A Sieve-SMM Estimator for Dynamic Models

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Abstract

This paper proposes a Sieve Simulated Method of Moments (Sieve-SMM) estimator for the parameters and the distribution of the shocks in nonlinear dynamic models where the likelihood and the moments are not tractable. An important concern with SMM, which matches sample with simulated moments, is that a parametric distribution is required but economic quantities that depend on this distribution, such as welfare and asset-prices, can be sensitive to misspecification. The Sieve-SMM estimator addresses this issue by flexibly approximating the distribution of the shocks with a Gaussian and tails mixture sieve. The asymptotic framework provides consistency, rate of convergence and asymptotic normality results, extending sieve theory to more general dynamics with latent variables. Monte-Carlo simulations illustrate the finite sample properties of the estimator. Two empirical applications highlight the importance of the distribution of the shocks. The first provides evidence of non-Gaussian shocks in macroeconomic data and their implications on welfare and the risk-free rate. The second finds that Gaussian estimates of stochastic volatility are significantly biased in exchange rate data because of fat tails.

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1 Introduction

Complex nonlinear dynamic models with an intractable likelihood or moments are increasingly common in economics. A popular approach to estimating these models is to match informative sample moments with simulated moments from a fully parameterized model using SMM. However, economic models are rarely fully parametric since theory usually provides little guidance on the distribution of the shocks. The Gaussian distribution is often used in applications but in practice, different choices of distribution may have different economic implications; this is illustrated below. Yet to address this issue, results on semiparametric simulation-based estimation are few.

This paper proposes a Sieve Simulated Method of Moments (Sieve-SMM) estimator for both the structural parameters and the distribution of the shocks and explains how to implement it. The dynamic models considered here have the form:

$$y_t = g_{obs}(y_{t-1}, x_t, \theta, f, u_t) \tag{1}$$

$$u_t = g_{latent}(u_{t-1}, \theta, f, e_t), \quad e_t \sim f \tag{2}$$

The observed outcome variable is y_t, x_t are exogenous regressors and u_t is an unobserved latent process. The unknown parameters include θ, a finite dimensional vector, and the distribution f of the shocks e_t. The functions g_{obs}, g_{latent} are known, or can be computed numerically, up to θ and f. The Sieve-SMM estimator extends the existing Sieve-GMM literature to more general dynamics with latent variables and the literature on sieve simulation-based estimation of some static models.

The estimator in this paper has two main building blocks: the first one is a sample moment function, such as the empirical characteristic function (CF) or the empirical CDF; infinite dimensional moments are needed to identify the infinite dimensional parameters. As in the finite dimensional case, the estimator simply matches the sample moment function with the simulated moment function. To handle this continuum of moment conditions, this paper adopts the objective function of Carrasco & Florens (2000); Carrasco et al. (2007a) in a semi-nonparametric setting.

The second building block is to nonparametrically approximate the distribution of the shocks using the method of sieves, as numerical optimization over an infinite dimension space is generally not feasible. Typical sieve bases include polynomials and splines which approximate smooth regression functions. Mixtures are particularly attractive to approximate densities for three reasons: they are computationally cheap to simulate from, they are known to have good approximation properties for smooth densities, and draws from the mixture sieve are shown to satisfy the L^2-smoothness regularity conditions of the moments required for the asymptotic results. Restrictions on the number of mixture components, the tails and the smoothness of the true density ensure that the bias is small relative to the variance so that valid inferences can be made in large

samples. To handle potentially fat tails, this paper introduces a Gaussian and tails mixture. The tail densities in the mixture are constructed to be easy to simulate from and also satisfy L^2-smoothness properties. The algorithm below summarizes the steps required to compute the estimator.

