for many singular integral operators. It is likely that Hölder regularity could be replaced with more general Besov classes which may exhibit less uniform regularity, at the expense of more difficult analysis of the quadrature approximation. In both of these cases, speed of the algorithm may be enhanced by pruning away those basis functions whose inner product with the kernel is below a small threshold. As the wavelet representation is often sparse, one may incur minimal error in the approximation of (B, A) while substantially reducing the size K of the matrix for which one calculates the generalized Schur decomposition, an order K^3 operation which dominates the quadratic time to evaluate and threshold a higher order wavelet representation.

On (ii): As described, the procedure represents each kernel in terms of a tensor product of multiresolution wavelet bases instead of a single multidimensional multiresolution analysis as advocated in Beylkin *et al.* (1991): while such a representation has desirable features for thresholding procedures, a tensor product representation of the operator ensures that functions in the domain and range space are represented in terms of the original d_j-dimensional orthonormal wavelet basis. For j with $d_j > 1$, either a tensor product wavelet basis or a multidimensional wavelet multiresolution analysis may be used in calculating the basis functions: the space spanned by a finite representation is identical. In practice, the multidimensional MRA is preferred computationally. The moment condition is assumed to hold for the one-dimensional wavelets generating the tensor product or multiresolution basis.

The requirement that both wavelet and scaling function have compact support, α vanishing moments, and generate an orthonormal basis strongly restricts the choice of wavelet class. The use of Coiflets (Beylkin *et al.* , 1991) (or certain mild generalizations, as in Wei (1998), which also maintain these properties) is in fact required to achieve optimal rates via the procedure described. The purpose of this assumption is to ensure that the operator can be represented directly in terms of the discrete wavelet transform of its evaluations at a set of points, effecting a 'one-point quadrature' scheme for the calculation of the coefficients of the representation. For more general classes of wavelets, the use of the discrete wavelet transform of the evaluation points of a smooth function to substitute for the projection onto a wavelet

basis results in an error which is of higher order than the error induced by restricting to a projection onto a finite basis.

Other classes of wavelets may be used if the projection is approximated by a multipoint quadrature scheme, as described in Beylkin *et al.* (1991) or Sweldens & Piessens (1994), at the cost of additional preprocessing before applying the discrete wavelet transform. If neither multipoint quadrature nor the use of Coiflets is acceptable, it is also possible to use interpolating wavelets, which do not form an orthogonal basis and result in a more complicated representation of $\pi^K(B_I, A_I)\pi^K$. General considerations regarding wavelet sampling are discussed in Mallat (2008). One case in which specialized classes of wavelets may be necessary is when the domain is not rectangular or is a subset of a non-Euclidean manifold, as may occur with geographic data restricted to an irregularly shaped geographic unit or on the surface of the Earth. In this case, a variety of alternative bases and sampling methods are available.

The procedure for constructing approximate projections $(\tilde{B}^K, \tilde{A}^K)$ using the Coiflet basis is laid out in Algorithm 2. Under the above conditions, it can easily be shown that one obtains rapid convergence of the approximation algorithm:

Theorem 2. *Let (B,A) and $(\tilde{B}^K, \tilde{A}^K)$ satisfy (2),(3), and (4). If $\bar{\alpha} = \min_{r,i,j} \frac{2\alpha_{r,ij}}{d_i+d_j}$ and $\bar{d} = \max_j 2d_j$, there exists $C > 0$ such that $\eta_K = O(J \max_{r,i,j}(K_i K_j)^{-\alpha_{r,ij}/(d_i+d_j)})$ and $\zeta_K = O(C^{\bar{d}} J \max_{r,i,j}(K_i K_j)^{-\alpha_{r,ij}/(d_i+d_j)})$. As a result, operator norm ϵ-approximations of h_x and g_x such that $\|h_K - h_x\|_{op} \leq \epsilon$ and $\|g_K - g_x\|_{op} \leq \epsilon$ can be calculated using a basis of $K = O(J(\frac{JC^{\bar{d}}}{\epsilon})^{\frac{1}{\bar{\alpha}}})$ functions in $O(J^{3+\frac{3}{\bar{\alpha}}} C^{\frac{3\bar{d}}{\bar{\alpha}}} \epsilon^{-\frac{3}{\bar{\alpha}}})$ operations.*

This result shows that a polynomial time approximation scheme is feasible for this class of models. Due to the accurate quadrature properties of compactly supported wavelet multiresolution analyses, the error from projection and the error from quadrature are of the same order in K, up to constants. While a curse of dimensionality exists with respect to the number of variables entering as arguments of the functions used as state variables, the fact that the functions are themselves infinite-dimensional objects does not impede impede approximation. Further, when the operators are reasonably smooth, as measured by Hölder exponent of the integral kernels, the rate of convergence can be quite rapid. If one is interested in the policy operators as a whole, rather than just their derivatives, this approach only provides a first order Taylor expansion. As a result, it provides accurate approximations within a local neighborhood of the nonstochastic steady state. In the case where the policy operators are continuously Fréchet differentiable, operator norm approximation of the first derivative ensures that the approximated Taylor expansions of the operators are ϵ-close to the true operators uniformly over an open neighborhood of this steady state.

Algorithm 2 Construction of $(\tilde{B}^K, \tilde{A}^K)$ using wavelet quadrature

Inputs: Block operators $(B, A) = (B_I, A_I) + (B_C, A_C)$ s.t. (B_C, A_C) is composed of integral operators $\int_{[0,1)^{d_j}} K_{r,ij}(x, y)[.]dy \ \forall i, j \in 1 \dots J, r \in \{B, A\}$ satisfying Condition 4(i), $\{K_j\}_{j=1}^J$ number of evaluation points for each block
Output: $(\tilde{B}^K, \tilde{A}^K)$ satisfying Condition 3

1. $[K_{r,ij}]_{s,t} \leftarrow \frac{1}{\sqrt{K_i K_j}} K_{r,ij}(x_s, y_t)$ for x_s, y_t on evenly spaced grids of size K_i, K_j over $[0,1)^{d_i}$, $[0,1)^{d_j}$ respectively, $\forall i, j, r$. Construct matrices to represent kernels of integral operators
2. $(\tilde{B}_C^K, \tilde{A}_C^K)_{r,ij} \leftarrow (\text{DWT}[(\text{DWT}[K_{r,ij}])^*])^* \ \forall i, j, r$ Construct approximate projection coefficients by discrete wavelet transform of rows then columns of $K_{r,ij}$, using Coiflet wavelets basis satisfying Condition 4(ii)
3. $(\tilde{B}_I^K, \tilde{A}_I^K) \leftarrow \pi^K (B_I, A_I) \pi^K$ Represent identity operators by $K_i \times K_j$ identity matrices
4. $(\tilde{B}^K, \tilde{A}^K) \leftarrow (\tilde{B}_I^K, \tilde{A}_I^K) + (\tilde{B}_C^K, \tilde{A}_C^K)$ Add components

Remark. The dependence on J, which in most applications has the interpretation of the number of independent functions which constitute the equilibrium objects (e.g., a value function, a distribution of individual states, and so on) and is usually a fixed feature of the model, will in general be conservative, as it is based on the worst case that all blocks of (B_C, A_C) contain an integral operator and that the difficulty of approximation of each operator, measured by $\frac{2\alpha_{r,ij}}{d_i + d_j}$, is roughly equal. If the row and column corresponding to subspace j for all but a subset S of subspaces do not contain an integral operator or contain only operators which are substantially smoother and so require fewer basis functions to approximate to ϵ accuracy, and only $K = O(S(\frac{JC^{\bar{d}}}{\epsilon})^{\frac{1}{\bar{\alpha}}})$ basis functions will be needed. This may be the case, for example, if one block contains an operator which is substantially harder to approximate than others (due to being higher-dimensional, less smooth, or both), in which case $S = 1$. In most applications, J is fixed and very small, though it could grow, for example, if some components are represented by a functional autoregressive model of high order.

6 Application, continued: Implementation and Evaluation

The above procedures may be applied to construct a linearized solution to the model of trade, migration, and economic geography of Section 3.

6.1 Steady State and Exact Projections

To go from the functional derivatives of the equilibrium conditions to a linear solution, it suffices to find projections onto a complete set of basis functions. The structure of the model makes that task particularly simple, because when the basis used is the standard Fourier basis of trigonometric polynomials, the projections can be calculated exactly without numerical integration. The structure can also be used to verify the conditions which ensure that projection approximations are valid. In fact, the structure allows even more to be said about the solution than can be inferred from (1). Because the model is block diagonal with respect to the Fourier basis, the solution operator can be calculated exactly for any input given by a Fourier basis function, and so for any bandlimited function.

Recalling from Section 3, the linearized equilibrium conditions in this model are given by

$$(B, A) = \left(- \begin{bmatrix} 0 & 0 & I \\ P & 0 & \beta P - \beta P P \\ 0 & \Gamma & 0 \end{bmatrix}, \begin{bmatrix} \frac{d\omega}{d\lambda} & I & \beta P \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \right)$$

in which $P[.] = \frac{1}{f} \int \exp(c(x'-x)+\beta \bar{V})[.]dx'$, Γ is likewise an integral operator, and $\frac{d\omega}{d\lambda}$ can be shown to be defined in terms of the composition of a number of convolution operators with respect to a Laplace distribution and their inverses. As can therefore be seen, the model is composed of identity and integral operators, exactly the structure needed for the projection approaches developed here to be valid. Moreover, examining the expressions for the derivatives of the economic geography model it can be seen that all of the integral operators are expressed in terms of convolution operators. By the convolution theorem, all convolution operators (and their inverses, as well as the identity) are diagonal in a Fourier basis, and so all operators can be expressed as a convolution with distributions, or equivalently, as multiplication of the Fourier transform of the input by a known function.

Because each functional derivative in the model is diagonal with respect to the Fourier transform, the model can be broken down into blocks corresponding to individual frequencies: there is no interaction across frequencies. Within a frequency, the linearized model can be written in terms of 3×3 matrices of derivatives of each component with respect to perturbations at that frequency. The exception is at frequency 0, where only derivatives with respect to V and ν are taken, as, by Parseval's theorem, functions $L_0^2(\mathbb{R})$ can be represented in the Fourier domain as sequences of Fourier coefficients with the coefficient at frequency 0 equal to 0.

Among other things, this block diagonal structure implies that Condition 1.(ii) regarding the modulus of continuity of the Schur decomposition holds so long as Condition 1.(i) holds.

Conditions 1.(i) and 1.(iii), requiring existence and uniqueness of a Schur decomposition into components inside and outside the unit circle with unstable subspace isomorphic to the space spanned by the jump variable (in this case V), may also be verified for any given set of parameters by ensuring the conditions hold for each finite-dimensional subsystem. In order for the system to have a locally stable rational expectations equilibrium, it must be the case that at each frequency, the system has two generalized eigenvalues inside the unit circle, corresponding to the predetermined variables ν and λ, and one generalized eigenvalue outside, corresponding to the jump variable V. Such a condition is not general: it requires restrictions on the parameter values to ensure that such a solution exists.

Impressionistically, because the value of a location is a weighted average of future wages (a function of population), and because the current population is a weighted average of past values, the system remains stable only if this mutual reinforcement is not too strong. Otherwise, at certain frequencies, at which more than one eigenvalue is unstable, the linearized model implies that value grows without bound and population does as well: this is the conclusion of Krugman (1996), which does not derive dynamics from forward-looking decisions. However, the stability condition on the eigenvalues is substantially weaker than the condition imposed by Krugman, that the impact of population on wages be negative for all frequencies. Positive feedback is consistent with stability of a rational expectations equilibrium so long as the effect on wages is expected to be temporary. Moreover, if the feedback is temporary, the population response is damped, and so the degree of mutual reinforcement is even lower. As a result, only frequencies where the parameterization implies that the feedback from population to wages is so large that no policy rule which eventually returns to steady state can be constructed are a problem for calculating a forward looking solution.

To consider which frequencies might be problematic, note that at extremely high frequencies, because convolution with a smooth density dampens high frequency fluctuations, the mutual reinforcement phenomenon is dampened and eventually disappears, so these frequencies are stable. Similarly, due to the dispersive forces in the geographic equilibrium model, at extremely low frequencies, increasing population actually reduces wages, ensuring stability. It is at intermediate frequencies where population growth and real wage growth are complementary, and parameters must be chosen so that at these frequencies the degree of complementarity is not so great as to prevent the mean-reversion induced by the dispersion of population due to idiosyncratic tastes from ensuring eventual return to uniformity after a temporary shock. This suggests that a parameterization of adjustment costs which ensures that medium to high frequency fluctuations are rapidly smoothed out is needed. However, degree of smoothing and size of adjustment costs have a nontrivial relationship. For quadratic

costs, a higher scale is equivalent to a smaller variance of the Gaussian flows and so results in less smoothing. However, while changing from quadratic to linear (in absolute distance) costs results in Laplace flows with substantially more movement to long distances as it lowers costs of moving large distances, it raises costs of moving short distances and so decreases mean reversion at medium to high frequencies. In practice, stability holds for a very broad range of parameter values.

Formally, at each frequency not equal to 0, the model is represented by a 3×3 block of the Bellman equation, the population transition, and the shock transition at that frequency. At a representative frequency ϕ, the model can be taken as a set of matrix equations in Fourier transform of the vector of endogenous functions at that frequency. The matrix of derivatives with respect to $(\hat{\lambda}_\phi, \hat{\nu}_\phi, \hat{V}_\phi)$ is

$$B_\phi = \begin{bmatrix} 0 & 0 & 1 \\ \hat{P}_\phi & 0 & \beta\hat{P}_\phi - \beta\hat{P}_\phi^2 \\ 0 & \hat{\Gamma}_\phi & 0 \end{bmatrix}$$

where $\hat{P}_\phi$ is the Fourier transform of $\frac{1}{f}\exp(c(x)+\beta\bar{V})$ evaluated at frequency ϕ and $\hat{\Gamma}_\phi$ is the Fourier transform of $\Gamma(x)$ evaluated at frequency ϕ. The matrix of derivatives with respect to $(\hat{\lambda}'_\phi, \hat{\nu}'_\phi, \hat{V}'_\phi)$ is

$$A_\phi = \begin{bmatrix} \frac{\hat{d\omega}}{d\lambda}_\phi & 1 & \beta\hat{P}_\phi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

where $\frac{\hat{d\omega}}{d\lambda}_\phi$ the Fourier transform of $\frac{d\omega}{d\lambda}$ at frequency ϕ, is derived in Appendix C.

Finally, at frequency 0, by dropping the transition equation which does not act over this frequency because perturbations of λ are restricted to lie in L_0^2, the space of functions integrating to 0, to ensure that the density λ integrates to 1, the system is represented by 2×2 blocks of derivatives with respect to $(\hat{\nu}_\phi, \hat{V}_\phi)$ and $(\hat{\nu}'_\phi, \hat{V}'_\phi)$ given by

$$(B_0, A_0) = \left(\begin{bmatrix} 0 & 1 \\ \hat{\Gamma}_0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \beta\hat{P}_0 \\ 1 & 0 \end{bmatrix} \right)$$

To construct an approximate solution from these projections, note that because the operator pairs are block diagonal, a fully upper triangular infinite-dimensional system can be constructed so long as each block can be placed in upper triangular form. Together, each pair of matrices forms a finite-dimensional linear rational expectations system which can be

evaluated by standard algorithms for calculating perturbative expansions of such systems, such as the Schmitt-Grohe & Uribe (2004) procedure, based on the algorithm of Klein (2000). Here, no changes need to be made to the finite-dimensional procedure: it is simply applied independently at each integer frequency ϕ. The derivatives of the policy functions are then given by the collection of derivatives at each frequency. For each $\phi \neq 0$, the policy functions $\hat{h}_\phi : (\hat{\lambda}_\phi, \hat{\nu}_\phi) \to (\hat{\lambda}'_\phi, \hat{\nu}'_\phi)$ and $\hat{g}_\phi : (\hat{\lambda}_\phi, \hat{\nu}_\phi) \to \hat{V}_\phi$ are given by 2×2 and 1×2 matrices. The first order approximate policy operators are then represented with respect to the Fourier basis as block-diagonal infinite matrices $\hat{h}$ and $\hat{g}$, with $\hat{h}_\phi$ and $\hat{g}_\phi$ on the diagonals, respectively, so that for general inputs in $L^2_0(\mathbb{R}) \times L^2(\mathbb{R})$, they may be represented as $h = \mathcal{F}^{-1}\hat{h}\mathcal{F}$ and $g = \mathcal{F}^{-1}\hat{g}\mathcal{F}$ where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ is the inverse Fourier transform.

For bandlimited perturbations, a finite representation h^K_x, g^K_x given by concatenating the first K frequencies is exact. More generally, the functional derivatives generated by taking an increasing finite collection of frequencies converge in the strong operator topology, and for any components which are compact, in the operator norm topology. Operator norm convergence follows from application of (1). To see that the conditions are met, note that for smooth adjustment costs and transition functions, $\hat{P}_\phi$, $\hat{\Gamma}_\phi$, and $\frac{\hat{d\omega}}{d\lambda}_\phi$ converge to 0, and so compactness and convergence of the projected derivatives in operator norm follows. Moreover, as $\phi \to \infty$, (B_ϕ, A_ϕ) converges to

$$(B^i_I, A^i_I) = \left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right),$$

and by the continuity of the generalized Schur decomposition with respect to perturbations, so do the policy functions at each frequency. It can be shown that the first derivatives of the policy functions $g^i_x = \hat{g}_\infty : (\hat{\lambda}_\infty, \hat{\nu}_\infty) \to \hat{V}_\infty$ and $h^i_x = \hat{h}_\infty : (\hat{\lambda}_\infty, \hat{\nu}_\infty) \to (\hat{\lambda}'_\infty, \hat{\nu}'_\infty)$ generated by calculating the finite-dimensional linear rational expectations solution for this pair are given by matrices which are identically 0, and so $h_K = h^K_x$ and $g_K = g^K_x$. As a result, by taking an increasing set of frequencies, the finite representation can be used to compute a response which is accurate uniformly over all input functions, and not just bandlimited ones.

It is possible to determine the rate of convergence, directly from the exact representations rather than by applying the rate results from (1). Note that perturbation results for generalized eigenvectors and eigenvalues imply a linear rate of convergence in the Frobenius norm of the perturbation (see Stewart & Sun (1990)), while sufficiently smooth functional forms for adjustment costs and for the transition operator for the exogenous shocks, and the exponential form chosen for trade costs, generate rates of convergence for the entries which are faster than linear in ϕ. As a result, given sufficiently smooth parameterizations,

the blocks of the policy function corresponding to each frequency converge at a rate comparable to the slowest rate of each of the components. So long as this converges to 0, this implies that the policy operators are compact (and if this rate is faster than linear, they are Hilbert-Schmidt), and so the policy operators given by taking an increasing finite sequence of blocks converge to the true policy functions in operator norm. One note regarding the form of this convergence is that the perturbation theorem for the Schur subspaces applies only under a separation condition on the generalized eigenvalues, while (B_I^i, A_I^i) has the generalized eigenvalues $(0, \infty, \infty)$. This implies that the blocks corresponding to forward and backward looking components are well separated, while within the block of backward looking components the eigenvalues are not asymptotically well separated and the generalized Schur vectors are not stable. However, the block itself is stable in the sense that the span of the Schur vectors converges, and so the policy functions, which are determined only by the sub-blocks of the Schur matrices, also converge.

Formally, this may be stated as

Lemma 2. *(i)* $g_x^i = \hat{g}_\infty = (0,0)$, $h_x^i = \hat{h}_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *Suppose*

$$\gamma_\phi := \left\|(B_\phi, A_\phi) - (B_I^i, A_I^i)\right\|_F \to 0$$

as $|\phi| \to \infty$. *Then* $\|\hat{g}_\phi - \hat{g}_\infty\|_F = O(\gamma_\phi^{\frac{1}{2}})$ *and* $\left\|\hat{h}_\phi - \hat{h}_\infty\right\|_F = O(\gamma_\phi^{\frac{1}{2}})$ *for large* $|\phi|$, *and so converge to 0 and*

$$h[\lambda(x), \nu(x)] = \hat{h}_0[\int \nu(x)dx] + \sum_{\phi \in \mathbb{Z}\setminus\{0\}} (\hat{h}_\phi \begin{bmatrix} \int \exp(-2\pi i\phi x)\lambda(x)dx \\ \int \exp(-2\pi i\phi x)\nu(x)dx \end{bmatrix}) \circ \begin{bmatrix} \exp(-2\pi i\phi x) \\ \exp(-2\pi i\phi x) \end{bmatrix}$$

and

$$g[\lambda(x), \nu(x)] = \hat{g}_0[\int \nu(x)dx] + \sum_{\phi \in \mathbb{Z}\setminus\{0\}} (\hat{g}_\phi \begin{bmatrix} \int \exp(-2\pi i\phi x)\lambda(x)dx \\ \int \exp(-2\pi i\phi x)\nu(x)dx \end{bmatrix}) \cdot \exp(-2\pi i\phi x)$$

are compact. (ii) Suppose in addition that $\gamma_\phi = O(|\phi|^{-(1+\epsilon)})$ *for some* $\epsilon > 0$. *Then* $h[\lambda(x), \nu(x)]$ *and* $g[\lambda(x), \nu(x)]$ *are Hilbert-Schmidt.*

Proof. See appendix. □

This result not only gives compactness and rates of convergence, it also implies that the approximated policy operators converge in a stronger norm, the Hilbert Schmidt norm. The demonstration that these operators are compact and Hilbert-Schmidt implies that in

principle, the policy function in this model could be consistently estimated from a time series of observations of (λ, ν) by procedures such as those described in Bosq (2000); Guillas (2001).