ALGORITHM: Computing the Sieve-SMM Estimator

Set a sieve dimension $k(n) \geq 1$ and a number of lags $L \geq 1$.
Compute $\hat{\psi}_n$, the Characteristic Function (CF) of $(y_t, \ldots, y_{t-L}, x_t, \ldots, x_{t-L})$.
for $s = 1, \ldots, S$ **do**
Simulate the shocks e_t^s from $f_{\omega,\mu,\sigma}$: a $k(n)$ component Gaussian and tails mixture distribution with parameters (ω, μ, σ).
Simulate artificial samples $(y_1^s, \ldots, y_n^s)$ at $(\theta, f_{\omega,\mu,\sigma})$ using e_t^s.
Compute $\hat{\psi}_n^s(\theta, f_{\omega,\mu,\sigma})$, the CF of the simulated data $(y_t^s, \ldots, y_{t-L}^s, x_t, \ldots, x_{t-L})$.
Compute the average simulated Characteristic Function $\hat{\psi}_n^S(\theta, f_{\omega,\mu,\sigma}) = \frac{1}{S}\sum_{s=1}^{S} \hat{\psi}_n^s(\theta, f_{\omega,\mu,\sigma})$.
Compute the objective function $\hat{Q}_n^S(\theta, f_{\omega,\mu,\sigma}) = \int \left|\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\theta, f_{\omega,\mu,\sigma})\right|^2 \pi(\tau) d\tau$.
Find the parameters $(\hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)$ that minimize $\hat{Q}_n^S$.

To illustrate the class of models considered and the usefulness of the mixture sieve for economic analysis, consider the first empirical application in section 6 where the growth rate of consumption $\Delta c_t = \log(C_t/C_{t-1})$ is assumed to follow the following process:

$$\Delta c_t = \mu_c + \rho_c \Delta c_{t-1} + \sigma_t e_{t,1}, \quad e_{t,1} \sim f \tag{3}$$

$$\sigma_t^2 = \mu_\sigma + \rho_\sigma \sigma_{t-1}^2 + \kappa_\sigma e_{t,2}, \quad e_{t,2} \sim \chi_1^2. \tag{4}$$

Compared to the general model (1)-(2), the Δc_t corresponds to the outcome y_t, the latent variable u_t is $(\sigma_t^2, e_{t,1})$ and the parameters are $\theta = (\mu_y, \rho_y, \mu_\sigma, \rho_\sigma, \kappa_\sigma)$. This very simple model, with a flexible distribution f for the shocks $e_{t,1}$, can explain the low level of the risk-free rate with a simple power utility and recent monthly data. In comparison, the Long-Run Risks models relies on more complex dynamics and recursive utilities (Bansal & Yaron, 2004) and the Rare Disasters literature involves hard to quantify very large, low frequency shocks (Rietz, 1988; Barro, 2006b). Empirically, the Sieve-SMM estimates of distribution of f in the model (3)-(4) implies both a 25% larger higher welfare cost of business cycle fluctuations and an annualized risk-free rate that is up to 4 percentage points lower than predicted by Gaussian shocks. Also, in this example the risk-free rate is tractable, up to a quadrature over σ_{t+1}, when using Gaussian mixtures:

$$r_t^{mixt} = -\log(\delta) + \gamma\mu_c + \gamma\rho_c \Delta c_t - \log\left(\sum_{j=1}^{k} \omega_j \mathbb{E}_t\left[e^{-\gamma\sigma_{t+1}\mu_j + \frac{\gamma^2}{2}\sigma_{t+1}^2[\sigma_j^2 - 1]}\right]\right).$$

In comparison, for a general distribution the risk-free rate depends on all moments but does not

necessarily have closed form. The mixture thus combines flexible econometric estimation with convenient economic modelling.[1]

As in the usual sieve literature, this paper provides a consistency result and derives the rate of convergence of the structural and infinite dimensional parameters, as well as asymptotic normality results for finite dimensional functionals of these parameters. While the results apply to both static and dynamic models alike, two important differences arise in dynamic models compared to the existing literature on sieve estimation: proving uniform convergence of the objective function and controlling the dynamic accumulation of the nonparametric approximation bias.

The first challenge is to establish the rate of convergence of the objective function for dynamic models. To allow for the general dynamics (1)-(2) with latent variables, this paper adapts results from Andrews & Pollard (1994) and Ben Hariz (2005) to construct an inequality for uniformly bounded empirical processes which may be of independent interest. It allows the simulated data to be non-stationary when the initial (y_0, u_0) is not taken from the ergodic distribution. It requires a geometric ergodicity condition as in Duffie & Singleton (1993). The boundedness condition is satisfied by the CF and the CDF for instance. Also, the inequality implies a larger variance than typically found in the literature.[2]

The second challenge is that in the model (1)-(2) the nonparametric bias accumulates dynamically. At each time period the bias appears because draws are taken from a mixture approximation instead of the true f_0, this bias is also transmitted from one period to the next since (y_t^s, u_t^s) depends on (y_{t-1}^s, u_{t-1}^s). To ensure that this bias does not accumulate too much, a decay condition is imposed on the DGP. For the consumption process (3)-(4), this condition holds if both $|\rho_y|$ and $|\rho_\sigma|$ are strictly less than 1. The resulting bias is generally larger than in static models and usual sieve estimation problems. Together, the increased variance and bias imply a slower rate of convergence for the Sieve-SMM estimates. Hence, in order to achieve the rate of convergence required for asymptotic normality, the Sieve-SMM requires additional smoothness of the true density f_0.

Monte-Carlo simulations illustrate the properties of the estimator and the effect of dynamics on the bias and the variance of the estimator. Two empirical applications highlight the importance of estimating the distribution of the shocks. The first is the example discussed above, and the second estimates a different stochastic volatility model on a long daily series of exchange rate data. The Sieve-SMM estimator suggests significant asymmetry and fat tails in the shocks, even after controlling for the time-varying volatility. As a result, commonly used parametric estimates

[1] Gaussian mixtures are also convenient in more complicated settings where the model needs to be solved numerically. For instance, all the moments of a Gaussian mixture are tractable and quadrature is easy so that it can be applied to both the perturbation method and the projection method (see e.g. Judd, 1996, for a review of these methods) instead of the more commonly applied Gaussian distribution.

[2] See Chen (2007, 2011) for a summary of existing results with iid and dependent data.

for the persistence are significantly downward biased which has implications for forecasting; this effect is confirmed by the Monte-Carlo simulations.

Related Literature

The Sieve-SMM estimator presented in this paper combines two literatures: sieves and the Simulated Method of Moments (SMM). This section reviews the existing methods and results in each literature to introduce the new challenges arising from the combined Sieve-SMM setting.

A key aspect to simulation-based estimation is the choice of moments $\hat{\psi}_n$. The Simulated Method of Moments (SMM) estimator of McFadden (1989) relies on unconditional moments, the Indirect Inference (IND) estimator of Gouriéroux et al. (1993) uses auxliary parameters from a simpler, tractable model and the Efficient Method of Moments (EMM) of Gallant & Tauchen (1996) uses the score of the auxiliary model. Simulation-based estimation has been applied to a wide array of economic settings: early empirical applications of these methods include the estimation of discrete choice models (Pakes, 1986; Rust, 1987), DSGE models (Smith, 1993) and models with occasionally binding constraints (Deaton & Laroque, 1992). More recent empirical applications include the estimation of earning dynamics (Altonji et al., 2013), of labor supply (Blundell et al., 2016) and the distribution of firm sizes (Gourio & Roys, 2014). Simulation-based estimation can also applied to models that are not fully specified as in Berry et al. (1995), these models are not considered in the Sieve-SMM estimation.

To achieve parametric efficiency a number of papers consider using nonparametric moments but they assumed the distribution f is known.[3] To avoid dealing with the nonparametric rate of convergence of the moments Carrasco et al. (2007a) use the continuum of moments implied by the CF. This paper uses a similar approach in a semi-nonparametric setting. Bernton et al. (2017) use the Wasserstein, or Kantorovich distance, between the empirical and simulated distributions. This distance relies on unbounded moments and is thus excluded from the analysis in this paper.