6.2 Calibration and Numerical Evaluation

To characterize the dynamics of population and values in the model, I calculate the first derivatives of the policy operators for a fixed set of parameters. For the adjustment cost function $c(x)$, in order to generate a Gaussian conditional distribution of population movements in steady state, I adopt a quadratic specification $c(x) = \frac{1}{2\sigma_c}x^2$, where σ_c parameterizes the cost of moving and is also the standard deviation of the conditional Gaussian distribution. For the kernel describing the persistence of the exogenous shocks $\Gamma(x)$, in order to ensure both stationarity and decay of coefficients to represent smooth diffusion of shocks from their initial locations, I choose a rescaled Gaussian pdf, $\Gamma(x) = \frac{k_\Gamma}{\sqrt{2\pi}\sigma_\Gamma}\exp(-\frac{1}{2\sigma_\Gamma^2}x^2)$, where $|k_\Gamma| < 1$ ensures stationarity at all frequencies and σ_Γ measures the speed at which shocks spread, or, more directly, how rapidly the autoregressive coefficient on each frequency goes to 0 as the frequency increases.

For the static equilibrium of the model, I borrow parameterizations from Krugman (1996), who considers the ranges $\sigma \in \{4, 5, 6\}$, $\mu \in \{0.2, 0.3, 0.4\}$. As within this range the qualitative behavior of the model is similar, all experiments reported are carried out with $\sigma = 4$, $\mu = 0.4$. While the trade cost parameter τ is left unspecified in the parameterization as it merely normalizes the unit of distance in the model, the relative values of τ, σ_Γ, and σ_c determine the characteristic length scales at which trade, productivity (or other shock) diffusion, and migration operate. Note however that because trade costs are specified as exponential, while migration and productivity diffusion follow a Gaussian and so squared exponential rate of increase in distance, that the numbers are not directly comparable. This specification implies that trade at long distances is relatively less costly than migration or diffusion of changes in the economic environment. While difficult to place on a comparable scale, this seems to be qualitatively reasonable for a global or national scale, with long-distance exchange relatively common while long distance migration is comparatively rare. For the purposes of simulations, and without any claim to represent empirically reasonable values, simulations set $\tau = 0.2$, $\sigma_\Gamma = 0.04$, and $\sigma_c = 0.05$, representing again fairly small trade costs and fairly slow diffusion of population and amenity value from an initial location. Along with a value of $k_\Gamma = 0.98$ and discount rate $\beta = 0.96$, these are designed to ensure that fluctuations in the spatial distribution of population and amenity values are persistent and that the model generates substantial variation in the expectations of future distributions.

While in principle a closed form is available for the policy functions at each frequency

for arbitrary parameter values, it is an unintuitive nonlinear function of the roots of a cubic polynomial, so instead we verify the stability conditions at each frequency numerically. By the stability of the system at (B_I^i, A_I^i) and the convergence of (B_ϕ, A_ϕ) to (B_I^i, A_I^i), it is sufficient to verify the eigenvalue condition for the finite set of frequencies where the derivatives differ by more than some small constant from (B_I^i, A_I^i).

In practice, and in contrast to the generically explosive limit generated by the ad-hoc dynamics imposed in Krugman (1996), only for relatively extreme parameterizations does the model with forward-looking decision making lack an equilibrium which is locally stable. The complementarity between wages and population at intermediate frequencies generated by agglomerative forces in the model and the substitutability at low frequencies generated by the dispersive forces are reflected in the cross-derivatives of the transition operator $\hat{h}$ mapping shocks to living standards and population this period to those next period. The complementarity and substitutability manifest as a positive coefficient in the map from the shock $\hat{\nu}_\phi$ to amenity value today to population tomorrow at intermediate frequencies and a negative coefficient at low frequencies, respectively. However, the presence of a positive coefficient does not generate explosive behavior if the shock itself is mean-reverting, as assumed, and the autonomous dynamics of population are also stable. Here, except when the elasticity of substitution across varieties σ is extremely low so the benefits of agglomeration in a region with large population and a large variety of goods is high, the natural smoothing of population across regions generated by heterogeneous idiosyncratic preferences is the dominant determinant of the speed of adjustment of population at a given frequency. As a result, even for very strong agglomerative forces, it is also necessary for adjustment costs of moving to be quite large before complementarities at some frequency dominate and generate dynamics which are locally unstable.

In part, this expresses an important difference between the myopic and forward-looking models. In the myopic case, even small complementarities result in a cumulative process which continues without bound, while in a forward looking setup, if the effects of such complementarities are transient, their impact on value and so on decisions is bounded and so is attenuated. From an economic perspective, forward-looking decisions respond less strongly to changes perceived as temporary, and so even in the presence of complementarities, regional shocks need not be destabilizing. To be fair, however, some of the difference also reflects the additional dispersive force provided by idiosyncratic preference shocks, though it's not clear how one would generate a smooth transition law as in Krugman (1996) even with myopic decision making without some other smoothing force.

6.2.1 Finite Domain

While solving the model on an infinite domain ensures a great deal of tractability, it has some disadvantages, of which lack of realism is a minor but nontrivial one. From the perspective of demonstrating existence, compactness of the domain permits the use of standard existence theorems which are unavailable on unbounded space. Further, when approximating the integrals via an expansion in basis functions other than trigonometric polynomials, as may be needed for nonperiodic variations of the model, it permits use of compactly-supported basis functions, such as B-splines or (certain classes of) wavelets, without requiring an unbounded number to cover the entire domain.

In the symmetric case, the loss of tractability is rather minor: by setting G to be a circle of circumference 1 with coordinates $x \in [0, 1)$ parameterizing the location[11] and changing the normalizing constant $\frac{\tau(\sigma-1)}{2}$ to $\frac{\tau(\sigma-1)}{2-2e^{-\tau(\sigma-1)/2}}$ in formulas (3.4) and (3.5), it can be easily seen that the steady state equilibrium remains uniformly distributed with $\bar{\lambda}(x) = \bar{\omega}(x) = \bar{W}(x) = \bar{T}(x) = 1 \ \forall x \in G$, and $\bar{V}(x)$ constant. The only material difference to the dynamics is that now instead of convolution with a Laplace or Gaussian distribution as the representation of the effect of population on wages or the dynamics of $\nu(x)$ or $\lambda(x)$ respectively, these operators are replaced by convolution with truncated (and recentered and renormalized) Laplace or Gaussian distributions, e.g. $\Gamma(x) = \frac{1}{1-2\Phi(\frac{1}{2})} \frac{k_\Gamma}{\sqrt{2\pi}\sigma_\Gamma} \exp(-\frac{1}{2\sigma_\Gamma^2}(x - \frac{1}{2})^2) 1[0 \leq x < 1]$. This reflects the economic structure of the problem: in a finite space, there is a finite maximum trade cost and finite maximum migration cost, and so a minimum impact of one location on another. Truncation does not change the ability to represent the operators as diagonal with respect to a Fourier basis, though now the result holds by the circular convolution theorem. The Fourier transform of a product is given by the convolution of the Fourier transforms, and so, by a change of variables, in the derivation of $\frac{d\omega}{d\lambda}$, $H(\phi)$ is replaced by $H(\phi) * \mathrm{Sinc}(\phi)$, where $\mathrm{Sinc}(\phi) = \frac{\sin \pi\phi}{\pi\phi}$ is the Fourier transform of $1[-\frac{1}{2} \leq x < \frac{1}{2}]$. While this convolution has no simple closed form expression, it is easily calculated numerically by quadrature.

For parameterizations with rapid increase in trade or migration costs over distance, this transformation has minimal effect, as the truncation only affects the far tails. For small trade or migration costs, it increases impact at some frequencies and decreases it at others, reflecting the periodicity induced by the circular shape. Numerical experiments suggest that even for relatively small costs, the impact of this change is limited. As a result, the main impact is on ensuring proper scaling and allowing testing approximate equilibrium computation using a wavelet basis.

To represent the circular convolutions with respect to a wavelet basis, the operators are

[11] By symmetry, the initial point 0 can be assigned to any arbitrary location.

first written in terms of the distance on a circle with $x' - x$ replaced with arc length along the diameter of the circle: $d(x', x) = \text{mod}(x' - x + \frac{1}{2}, 1) - \frac{1}{2}$ is the distance between points x', $x \in [0, 1)$ on the circumference. For example $\Gamma[\nu](x') = \int_0^1 \Gamma(d(x', x))[\nu(x)]dx$ describes the value of the amenity value $\nu'(x')$ next period at each point $x' \in [0, 1)$ given an initial distribution $\nu(x)$. Construction of wavelet approximations consists of sampling the kernels (e.g. $\Gamma(d(x', x))$) at an evenly spaced grid of $K \times K$ points on $[0, 1) \times [0, 1)$ and applying the discrete wavelet transform to the rows and columns of the resulting matrix. The kernels used in this model are infinitely differentiable at most values of x, y but nondifferentiable at $d(x, y) = 0.5$ due to the finite domain creating a maximal possible level of trade or migration costs at the antipodal location on the circle where counterclockwise or clockwise movements meet. For the exponential trade costs, there is also a point of nondifferentiability at $d(x, y) = 0$. For wavelet representations, there is a tradeoff between vanishing moments to represent the smooth parts parsimoniously and width of the scaling function which creates distortions at nonsmooth points. Although higher order Coiflets will achieve faster rates asymptotically, for finite values of K, lower order Coiflets may yield better performance, which is borne out in numerical experiments. As a compromise, level 3 Coiflets are used in all simulations and evaluations.

Two additional sets of approximations are made beyond those described in (2). To ensure that perturbations to the population distribution $\lambda(x)$ remain in the space of mean 0 functions, the wavelet representations of operators acting on this space are orthogonalized with respect to the the wavelet representation of the constant function. While for Haar wavelets this demeaning is exact, for other bases it yields a representation which is approximately orthogonal to constants. Rather than defining the exact kernel for $\frac{d\omega}{d\lambda}$ and applying the wavelet transform to it directly, because it is composed of convolutions with a Laplace distribution and identity operators, it may be constructed by applying the products and inverses of the wavelet representations of these operators. Because all applications are continuous (note that the inverse is applied to a sum of identity and convolution operators and so is well posed), so long as the convolution operators themselves are consistently approximated, $\frac{d\omega}{d\lambda}$ is as well.

6.3 Results

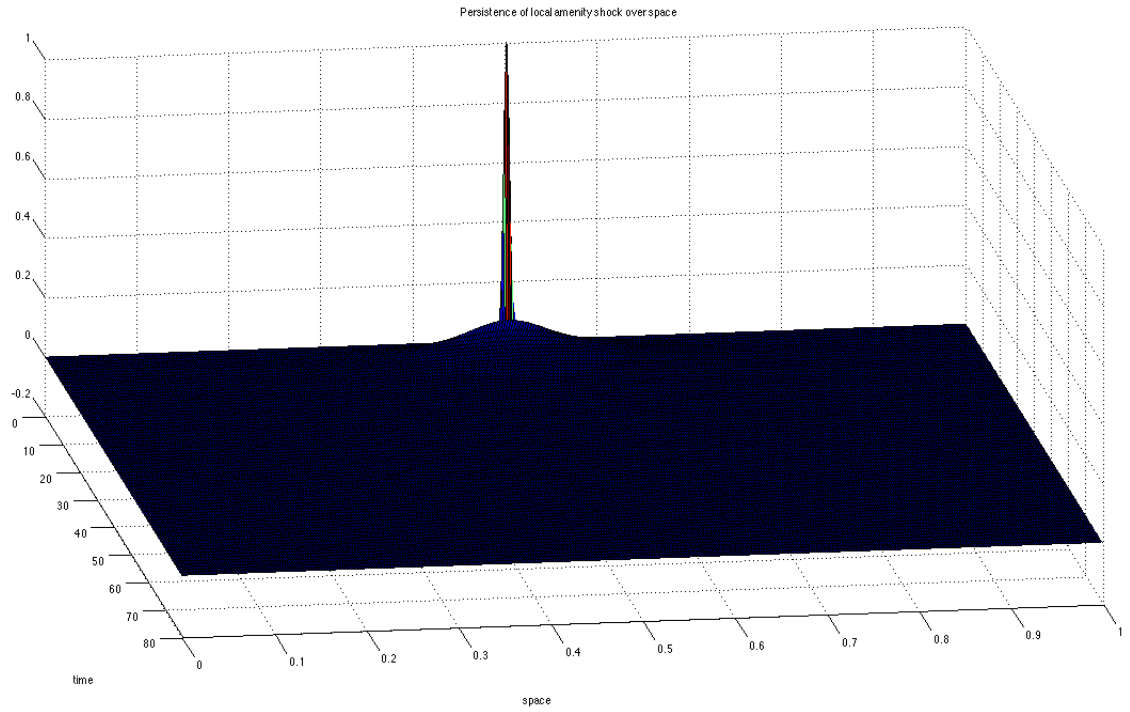
To evaluate the approximation algorithm, several numerical comparisons are performed using both the Fourier and the wavelet representations of the model, at different levels of K. Accuracy can be compared for impulse responses to function-valued shocks, as well as for simulations. Although the Fourier representation is exact in principle, at least for bandlim-

ited functions, when restricted to a finite domain, the value of the Fourier coefficient at each frequency must be computed by quadrature due to the convolution with the sinc function. Nevertheless, the Fourier and wavelet methods appear to exhibit a high degree of agreement, whether expressed in squared error norm over the grid points (a proxy for L^2 norm, controlled by the theory) or in maximum norm over grid points (not controlled by the theory). Error is largest for components which are strongly impacted by the finite diameter of the geography, and declines for parameter values which ensure that the cutoff has limited effect on the representation, suggesting that the numerical error induced by quadrature may be a non-negligible factor contributing to the discrepancy between Fourier and wavelet representations. For the Fourier representations, integer frequencies $\frac{-K}{2}$ to $\frac{K}{2}$ are used for each of $J = 3$ functions $\nu(x)$, $\lambda(x)$, and $V(x)$, giving $3 \times (K + 1)$ basis functions, for symmetry, while for wavelets K grid points are used to represent the scaling function coefficients for each function, with K given by a power of 2.

To describe the behavior of the model, first consider the impulse response to a smooth but spatially localized shock $\varepsilon(x)$ to the amenity value of locations, a scaled Gaussian spike centered at location 0.5, with functional form $\exp(50000(x - 0.5)^2)$. This may represent a nearly exactly localized improvement, as might occur in response to a local policy initiative or favorable productivity shock. As can be seen in Figure (6.1), the response of amenity value over time and space, calculated from $K = 1024$ using the Fourier representation, in spite of the high persistence parameter k_Γ and the relatively small standard deviation of the diffusion kernel σ_Γ, this shock spreads out rapidly from the initial location and diffuses from a local region to an eventually larger and larger area. Note that while the space coordinate is represented on a line segment, the model is defined over a circle, so the edges are connected.

The population response, displayed in Figure (6.2), follows the amenity shock but is much more dispersed, and responds slowly, peaking over 10 periods later and then declining gradually. The population in regions far from the center declines, as people move towards the more desirable area, with a nadir over 20 periods later. Despite the slow speed of adjustment, movements begin the first period after the shock, as individuals anticipate the spread of the amenity over space and the possibility of moving in the future to more desirable areas, which are desirable in part because they provide the option value of moving even close to the center in future at lower cost and so taking advantage of the improved amenity there. This is displayed clearly in the plot of welfare, $V_t(x)$ in Figure (6.2), which jumps immediately, with peak at the location of the shock but high values substantially more broadly dispersed, with a nontrivial jump in welfare over the entire domain, as even regions for which the value of the shock immediately and in the first few periods is essentially negligible face the prospect of higher welfare in the future as the amenity spreads out and population moves to regions

Figure 6.1: Impulse Response of $\nu_t(x)$ to $\varepsilon_0(x) = \exp(50000(x - 0.5)^2)$



positively affected by the shock.

Relative accuracy of the Fourier and wavelet representations of the model for the above shock are measured in Table (1), for $K = 256$, 512, and 1024, for the maximum error at any grid point over 80 periods of the impulse response. Figures (D.1) and (D.2) represent the Euclidean norm difference (over an evenly spaced grid) at each time point between the wavelet and Fourier representations at the different values of K, a proxy for the L^2 norm. Note that even for $K = 512$, the errors are already extremely small, with maximum pointwise error on the order of 10^{-7} or smaller for $\nu_t(x)$ a function with values ranging from 0 to 1, and 10^{-8} for $\lambda_t(x)$ and $V_t(x)$, functions with range of about 0.1. The order of this error decreases significantly for $K = 1024$, both for maximum and squared average error.

The clock time to compute the wavelet solutions, also displayed in Table (1), is relatively fast and increases roughly in cubic proportion to K, taking under two minutes for $K = 512$, including producing all figures and evaluation metrics, coded in Matlab using the default QZ function on a 2011 Macbook Pro with 2.8 GHz Intel i7 processor and 2 GB RAM. This level of speed and accuracy on a far from state of the art setup suggests that the procedure may be useful in applications where it is applied repeatedly, for example to estimate parameters. The Fourier representation takes only a few seconds for any K, which should be expected as it allows calculating solutions for each frequency separately and so takes time linear in

Figure 6.2: Impulse Responses of $\lambda_t(x)$ & $V_t(x)$ to $\varepsilon_0(x) = \exp(50000(x-0.5)^2)$

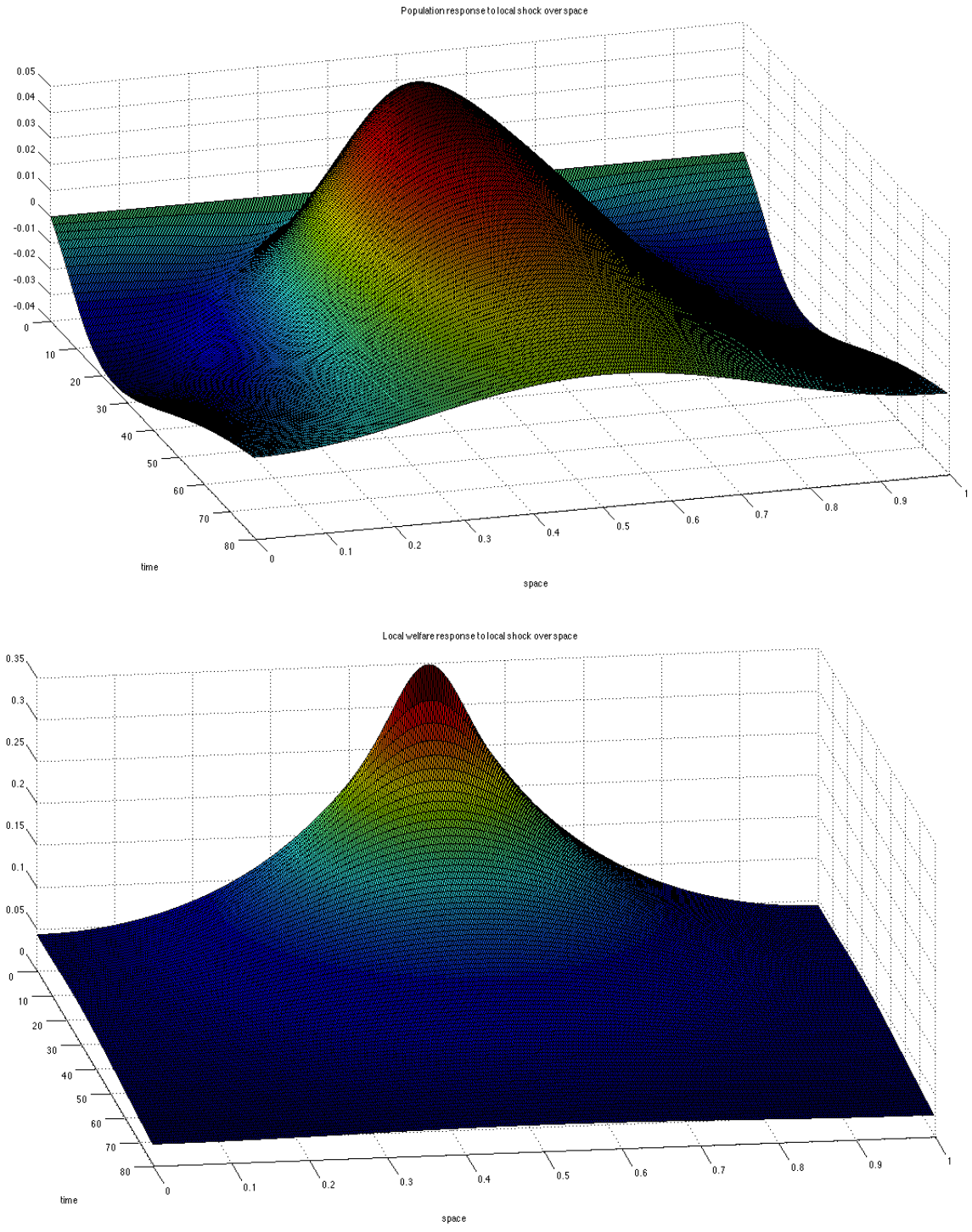


Table 1: Numerical IRF Discrepancy, Fourier vs. Wavelet Representations

K	max pointwise, ν_t	max pointwise, λ_t	max pointwise, V_t	Running Time (seconds)
256	0.0107	3.9549e-07	1.9362e-06	11.607549
512	3.4594e-07	5.0737e-08	7.6597e-08	96.187571
1024	8.9301e-11	1.2976e-08	1.9643e-08	376.833220

K. This feature is only a result of the special structure of this model and is not likely to generalize.

To consider the behavior of the model in response to more complex patterns of input, I use it to produce simulated time paths. The shocks $\varepsilon_t(x)$ are drawn from a spatially correlated Gaussian process, a simulated fractional Brownian motion (started at 0) with Hurst parameter 0.7 and so a degree of Hölder regularity no greater than 0.7. Wavelet quadrature is easily capable of representing functions with this degree of regularity and so the simulations are drawn from the representation of the model with respect to a wavelet basis, with $K = 512$. Time paths are displayed in Figures (6.3) and (6.4).

One feature which stands out is the low degree of smoothness of $\nu_t(x)$, the persistent shock process, and $V_t(x)$, the welfare of residents at each location x, in contrast to the fairly high degree of smoothness of population movements $\lambda_t(x)$. This contrast is as should be expected, because $V_t(x)$ is a jump variable, and so adjusts immediately to reflect changes in the state, while population is a predetermined variable, and so changes only in response to expected future changes in welfare, which, because shocks to amenity value are expected to be smoothed out over time, substantially discounts the high frequency variations which impart roughness to the spatial distribution of current welfare. This is in line with standard reasoning for rational expectations decision problems: because moving is costly, transitory variation, expressed by the rough local movements in amenity values, has minimal effect on forward looking decisions. In contrast, low frequency changes, which are expected to be more persistent, do induce population movements, and the simulation does show periods of time where there are large population movements between regions. The simulation also exemplifies the expressive power of functional methods, as it allows description of the welfare and behavioral consequences of extremely finely detailed patterns of aggregate shocks, which would be difficult to express even with smooth nonparametric function representations, let alone low-dimensional parametric approximations.

Figure 6.3: Simulated Geographic Equilibrium: Amenities and Welfare

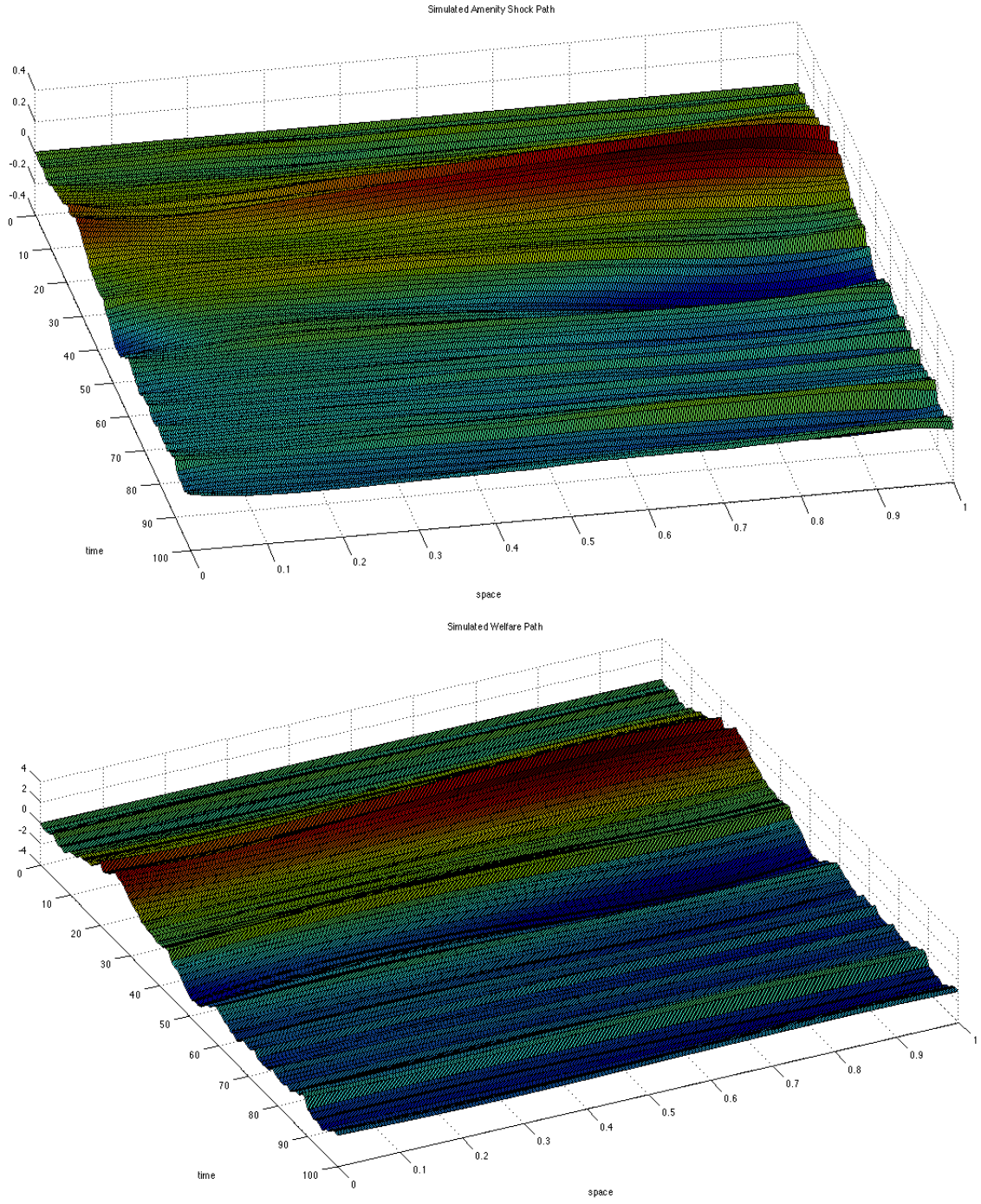
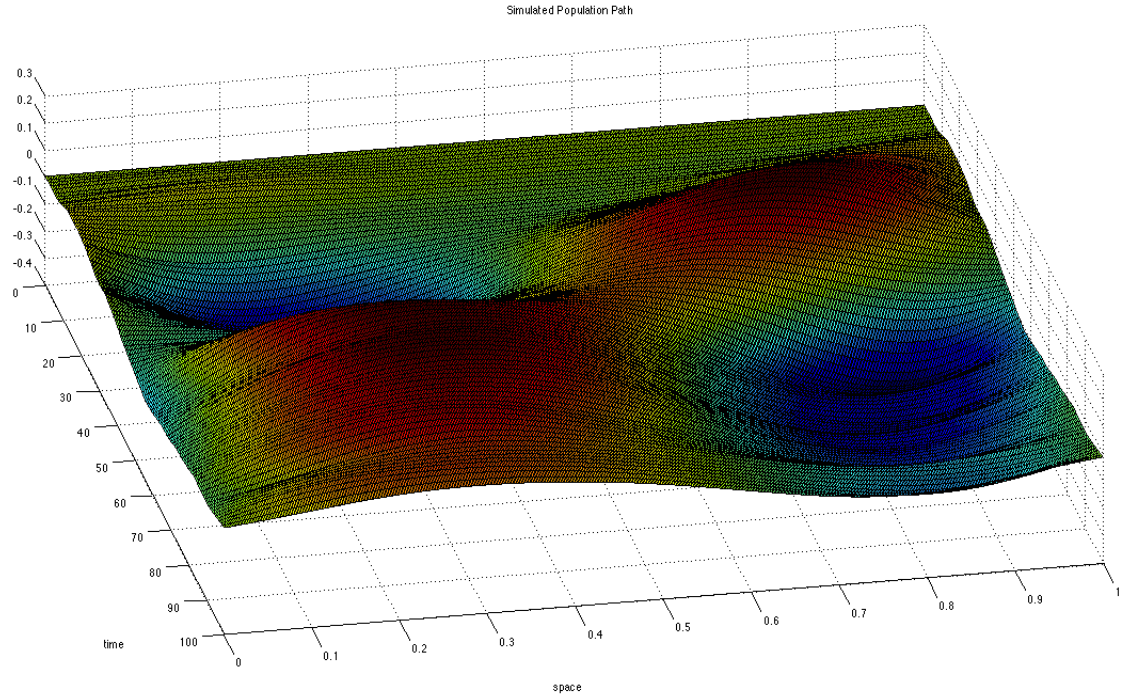


Figure 6.4: Simulated Geographic Equilibrium: Population



7 Conclusion

The idea that heterogeneity matters for economic outcomes, not only at the individual level but through the set of interdependencies linking behavior at the individual level to the environment faced by others, is a core principle in economics. Function-valued stochastic processes, by describing how patterns of heterogeneity change over time and relate to other variables, provide an analytical framework in which these interdependencies can be modeled and evaluated directly rather than considering only aggregate variables. While describing economic decision making in these environments can be challenging due to the high dimension of the relevant variables, a substantial amount of information can be recovered by describing the problem locally near a point where infinite-dimensional uncertainty disappears. A linearized solution allows consideration of responses to any possible pattern or shape that can be considered, accurately representing the behavior of the system in an infinite-dimensional set of possible inputs. Moreover, for many systems, this response can be calculated quickly and accurately, uniformly over all possible directions by projection representations of the functional derivatives of the system.

The dynamics of economic interactions over space, typically challenging to describe due to the fact that people in different locations must respond differently to the geographic patterns of economic activity induced by trade and spatially inhomogeneous regional disturbances,

provide a demonstration of the rich patterns of relationships that can be captured by allowing decisions and distributions to respond to the precise geographic pattern of shocks. Responses can differ substantially based on distance, but also based on expectations of perceived future spatial distributions. Although spatial interactions provide a case which illustrates the full importance of allowing for response to potentially arbitrarily shaped patterns of heterogeneity, the function-valued approach seems promising for a wide variety of applications. These include understanding the mechanisms behind the dynamics of income and wealth inequality over business cycles, analyzing both through the relationship with capital markets, as has been explored in existing studies of incomplete markets models with aggregate shocks, as well as other potential economic mechanisms and policies. They may also be useful for studying a variety of patterns of interaction which depend on the entire shape of the distribution of heterogeneity, such as matching markets in labor or other contexts or interactions through a social or economic network.

While for some applications, existing methods may be used to characterize the dynamics of economic heterogeneity, albeit without explicit guarantees of accuracy, the function-valued approach may still be desirable as a framework for data analysis. By explicitly allowing the model to incorporate uncertainty of arbitrary shape, the models described allow a complete characterization of the variation in micro and macroeconomic data and open the possibility of comparing the model directly to cross-sectional micro data. Because linearized function-valued models generate dynamics consistent with functional linear processes, estimation and inference methods from functional data analysis may be applied to evaluate them empirically. Given the speed and accuracy of the solution methods, they may also open up the possibility of using functional data methods to perform full information structural estimation of models with heterogeneous agents.

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A Existence of a generalized Schur decomposition for pairs of bounded operators

The construction of a solution for the linear expectational difference equation defined by a linear or linearized rational expectations model in finite dimensions relies on the ability to partition the state space and the equilibrium equations into 'stable' and 'unstable' components which may be treated separately. This is generally achieved by either a Jordan decomposition, generating block-diagonal matrices, as in Blanchard & Kahn (1980) or by a generalized Schur decomposition, generating upper-triangular matrices,[12] as in Klein (2000). In practice, the latter has become preferred, as the Jordan decomposition of a matrix is not in general continuous while the generalized Schur decomposition, which is generated by unitary matrices, exhibits numerical stability in theory and practice. Such stability is particularly desirable in the infinite-dimensional case, as closed form solutions for the eigenfunctions are not in general feasible and finite-dimensional numerical procedures must by necessity induce some error into the representation of the operator pair of interest.

While generalization of the Jordan decomposition to infinite-dimensional operator pairs is well established (Kato, 1976; Gohberg *et al.* , 1990, Ch IV) and the Schur decomposition for a single infinite-dimensional operator has also been defined (Gohberg *et al.* , 1990, Ch II.3), an analogue of the generalized Schur decomposition for pairs of infinite-dimensional linear operators has not, to the best of my knowledge, been described. As in the case of the Schur decomposition of a single operator, extension to the infinite-dimensional case is slightly delicate, as the existence of the Schur or generalized Schur decomposition is based on an iterative construction which extends only in certain cases to an uncountable state space. In particular, the Schur decomposition may be extended to compact operators but not to arbitrary bounded or closed operators, for which a Jordan decomposition exists but a Schur decomposition may not. For the purposes of constructing an analogy of the generalized Schur decomposition which permits extension of rational expectation solution procedures, there are at least two ways around this difficulty. The first, and simplest, is to note that while solution

[12]This decomposition is often referred to as the QZ decomposition, in reference to the QZ algorithm often used to compute it. See Golub & van Loan (1996).

requires splitting the domain into 'forward' and 'backward' subspaces, for a stationary solution there is no requirement that the restriction of the operator to these subspaces itself take upper triangular form. Instead, one can construct a block upper triangular decomposition which preserves the desirable feature of being generated by unitary transformation while eschewing the necessity to make restrictive compactness assumptions. Alternately, one may construct a generalized Schur decomposition analogously to the infinite-dimensional Schur decomposition, which does preserve an upper-triangular structure within blocks, under a modified and so slightly less onerous compactness condition than in the single operator case. In the following, I show existence of a blockwise decomposition under general conditions, and also decomposition which is upper triangular within blocks under a condition on compactness of certain transformations of the operator pair which does not imply that both operators are compact, and in particular allows the pertinent example of the standard eigenvalue problem in which one of the operators in the pair is the identity operator, which is not compact on an infinite-dimensional space. This construction also has the advantage that it implies compactness of certain Schur components and so generates a solution for the law of motion which is itself compact.

Formally, let (M, G) be a pair of bounded linear operators acting between complex Hilbert spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$, i.e. $M \in \mathcal{L}(\mathcal{H}_X \to \mathcal{H}_Y)$ $G \in \mathcal{L}(\mathcal{H}_X \to \mathcal{H}_Y)$. Following Gohberg *et al.* (1990), define the spectrum $\sigma(M, G)$ as those $\lambda \in \mathbb{C}$ such that $\lambda G - M$ is not invertible, accompanied by the point ∞ if and only if G does not have bounded inverse, and the resolvent set $\rho(M, G)$ as $\mathbb{C}_\infty \backslash \sigma(M, G)$, where $\mathbb{C}_\infty$ is the extended complex plane with the standard topology (see Conway (1978, Ch. 1 S. 6)).

Definition 4. An operator pair is said to be Γ-regular (with respect to a set Γ) if for some nonempty subset $\Gamma \subset \mathbb{C}_\infty$, $\Gamma \subset \rho(M, G)$.

Assume Γ is a Cauchy contour (c.f. Gohberg *et al.* (1990, p.6)) with inner domain Δ_+ and outer domain Δ_-, and that (M, G) is Γ-regular. For concreteness, we will often take Γ to be the positively oriented complex unit circle, in which case Γ-regularity means that the spectrum does not contain λ such that $|\lambda| = 1$. From a modeling perspective, this ensures stationarity by ruling out unit roots; this particular choice is not required to ensure existence of a generalized Schur decomposition. By Gohberg *et al.* (1990) Theorem IV.1.1, the above assumptions ensure the existence of (possibly oblique) projection operators π_1: $\mathcal{H}_X \to \mathcal{H}_X$ and π_2: $\mathcal{H}_Y \to \mathcal{H}_Y$ which partition $\mathcal{H}_X$ and $\mathcal{H}_Y$ into Im $\pi_1 \oplus$ Ker π_1 and Im $\pi_2 \oplus$ Ker π_2 respectively, and the operator pair (M, G) into components

$$(M, G) = \left(\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \right) : \text{ Im } \pi_1 \oplus \text{Ker } \pi_1 \to \text{Im } \pi_2 \oplus \text{Ker } \pi_2 \qquad \text{(A.1)}$$

such that (M_1, G_1) and (M_2, G_2) are Γ-regular, $\sigma(M_1, G_1) = \sigma(M, G) \cap \Delta_+$ and $\sigma(M_2, G_2) = \sigma(M, G) \cap \Delta_-$. In words, this says one can separate the pair into a component with spectrum inside some domain and a component with spectrum outside.

Assume in addition that $0 \in \Delta_+$ and $\infty \in \Delta_-$. By the above result and the definition of the resolvent, this implies that G_1 and M_2 are invertible on their respective domains. In particular, $E = \begin{pmatrix} G_1^{-1} & 0 \\ 0 & M_2^{-1} \end{pmatrix}$: Im $\pi_2 \oplus$ Ker $\pi_2 \to$ Im $\pi_1 \oplus$ Ker π_1 is a bounded invertible operator and we may define the partition

$$(EM, EG) = \left(\begin{bmatrix} \Omega_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \right) : \text{ Im } \pi_1 \oplus \text{Ker } \pi_1 \to \text{Im } \pi_1 \oplus \text{Ker } \pi_1 \qquad \text{(A.2)}$$

where $\Omega_1 = G_1^{-1} M_1$ and $\Omega_2 = M_2^{-1} G_2$. These operators have the following relationship with (M_1, G_1) and (M_2, G_2):

Lemma 3. *$\sigma(M_1, G_1) = \sigma(\Omega_1)$, and $\frac{1}{\lambda} \in \sigma(\Omega_2)$ if and only if $\lambda \in \sigma(M_2, G_2)$ (where $\frac{1}{\infty}$ may be defined to equal 0)*

Proof. Suppose λ is in the resolvent set of Ω_1. Then $\Omega_1 - \lambda I_1$ has some bounded inverse Z. Then $-ZG_1^{-1}$ satisfies $-ZG_1^{-1}(\lambda G_1 - M_1) = Z(\Omega_1 - \lambda I_1) = I_1$ and $-(\lambda G_1 - M_1)ZG_1^{-1} = -G_1 G_1^{-1}(\lambda G_1 - M_1)ZG_1^{-1} = G_1(\Omega - \lambda I_1)ZG_1^{-1} = G_1 G_1^{-1} = I_1$, so $\lambda \in \rho(M_1, G_1)$. That is, $\rho(\Omega_1) \subset \rho(M_1, G_1)$. Next, suppose $\lambda \in \rho(M_1, G_1)$. Then $\lambda G_1 - M_1$ has a bounded inverse Z, and $-ZG_1$ satisfies $-ZG_1(\Omega - \lambda I_1) = Z(\lambda G_1 - M_1) = I_1$ and $-(\Omega - \lambda I_1)ZG_1 = -G_1^{-1} G_1(\Omega - \lambda I_1)ZG_1 = G_1^{-1}(\lambda G_1 - M_1)ZG_1 = G_1^{-1} G_1 = I_1$, and so $\rho(M_1, G_1) \subset \rho(\Omega_1)$. Combining, $\rho(M_1, G_1) = \rho(\Omega_1)$ and so $\sigma(M_1, G_1) = \sigma(\Omega_1)$. Similar calculations show $\frac{1}{\lambda} \in \sigma(\Omega_2)$ if and only if $\lambda \in \sigma(M_2, G_2)$. If $\infty \in \sigma(M_2, G_2)$, G_2 is not invertible and so $M_2^{-1} G_2 - \frac{1}{\infty} I_2 = M_2^{-1} G_2$ must also have nontrivial kernel, and so be noninvertible. □

With this notation, it is possible to characterize conditions under which the operator pair (M, G) has a generalized Schur decomposition. As our construction makes use of complete orthonormal bases, we assume now that (M, G) are operators between *separable* Hilbert spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$.

Lemma 4. *Let (M, G) be a pair of bounded operators $M \in \mathcal{L}(\mathcal{H}_X \to \mathcal{H}_Y)$ $G \in \mathcal{L}(\mathcal{H}_X \to \mathcal{H}_Y)$ Γ-regular with respect to a Cauchy curve with inner domain Δ_+ such that $0 \in \Delta_+$ and outer domain Δ_- such that $\infty \in \Delta_-$. Define projectors π_1 and π_2 as in A.1 with respect to Γ. Then, there exist unitary operators $Q = [Q^1, Q^2]$: $Im\pi_2 \oplus \mathcal{H}_Y / Im\pi_2 \to F_1 \oplus F_2$ and $P = [P^1, P^2]$: $Im\pi_1 \oplus \mathcal{H}_X / Im\pi_1 \to E_1 \oplus E_2$ such that (M, G) has the following block-wise generalized Schur decomposition*

$$(QMU^*, QGU^*) = \left(\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix} \right)$$

from $E_1 \oplus E_2 \to F_1 \oplus F_2$

where E_1*,* E_2*,* F_1*, and* F_2 *are spaces such that there exist linear isometric isomorphisms from* $\mathrm{Im}\pi_1 \to E_1$*,* $\mathcal{H}_X/\mathrm{Im}\pi_1 \to E_2$*,* $\mathrm{Im}\pi_2 \to F_1$*, and* $\mathcal{H}_X/\mathrm{Im}\pi_2 \to F_2$*, respectively. Further,* $\sigma(M_{11}, G_{11}) = \sigma(M_1, G_1) = \sigma(M, G) \cap \Delta_+$ *and* $\sigma(M_{22}, G_{22}) = \sigma(M_2, G_2) = \sigma(M, G) \cap \Delta_-$*.*

Remark. The precise identity of the spaces E_1, E_2, F_1, and F_2 need not be considered for this result. However, a canonical choice of spaces would be to allow $E_1 = \mathrm{Im}\pi_1$, $E_2 = \mathcal{H}_X/\mathrm{Im}\pi_1$, $F_1 = \mathrm{Im}\pi_2$, $F_2 = \mathcal{H}_Y/\mathrm{Im}\pi_2$, in which case the Schur decomposition acts on the same space as (M, G).