General asymptotic results are given by Pakes & Pollard (1989) for SMM with iid data and Lee & Ingram (1991); Duffie & Singleton (1993) for time-series. Gouriéroux & Monfort (1996) provide an overview of existing results for a large number of simulation-based estimation methods.

While most of the literature discussed so far deals with fully parametric SMM models, there are a few papers concerned with sieve simulation-based estimation. Bierens & Song (2012) pro-

[3] See e.g. Gallant & Tauchen (1996); Fermanian & Salanié (2004); Kristensen & Shin (2012); Gach & Pötscher (2010); Nickl & Pötscher (2011).

vide a consistency result for Sieve-SMM estimation of a static first-price auction model.[4] Newey (2001) uses a sieve simulated IV estimator for a measurement error model and proves consistency as both n and S go to infinity. These papers only consider specific static models and only provide limited asymptotic results. Furthermore, they consider sampling methods for the simulations that are very computationally costly (see section 2.3 for a discussion). Additionally, an incomplete working paper by Blasques (2011) uses the high-level conditions in Chen (2007) for a "Semi-NonParametric Indirect Inference" estimator. These conditions are very difficult to verify in practice and additional results are needed to handle the dynamics.[5]

An alternative to using sieves in SMM estimation involves using more general parametric families to model the first 3 or 4 moments flexibly. Ruge-Murcia (2012, 2017) considers the skew Normal and the Generalized Extreme Value distributions to model the first 3 moments of productivity and inflation shocks. Gospodinov & Ng (2015); Gospodinov et al. (2017) use the Generalized Lambda famility to flexibly model the first 4 moments of the shocks in a non-invertible moving avergage and a measurement error model. However, in applications where the moments depend on the full distribution of the shocks, which is the case if the data y_t is non-separable in the shocks e_t, then the estimates $\hat{\theta}_n$ will be sensitive to the choice of parametric family. Also, quantities of interest such as welfare estimates and asset prices that depend on the full distribution will also be sensitive to the choice of parametric family.

Another related literature is the sieve estimation of models defined by moment conditions. These models can be estimated using either Sieve-GMM, Sieve Empirical Likelihood or Sieve Minimum Distance (see Chen, 2007, for a review). Applications include nonparametric estimation of mean instrumental variables regressions[6], of quantile instrumental variables regressions,[7] and the semi-nonparametric estimation of asset pricing models,[8] for instance. Existing results cover the consistency and the rate of convergence of the estimator as well as asymptotic normality of functional of the parameters for both iid and dependent data. Recent general asymptotic results include Chen & Pouzo (2012, 2015) for iid data and Chen & Liao (2015) for dependent data.

In the empirical Sieve-GMM literature, an application closely related to the dynamics encoun-

[4] In order to do inference on f, they propose to invert a simulated version of Bierens (1990)'s ICM test statistic. A recent working paper by Bierens & Song (2017) introduces covariates in the same auction model and gives an asymptotic normality result for the coefficients $\hat{\theta}_n$ on the covariates.

[5] Also, to avoid using sieves and SMM in moment conditions models that are tractable up to a latent variable, Schennach (2014) proposes an Entropic Latent Variable Integration via Simulation (ELVIS) method to build estimating equations that only involve the observed variables. Dridi & Renault (2000) propose a Semi-Parametric Indirect Inference based on a partial encompassing principle.

[6] See e.g. Hall & Horowitz (2005); Carrasco et al. (2007b); Blundell et al. (2007); Darolles et al. (2011); Horowitz (2011).

[7] See e.g. Chernozhukov & Hansen (2005); Chernozhukov et al. (2007); Horowitz & Lee (2007).

[8] See e.g. Hansen & Richard (1987); Chen & Pouzo (2009); Chen et al. (2013); Christensen (2017).

tered in this paper appears in Chen et al. (2013). The authors show how to estimate an Euler equation with recursive preferences when the value function is approximated using sieves. Recursive preferences require a filtering step to recover the latent variable. This implies that the moments depend on the whole history of the data $(y_t, \ldots, y_1)$. However, general results based on coupling results (see e.g. Doukhan et al., 1995; Chen & Shen, 1998) do not apply to this class of moments. The authors use a Bootstrap for inference without formal asymptotic results.

Notation

The following notation and assumptions will be used throughout the paper: the parameter of interest is $\beta = (\theta, f) \in \Theta \times \mathcal{F} = \mathcal{B}$. The finite dimensional parameter space Θ is compact and the infinite dimensional set of densities $\mathcal{F}$ is possibly non-compact. The sets of mixtures satisfy $\mathcal{B}_k \subseteq \mathcal{B}_{k+1} \subseteq \mathcal{B}$, k is the data dependent dimension of the sieve set $\mathcal{B}_k$. The dimension k increases with the sample size: $k(n) \to \infty$ as $n \to \infty$. Using the notation of Chen (2007), $\Pi_{k(n)} f$ is the mixture approximation of the density f. The vector of shocks e has dimension $d_e \geq 1$ and density f. The total variation distance between two densities is $\|f_1 - f_2\|_{TV} = 1/2 \int |f_1(e) - f_2(e)| de$ and the supremum (or sup) norm is $\|f_1 - f_2\|_\infty = \sup_{e \in \mathbb{R}^{d_e}} |f_1(e) - f_2(e)|$. For simplification, the following convention will be used $\|\beta_1 - \beta_2\|_{TV} = \|\theta_1 - \theta_2\| + \|f_1 - f_2\|_{TV}$ and $\|\beta_1 - \beta_2\|_\infty = \|\theta_1 - \theta_2\| + \|f_1 - f_2\|_\infty$, where $\|\theta\|$ and $\|e\|$ correspond the Euclidian norm of θ and e respectively. $\|\beta_1\|_m$ is a norm on the mixture components: $\beta_1\|_m = \|\theta\| + \|(\omega, \mu, \sigma)\|$ where $\|\cdot\|$ is the Euclidian norm and (ω, μ, σ) are the mixture parameters. For a functional ϕ, its pathwise, or Gâteau, derivative at β_1 in the direction β_2 is $\frac{d\phi(\beta_1)}{d\beta}[\beta_2] = \left.\frac{d\phi(\beta_1 + \varepsilon\beta_2)}{d\varepsilon}\right|_{\varepsilon=0}$, it will be assumed to be continuous in β_1 and linear in β_2. For two sequences a_n and b_n, the relation $a_n \asymp b_n$ implies that there exists $0 < c_1 \leq c_2 < \infty$ such that $c_1 a_n \leq b_n \leq c_2 a_n$ for all $n \geq 1$.

Structure of the Paper

The paper is organized as follows: Section 2 introduces the Sieve-SMM estimator, explains how to implement it in practice and provides important properties of the mixture sieve. Section 3 gives the main asymptotic results: under regularity conditions, the estimator is consistent. Its rate of convergence is derived, and under further conditions, finite dimensional functionals of the estimates are asymptotically normal. Section 4 provides two extensions, one to include auxiliary variables in the CF and another to allow for dynamic panels with small T. Section 5 provides Monte-Carlo simulations to illustrate the theoretical results. Section 6 gives empirical examples for the estimator. Section 7 concludes. Appendix A gives some information about the CF and details on how to compute the estimator in practice. Appendix B provides the proofs to the main

results. The online supplement includes:[9] Appendix C which provides results for more general moment functions and sieve bases and Appendix D which provides the proofs for these results.

2 The Sieve-SMM Estimator

This section introduces the notation used in the remainder of the paper. It describes the class of DGPs considered in the paper and describes the DGP of the leading example in more details. It discusses the choice of mixture sieve, moments and objective function as well as some important properties of the mixture sieve. The running example used throughout the analysis is based on the empirical applications of section 6.