Proof. We generate Q and U constructively, then verify their properties. Choose a complete orthonormal basis on $\mathrm{Im}\pi_1$, denoted $\{u_{1i}\}_{i=1}^{\infty}$ and then a complete orthonormal basis on the orthogonal complement of $\mathrm{Im}\pi_1$ in $\mathcal{H}_1$, denoted $\{u_{2i}\}_{i=1}^{\infty}$. The eigenvectors are not, in general, such a basis, because Ω_1 and Ω_2 are not assumed self-adjoint and so nothing requires their eigenvectors to be orthogonal vectors. Then, U_1 is the operator $\sum_{i=1}^{\infty} \langle u_{1i}, . \rangle \, e_i^1$ where $\{e_i^1\}_{i=1}^{\infty}$ are an arbitrary orthonormal basis on E_1, a space isometrically isomorphic to $\mathrm{Im}\pi_1$, U_2 is the operator $\sum_{i=1}^{\infty} \langle u_{2i}, . \rangle \, e_i^2$ where $\{e_i^2\}_{i=1}^{\infty}$ are an arbitrary orthonormal basis on E_2, a space isometrically isomorphic to $\mathcal{H}_1/\mathrm{Im}\pi_1$. Likewise, choose a complete orthonormal basis $\{q_{1i}\}_{i=1}^{\infty}$ for the image of (M_1, G_1) and a complete orthonormal basis for the orthogonal complement of this space in $\mathcal{H}_Y$, $\{q_{2i}\}_{i=1}^{\infty}$. We define $Q_1 = \sum_{i=1}^{\infty} \langle q_{1i}, . \rangle \, f_i^1$ and $Q_2 = \sum_{i=1}^{\infty} \langle q_{2i}, . \rangle \, f_i^2$, for $\{f_i^1\}_{i=1}^{\infty}$ and $\{f_i^2\}_{i=1}^{\infty}$ orthonormal bases of F_1 and F_2, spaces isometrically isomorphic to the domains of Q_1 and Q_2 respectively.

Next, we show that these induce an upper triangular decomposition. We define

$$\left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \right) = \left(\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} M \begin{bmatrix} U_1^* & U_2^* \end{bmatrix}, \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} G \begin{bmatrix} U_1^* & U_2^* \end{bmatrix} \right)$$

Using A.1, we have that $(BU_1^*, AU_1^*) = (B_1U_1^*, A_1U_1^*)$ since the range of U_1^* is $\mathrm{Im}\pi_1$, and the restriction of (M, G) to this space is (M_1, G_1). Then, since the domain of Q_2 is orthogonal to $\mathrm{Im}(M_1, G_1)$, we have $(M_{21}, G_{21}) = (0, 0)$, so this is a triangular decomposition.

To characterize the spectrum of the decomposition, first note that $\sigma(M_1, G_1) = \sigma(M, G) \cap \Delta_+$ and $\sigma(M_2, G_2) = \sigma(M, G) \cap \Delta_-$ by Gohberg *et al.* (1990) Theorem IV.1.1. (M_{11}, G_{11}) may be written as $(Q_1MU_1^*, Q_1GU_1^*) = (Q_1M_1U_1^*, Q_1G_1U_1^*)$. Consider $\gamma \in \rho(M_1, G_1)$. Then

$\gamma M_{11} - G_{11} = \gamma Q_1 M_1 U_1^* - Q_1 G_1 U_1^* = Q_1(\gamma M_1 - G_1)U_1^*$, which has inverse $U_1(\gamma M_1 - G_1)^{-1}Q_1^*$ which is bounded since $(\gamma M_1 - G_1)^{-1}$ is bounded, by definition of the resolvent set, and U_1 and Q_1^* are since they are unitary by construction. So $\sigma(M_{11}, G_{11}) \subset \sigma(M_1, G_1) = \sigma(M, G) \cap \Delta_+$.

Characterization of the spectrum of (M_{22}, G_{22}) requires a bit more care. (M_{22}, G_{22}) may be written as

$$\begin{aligned}(Q_2 M U_2^*, Q_2 G U_2^*) &= (Q_2 M(\pi_1 + (I - \pi_1))U_2^*, Q_2 G(\pi_1 + (I - \pi_1))U_2^*) \\ &= (Q_2 M_1 \pi_1 U_2^*, Q_2 G_1 \pi_1 U_2^*) + (Q_2 M_2(I - \pi_1)U_2^*, Q_2 G_2(I - \pi_1)U_2^*) \\ &= (Q_2 M_2(I - \pi_1)U_2^*, Q_2 G_2(I - \pi_1)U_2^*)\end{aligned}$$

where the second line follows from A.1 and the final line follows from the fact that the domain of Q_2 is orthogonal to the range of (M_1, G_1). Consider $\gamma \in \rho(M_2, G_2)$. By definition of the resolvent, $T(\gamma) := (\gamma M_2 - G_2)^{-1}$ is a bounded operator for all such γ. Then $\gamma M_{22} - G_{22} = Q_2(\gamma M_2 - G_2)(I - \pi_1)U_2^*$. I claim that $U_2 T(\gamma) Q_2^*$ is a bounded inverse of $\gamma M_{22} - G_{22}$. To see this, note that $Q_2^* Q_2$ is equal to $I_{\mathcal{H}_Y/\mathrm{Im}\pi_2}$ and $U_2^* U_2 = I_{\mathcal{H}_X/\mathrm{Im}\pi_1}$. As a result, we have $U_2 T(\gamma) Q_2^* Q_2 (\gamma M_2 - G_2)(I - \pi_1)U_2^* = U_2(I - \pi_1)U_2^* = U_2 U_2^* = I_{\mathcal{H}_X/\mathrm{Im}\pi_1}$, where we use the fact that $U_2 \pi_1 = 0$ since U_2 has domain orthogonal to the image of π_1. By the identity $(M_2, G_2)(I - \pi_1) = (I - \pi_2)(M_2, G_2)$, $Q_2(\gamma M_2 - G_2)(I - \pi_1)U_2^* U_2 T(\gamma) Q_2^* = Q_2(I - \pi_2)(\gamma M_2 - G_2)U_2^* U_2 T(\gamma) Q_2^* = Q_2(I - \pi_2)Q_2^* = I_{\mathcal{H}_Y/\mathrm{Im}\pi_2}$, since $Q_2 \pi_2 = 0$. As a result, $\sigma(M_{22}, G_{22}) \subset \sigma(M_2, G_2) = \sigma(M, G) \cap \Delta_-$.

To show the reverse inclusion, note that $\sigma(M, G) = \sigma(QMU^*, QGU^*)$ by unitarity of Q and U. Next, we show that $\sigma(QMU^*, QGU^*) = \sigma(M_{11}, G_{11}) \cup \sigma(M_{22}, G_{22})$. Since Δ_+ and Δ_- are disjoint, $\sigma(M_{11}, G_{11}) \subset \sigma(M, G) \cap \Delta_+$, and $\sigma(M_{22}, G_{22}) \subset \sigma(M, G) \cap \Delta_-$, this implies that $\sigma(M_{11}, G_{11}) = \sigma(M, G) \cap \Delta_+$ and $\sigma(M_{22}, G_{22}) = \sigma(M, G) \cap \Delta_-$, as claimed. To show this, consider $\gamma \in \rho(M_{11}, G_{11}) \cap \rho(M_{22}, G_{22})$. Then $\gamma QMU^* - QGU^* = \begin{bmatrix} \gamma M_{11} - G_{11} & \gamma M_{12} - G_{12} \\ 0 & \gamma M_{22} - G_{22} \end{bmatrix}$ has bounded inverse given by

$$\begin{bmatrix} (\gamma M_{11} - G_{11})^{-1} & -(\gamma M_{11} - G_{11})^{-1}(\gamma M_{12} - G_{12})(\gamma M_{22} - G_{22})^{-1} \\ 0 & (\gamma M_{22} - G_{22})^{-1} \end{bmatrix}$$

and so $\sigma(QMU^*, QGU^*) \subset \sigma(M_{11}, G_{11}) \cup \sigma(M_{22}, G_{22})$. Next, suppose $\gamma \in \sigma(M_{11}, G_{11})$ and assume for contradiction that $\gamma \in \rho(QMU^*, QGU^*)$, and so $\gamma QMU^* - QGU^*$ has some bounded inverse $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \gamma M_{11} - G_{11} & \gamma M_{12} - G_{12} \\ 0 & \gamma M_{22} - G_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ and so $a(\gamma M_{11} - G_{11}) = I$, implying that $\gamma M_{11} - G_{11}$ has bounded inverse a, a contradiction. Sim-

ilarly, if $\gamma \in \sigma(M_{22}, G_{22})$, then if $\gamma QMU^* - QGU^*$ had some bounded inverse $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\begin{bmatrix} \gamma M_{11} - G_{11} & \gamma M_{12} - G_{12} \\ 0 & \gamma M_{22} - G_{22} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ would equal $\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$, implying $(\gamma M_{22} - G_{22})d = I$, which is assumed false. As a result, $\sigma(QMU^*, QGU^*) \supset \sigma(M_{11}, G_{11}) \cup \sigma(M_{22}, G_{22})$, and the claim is shown. □

Slightly stronger assumptions than used in the above can yield stronger results. In particular, the assumption of compactness of Ω_1 and Ω_2 may permit the block triangular decomposition to be extended to a triangular decomposition within each block, as in the infinite-dimensional Schur decomposition in Gohberg *et al.* (1990). This provides a link to the finite-dimensional method, but is nowhere necessary for the application of the decomposition considered. However, compactness of the components does provide a useful sufficient condition for the necessary conditions, and also ensures the compactness of the solution operators, which is a condition commonly imposed for the validity of estimators of infinite-dimensional operators: see, e.g., Bosq (2000).

If this refinement is not needed, we may instead operate under a strictly weaker assumption: viz. that the spectrum of (M, G) is bounded away from Γ. To see that this is weaker, note that compactness implies that the unique accumulation point of the spectrum is at 0, and so by 3, the spectra of Ω_1 and Ω_2 and, as a result, of (M, G) must neither be inside of Γ or have limit point in Γ. Formally, we define a block triangular decomposition as follows. For notational convenience and analogy to the finite-dimensional case, we take the decomposition to be defined as a pair on $\mathcal{L}(\mathcal{H}_X \to \mathcal{H}_Y)$ rather than over isometrically isomorphic spaces.

Lemma 5. *Let (M, G) be a pair of bounded operators $M \in \mathcal{L}(\mathcal{H}_X \to \mathcal{H}_Y)$ $G \in \mathcal{L}(\mathcal{H}_X \to \mathcal{H}_Y)$ Γ-regular with respect to a Cauchy curve with inner domain Δ_+* such that $0 \in \Delta_+$ *and outer domain Δ_- such that $\infty \in \Delta_-$. Define projectors π_1 and π_2 as in A.1 with respect to Γ and Ω_1 and Ω_2 as in A.2. Suppose in addition that Ω_1 and Ω_2 are compact operators. Then, there exist unitary operators $Q = [Q^1, Q^{1\perp}, Q^2, Q^{2\perp}] : \ F_1 \oplus F_1^\perp \oplus F_2 \oplus F_2^\perp \to F_1 \oplus F_1^\perp \oplus F_2 \oplus F_2^\perp$ and $P = [P^1, P^{1\perp}, P^2, P^{2\perp}] : \ E_1 \oplus E_1^\perp \oplus E_2 \oplus E_2^\perp \to E_1 \oplus E_1^\perp \oplus E_2 \oplus E_2^\perp$ such that (M, G) has the following (generalized Schur) decomposition*

$$(M,G) = \left(\begin{bmatrix} M_{11} & M_{11}^{off} & M_{12} & . \\ 0 & M_{11}^{\perp} & . & . \\ 0 & 0 & M_{22} & M_{22}^{off} \\ 0 & 0 & 0 & M_{22}^{\perp} \end{bmatrix}, \begin{bmatrix} G_{11} & G_{11}^{off} & G_{12} & . \\ 0 & G_{11}^{\perp} & . & . \\ 0 & 0 & G_{22} & G_{22}^{off} \\ 0 & 0 & 0 & G_{22}^{\perp} \end{bmatrix} \right)$$

from $E_1 \oplus E_1^{\perp} \oplus E_2 \oplus E_2^{\perp} \to F_1 \oplus F_1^{\perp} \oplus F_2 \oplus F_2^{\perp}$

where $E_1, E_1^{\perp}, E_2, E_2^{\perp}$ *and* $F_1, F_1^{\perp}, F_2, F_2^{\perp}$ *are closed linear subspaces of* $\mathcal{H}_X$ *and* $\mathcal{H}_Y$*, respectively. Further, with respect to the orthonormal bases* $\{\tilde{p}_i^1\}_{i=1}^{\infty}$ *of* E_1 *and* $\{\tilde{q}_i^1\}_{i=1}^{\infty}$ *of* F_1 *generating the rows of* P^1 *and* Q^1*, respectively,* (M_{11}, G_{11}) *are upper triangular with* $(M_{11})_{jj}/(G_{11})_{jj} = \lambda_j$ *where* λ_j *is the* j^{th} *nonzero generalized eigenvalue (in some arbitrary fixed order) repeated a number of times equal to its multiplicity in* $\sigma(M_1, G_1)$*, and similarly with respect to the orthonormal bases* $\{\tilde{p}_i^2\}_{i=1}^{\infty}$ *of* E_2 *and* $\{\tilde{q}_i^2\}_{i=1}^{\infty}$ *of* F_2 *generating the rows of* P^2 *and* Q^2*, respectively,* (M_{22}, G_{22}) *are upper triangular with* $(M_{22})_{jj}/(G_{22})_{jj} = \lambda_j$ *where* λ_j *is the* j^{th} *finite generalized eigenvalue repeated a number of times equal to its multiplicity in* $\sigma(M_2, G_2)$*. In addition,* $\sigma(M_{11}^{\perp}, G_{11}^{\perp}) \subset \{0\}$ *and* $\sigma(M_{22}^{\perp}, G_{22}^{\perp}) \subset \{\infty\}$*.*

Remark. $(G_{11}^{\perp})^{-1} M_{11}^{\perp}$ and $(M_{22}^{\perp})^{-1} G_{22}^{\perp}$ are examples of Volterra operators, as they are compact and quasinilpotent (with spectrum equal to zero only). As a result, they may be shown to be unitarily equivalent to a particular continuous analogue of an upper-triangular operator with respect to a (not necessarily countable) increasing chain of projections on subspaces of $\mathcal{H}_X$ (Gohberg *et al.* , 1993, Thm. XXI.1.5). In principle, a fully triangular representation of (M, G) in which $(M_{11}^{\perp}, G_{11}^{\perp})$ and $(M_{22}^{\perp}, G_{22}^{\perp})$ are also upper-triangular with respect to some chain of subspaces could be generated via an analogue for operator pairs of Gohberg *et al.* (1993, Thm. XXI.1.2). Such a decomposition is unnecessary for our purposes, as block-triangular structure is sufficient for representing a solution of the equilibrium conditions and the approximation techniques to be used do not take advantage of the continuous structure provided by the more intricate decomposition.

Proof. Begin by noting that if $P_{\{\lambda_i\}}$ is a projector onto an eigenspace of Ω_1 corresponding to nonzero eigenvalue λ_i (sorted in arbitrary but fixed order), it is also a projector onto an eigenspace of (M_1, G_1) corresponding to the same eigenvalue. By compactness, any nonzero element of the spectrum of Ω_1 is isolated and an eigenvalue, and by equality of spectra corresponds to an isolated point in the spectrum $\sigma(M_1, G_1)$. As a result, one may write the projector onto the eigenspace associated with λ_i of Ω_1 as $P_{\{\lambda_i\}}^{\Omega_1} := \frac{1}{2\pi\iota} \int_{\Gamma_{\lambda_i}} (\zeta I_1 - \Omega_1)^{-1} d\zeta$, where Γ_{λ_i} is a closed Cauchy curve enclosing λ_i, and the projector onto the space associated

with element λ_i of the spectrum of pair (M_1, G_1) as $P^{(M_1,G_1)}_{\{\lambda_i\}} := \frac{1}{2\pi\iota}\int_{\Gamma_{\lambda_i}}(\zeta G_1 - M_1)^{-1}G_1 d\zeta$ (See Gohberg *et al.* (1990, Ch. I.2 and IV.1)). Since $\Omega_1 = G_1^{-1}M_1$,

$$\begin{aligned} P^{(M_1,G_1)}_{\{\lambda_i\}} = \frac{1}{2\pi\iota}\int_{\Gamma_{\lambda_i}}(\zeta G_1 - M_1)^{-1}G_1 d\zeta &= \frac{1}{2\pi\iota}\int_{\Gamma_{\lambda_i}}((G_1G_1^{-1})(\zeta G_1 - M_1))^{-1}G_1 d\zeta \\ &= \frac{1}{2\pi\iota}\int_{\Gamma_{\lambda_i}}((G_1)(\zeta I_1 - \Omega_1))^{-1}G_1 d\zeta \\ &= \frac{1}{2\pi\iota}\int_{\Gamma_{\lambda_i}}(\zeta I_1 - \Omega_1)^{-1}G_1^{-1}G_1 d\zeta \\ &= P^{\Omega_1}_{\{\lambda_i\}} \end{aligned} \tag{A.3}$$

Compactness also guarantees that the dimension of the image of $P_{\{\lambda_i\}}$ is finite (Gohberg *et al.* , 1990, Thm II.3.2), and so by equality of spectra, the subspaces associated with points not equal to zero in the spectrum of (M_1, G_1) are all finite-dimensional. As a result, we may choose for each i, a finite set of basis vectors, of cardinality k_i, for the space $\mathrm{Im}P_{\{\lambda_i\}}$ and a basis for the image of the pair $(MP_{\{\lambda_i\}}, GP_{\{\lambda_i\}})$ which must be of dimension k_i as $GP_{\{\lambda_i\}}$ must be of full rank since G_1 is. In particular, as on this space the operator pair has a representation as a pair of $k_i \times k_i$-dimensional matrices, we may without loss of generality use orthonormal basis vectors $\{q^1_{i1}, \ldots, q^1_{ik_i}\}$ for the image of $(MP_{\{\lambda_i\}}, GP_{\{\lambda_i\}})$ and $\{p^1_{i1}, \ldots, p^1_{ik_i}\}$ for $\mathrm{Im}P_{\{\lambda_i\}}$ such that with respect to these bases, M and G are upper triangular with diagonal elements of M and G identically equal to σ_i and τ_i, respectively, where $\frac{\sigma_i}{\tau_i} = \lambda_i$. Such a representation exists by the generalized Schur decomposition for finite-dimensional matrix pairs (Stewart & Sun, 1990, Th. VI.1.9). Note that while these basis vectors are orthogonal within each block, in general $\mathrm{Im}P_{\{\lambda_i\}}$ is not necessarily orthogonal to $\mathrm{Im}P_{\{\lambda_j\}}$ for $i \neq j$ as these are oblique, not orthogonal projections.

For Ker π_1, compactness of Ω_2 permits an analogous construction of a countable sequence of finite-dimensional eigenprojections associated to isolated points of the spectrum, with the difference that the projection onto the space associated with point λ_i in the spectrum $\sigma(M_2, G_2)$ is equal to projection associated with nonzero eigenvalue $\frac{1}{\lambda_i} \in \sigma(\Omega_2)$. That is to say, in the notation above, $P^{(M_2,G_2)}_{\{\lambda_i\}} = P^{\Omega_2}_{\{\frac{1}{\lambda_i}\}}$ for $\lambda_i \neq \infty$. Since Im $P^{\Omega_2}_{\{\frac{1}{\lambda_i}\}}$ is a finite-dimensional subspace of dimension k_i, and since $MP^{(M_2,G_2)}_{\{\lambda_i\}}$ is full rank since M_2 is, we may define sets of orthonormal basis vectors $\{q^2_{i1}, \ldots, q^2_{ik_i}\}$ on the image of $(MP^{(M_2,G_2)}_{\{\lambda_i\}}, GP^{(M_2,G_2)}_{\{\lambda_i\}})$ and $\{p^2_{i1}, \ldots, p^2_{ik_i}\}$ on $\mathrm{Im}P^{(M_2,G_2)}_{\{\lambda_i\}}$ such that with respect to these basis vectors, (M, G) has a representation as a pair of $k_i \times k_i$ upper-triangular Schur matrices with diagonal elements identically equal to the corresponding eigenvalue pair $\{\sigma_i, \tau_i\}$ where $\frac{\sigma_i}{\tau_i} = \lambda_i$.

For the space Im $\pi_1 \backslash \overline{\mathrm{Span}}\{p^1_{11}, \ldots, p^1_{1k_1}, p^1_{21,\ldots}\}$, choose choose an arbitrary complete or-

thonormal basis, say $\{p_1^{1\perp}, p_2^{1\perp}, \ldots\}$ and for Im $\pi_2 \backslash \overline{\text{Span}}\{q_{11}^1, \ldots, q_{1k_1}^1, q_{21,\ldots}^1\}$, choose a basis $\{q_1^{1\perp}, q_2^{1\perp}, \ldots\}$. Likewise, for the space Ker $\pi_1 \backslash \overline{\text{Span}}\{p_{11}^2, \ldots, p_{1k_1}^2, p_{21,\ldots}^2\}$, choose choose an arbitrary complete orthonormal basis, say $\{p_1^{2\perp}, p_2^{2\perp}, \ldots\}$ and for Ker $\pi_2 \backslash \overline{\text{Span}}\{q_{11}^2, \ldots, q_{1k_1}^2, q_{21,\ldots}^2\}$ choose a basis $\{q_1^{2\perp}, q_2^{2\perp}, \ldots\}$. These bases may in general be infinite dimensional and are not necessarily orthogonal to the bases defined for other spaces. To produce the stated decomposition, these bases will be used to construct an orthogonal basis with the desired properties.