Example 1 (Stochastic Volatility Models)**.** *In both empirical applications, y_t follows an AR(1) process with log-normal stochastic volatility*

$$y_t = \mu_y + \rho_y y_{t-1} + \sigma_t e_{t,1}.$$

The first empirical application estimates a linear volatility process:

$$\sigma_t^2 = \mu_\sigma + \rho_\sigma \sigma_{t-1}^2 + \kappa_\sigma e_{t,2}$$

where $e_{t,2} \sim \chi_1^2$. The second empirical application estimates a log-normal stochastic volatility process:

$$\log(\sigma_t) = \mu_\sigma + \rho_\sigma \log(\sigma_{t-1}) + \kappa_\sigma e_{t,2}.$$

where $e_{t,2} \overset{iid}{\sim} \mathcal{N}(0,1)$. In both applications $e_{t,1} \overset{iid}{\sim} f$ with the restrictions $\mathbb{E}(e_{t,1}) = 0$ and $\mathbb{E}(e_{t,1}^2) = 1$. The first application approximates f with a mixture of Gaussian distributions, the second adds two tail components to model potential fat tails.

Stochastic volatility (SV) models in Example 1 are intractable because of the latent volatility. With log-normal volatility, the model becomes tractable after taking the transformation $\log([y_t - \mu_y - \rho_y y_{t-1}]^2)$ (see e.g. Kim et al., 1998) and the problem can be cast as a deconvolution problem (Comte, 2004). However, the transformation removes all the information about asymmetries in f, which turn out to empirically significant (see section 6). In the parametric case, alternatives to using the transformation involve Bayesian simulation-based estimators such as the Particle Filter and Gibbs sampling or EMM for frequentist estimation.

[9] The online supplement can be found at http://jjforneron.com/SieveSMM/Supplement.pdf.

2.1 Sieve Basis - Gaussian and Tails Mixture

The following definition introduces the Gaussian and tails mixture sieve that will be used in the paper. It combines a simple Gaussian mixture with two tails densities which model asymmetric fat tails parametrically. Drawing from this mixture is computationally simple: draw uniforms and gaussian random variables, switch between the Gaussians and the tails depending on the uniform and the mixture weights ω. The tail draws are a simple function of uniform random variables.

Definition 1 (Gaussian and Tails Mixture)**.** *A random variable e follows a k component Gaussian and Tails mixture if its density has the form:*

$$f_{\omega,\mu,\sigma}(e) = \sum_{j=1}^{k} \frac{\omega_j}{\sigma_j}\phi(\frac{e-\mu_j}{\sigma_j}) + \frac{\omega_{k+1}}{\sigma_{k+1}}\mathbb{1}_{e\leq\mu_{k+1}} f_L\left(\frac{e-\mu_{k+1}}{\sigma_{k+1}}\right) + \frac{\omega_{k+2}}{\sigma_{k+2}}\mathbb{1}_{e\geq\mu_{k+2}} f_R\left(\frac{e-\mu_{k+2}}{\sigma_{k+2}}\right)$$

where ϕ is the standard Gaussian density and its left and right tail components are

$$f_L(e,\xi_L) = (2+\xi_L)\frac{|e|^{1+\xi_L}}{[1+|e|^{2+\xi_L}]^2} \quad \text{for } e\leq 0, \quad f_R(e,\xi_R) = (2+\xi_R)\frac{e^{1+\xi_R}}{[1+e^{2+\xi_R}]^2} \quad \text{for } e\geq 0$$

with $f_L(e,\xi_L) = 0$ for $e \geq 0$ and $f_R(e,\xi_L) = 0$ for $e \leq 0$. To simulate from the Gaussian and tails mixture, draw $Z_1,\ldots,Z_k \overset{iid}{\sim} \mathcal{N}(0,1)$, $\nu,\nu_L,\nu_R \overset{iid}{\sim} \mathcal{U}_{[0,1]}$ and compute $Z_{k+1} = -\left(\frac{1}{\nu_L}-1\right)^{\frac{1}{2+\xi_L}}$ and $Z_{k+2} = \left(\frac{1}{\nu_R}-1\right)^{\frac{1}{2+\xi_R}}$. Then, for $\omega_0 = 0$:

$$e = \sum_{j=1}^{k+2} \mathbb{1}_{\nu\in[\sum_{l=0}^{j-1}\omega_l, \sum_{l=0}^{j}\omega_l]}\left(\mu_j + \sigma_j Z_j\right)$$

follows the Gaussian and tails mixture $f_{\omega,\mu,\sigma}$.

For application where fat tails are deemed unlikely, as in the first empirical application, the weights $\omega_{k+1},\omega_{k+2}$ can be set to zero to use a Gaussian mixture. If $\frac{\omega_{k+1}}{\sigma_{k+1}} \neq 0$ and $\frac{\omega_{k+2}}{\sigma_{k+2}} \neq 0$ then the left and right tails satisfy:

$$f_L(e) \overset{e\to-\infty}{\sim} |e|^{-3-\xi_L}, \quad f_R(e) \overset{e\to+\infty}{\sim} e^{-3-\xi_R}.$$

If $\xi_L,\xi_R \geq 0$ then draws from the tail components have finite expectation, they also have finite variance if $\xi_L,\xi_R \geq 1$. More generally, for the j-th moment to be finite, $j \geq 1, \xi_L,\xi_R \geq j$ is necessary. Gallant & Nychka (1987) also add a parametric component to model fat tails by using a mixture of a Hermite polynomial with a Student density. However, neither the Hermite polynomial nor the Student t-distribution have closed-form quantiles, which is not practical for simulation. Here, the densities f_L, f_R are constructed to be easy to simulated from.

The indicator function $\mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1} \omega_l, \sum_{l=0}^{j} \omega_l]}$ introduces discontinuities in the parameter ω. Standard derivative-free optimization routines such as the Nelder-Mead algorithm (Nelder & Mead, 1965) as implemented in the NLopt library of Johnson (2014) can handle this estimation problem as illustrated in section 5.[10]

In the finite mixture literature, mixture components are known to be difficult to identify because of possible label switching and the likelihood is globally unbounded.[11] Using the characteristic function rather than the likelihood resolves the unbounded likelihood problem as discussed in Yu (1998). More importantly, the object of interest in this paper is the mixture density $f_{\omega,\mu,\sigma}$ itself rather than the mixture components. As a result, permutations of the mixture components are not a concern, since they do not affect the resulting mixture density $f_{\omega,\mu,\sigma}$.

2.2 Moments - Empirical Characteristic Function and Objective Function

As in the parametric case, the moments need to be informative enough to identify the parameters. In Sieve-SMM estimation, the parameter $\beta = (\theta, f)$ is infinite dimensional so that no finite dimensional vector of moments could possibly identify β. As a result, this paper relies on moment functions which are themselves infinite dimensional.

The leading choice of moment function in this paper is the empirical characteristic function for the joint vector of lagged observations $(\mathbf{y}_t, \mathbf{x}_t) = (y_t, \ldots, y_{t-L}, x_t, \ldots, x_{t-L})$:

$$\hat{\psi}_n(\tau) = \frac{1}{n} \sum_{t=1}^{n} e^{i\tau'(\mathbf{y}_t, \mathbf{x}_t)}, \ \forall \tau \in \mathbb{R}^{d_\tau}$$

where i is the imaginary number such that $i^2 = -1$.[12] The CF is one-to-one with the joint distribution of $(\mathbf{y}_t, \mathbf{x}_t)$, so that the model is identified by $\hat{\psi}_n(\cdot)$ if and only if the distribution of $(\mathbf{y}_t, \mathbf{x}_t)$ identifies the true β_0. Using lagged variables allows to identify the dynamics in the data. Knight & Yu (2002) show how the characteristic function can identify parametric dynamic models. Some useful properties of the CF are given in Appendix A.1.