To produce the desired decomposition, order the sets of vectors as $(\{p_{11}^1, \ldots, p_{1k_1}^1\}, \{p_{21}^1, \ldots, p_{2k_2}^1\}, \ldots, \{p_1^{1\perp}, p_2^{1\perp}, \ldots\}, \ldots, \{p_{11}^2, \ldots, p_{1k_1}^2\}, \{p_{21}^2, \ldots, p_{2k_2}^2\}, \ldots, \{p_1^{2\perp}, p_2^{2\perp}, \ldots\}, \ldots)$ and $(\{q_{11}^1, \ldots, q_{1k_1}^1\}, \{q_{21}^1, \ldots, q_{2k_2}^1\}, \ldots, \{q_1^{1\perp}, q_2^{1\perp}, \ldots\}, \ldots, \{q_{11}^2, \ldots, q_{1k_1}^2\}, \{q_{21}^2, \ldots, q_{2k_2}^2\}, \ldots, \{q_1^{2\perp}, q_2^{2\perp}, \ldots\}$ and apply Gram-Schmidt orthonormalization to the countable sequences to produce a pair of orthonormal bases $\{\tilde{p}_{11}^1, \ldots, \tilde{p}_{1k_1}^1, \tilde{p}_{21,}^1 \ldots, \tilde{p}_1^{1\perp}, \ldots, \tilde{p}_{11}^2, \ldots, \tilde{p}_{1k_1}^2, \tilde{p}_{21,}^2 \ldots, \tilde{p}_1^{2\perp}, \ldots\}$ and $\{\tilde{q}_{11}^1, \ldots, \tilde{q}_{1k_1}^1, \tilde{q}_{21,}^1 \ldots, \tilde{q}_1^{1\perp}, \ldots, \tilde{q}_{11}^2, \ldots, \tilde{q}_{1k_1}^2, \tilde{q}_{21,}^2 \ldots, \tilde{q}_1^{2\perp}, \ldots\}$ of $\mathcal{H}_X$ and $\mathcal{H}_Y$ respectively. We may then define $E_1 = \overline{\text{Span}}\{\tilde{p}_{11}^1, \ldots, \tilde{p}_{1k_1}^1, \tilde{p}_{21,}^1 \ldots\}$, $F_1 = \overline{\text{Span}}\{\tilde{q}_{11}^1, \ldots, \tilde{q}_{1k_1}^1, \tilde{q}_{21,}^1 \ldots\}$, $E_1^\perp = \overline{\text{Span}}\{\tilde{p}_1^{1\perp}, \tilde{p}_2^{1\perp}, \ldots\}$, $F_1^\perp = \overline{\text{Span}}\{\tilde{q}_1^{1\perp}, \tilde{q}_2^{1\perp}, \ldots\}$, $E_2 = \overline{\text{Span}}\{\tilde{p}_{11}^2, \ldots, \tilde{p}_{1k_1}^2, \tilde{p}_{21,}^2 \ldots\}$, $F_2 = \overline{\text{Span}}\{\tilde{q}_{11}^2, \ldots, \tilde{q}_{1k_1}^2, \tilde{q}_{21,}^2 \ldots\}$ $E_2^\perp = \overline{\text{Span}}\{\tilde{p}_1^{2\perp}, \tilde{p}_2^{2\perp}, \ldots\}$, and $F_2^\perp = \overline{\text{Span}}\{\tilde{q}_1^{2\perp}, \tilde{q}_2^{2\perp}, \ldots\}$, and decompose (M, G) into its restrictions to these spaces. We may define P and Q as the unitary operators whose rows are given by the basis vectors. That is, let $P^1 = \sum_{i,j=1}^\infty \langle \tilde{p}_{ij}^1, . \rangle \, \tilde{p}_{ij}^1$, $Q^1 = \sum_{i,j=1}^\infty \langle \tilde{q}_{ij}^1, . \rangle \, \tilde{q}_{ij}^1$, $P^{1\perp} = \sum_{i=1}^\infty \langle \tilde{p}_i^{1\perp}, . \rangle \, \tilde{p}_i^{1\perp}$, $Q^{1\perp} = \sum_{i=1}^\infty \langle \tilde{q}_i^{1\perp}, . \rangle \, q_i^{1\perp}$, $P^2 = \sum_{i,j=1}^\infty \langle \tilde{p}_{ij}^2, . \rangle \, \tilde{p}_{ij}^2$, $Q^2 = \sum_{i,j=1}^\infty \langle \tilde{q}_{ij}^2, . \rangle \, \tilde{q}_{ij}^2$, $P^{2\perp} = \sum_{i=1}^\infty \langle \tilde{p}_i^{2\perp}, . \rangle \, \tilde{p}_i^{2\perp}$, and $Q^{2\perp} = \sum_{i=1}^\infty \langle \tilde{q}_i^{2\perp}, . \rangle \, q_i^{2\perp}$.

I claim that with respect to these bases, (M_{11}, G_{11}), has the desired properties. The proof of this fact follows by induction. Denote $P_m = \sum_{i,j=1}^m \langle \tilde{p}_{ij}^1, . \rangle \, \tilde{p}_{ij}^1$ and $Q_m = \sum_{i,j=1}^m \langle \tilde{q}_{ij}^1, . \rangle \, \tilde{q}_{ij}^1$. To show (M_{11}, G_{11}) are upper-triangular with respect to this basis, it suffices to show $(I - Q_s)MP_s = (I - Q_s)GP_s = 0$ for all $s \in \mathbb{N}$. It then also follows that $(I - Q^1)MP^1 = (I - Q^1)GP^1 = 0$, and so the $(2,1)$, $(3,1)$ and $(4,1)$ elements of M and G are indeed 0 as claimed. To see this, note that by definition of a closed span, for any $x \in E_1$, for all $\delta > 0$, $\exists s$ such that $\|P_s x - x\| < \delta$. Since M and G are continuous, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\|z\| < \delta$ implies $\|Mz\| < \epsilon$, $\|Gz\| < \epsilon$, and so for any $x \in \mathcal{H}_X$, $\exists s \in \mathbb{N}$ s.t. $\|(I - Q^1)MP^1 x\| = \|(I - Q^1)MP_s x\| + \|(I - Q^1)M(P^1 - P_s)x\| < \epsilon$ and similarly $\|(I - Q^1)GP^1 x\| < \epsilon$.

Begin by showing that the first step of the induction chain holds. By construction of the generalized Schur decomposition for the finite-dimensional matrix pair, $\tilde{q}_{11}^1 = q_{11}^1 = \frac{1}{\|G\tilde{p}_{11}^1\|} G\tilde{p}_{11}^1$ and so $(I - Q_1)GP_1 = 0$ and likewise, since $\tilde{p}_{11}^1$ satisfies $M\tilde{p}_{11}^1 = \lambda_1 G\tilde{p}_{11}^1 = \lambda_1 \|G\tilde{p}_{11}^1\| \, \tilde{q}_{11}^1$, $(I - Q_1)MP_1 = 0$. Next, for arbitrary index $s = k \times \ell$ assume the inductive hypothesis $(I - Q_{s-1})MP_{s-1} = (I - Q_{s-1})GP_{s-1} = 0$. By the Gram-Schmidt process, $\tilde{p}_s^1 = \frac{1}{\|(I - P_{s-1})p_s^1\|}(I - P_{s-1})p_s^1$. Since p_s^1 is a generalized Schur vector of a finite-dimensional

matrix pair,

$$q_s^1 = \frac{1}{\|(I - \sum_{j=1}^{\ell-1} \langle q_{k,j}^1, . \rangle q_{k,j}^1) G p_s^1\|} (I - \sum_{j=1}^{\ell-1} \langle q_{k,j}^1, . \rangle q_{k,j}^1) G p_s^1 \ (*)$$

and

$$(I - \sum_{j=1}^{\ell-1} \langle q_{k,j}^1, . \rangle q_{k,j}^1) M p_s^1 = \lambda_k (I - \sum_{j=1}^{\ell-1} \langle q_{k,j}^1, . \rangle q_{k,j}^1) G p_s^1 \ (**)$$

, or, in words, p_s^1 is a generalized eigenvector of the matrix pair on the space orthogonal to previous generalized Schur vectors within the block. Now consider $(I - Q_s)G\tilde{p}_s^1 = \frac{1}{\|(I-P_{s-1})p_s^1\|}(I - Q_s)G(I - P_{s-1})p_s^1 = \frac{1}{\|(I-P_{s-1})p_s^1\|}(I - Q_s)Gp_s^1$ by the inductive hypothesis. By $(*)$, $Gp_s^1 \in \text{span}\{q_{k,1}^1, \ldots, q_s^1\} \subset \text{span}\{\tilde{q}_1^1, \ldots, \tilde{q}_s^1\}$ so $(I - Q_s)G\tilde{p}_s^1 = 0$, and since by the inductive hypothesis $(I - Q_s)G\tilde{p}_m^1 = 0$ for $m < s$, $(I - Q_s)GP_s = 0$. Similarly, by $(**)$ and the inductive hypothesis, $(I - Q_s)M\tilde{p}_s^1 = 0$, so it is also the case that $(I - Q_s)MP_s = 0$. By induction, $(I - Q_s)MP_s = (I - Q_s)GP_s = 0$ for all $s \in \mathbb{N}$.

To show that diagonals of (M_{11}, G_{11}) are the generalized eigenvalues, note that the s^{th} diagonal elements with respect to this basis are given by $\langle M\tilde{p}_s^1, \tilde{q}_s^1 \rangle$ and $\langle G\tilde{p}_s^1, \tilde{q}_s^1 \rangle$. Since $\tilde{p}_s^1 = \frac{1}{\|(I-P_{s-1})p_s^1\|}(I - P_{s-1})p_s^1$, $\tilde{q}_s^1 = \frac{1}{\|(I-Q_{s-1})q_s^1\|}(I - Q_{s-1})q_s^1$, $(I - Q_{s-1})M(I - P_{s-1}) = (I - Q_{s-1})M$ by triangularity, and Q_{s-1} is idempotent and self-adjoint since it is an orthogonal projection, $\langle M\tilde{p}_s^1, \tilde{q}_s^1 \rangle = \frac{1}{\|(I-P_{s-1})p_s^1\|} \langle Mp_s^1, \tilde{q}_s^1 \rangle$, and similarly $\langle G\tilde{p}_s^1, \tilde{q}_s^1 \rangle = \frac{1}{\|(I-P_{s-1})p_s^1\|} \langle Gp_s^1, \tilde{q}_s^1 \rangle$. By the finite-dimensional generalized Schur decomposition, $(**)$ holds, and so $\langle M\tilde{p}_s^1, \tilde{q}_s^1 \rangle / \langle G\tilde{p}_s^1, \tilde{q}_s^1 \rangle = \lambda_k$, and so (M_{11}, G_{11}) has the generalized eigenvalues along the diagonals as desired.

To demonstrate that the $(3, 2)$ and $(4, 2)$ blocks of (M, G) are equal to 0 is equivalent to requiring that $(I - [Q^1, Q^{1\perp}])M[P^1, P^{1\perp}] = (I - [Q^1, Q^{1\perp}])G[P^1, P^{1\perp}] = 0$. Because $(\{p_{11}^1, \ldots, p_{1k_1}^1\}, \{p_{21}^1, \ldots, p_{2k_2}^1\}, \ldots, \{p_1^{1\perp}, p_2^{1\perp}, \ldots\}, \ldots)$ span Im π_1 and $(\{q_{11}^1, \ldots, q_{1k_1}^1\}, \{q_{21}^1, \ldots, q_{2k_2}^1\}, \ldots, \{q_1^{1\perp}, q_2^{1\perp}, \ldots\}, \ldots)$ span Im π_2, we have by A.1 that Im $M[P^1, P^{1\perp}] \subset$ Im $\pi_2 =$ Im $[Q^1, Q^{1\perp}]$ and Im $G[P^1, P^{1\perp}] \subset$ Im $\pi_2 =$ Im $[Q^1, Q^{1\perp}]$ so $M[P^1, P^{1\perp}] = [Q^1, Q^{1\perp}]M[P^1, P^{1\perp}]$ and $G[P^1, P^{1\perp}] = [Q^1, Q^{1\perp}]G[P^1, P^{1\perp}]$ so orthogonality holds.

The proof of the upper-triangular structure of (M_{22}, G_{22}) proceeds similarly to the above, by induction. Denote $P_m^2 = \sum_{i,j=1}^m \langle \tilde{p}_{ij}^2, . \rangle \tilde{p}_{ij}^2$ and $Q_m^2 = \sum_{i,j=1}^m \langle \tilde{q}_{ij}^2, . \rangle \tilde{q}_{ij}^2$. Further, denote $Q_m^u = [Q^1, Q^{1\perp}, Q_m^2]$ the projection onto the set of basis vectors of $\mathcal{H}_Y$ up to $\tilde{q}_m^2$ and similarly $P_m^u = [P^1, P^{1\perp}, P_m^2]$. To show (A_{22}, B_{22}) are upper-triangular with respect to this basis, it suffices to show $(I - Q_s^u)MP_s^2 = (I - Q_s^u)GP_s^2 = 0$ for all $s \in \mathbb{N}$. It then also follows by analogous $\delta - \epsilon$ argument that $(I - [Q^1, Q^{1\perp}, Q^2])MP^2 = (I - [Q^1, Q^{1\perp}, Q^2])GP^2 = 0$, and so the $(4, 3)$ elements of M and G are 0 as claimed. The proof is essentially identical to that for (M_{11}, G_{11}) except that all vectors are orthogonalized with respect to previous basis

vectors, and the generalized Schur form of each matrix pair constructs q_{ij}^2 from M instead of G, as on Ker π_1 the spectrum excludes 0 and so M_2 is guaranteed to be invertible while G_2 is not.

Begin by showing the first step of the induction for (M_{22}, G_{22}). By construction of the generalized Schur decomposition for the finite-dimensional matrix pair, $\tilde{q}_{11}^2 = (I - [Q^1, Q^{1\perp}])q_{11}^2 = \frac{(I-[Q^1,Q^{1\perp}])}{\|Mp_{11}^2\|} Mp_{11}^2$ while $\tilde{p}_{11}^2 = (I - [P^1, P^{1\perp}])p_{11}^2$. As shown above, $(I - [Q^1, Q^{1\perp}])M[P^1, P^{1\perp}] = 0$ and so $\tilde{q}_{11}^2 = \frac{(I-[Q^1,Q^{1\perp}])}{\|Mp_{11}^2\|} M\tilde{p}_{11}^2$ and so $(I - Q_1^u)MP_1^u = 0$. Likewise, since $\tilde{p}_{11}^2$ satisfies $(I - [Q^1, Q^{1\perp}])Gp_{11}^2 = \frac{1}{\lambda}(I - [Q^1, Q^{1\perp}])Mp_{11}^2 = \frac{1}{\lambda}\|Mp_{11}^2\|\,\tilde{q}_{11}^2$, $(I - Q_1^u)BP_1^u = 0$ also. This shows that the first step of the induction holds: the continuation proceeds as for (M_{11}, G_{11}) except switching the order of M and G. Similarly, the presence of the eigenvalues along the diagonals is shown in a completely analogous manner.

It remains to show that $(M_{11}^\perp, G_{11}^\perp)$ satisfies $\sigma(M_{11}^\perp, G_{11}^\perp) \subset \{0\}$. In this, I follow Gohberg *et al.* (1990, Lemma II.3.4) closely. By construction, $(M_{11}^\perp z, G_{11}^\perp z) = (Q_1^\perp M_1 z, Q_1^\perp G_1 z)$ for all $z \in E_1^\perp$. By assumption, G is a bounded operator, so G_1 must be also and so $G_1^{-1}Q_1^\perp G_1$ must be as well. Since the compact operators are a closed ideal within the algebra of bounded operators on a Banach space (see, e.g. (Carl & Stephani, 1990)) and $G_1^{-1}M_1$ is compact by assumption, $G_1^{-1}Q_1^\perp M_1 = G_1^{-1}Q_1^\perp G_1 G_1^{-1}M_1$ is compact also, as is $\Omega_1^\perp := P_1^\perp G_1^{-1}Q_1^\perp M_1 P_1^\perp$, its restriction to $E_1^\perp$. Suppose for contradiction that μ is a nonzero element of $\sigma(M_{11}^\perp, G_{11}^\perp)$. Then by reasoning entirely analogous to 3, $\sigma(M_{11}^\perp, G_{11}^\perp) = \sigma(\Omega_1^\perp)$ and so by compactness μ is an isolated point in the spectrum of $\Omega_1^\perp$. Further, $\Omega_1^{\perp *}$ must have $\bar{\mu} \in \sigma(\Omega_1^{\perp *})$ as a nonzero point in the spectrum, and so by compactness, it must be an isolated point in the spectrum associated with (at least one) nonzero eigenvector, which we will call $x_0 \in E_1^\perp$. The upper triangular decomposition of (M_1, G_1) may be used to show $\Omega_1^{\perp *} = P_1^\perp (G_1^{-1}M_1)^* P_1^\perp$. To see this, note that multiplication of the the upper triangular decomposition of M_1 by the inverse of the upper triangular decomposition of G_1 yields

$$(G_1^{-1}M_1)^* = \begin{pmatrix} G_{11}^{-1}M_{11} & -G_{11}^{-1}G_{11}^{off}G_{11}^{\perp -1}M_{11}^{off} \\ 0 & G_{11}^{\perp -1}M_{11}^\perp \end{pmatrix}^* = \begin{pmatrix} (G_{11}^{-1}M_{11})^* & 0 \\ (-G_{11}^{-1}G_{11}^{off}G_{11}^{\perp -1}M_{11}^{off})^* & \Omega_1^{\perp *} \end{pmatrix}$$

and so $\Omega_1^{\perp *} = P_1^\perp (G_1^{-1}M_1)^* P_1^\perp$ as claimed. As a result, x_0 is also an eigenvector of compact operator $(G_1^{-1}M_1)^*$ associated with eigenvalue $\bar{\mu}$, and so $x_0 \in E_1^\perp \cap \text{Im } P_{\{\bar{\mu}\}}^{(G_1^{-1}M_1)^*}$.

However, we know also by A.3 that $\text{Im } P_{\{\mu\}}^{G_1^{-1}M_1} = \text{Im } P_{\{\mu\}}^{(M_1,G_1)} \subset E_1$, and by orthogonality of the decompositions, $E_1^\perp$ is orthogonal to $\text{Im } P_{\{\mu\}}^{G_1^{-1}M_1}$, and so must be a subset of $\text{Ker } (P_{\{\mu\}}^{G_1^{-1}M_1})^*$. Since this is an isolated eigenvalue of an operator on a Hilbert space, Gohberg *et al.* (1990, Prop I.2.5) gives that $(P_{\{\mu\}}^{G_1^{-1}M_1})^* = P_{\{\bar{\mu}\}}^{(G_1^{-1}M_1)^*}$, and so $E_1^\perp \subset \text{Ker } P_{\{\bar{\mu}\}}^{(G_1^{-1}M_1)^*}$. This

contradicts the previous assertion that there is a nonzero element x_0 in $E_1^{\perp} \cap \mathrm{Im}\ P_{\{\bar{\mu}\}}^{(G_1^{-1}M_1)^*}$ and so the original assertion that there is some $\mu \neq 0$ in $\sigma(M_{11}^{\perp}, G_{11}^{\perp})$.

The proof that $(M_{22}^{\perp}, G_{22}^{\perp})$ satisfies $\sigma(M_{22}^{\perp}, G_{22}^{\perp}) \subset \{\infty\}$ is essentially similar to the above, except using $((I - [Q^1, Q^{1\perp}])M(I - [P^1, P^{1\perp}]), (I - [Q^1, Q^{1\perp}])G(I - [P^1, P^{1\perp}]))$ in place of (M_1, G_1) and reversing the order of M and G. □

B Perturbation Theory for the Generalized Schur Decomposition

Perturbation for generalized Schur subspaces associated with a subset of the spectrum is covered in Stewart (1973) for perturbations measured in Frobenius norm. In this section, I extend the results to perturbation in operator norm. In addition to bounds on the error in terms of the subspace angle between the approximate and true deflating subspaces, this section will also consider approximation of the Rayleigh components of the operator pair corresponding to these subspaces. First, set up the generalized Schur subspace approximation problem exactly as in Stewart (1973).

Let $(A, B) \in \mathcal{L}(\mathcal{H}_1 \to \mathcal{H}_2, \mathcal{H}_1 \to \mathcal{H}_2)$ and unitary operators $X = (X_1, X_2)\ \mathcal{H}_1 \to \mathcal{H}_1$ and $Y = (Y_1, Y_2)\ \mathcal{H}_2 \to \mathcal{H}_2$ decompose (A, B) as

$$(Y^*AX, Y^*BX) = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \right)$$

To find a perturbation bound, we search for the minimal rotations

$$U_X = \begin{pmatrix} I & -P^* \\ P & I \end{pmatrix} \begin{pmatrix} (I + P^*P)^{-1/2} & 0 \\ 0 & (I + PP^*)^{-1/2} \end{pmatrix}$$

$$U_Y = \begin{pmatrix} I & -Q^* \\ Q & I \end{pmatrix} \begin{pmatrix} (I + Q^*Q)^{-1/2} & 0 \\ 0 & (I + QQ^*)^{-1/2} \end{pmatrix}$$

such that $X' = (X_1', X_2') = XU_X$ and $Y' = (Y_1', Y_2') = YU_Y$ generate subspaces $\mathscr{R}(X_1') = \mathscr{X} \subset \mathcal{H}_1$ and $\mathscr{R}(Y_1') = \mathscr{Y} \subset \mathcal{H}_2$ which form a deflating pair of (A, B). A pair of subspaces $\mathscr{X}, \mathscr{Y}$ form a deflating pair if and only if $(A_{21}', B_{21}') = (0, 0)$. This is equivalent to

$$\begin{aligned} QA_{11} - A_{22}P &= A_{21} - QA_{12}P \\ QB_{11} - B_{22}P &= B_{21} - QB_{12}P \end{aligned} \tag{B.1}$$

In order to find (Q, P) which satisfy the above condition and are small relative to perturbations in operator norm, define a norm over the space of operator pairs over subspaces conformable to the pair (Q, P) as the largest operator norm of an operator in the pair, i.e.

$$\|(Q, P)\|_{\mathcal{B}} = \max(\|Q\|, \|P\|)$$

If we can show that the conditions of Stewart (1973) Theorem 3.1 are satisfied for B.1 using this norm, then this theorem will provide a bound on the operator norm of the rotation needed to generate such a decomposition. Define

$$T(Q, P) = \begin{pmatrix} QA_{11} - A_{22}P & QB_{11} - B_{22}P \end{pmatrix}$$

$$g = \begin{pmatrix} A_{21} & B_{21} \end{pmatrix}$$

$$\varphi(Q, P) = \begin{pmatrix} QA_{12}P & QB_{12}P \end{pmatrix}$$

To show a quadratic bound for $\varphi(Q, P)$, begin with the first term:

$$\begin{aligned} \|\varphi_1(Q, P)\| &\leq \|Q\| \, \|P\| \, \|A_{12}\| \\ &\leq \|(Q, P)\|_{\mathcal{B}}^2 \, \|A_{12}\| \end{aligned}$$

Combining with identical calculations for the second term yields quadratic bound

$$\|\varphi(Q, P)\|_{\mathcal{B}} \leq \eta \|(Q, P)\|_{\mathcal{B}}^2 \tag{B.2}$$

where

$$\eta = \|(A_{12}, B_{12})\|_{\mathcal{B}}$$

To demonstrate the Lipschitz property for this operator, again note

$$\begin{aligned} \|\varphi_1(Q, P) - \varphi_1(\tilde{Q}, \tilde{P})\| &\leq \left\|Q - \tilde{Q}\right\| \|P\| \, \|A_{12}\| + \left\|\tilde{Q}\right\| \left\|P - \tilde{P}\right\| \|A_{12}\| \\ &\leq 2 \max(\|(Q, P)\|_{\mathcal{B}}, \|(\tilde{Q}, \tilde{P})\|_{\mathcal{B}}) \|(Q - \tilde{Q}, P - \tilde{P})\|_{\mathcal{B}} \, \|A_{12}\| \end{aligned}$$

Combining with identical calculations for the second term gives Lipschitz condition

$$\|\varphi(Q, P) - \varphi(\tilde{Q}, \tilde{P})\|_B \leq 2\eta \max(\|(Q, P)\|_{\mathcal{B}}, \|(\tilde{Q}, \tilde{P})\|_{\mathcal{B}}) \|(Q - \tilde{Q}, P - \tilde{P})\|_{\mathcal{B}} \tag{B.3}$$

These demonstrate that conditions (i) and (ii) of Theorem 3.1 in Stewart (1973) continue to hold for the norm $\|.\|_{\mathcal{B}}$

Again defining

$$\gamma = \|g\|_{\mathcal{B}}$$

$$\delta = \|T^{-1}\|_{\mathcal{B}}^{-1}$$

one obtains

Lemma 6. *Suppose $T(Q,P) = g - \varphi(Q,P)$ with T, g, and φ defined as above, where φ satisfies the quadratic bound and Lipschitz conditions. Let $\delta > 0$ and $\gamma\eta/\delta^2 < 1/4$. Then*

$$\| \begin{pmatrix} Q, & P \end{pmatrix} \|_{\mathcal{B}} < 2\frac{\gamma}{\delta}$$

To determine precisely how the above theorem imposes bounds on errors in Schur subspaces, it is necessary to examine the stability properties of the term δ. Define

$$\text{dif}(A,B) = \text{dif}(\begin{matrix} A_{11} & B_{11} \\ A_{22} & B_{22} \end{matrix}) = \|T^{-1}\|_{\mathcal{B}}^{-1} \tag{B.4}$$

Note that this operator depends on only the block diagonal terms of the pair (A,B). Define the perturbation $(E,F) \in \mathcal{L}(\mathcal{H}_1 \to \mathcal{H}_2, \mathcal{H}_1 \to \mathcal{H}_2)$ and define a partition of the operator conformable with that of (A,B) by $(E_{ij}, F_{ij}) = (Y_i^H E X_j, Y_i^H F X_j)$. We would like to define a bound on the term

$$\text{dif}(A+E, B+F) = \text{dif}(\begin{matrix} A_{11}+E_{11} & B_{11}+F_{11} \\ A_{22}+F_{22} & B_{22}+F_{22} \end{matrix})$$

Using the alternate characterization $\text{dif}(A,B) = \inf_{\|Z\|_{\mathcal{B}}=1} \|T(Z)\|_{\mathcal{B}}$ where $Z \in \mathcal{B}$, one can derive lower and upper bounds

$$\text{dif}(A,B) + \nu(E,F) \geq \text{dif}(A+E, B+F) \geq \text{dif}(A,B) - \nu(E,F)$$

where

$$\nu(E,F) = \max(\|E_{11}\| + \|E_{22}\|, \|F_{11}\| + \|F_{22}\|)$$

Combing this bound with the previous lemma, obtain

Theorem 3. *Let (A, B) and $(E, F) \in \mathcal{L}(\mathcal{H}_1 \to \mathcal{H}_2, \mathcal{H}_1 \to \mathcal{H}_2)$ and $X = (X_1, X_2)$ $\mathcal{H}_1 \to \mathcal{H}_1$ and $Y = (Y_1, Y_2)$ $\mathcal{H}_2 \to \mathcal{H}_2$ be unitary operators such that $\mathscr{R}(X_1)$ and $\mathscr{R}(Y_1)$ form a deflating pair of subspaces for the operator pair (A, B). Suppose these operators partition the pairs such that*

$$(Y^H A X, Y^H B X) = \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \right)$$

$$(Y^H E X, Y^H F X) = \left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \right)$$

Define

$$\delta = \mathit{dif}(A, B) - \nu(E, F)$$

along with

$$\gamma = \| \begin{pmatrix} E_{21} & F_{21} \end{pmatrix} \|_{\mathcal{B}}$$

and

$$\eta = \|(A_{12} + E_{12}, B_{12} + F_{12})\|_{\mathcal{B}}$$

Suppose $\delta > 0$ and $\gamma\eta/\delta^2 < 1/4$. Then
Then there is a pair of operators (Q, P) with

$$\left\| \begin{pmatrix} Q, & P \end{pmatrix} \right\|_{\mathcal{B}} \leq \frac{2\gamma}{\delta}$$

such that
$X_1' = (X_1 + X_2 P)(I + P^* P)^{-1/2}$
$Y_1' = (Y_1 + Y_2 Q)(I + Q^* Q)^{-1/2}$
and $\mathscr{R}(X_1')$ and $\mathscr{R}(Y_1')$ form a pair of deflating subspaces for $(A + E, B + F)$.

This is essentially identical to Theorem 5.7 of Stewart (1973) aside from the definition of the norms via which the terms are defined and the resulting difference in the lower bound on δ.

Via Theorem 2.7 in Stewart (1973), we know that

$$\| \sin \Theta(\mathscr{R}(X_1), \mathscr{R}(X_1')) \| \leq \| \tan \Theta(\mathscr{R}(X_1), \mathscr{R}(X_1')) \| = \|P\|$$

$$\| \sin \Theta(\mathscr{R}(Y_1), \mathscr{R}(Y_1')) \| \leq \| \tan \Theta(\mathscr{R}(Y_1), \mathscr{R}(Y_1')) \| = \|Q\|$$

both of which are less than $\left\| \begin{pmatrix} Q, & P \end{pmatrix} \right\|_{\mathcal{B}}$. As a result, we have the following corollary

Corollary 1. *Suppose (A, B), (E, F), X and Y satisfy the conditions of the theorem above.*

Then the operator pair $(A+E, B+F)$ *has a right generalized Schur subspace* $\mathscr{R}(X_1')$ *such that* $\|Proj_{X_1'} - Proj_{X_1}\|_2 \leq \frac{2\gamma}{\delta}$ *and associated left generalized Schur subspace* $\mathscr{R}(Y_1')$ *such that* $\|Proj_{Y_1'} - Proj_{Y_1}\|_2 \leq \frac{2\gamma}{\delta}$

As a result, for appropriately small approximation error in the operator pair of interest, a fixed, well-separated, primary generalized Schur subspace (and associated generalized Schur functions or vectors whose range spans it) of the perturbed pair differs by an amount which is on the order of the operator norm of the perturbation from the corresponding true subspace (and associated functions). This dependence on the order of the operator norm of the error may be particularly useful in the case of large or infinite-dimensional subspaces, for which the Frobenius norm of the error may increase as the square root of the dimension of the subspace. One loses, however, the set of sharp characterizations of the difference term δ in terms of spectral properties of the operator to be approximated which may be obtained when it is defined via the Frobenius norm. This seems necessary in general, however, as the Frobenius or Hilbert-Schmidt norm may fail to be finite in the infinite-dimensional case for otherwise well-behaved operators.

To bound the approximation error in the components (A_{11}, B_{11}) induced by an approximation, it is helpful to introduce an additional pair of subspaces to correspond to the right deflating pair $\mathscr{R}(X_1)$ and $\mathscr{R}(Y_1)$. Defining (X_1, X_2) and (Y_1, Y_2) as above so $\mathscr{R}(X_1)$ and $\mathscr{R}(Y_1)$ form a deflating pair, we look for operators V_1 and U_2 and R and S with $V_1 = Y_1 + Y_2 R^*$ and $U_2 = X_2 - X_1 S$ to solve

$$
\begin{aligned}
(V_1, Y_2)^* A (X_1, U_2) &= \begin{pmatrix} I & R \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \\
(V_1, Y_2)^* B (X_1, U_2) &= \begin{pmatrix} I & R \\ 0 & I \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix} = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \quad \text{(B.5)}
\end{aligned}
$$

This holds if there exist S, R such that

$$
\begin{aligned}
A_{11} S - R A_{22} &= A_{12} \\
B_{11} S - R B_{22} &= B_{12}
\end{aligned}
$$

Theorem 5.9 in Stewart (1973) notes that if T is nonsingular, there exist S and R which solve this equation, and so (X_1, U_2) and (V_1, Y_2), which are not in general unitary, though are nonsingular, block diagonalize (A, B). Further, by the definitions of V_1 and U_2, one has $\|V_1\| = \|\sec\Theta(\mathscr{R}(V_1), \mathscr{R}(Y_1)\| = \|\sec\Theta_1\|$ and $\|U_2\| = \|\sec\Theta(\mathscr{R}(U_2), \mathscr{R}(X_2)\| = \|\sec\Theta_2\|$.

This block diagonalization can be used along with the perturbation formula to construct

bounds on the approximation error in (A_{11}, B_{11}). Consider a perturbation (E, F) of (A, B) and define

$$((V_1, Y_2)^* E(X_1, U_2), (V_1, Y_2)^* F(X_1, U_2)) = \left(\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}, \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \right)$$

so that perturbed operator pair satisfies

$$((V_1, Y_2)^*(A+E)(X_1, U_2), (V_1, Y_2)^*(B+F)(X_1, U_2)) = \left(\begin{pmatrix} A_{11}+E_{11} & E_{12} \\ E_{21} & A_{22}+E_{22} \end{pmatrix}, \begin{pmatrix} B_{11}+F_{11} & F_{12} \\ F_{21} & B_{22}+F_{22} \end{pmatrix} \right) \quad \text{(B.6)}$$

then, following Stewart & Sun (1990) VI.2.15, we have

Theorem 4. *Define*

$$\delta = dif(A, B) - \max(\|E_{11}\| + \|E_{22}\|, \|F_{11}\| + \|F_{22}\|)$$

along with

$$\gamma = \| \begin{pmatrix} E_{21} & F_{21} \end{pmatrix} \|_{\mathcal{B}}$$

and

$$\eta = \|(E_{12}, F_{12})\|_{\mathcal{B}}$$

Suppose $\delta > 0$ *and* $\gamma\eta/\delta^2 < 1/4$.
Then there is a pair of operators (Q, P) *with*

$$\left\| \begin{pmatrix} Q, & P \end{pmatrix} \right\|_{\mathcal{B}} \leq \frac{2\gamma}{\delta}$$

such that

$X_1' = X_1 + U_2 P$
$Y_2' = Y_2 + V_1 Q^*$
satisfy

$$((V_1, Y_2')^*(A+E)(X_1', U_2), (V_1, Y_2')^*(B+F)(X_1', U_2)) = \left(\begin{pmatrix} A_{11}+E_{11}+E_{12}P & E_{12} \\ 0 & A_{22}+E_{22}+QE_{12} \end{pmatrix}, \begin{pmatrix} B_{11}+F_{11}+F_{12}P & F_{12} \\ 0 & B_{22}+F_{22}+QF_{12} \end{pmatrix} \right) \quad \text{(B.7)}$$

and so (A'_{11}, B'_{11}) =$(A_{11} + E_{11} + E_{12}P, B_{11} + F_{11} + F_{12}P)$ *form the generalized Rayleigh quotients of the perturbed operator pair, and as a result, we have*

$$\|A_{11} - A'_{11}\| \leq \|E_{11} + E_{12}P\| \leq \|E_{11}\| + \|E_{12}\|\frac{2\gamma}{\delta}$$

$$\|B_{11} - B'_{11}\| \leq \|F_{11} + F_{12}P\| \leq \|F_{11}\| + \|F_{12}\|\frac{2\gamma}{\delta}$$

Proof. Existence of a unique solution (Q, P) with the specified properties follows if there exist (Q, P) such that left multiplying (B.6) by $\begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}$ and right multiplying by $\begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$ sets the lower left elements in (B.7) to 0. This holds if there is unique solution to

$$\begin{pmatrix} Q(A_{11} + E_{11}) + (A_{22} + E_{22})P \\ Q(B_{11} + F_{11}) + (B_{22} + F_{22})P \end{pmatrix} = \begin{pmatrix} E_{21} \\ F_{21} \end{pmatrix} + \begin{pmatrix} QE_{12}P \\ QF_{12}P \end{pmatrix}$$

Existence of a unique solution here follows from application of Theorem 3.1 in Stewart (1973), the Lipschitz and norm bound shown for the quadratic component above, and the lower bound on δ which lower bounds the minimum singular value of the lefthand side. □

C Proofs

C.1 Section 2 Proof

Proof. of (1). Suppose $h(x, z) := h(x, \sigma)+\sigma z$ is a measurable function from $(\mathcal{B}_x \times \mathcal{B}_z, \Sigma_x \otimes \Sigma_z)$, the product space of $\mathcal{B}_x \times \mathcal{B}_z$ equipped with a product sigma field, to $(\mathcal{B}_x, \Sigma_x)$. We want conditions on the space, the function, and the sigma fields such that it induces a measurable stochastic process on the product space of $\mathcal{B}_x$. We may assume z is drawn independently of x according to measure μ^z on $(\mathcal{B}_z, \Sigma_z)$, and may ask for the initial distribution of x to be given by μ^x. For each x, we can define the pushforward measure on $(\mathcal{B}_x, \Sigma_x)$ by $\mu_x^{x'}(f(x')) := \mu^z(f(h(x, \sigma) + \sigma z)$ for any $f \in \mathcal{M}^+(\mathcal{B}_x, \Sigma_x, \bar{\mathbb{R}}, \mathscr{B}(\bar{\mathbb{R}}))$ nonnegative measurable functions from x to the real line equipped with the Borel sigma field. If the family $(\mu_x^{x'})_x$ of measures satisfies $x \to \mu_x^{x'}(A)$ is a measurable function from $(\mathcal{B}_x, \Sigma_x) \to (\bar{\mathbb{R}}, \mathscr{B}(\bar{\mathbb{R}}))$ for any $A \in \Sigma_x$, then this is a probability kernel and by, e.g, the Ionescu Tulcea extension theorem, the family induces a measurable stochastic process for x_t on the countable product space $\otimes_{t=1}^{\infty}(\mathcal{B}_x, \Sigma_x)$.

To show measurability of the family of measures $(\mu_x^{x'})_x$, consider a λ-class argument. The measure μ^z maps the class of measurable rectangles $\{x \in A^1, z \in A^2\}$ for $A^1 \in \Sigma_x, A^2 \in \Sigma_z$

to (nonnegative multiples of) indicators of sets Σ_x, which are therefore measurable. The class of measurable rectangles generates the product sigma field $\Sigma_x \otimes \Sigma_z$ and is stable under pairwise intersections. The class of bounded nonnegative functions $f(x,z) \in \mathcal{M}^+(\mathcal{B}_x \times \mathcal{B}_z, \Sigma_x \otimes \Sigma_z, \bar{\mathbb{R}}, \mathscr{B}(\bar{\mathbb{R}}))$ such that $\mu^z f(x,z)$ is $(\mathcal{B}_x, \Sigma_x)$ measurable can be shown to form a λ-cone (Pollard, 2002, 2.11 Def. <43>) and so by these facts (Pollard, 2002, 2.11 Lemma <44>), μ^z maps $\mathcal{M}^+(\mathcal{B}_x \times \mathcal{B}_z, \Sigma_x \otimes \Sigma_z, \bar{\mathbb{R}}, \mathscr{B}(\bar{\mathbb{R}}))$ to $\mathcal{M}^+(\mathcal{B}_x, \Sigma_x, \bar{\mathbb{R}}, \mathscr{B}(\bar{\mathbb{R}}))$. In particular, let $h(x,z)$ be $\mathcal{B}_x \times \mathcal{B}_z, \Sigma_x \otimes \Sigma_z \to \mathcal{B}_x, \Sigma_x$ measurable, then $\mu^z(f(h(x,z))$ is $(\mathcal{B}_x, \Sigma_x)$ measurable for any $f \in \mathcal{M}^+(\mathcal{B}_x, \Sigma_x, \bar{\mathbb{R}}, \mathscr{B}(\bar{\mathbb{R}}))$ and in particular, $x \to \mu_x^{x'}(A)$ is a measurable function from $(\mathcal{B}_x, \Sigma_x) \to (\bar{\mathbb{R}}, \mathscr{B}(\bar{\mathbb{R}}))$ for any $A \in \Sigma_x$. As a result, $(\mu_x^{x'})_x$ is a probability kernel.

To construct a measurable stochastic process, consider the i.i.d. sequence $\{z_t\}_{t=0}^{\infty}$ such that z_t each have identical marginal measure μ_t^z and, beginning with initial measure μ^x, construct the sequence of probability kernels on $\otimes_{t=1}^{\infty}(\mathcal{B}_x, \Sigma_x)$ by iterating the identical kernels defined by $\mu_{xt}^{x'}(f(x')) := \mu_t^z(f(h(x,z))$. This generates a sequence $x_0 \sim \mu_0^x$, $x_t = h(x_{t-1}, z_t)$. By the Ionescu Tulcea extension theorem, the sequence of kernels induces a measurable stochastic process on the countable product space $\otimes_{t=1}^{\infty}(\mathcal{B}_x, \Sigma_x)$ with finite-dimensional distributions generated by the iterated probability kernels. Note that the only assumptions made on $(\mathcal{B}_x, \Sigma_x)$, $(\mathcal{B}_z, \Sigma_z)$ and $h(x,z)$ are that $h(x,z)$ is jointly measurable from the product sigma field over x and z to the sigma field over x. In particular, because the probability kernel was constructed explicitly, no topological assumptions needed to be made on the spaces or sigma fields, as are usually required to invoke the Kolmogorov extension theorem. This permits, among other constructions, the use of nonseparable function spaces or non-Borel sigma fields, which may alleviate some difficulties when working in infinite-dimensional space.

By measurability of $g(x,\sigma)$ and F, the measurability of the probability kernels defining the conditional distribution of the random variables $y_t = g(x_t, \sigma)$ and $F(x_t, g(x_t,\sigma), h(x_t,\sigma) + \sigma z_{t+1}, g(h(x_t,\sigma) + \sigma z_{t+1}, \sigma)$ given x and from there the corresponding stochastic processes can be established in an analogous fashion, ensuring that (x_t, y_t) is product measurable and $\mathbb{E}F(x, g(x,\sigma), h(x,\sigma) + \sigma\eta z', g(h(x,\sigma) + \sigma\eta z', \sigma), \sigma)$ coincides with the conditional expectation of $F(x_t, g(x_t,\sigma), h(x_t,\sigma) + \sigma\eta z_{t+1}, g(h(x_t,\sigma) + \sigma\eta z_{t+1}, \sigma), \sigma)$ at time t given $x_t = x$, as claimed. □

C.2 Section 3 Results

We apply the implicit function theorem to calculate $\frac{d\omega}{d\lambda}$. Taking derivatives of equations (3.3),(3.4), (3.5), and (3.6) evaluated at the uniform steady state, obtain

$$
\begin{aligned}
\frac{dY}{d\lambda} &= \mu[.] \\
\frac{dY}{dw} &= \mu[.] \\
\frac{dT}{d\lambda} &= \left[\frac{\tau(\sigma-1)}{2}\right]^{\frac{1}{1-\sigma}} \frac{1}{1-\sigma}\left[\int_G e^{\tau(1-\sigma)|x-z|}dz\right]^{\frac{\sigma}{1-\sigma}} \int_G [.]e^{\tau(1-\sigma)|x-z|}dz \\
\frac{dT}{dw} &= \left[\frac{\tau(\sigma-1)}{2}\right]^{\frac{1}{1-\sigma}} \frac{1}{1-\sigma}\left[\int_G e^{\tau(1-\sigma)|x-z|}dz\right]^{\frac{\sigma}{1-\sigma}} (1-\sigma)\int_G [.]e^{\tau(1-\sigma)|x-z|}dz \\
\frac{dw}{dY} &= \left[\frac{\tau(\sigma-1)}{2}\right]^{\frac{1}{\sigma}} \frac{1}{\sigma}\left[\int_G e^{-\tau(\sigma-1)|x-z|}dz\right]^{\frac{1-\sigma}{\sigma}} \int_G [.]e^{-\tau(\sigma-1)|x-z|}dz \\
\frac{dw}{dT} &= \left[\frac{\tau(\sigma-1)}{2}\right]^{\frac{1}{\sigma}} \frac{1}{\sigma}\left[\int_G e^{-\tau(\sigma-1)|x-z|}dz\right]^{\frac{1-\sigma}{\sigma}} (\sigma-1)\int_G [.]e^{-\tau(\sigma-1)|x-z|}dz \\
\frac{d\omega}{dw} &= [.] \\
\frac{d\omega}{dT} &= -\mu[.]
\end{aligned}
$$

By the chain rule, we can express the derivative of the real wage with respect to the population distribution as

$$\frac{d\omega}{d\lambda} = \frac{dw}{d\lambda} - \mu(\frac{dT}{dw}\frac{dw}{d\lambda} + \frac{dT}{d\lambda}) \tag{C.1}$$

where by the implicit function theorem in Banach space and the chain rule repeatedly applied,

$$\frac{dw}{d\lambda} = (I - \frac{dw}{dY}\frac{dY}{dw} - \frac{dw}{dT}\frac{dT}{dw})^{-1}(\frac{dw}{dY}\frac{dY}{d\lambda} + \frac{dw}{dT}\frac{dT}{d\lambda}).$$

C.3 Section 4 Proofs

Proof. of Lemma 1 Unitarity of U provides the following facts: since $U^* = U^{-1}$, we have $U^*U = I$. Decomposing U into U_{11}, U_{12}, U_{21}, and U_{22} obtain

$$\begin{bmatrix} U_{11}^*U_{11} + U_{21}^*U_{21} & U_{11}^*U_{12} + U_{21}^*U_{22} \\ U_{12}^*U_{11} + U_{22}^*U_{21} & U_{12}^*U_{12} + U_{22}^*U_{22} \end{bmatrix} = I = \begin{bmatrix} I_x & 0 \\ 0 & I_y \end{bmatrix}$$

Where $I_x = \varphi^{X*}\varphi^X$ and $I_y = \varphi^{Y*}\varphi^Y$ are the identity operators on $\mathcal{H}_x$ and $\mathcal{H}_y$ respectively.

To see this more formally, consider $U_{11}^* U_{12} + U_{21}^* U_{22}$. It can be written as

$$\varphi^{X*} U_1^* U_1 \varphi^Y + \varphi^{X*} U_2^* U_2 \varphi^Y = \varphi^{X*}(U_1^* U_1 + U_2^* U_2)\varphi^Y$$
$$= \varphi^{X*}\varphi^Y = 0$$

Equivalent calculations describe the other identities.

Using these identities we can express

$$\begin{aligned}
(U_{11} + U_{12}g_x)^*(U_{11} + U_{12}g_x) &= U_{11}^* U_{11} + U_{11}^* U_{12} g_x + g_x^* U_{12}^* U_{11} + g_x^* U_{12}^* U_{12} g_x \\
&= I_x - U_{21}^* U_{21} - U_{21}^* U_{22} g_x + g_x^* U_{22}^* U_{21} + g_x^*(I_y - U_{22}^* U_{22}) g_x \\
&= I_x - U_{21}^* U_{21} + U_{21}^* U_{22} U_{22}^* (U_{22} U_{22}^*)^{-1} U_{21} \\
&\quad + U_{21}^* (U_{22} U_{22}^*)^{-1*} U_{22} U_{22}^* U_{21} + g_x^* I_y g_x - g_x^* U_{22}^* U_{22} g_x \\
&= I_x - U_{21}^* U_{21} + U_{21}^* U_{21} + U_{21}^* U_{21} + g_x^* I_y g_x - U_{21}^* U_{21} \\
&= I_x + g_x^* I_y g_x \\
&= I_x + {g_x}^* g_x
\end{aligned}$$

As a result, post-multiplying by $(U_{11} + U_{12}g_x)^{-1}$ and inverting $(I_x + {g_x}^* g_x)$, obtain

$$(U_{11} + U_{12}g_x)^{-1} = (I_x + {g_x}^* g_x)^{-1}(U_{11} + U_{12}g_x)^*$$

□

C.4 Section 5 Proofs

Proof. Of Theorem (1). The proof proceeds in two steps: first, showing that the generalized Schur decomposition is continuous with respect to the approximation, and then showing the policy operators are continuous in the generalized Schur decomposition.

First, note that

$$\begin{aligned}
&\|(\tilde{B}^K,\tilde{A}^K)+(I-\pi^K)(B_I,A_I)(I-\pi^K)-(B,A)\|_{\mathcal{B}}\leq\\
&\quad\|(\tilde{B}^K,\tilde{A}^K)-(B^K,A^K)\|_{\mathcal{B}}+\|(B^K,A^K)+(I-\pi^K)(B_I,A_I)(I-\pi^K)-(B,A)\|_{\mathcal{B}}=\\
&\qquad\|(\tilde{B}^K,\tilde{A}^K)-(B^K,A^K)\|_{\mathcal{B}}+\\
&\|(I-\pi^K)(B_I,A_I)(I-\pi^K)-(I-\pi^K)(B,A)(I-\pi^K)-(I-\pi^K)(B,A)\pi^K-\pi^K(B,A)(I-\pi^K)\|_{\mathcal{B}}=\\
&\|(\tilde{B}^K,\tilde{A}^K)-(B^K,A^K)\|_{\mathcal{B}}+\|-(I-\pi^K)(B_C,A_C)(I-\pi^K)-(I-\pi^K)(B_C,A_C)\pi^K-\pi^K(B_C,A_C)(I-\pi^K)\|_{\mathcal{B}}\\
&\quad=\|(\tilde{B}^K,\tilde{A}^K)-(B^K,A^K)\|_{\mathcal{B}}+\|\pi^K(B_C,A_C)\pi^K-(B_C,A_C)\|_{\mathcal{B}}\leq\zeta_K+\eta_K
\end{aligned}$$

where the third expression follows from the decomposition of $(B,A)=(B_I,A_I)+(B_{C,},A_C)$, and the construction of π^K so that $(I-\pi^K)(B_I,A_I)\pi^K=0$ and $\pi^K(B_I,A_I)(I-\pi^K)=0$.

The consistency of the approximation of (B,A) implies consistency of the components of the Schur decomposition by (3) and (4) and the bound on $\mathrm{dif}(B,A)$. Note that the generalized Schur decomposition of $(\tilde{B}^K,\tilde{A}^K)$ separately is equivalent to (one ordering of) the generalized Schur decomposition of their sum. More precisely,

$$\begin{bmatrix} \tilde{Q}_1^{*K} & 0 & ,\tilde{Q}_2^{*K} & 0\\ 0 & Q_1^{*I} & 0 & Q_2^{*I}\end{bmatrix}\begin{bmatrix} \tilde{T}_{11}^K & 0 & \tilde{T}_{12}^K & 0\\ 0 & T_{11}^I & 0 & T_{12}^I\\ 0 & 0 & \tilde{T}_{22}^K & 0\\ 0 & 0 & 0 & T_{22}^I\end{bmatrix},\begin{bmatrix} \tilde{S}_{11}^K & 0 & \tilde{S}_{12}^K & 0\\ 0 & S_{11}^I & 0 & S_{12}^I\\ 0 & 0 & \tilde{S}_{22}^K & 0\\ 0 & 0 & 0 & S_{22}^I\end{bmatrix}\begin{bmatrix} \tilde{U}_{11}^K & 0 & \tilde{U}_{12}^K & 0\\ 0 & U_{11}^I & 0 & U_{12}^I\\ \tilde{U}_{21}^K & 0 & \tilde{U}_{22}^K & 0\\ 0 & U_{21}^I & 0 & U_{22}^I\end{bmatrix}$$

where an I superscript indicates a component corresponding to the Schur decomposition on $\mathrm{Ker}\,\pi^K$ of (B^I,A^I), is a generalized Schur decomposition of $(\tilde{B}^K,\tilde{A}^K)+(I-\pi^K)(B_I,A_I)(I-\pi^K)$ corresponding to curve Γ. Note that by operator norm convergence, for sufficiently large K, $\sigma_{\min}(\lambda\tilde{B}^K-\tilde{A}^K+(I-\pi^K)(\lambda B_I-,A_I)(I-\pi^K))\geq\sigma_{\min}(\lambda B-A)-2(\zeta_K+\eta_K)>0$ uniformly in $\lambda\in\Gamma$ by Weyl's inequality and the compactness of Γ and so $(\tilde{B}^K,\tilde{A}^K)+(I-\pi^K)(B_I,A_I)(I-\pi^K)$ is Γ-regular and so the generalized Schur decomposition described exists.

To bound $\|g_K-g_X\|$, note

$$g_K=-(\tilde{U}_{22}^K)^{-1}\tilde{U}_{21}-(U_{22}^I)^{-1}U_{21}^I=-\begin{pmatrix}\tilde{U}_{22}^K & 0\\ 0 & U_{22}^I\end{pmatrix}^{-1}\begin{pmatrix}\tilde{U}_{21}^K & 0\\ 0 & U_{21}^I\end{pmatrix}=-(\tilde{U}_{22})^{-1}\tilde{U}_{21}.$$

By (3),

$$\begin{aligned}\|\tilde{U}_1 - U_1\| &\leq \|U_1\|\|I - (I + P^*P)^{-\frac{1}{2}}\| + \|P\| \left\|U_2(I + P^*P)^{-\frac{1}{2}}\right\| \\ &\leq C\|P\| + o(\|P\|) \leq C\frac{2(\zeta_K + \eta_K)}{\delta}\end{aligned}$$

for some constant $C < 2+\epsilon$ for any ϵ, for K sufficiently large, where $\delta > 0$ by the assumption that dif(B,A)>0. As a result, by invertibility of U_{22}, Weyl's inequality, and the triangle inequality, $\left\|-\tilde{U}_{22}^{-1}\tilde{U}_{21} + U_{22}^{-1}U_{21}\right\| \leq C\frac{2(\zeta_K+\eta_K)}{\delta}$ for some constant C for K large enough.

Similarly, we have

$$\begin{aligned}h_K = &\left(\begin{pmatrix} \tilde{U}_{11}^K & 0 \\ 0 & U_{11}^I \end{pmatrix} - \begin{pmatrix} \tilde{U}_{12}^K & 0 \\ 0 & U_{12}^I \end{pmatrix}\begin{pmatrix} \tilde{U}_{22}^K & 0 \\ 0 & U_{22}^I \end{pmatrix}^{-1}\begin{pmatrix} \tilde{U}_{21}^K & 0 \\ 0 & U_{21}^I \end{pmatrix}\right)^{-1} * \\ &\begin{pmatrix} \tilde{S}_{11}^K & 0 \\ 0 & S_{11}^I \end{pmatrix}^{-1}\begin{pmatrix} \tilde{T}_{11}^K & 0 \\ 0 & T_{11}^I \end{pmatrix} * \\ &\left(\begin{pmatrix} \tilde{U}_{11}^K & 0 \\ 0 & U_{11}^I \end{pmatrix} - \begin{pmatrix} \tilde{U}_{12}^K & 0 \\ 0 & U_{12}^I \end{pmatrix}\begin{pmatrix} \tilde{U}_{22}^K & 0 \\ 0 & U_{22}^I \end{pmatrix}^{-1}\begin{pmatrix} \tilde{U}_{21}^K & 0 \\ 0 & U_{21}^I \end{pmatrix}\right) \\ &= (\tilde{U}_{11} + \tilde{U}_{12}g_K)^{-1}(\tilde{S}_{11})^{-1}\tilde{T}_{11}(\tilde{U}_{11} + \tilde{U}_{12}g_K)\end{aligned}$$

Applying the triangle inequality, (3), (4), and convergence of g_K, this implies that for some constant C, for K large enough, $\|h_K - h_x\|_{op} \leq C\frac{\zeta_K+\eta_K}{\delta}$, as claimed. □

A demonstration that for appropriately smooth functions wavelet representations provide the necessary error control to ensure consistency follows from some standard estimates regarding wavelet coefficients.

Proof. of Theorem (2). First we demonstrate bounds on η_K, the error induced by truncating to a K term wavelet series, using results on wavelet coefficients and operator norm bounds from Johnstone (2013), then bounds on ζ_K, the error induced by calculating the inner products with the wavelet basis by quadrature using quadrature error estimates from Beylkin *et al.* (1991).

First, denoting the blocks of (B_C, A_C) as $K_{r,ij}$ $\max\{\left\|B_C^K - B_C\right\|_{op}, \left\|A_C^K - A_C\right\|_{op}\} \leq J\max_{r,i,j}\left\|K_{r,ij}^K - K_{r,ij}\right\|_{op}$ by definition of operator norm. Because an orthonormal basis is used, $\pi^K(B_I, A_I)\pi^K$ is simply expressed in terms of identity matrices on this space, and so can be evaluated exactly.

The projection of $K_{r,ij}$ onto the space of the first $K_i \times K_j$ wavelet coefficients can be

expressed using the inner product with the tensor product over the first $K_i \times K_j$ orthonormal basis functions $\{\phi_k\}_{k=1}^{K_i}$ and $\{\phi_k\}_{k=1}^{K_j}$ as

$$\pi^{K_i} K_{r,ij} \pi^{K_j}[f(y)] = \sum_{k=1}^{K_j} \sum_{l=1}^{K_i} \langle K_{r,ij}(x,y), \phi_k(x)\phi_l(y)\rangle \langle \phi_k(y), [f(y)]\rangle \phi_l(x)$$

$= \int \hat{K}_{r,ij}(x,y) f(y) dy$ where $\hat{K}_{r,ij}(x,y) = \sum_{k=1}^{K_j} \sum_{l=1}^{K_i} \langle K_{r,ij}(x,y), \phi_k(x)\phi_l(y)\rangle \phi_k(y)\phi_l(x)$ is the $K_i \times K_j$ term projection of the kernel of the integral operator onto the wavelet basis. Since $K_{r,ij}(x,y) \in \Lambda^{\alpha_{r,ij}}([0,1]^{d_i} \times [0,1]^{d_j})$ and ϕ_k are a standard wavelet basis, we can use norm bounds to control the error in this projection. Sup norm bounds available in Chen & Christensen (2015), show that under the $\alpha_{r,ij}-$Hölder assumption,

$$\left\| \hat{K}_{r,ij}(x,y) - K_{r,ij}(x,y) \right\|_{L^\infty([0,1]^{d_i} \times [0,1]^{d_j})} = O((K_i K_j)^{-\alpha_{r,ij}/(d_i+d_j)})$$

when wavelets satisfying (4) are used. In particular, adapting the proof of their Lemma 2.4, letting

$$\ell_{K_i K_j} = \sup_{f \in L^\infty([0,1]^{d_i} \times [0,1]^{d_j})} \left\| \sum_{k=1}^{K_j} \sum_{l=1}^{K_i} \langle f(x,y), \phi_k(x)\phi_l(y)\rangle \phi_k(y)\phi_l(x) \right\|_{L^\infty} / \left\| f(x,y) \right\|_{L^\infty}$$

be the Lebesgue constant for the tensor product wavelet basis

$$\left\| \hat{K}_{r,ij}(x,y) - K_{r,ij}(x,y) \right\|_{L^\infty([0,1]^{d_i} \times [0,1]^{d_j})} \quad \leq \quad (1 \; + \; \ell_{K_i K_j}) O((K_i K_j)^{-\alpha_{r,ij}/(d_i+d_j)}),$$

and by their Theorem 5.1 applied in the case of uniform density, $\ell_{K_i K_j}$ is bounded uniformly in K_i and K_j.

By compactness of the domain, we have

$$\begin{aligned} \int \left| \hat{K}_{r,ij}(x,y) - K_{r,ij}(x,y) \right| dx &\leq C \left\| \hat{K}_{r,ij}(x,y) - K_{r,ij}(x,y) \right\|_{L^\infty([0,1]^{d_i} \times [0,1]^{d_j})} \\ \int \left| \hat{K}_{r,ij}(x,y) - K_{r,ij}(x,y) \right| dy &\leq C \left\| \hat{K}_{r,ij}(x,y) - K_{r,ij}(x,y) \right\|_{L^\infty([0,1]^{d_i} \times [0,1]^{d_j})} \end{aligned}$$

almost surely, so by Young's inequality (Johnstone, 2013, Theorem C.26)

$$\sup_{\|f\|=1} \left\| \int (\hat{K}_{r,ij}(x,y) - K_{r,ij}(x,y)) f(y) dy \right\| \leq C \left\| \hat{K}_{r,ij}(x,y) - K_{r,ij}(x,y) \right\|_{L^\infty([0,1]^{d_i} \times [0,1]^{d_j})} \leq (K_i K_j)^{-\alpha_{r,ij}/(d_i+d_j)}$$

As this holds for each r, i, j, we have

$$\eta_K = \max\{\left\|B_C^K - B_C\right\|_{op}, \left\|A_C^K - A_C\right\|_{op}\} \leq O(J \max_{r,i,j} (K_i K_j)^{-\alpha_{r,ij}/(d_i+d_j)})$$

as claimed, by bounding the operator norm by the Frobenius norm of the $J \times J$ matrix with i, j element equal to the operator norm of the i, j block.

To use this result to bound the number of basis functions needed to obtain a total operator norm error of order ϵ, letting $\bar{\alpha} = \min_{r,ij} \frac{2\alpha_{r,ij}}{d_i+d_j}$, by setting $\{K_j\}_{j=1}^J$ all equal and proportional to $(\frac{J}{\epsilon})^{\frac{1}{\bar{\alpha}}}$, obtain $J \max_{r,i,j} (K_i K_j)^{-\alpha_{r,ij}/(d_i+d_j)} = O(\epsilon)$. This results in a basis of size $K = \sum_{j=1}^J K_j$ proportional to $J(\frac{J}{\epsilon})^{\frac{1}{\bar{\alpha}}}$ as claimed.

Next, bound ζ_K, the error induced by approximating each integral operator in (B_C, A_C) by a matrix with entries given by the discrete wavelet transform of $K_{r,ij}(x_s, y_t)$. For convenience, define the level of the d_j-tensor product of multiresolution analyses of Im π^{K_j} in each dimension as $\{n_{jp}\}_{p=1}^{d_j}$, and let the total number of basis functions in the tensor product basis satisfy $K_j = \Pi_{p=1}^{d_j} 2^{n_{jp}}$.[13] The discrete wavelet transform in one dimension is a unitary mapping on the space spanned by the scaling functions $\phi_{n_j,s} := 2^{-n_j/2}\phi(2^{-n_j}x - s + 1)$ at multiresolution level n_j from vectors whose entries are inner products with these scaling functions to vectors whose entries are inner products with the orthonormal wavelet basis spanning the same space, and in multiple dimensions it maps the tensor product of scaling functions representation to the tensor product of wavelets representation. As the operator norm is unitarily invariant, it therefore suffices to bound the operator norm error in terms of the error in the representation defined in terms of scaling function coefficients. By the compact support, vanishing moment condition, and Hölder exponent bound, Beylkin *et al.* (1991) show by a Taylor expansion argument that if a scaling function with the property $\int \phi(x+\tau)x^m dx = 0$ for all integers $m \leq \alpha + 1$, for some integer τ, is used, then any $f(x) \in \Lambda^\alpha[0,1]$ satisfies $2^{-n/2} f(2^{-n}(k-1+\tau)) = \int f(x)\phi_{n,k}(x)dx + O(2^{-n(\alpha+\frac{1}{2})})$ uniformly in k, and for multivariate functions $f(x^1, \ldots, x^d) \in \Lambda^\alpha[0,1]^d$, a straightforward extension

[13] One can avoid restricting to powers of 2 by using a larger number of functions at the finest level, at the cost of more cumbersome notation. The order of all asymptotic results remains the same.

shows

$$(2^{-n_1/2}\dots 2^{-n_d/2})f(2^{-n}(k_1-1+\tau),\dots,2^{-n}(k_d-1+\tau)) = \int\dots\int f(x_1,\dots,x_d)\phi_{n,k^1}(x_1)\dots\phi_{n,k^d}(x_d)dx_1\dots dx_d + O(\prod_{p=1}^{d}2^{-n_p/2}\sum_{p=1}^{d}2^{-\alpha n_p})$$

Applying this to $\frac{1}{\sqrt{K_iK_j}}K_{r,ij}(x_s,y_t)$ we see that its entries satisfy

$$\left|\frac{1}{\sqrt{K_iK_j}}K_{r,ij}(x_s,y_t) - \left\langle K_{r,ij}(x,y), \Pi_{p=1}^{d_i}\phi_{n_{ip},s+\tau}(x_p)\Pi_{p=1}^{d_j}\phi_{n_{jp},t+\tau}(y_p)\right\rangle\right| = O((\prod_{p=1}^{d_i}2^{-n_{ip}/2}\prod_{p=1}^{d_j}2^{-n_{jp}/2})(\sum_{p=1}^{d_i}2^{-n_{ip}\alpha_{r,ij}} + \sum_{p=1}^{d_j}2^{-n_{jp}\alpha_{r,ij}}))$$

uniformly in s,t. To control the operator norm error induced by this approximation to the matrix of scaling function coefficients, we again use Young's inequality, combined with the fact that the scaling functions $\phi_{n_j,s}$ are rescaled translations of a single bounded and compactly supported function over a regular grid, to bound the operator norm error in the quadrature approximation of the finite projection of $K_{r,ij}(x,y)$ onto a finite tensor product wavelet basis. In particular, denoting $\theta_{ijst} := \left\langle K_{r,ij}(x,y), \Pi_{p=1}^{d_i}\phi_{n_{ip},s+\tau}(x_p)\Pi_{p=1}^{d_j}\phi_{n_{jp},t+\tau}(y_p)\right\rangle$ and $\hat{\theta}_{ijst} := \frac{1}{\sqrt{K_iK_j}}K_{r,ij}(x_s,y_t)$ the L^∞ norm error induced by quadrature in the $K_i \times K_j$ term representation of the kernel is equal to

$$\sup_{x,y\in[0,1]^{d_i}\times[0,1]^{d_j}}\left|\sum_{s=1}^{K_i}\sum_{t=1}^{K_j}(\theta_{ijst}-\hat{\theta}_{ijst})\Pi_{p=1}^{d_i}\phi_{n_{ip},s+\tau}(x_p)\Pi_{p=1}^{d_j}\phi_{n_{jp},t+\tau}(y_p)\right|$$

As noted in Chen & Christensen (2015, Section 6), by the assumption that the one-dimensional scaling function ϕ has support within a compact interval, with length no greater than $3N+1$ for a fixed integer N (depending order of the wavelet used), at most $3N+1$ scaling functions at any fixed level n_j may overlap on any set of positive Lebesgue measure, and so over the d_i+d_j-dimensional tensor product space, no point x,y is covered by more than $(3N+1)^{d_i+d_j}$

scaling functions.[14] As a result

$$\sup_{x,y\in[0,1]^{d_i}\times[0,1]^{d_j}}\left|\sum_{s=1}^{K_i}\sum_{t=1}^{K_j}(\hat{\theta}_{ijst}-\theta_{ijst})\prod_{p=1}^{d_i}\phi_{n_{ip},s+\tau}(x_p)\prod_{p=1}^{d_j}\phi_{n_{jp},t+\tau}(y_p)\right|$$

$$\leq (3N+1)^{d_i+d_j}\max_{s,t}\left|\hat{\theta}_{ijst}-\theta_{ijst}\right|\sup_{x,y}\left|\prod_{p=1}^{d_i}\phi_{n_{ip},s+\tau}(x_p)\prod_{p=1}^{d_j}\phi_{n_{jp},t+\tau}(y_p)\right|$$

$$\leq (3N+1)^{d_i+d_j}O(\prod_{p=1}^{d_i}2^{-n_{ip}/2}\prod_{p=1}^{d_j}2^{-n_{jp}/2})(\sum_{p=1}^{d_i}2^{-n_{ip}\alpha_{r,ij}}+\sum_{p=1}^{d_j}2^{-n_{jp}\alpha_{r,ij}}))\prod_{p=1}^{d_i}2^{n_{ip}/2}\prod_{p=1}^{d_j}2^{n_{jp}/2}\sup_x|\phi(x)|$$

$$= O((3N+1)^{d_i+d_j}(\sum_{p=1}^{d_i}2^{-n_{ip}\alpha_{r,ij}}+\sum_{p=1}^{d_j}2^{-n_{jp}\alpha_{r,ij}}))$$

by boundedness and the definition of $\phi_{n_{jp},s}$. When the number of basis functions used in each dimension is identical for all dimensions $p = 1 \dots d_i$ and $1 \dots d_j$, this term is bounded by

$$O((3N+1)^{d_i+d_j}(d_i+d_j)(K_iK_j)^{-\alpha_{r,ij}/(d_i+d_j)})$$

This is the same order as the projection result, except for a multiplicative constant depending on dimension. Let $\bar{d} = \max_j 2d_j$. Then, if the number of basis functions is set so that $\{K_j\}_{j=1}^J$ all equal and proportional to $(\frac{(3N+1)^{\bar{d}}\bar{d}J}{\epsilon})^{\frac{1}{\bar{\alpha}}}$, the above bound along with Young's inequality gives an operator norm error bound bound for each block no greater than $O(\frac{\epsilon}{J})$. With each of J^2 blocks bounded by no more than this quantity, obtain the bound

$$\zeta_K \leq O(\epsilon)$$

exactly as claimed. □

C.5 Section 6 Proofs

Proposition. *Derivation of* $\frac{\hat{d\omega}}{d\lambda}_{\phi}$*:* $\frac{\hat{d\omega}}{d\lambda}_{\phi} = (1-\mu H(\phi))\frac{\mu H(\phi)-H(\phi)^2}{\sigma-\mu H(\phi)-(\sigma-1)H(\phi)^2}+\frac{\mu}{\sigma-1}H(\phi)$*, with* $H(\phi) := \frac{(\sigma-1)^2}{(\sigma-1)^2+\tau^{-2}\phi^2}$

Proof. $\frac{d\omega}{d\lambda}$ is shown in Equation (C.1) to equal $\frac{dw}{d\lambda}-\mu(\frac{dT}{dw}\frac{dw}{d\lambda}+\frac{dT}{d\lambda})$ which is a composition of convolution operators and their inverses and so can also be expressed as multiplication by

[14]The vanishing moments property characterizing Coiflets also requires that the length of the filter defining the scaling function be longer by a factor of 1.5 than the filter for the corresponding standard Daubechies wavelet. This results in an larger constant in front of the quadrature error and the running time of the discrete wavelet transform, but does not affect the rate of convergence.

the Fourier transform of some function. To construct the Fourier transform of the function, simplify the integrals in equations (3.3),(3.4), (3.5), and (3.6) and denote

$$H(\phi) = \frac{(\sigma-1)^2}{(\sigma-1)^2 + \tau^{-2}\phi^2}$$

the Fourier transform of the Laplace distribution in the convolution operator

$$\frac{\tau(1-\sigma)}{2}\int_G [.]e^{\tau(1-\sigma)|x-z|}dz.$$

This yields the formulas $\frac{\hat{dw}}{dT} = \frac{\sigma-1}{\sigma}H$, $\frac{\hat{dw}}{dY} = \frac{1}{\sigma}H$, $\frac{\hat{dT}}{d\lambda} = \frac{1}{1-\sigma}H$, $\frac{dT}{dw} = H$. Substituting into the expressions for partial derivatives, obtain $\frac{\hat{dw}}{d\lambda} = \frac{-\frac{\mu}{\sigma}H + \frac{1}{\sigma}H^2}{1-\frac{\mu}{\sigma}H - \frac{\sigma-1}{\sigma}H^2}$ and

$$\frac{\hat{d\omega}}{d\lambda}_{\phi} = (1-\mu H(\phi))\frac{\mu H(\phi) - H(\phi)^2}{\sigma - \mu H(\phi) - (\sigma-1)H(\phi)^2} + \frac{\mu}{\sigma-1}H(\phi). \tag{C.2}$$

This is almost the same as Krugman (1996)'s equation (A.44) for this term, but differs slightly due to what appears to be an algebra error in the text. □

Proof. of (2). The proof applies the machinery and notation of Stewart (1973). While rates of convergence are obtained, no attempt is made to ensure that these are optimal. First, note that $\gamma_\phi = \|(B_\phi, A_\phi) - (B_I^i, A_I^i)\|_F \to 0$ by assumption, and so all submatrices also converge at least as rapidly in Frobenius norm. Next note that (B_I^i, A_I^i) has generalized Schur decomposition

$$\left(Q^{*\infty}\begin{bmatrix} S_{11}^\infty & S_{22}^\infty \\ 0 & S_{22}^\infty \end{bmatrix}\begin{bmatrix} U_{11}^\infty & U_{12}^\infty \\ U_{21}^\infty & U_{22}^\infty \end{bmatrix}, Q^{*\infty}\begin{bmatrix} T_{11}^\infty & T_{22}^\infty \\ 0 & T_{22}^\infty \end{bmatrix}\begin{bmatrix} U_{11}^\infty & U_{12}^\infty \\ U_{21}^\infty & U_{22}^\infty \end{bmatrix}\right)$$

where $(S^\infty, T^\infty) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}\right)$ and $U^\infty = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Applying standard formulas for policy functions, obtain $\hat{g}_\infty = -U_{22}^{\infty -} U_{21}^\infty = (0,0)$ and

$$\hat{h}_\infty = (I_2 + \hat{g}_\infty^* \hat{g}_\infty)^{-1}(\begin{pmatrix} I_2 \\ \hat{g}_\infty \end{pmatrix}^* U_1^{\infty *} S_{11}^{\infty -1} T_{11}^\infty U_1^\infty \begin{pmatrix} I_2 \\ \hat{g}_\infty \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

As generalized eigenvalues corresponding to the stable subspace are equal to 0 and the generalized eigenvalue corresponding to the unstable subspace is ∞, the measure of subspace

separation defined in Stewart (1973), which ensures that Schur subspaces are numerically stable, is given by $\delta = \text{dif}(S_{11}^{\infty}, T_{11}^{\infty}, S_{22}^{\infty}, T_{22}^{\infty}) > 0$. As a result, by Stewart (1973), Theorem 5.7 and 5.3, $\left\|\sin\Theta(U_1^{\infty *}, U_1^{\phi *})\right\|_F \leq 2\frac{\gamma_\phi}{\delta - 2\gamma_\phi}$ for γ_ϕ small enough, and similarly for U_2^{∞}, where $\Theta(U_1^{\infty *}, U_1^{\phi *})$ is the matrix of principal angles between the span of $U_1^{\infty *}$ and $U_1^{\phi *}$. While this does not imply that $\left\|U_1^{\infty *} - U_1^{\phi *}\right\|_F \to 0$, as the span does not uniquely define the basis, it does imply, because $U_2^{\phi *}$ and $U_2^{\infty *}$ have a one-dimensional span and norm 1, that

$$\begin{aligned}
\left\|U_2^{\phi *} - U_2^{\infty *}\right\|_F^2 &= 2 - 2\left|\cos\Theta(U_2^{\phi *}, U_2^{\infty *})\right| \\
&= 2 - 2\sqrt{1 - \sin^2\Theta(U_2^{\phi *}, U_2^{\infty *})} \\
&\leq 2 - 2\sqrt{1 - (\frac{2\gamma_\phi}{\delta - 2\gamma_\phi})^2} = O(\gamma_\phi) \to 0
\end{aligned}$$

. Since $U_{22}^{\infty} = 1$ is invertible, the policy function $\hat{g}_\phi = -U_{22}^{\phi -} U_{21}^{\phi}$ therefore satisfies the bound $\|\hat{g}_\phi - \hat{g}_\infty\|_F^2 \leq O(\gamma_\phi) \to 0$, as claimed.

Further, it is possible to show that for each ϕ, there exists a unitary (2×2) transformation R_ϕ of $U_1^{\infty *}$ such that $\left\|U_1^{\infty *} R_\phi - U_1^{\phi}\right\|_F \to 0$. Applying the definition of principal angles, for each ϕ there exist unitary matrices $R_\phi^1 = [R_{\phi 1}^1, R_{\phi 2}^1]$ and $R_\phi^2 = [R_{\phi 1}^2, R_{\phi 2}^2]$ such that $[\cos\Theta(U_1^{\infty *}, U_1^{\phi *})]_{11} = \left\langle U_1^{\infty *} R_{\phi 1}^1, U_1^{\phi *} R_{\phi 1}^2 \right\rangle$ and $[\cos\Theta(U_1^{\infty *}, U_1^{\phi *})]_{22} = \left\langle U_1^{\infty *} R_{\phi 2}^1, U_1^{\phi *} R_{\phi 2}^2 \right\rangle$, so

$$\begin{aligned}
\left\|U_1^{\infty *} R_\phi - U_1^{\phi}\right\|_F^2 &:= \left\|U_1^{\infty *} R_\phi^1 R_\phi^{2*} - U_1^{\phi}\right\|_F^2 \\
&= \left\|U_1^{\infty *} R_\phi^1 - U_1^{\phi} R_\phi^2\right\|_F^2 \\
&= 2(1 - [\cos\Theta(U_1^{\infty *}, U_1^{\phi *})]_{11} + 1 - [\cos\Theta(U_1^{\infty *}, U_1^{\phi *})]_{22}) \\
&\leq 4 - 4\sqrt{1 - (\frac{2\gamma_\phi}{\delta - 2\gamma_\phi})^2} = O(\gamma_\phi) \to 0.
\end{aligned}$$

Equivalent results show that for a different unitary transform R_ϕ^Q, $\left\|R_\phi^Q Q_1^{\infty} - U_1^{\phi}\right\|_F^2 = O(\gamma_\phi)$. Combining these results and applying the triangle inequality,

$$\left\|S_{11}^{\phi} - S_{11}^{\infty(\phi)}\right\|_F := \left\|S_{11}^{\phi} - R_\phi^Q Q_1^{\infty} A_\infty U_1^{\infty *} R_\phi\right\|_F \leq O(\gamma_\phi^{\frac{1}{2}})$$

and

$$\left\|T_{11}^{\phi} - T_{11}^{\infty(\phi)}\right\|_F := \left\|T_{11}^{\phi} - R_\phi^Q Q_1^{\infty} B_\infty U_1^{\infty *} R_\phi\right\|_F \leq O(\gamma_\phi^{\frac{1}{2}})$$

also, gives convergence of the generalized Schur components of the finite order matrices along a triangular array to unitary transformations $(S_{11}^{\infty(\phi)}, T_{11}^{\infty(\phi)})$ of the generalized Schur

components of the limit pencil. Noting that unitary transformations leave singular values unaffected and that S_{11}^{∞} is invertible, S_{11}^{ϕ} is also asymptotically invertible, so by Weyl's inequality

$$\left\|S_{11}^{\phi-1} - S_{11}^{\infty(\phi)-1}\right\|_F \leq \left\|S_{11}^{\phi-1}\right\|_{op} \left\|S_{11}^{\infty(\phi)-1}\right\|_{op} \left\|S_{11}^{\phi} - S_{11}^{\infty(\phi)}\right\|_F \leq O(\gamma_{\phi}^{\frac{1}{2}}).$$

Using the unitarity of R_ϕ and applying the triangle inequality, one can see that

$$\begin{aligned}\left\|U_1^{\phi*} S_{11}^{\phi-1} T_{11}^{\phi} U_1^{\phi} - U_1^{\infty*} S_{11}^{\infty-1} T_{11}^{\infty} U_1^{\infty}\right\|_F &= \left\|U_1^{\phi*} S_{11}^{\phi-1} T_{11}^{\phi} U_1^{\phi} - U_1^{\infty*} R_\phi S_{11}^{\infty(\phi)-1} T_{11}^{\infty(\phi)} R_\phi^* U_1^{\infty}\right\|_F \\ &\leq O(\gamma_{\phi}^{\frac{1}{2}}),\end{aligned}$$

and so the fact that Schur vectors do not converge does not affect the convergence of the policy function, which is invariant to unitary transformations of these vectors. Finally, defining

$$\hat{h}_\phi = (I_2 + \hat{g}_\phi^* \hat{g}_\phi)^{-1}(\begin{pmatrix} I_2 \\ \hat{g}_\phi \end{pmatrix}^* U_1^{\phi*} S_{11}^{\phi-1} T_{11}^{\phi} U_1^{\phi} \begin{pmatrix} I_2 \\ \hat{g}_\phi \end{pmatrix})$$

the above results and the triangle inequality imply that $\left\|\hat{h}_\phi - \hat{h}_\infty\right\|_F \leq O(\gamma_{\phi}^{\frac{1}{2}})$.

To show compactness, it suffices to show that the singular values converge to 0. As $g[.]$ and $h[.]$ are block-diagonal, it suffices to show that the operator norm of each block converges to 0. As the operator norm is bounded by the Frobenius norm, each block has operator norm at most $O(\gamma_{\phi}^{\frac{1}{2}}) \to 0$ and so compactness holds.

(ii) To show that an $h[.]$ is Hilbert Schmidt, $\text{Tr}(h^*h) < \infty$, it suffices to show that the sum of squared singular values converges. As the sum of squared singular values for each block is equal to the square of its Frobenius norm, which is $O(\gamma_\phi)$ for large $|\phi|$, convergence holds so long as $\sum_{\phi=n}^{\infty} \gamma_\phi < \infty$ for some finite n. Superlinear convergence $\gamma_\phi = O(|\phi|^{-(1+\epsilon)})$ for some $\epsilon > 0$ is sufficient for this sum to be finite. □

D Additional Figures

The following figures display the Euclidean norm of pointwise errors over a grid in impulse responses calculated by Fourier and Wavelet basis at each time period for different numbers of basis functions K. For details, see Section 5.

Figure D.1: Euclidean Discrepancy, K=512

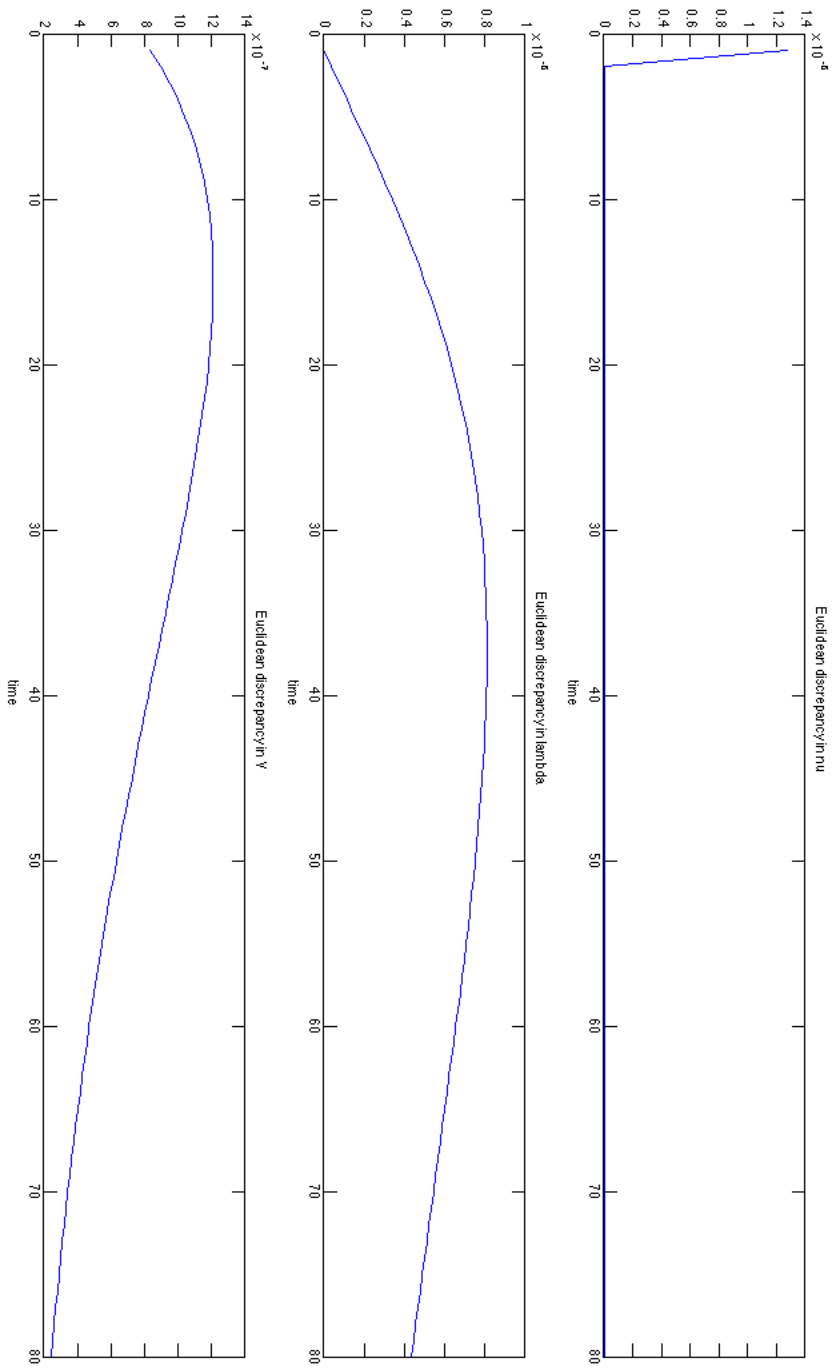

Figure D.2: Euclidean Discrepancy, K=1024