Besides the CF, another choice of bounded moment function is the CDF. While the CF is a smooth transformation of the data, the empirical CDF has discontinuities at each point of support of the data $(\mathbf{y}_t, \mathbf{x}_t)$ which could make numerical optimization more challenging. Also, the CF around $\tau = 0$ summarizes the information about the tails of the distribution (see Ushakov, 1999, page 30). This information is thus easier to extract from the CF than the CDF. The main results of

[10] The NLopt library is available for C++, Fortran, Julia, Matlab, Python and R among others.

[11] See e.g. McLachlan & Peel (2000) for a review of estimation, identification and applications of finite mixtures. See also Chen et al. (2014b) for some recent results.

[12] The moments can also be expressed in terms of sines and cosines since $e^{i\tau'(\mathbf{y}_t, \mathbf{x}_t)} = \cos(\tau'(\mathbf{y}_t, \mathbf{x}_t)) + i \sin(\tau'(\mathbf{y}_t, \mathbf{x}_t))$.

this paper can be extended to any bounded moment function satisfying a Lipschitz condition.[13]

Since the moments are infinite dimensional, this paper adopts the objective function of Carrasco & Florens (2000); Carrasco et al. (2007a) to handle the continuum of moment conditions:[14]

$$\hat{Q}_n^S(\beta) = \int \left| \hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta) \right|^2 \pi(\tau) d\tau. \tag{5}$$

The objective function is a weighted average of the square norm between the empirical $\hat{\psi}_n$ and the simulated $\hat{\psi}_n^S$ moment functions. As discussed in Carrasco & Florens (2000) and Carrasco et al. (2007a), using the continuum of moments avoids the problem of constructing an increasing vector of moments. The weighting density π is chosen to be the multivariate normal density for the main results. Other choices for π are possible as long as it has full support and is such that $\int \sqrt{\pi(\tau)} d\tau < \infty$. As an example, the exponential distribution satisfies these two conditions, while the Cauchy distribution does not satisfy the second. In practice, choosing π to be the Gaussian density with same mean and variance as $(\mathbf{y_t}, \mathbf{x}_t)$ gave satisfying results in sections 5 and 6.[15] In the appendix, the results allow for a bounded linear operator B which plays the role of the weight matrix W in SMM and GMM as in Carrasco & Florens (2000). Carrasco & Florens (2000); Carrasco et al. (2007a) provide theoretical results for choosing and approximating the optimal operator B in the parametric setting. Similar work is left to future research in this semi-nonparametric setting.

Given the sieve basis, the moments and the objective function, the estimator $\hat{\beta}_n = (\hat{\theta}_n, \hat{f}_n)$ is defined as an approximate minimizer of $\hat{Q}_n^S$:

$$\hat{Q}_n^S(\hat{\beta}_n) \leq \inf_{\beta \in \mathcal{B}_{k(n)}} \hat{Q}_n^S(\beta) + O_p(\eta_n) \tag{6}$$

where $\eta_n \geq 0$ and $\eta_n = o(1)$ corresponds to numerical optimization and integration errors. Indeed, since the integral in (5) needs to be evaluated numerically, some form of numerical integration is required. Quadrature and sparse quadrature were found to give satisfying results when $\dim(\tau)$ is not too large (less than 4). For larger dimensions, quasi-Monte-Carlo integration using either the Halton or Sobol sequence gave satisfying results.[16] All Monte-Carlo simulations and empirical results in this paper are based on quasi-Monte-Carlo integration. Additional details on the computation of the objective function are given in Appendix A.2.

[13] Appendix C allows for more general non-Lipschitz moment functions and other sieve bases. However, the conditions required for these results are more difficult to check.

[14] Carrasco & Florens (2000) provide a general theory for GMM estimation with a continuum of moment conditions. They show how to efficiently weight the continuum of moments and propose a Tikhonov (ridge) regularization approach to invert the singular variance-covariance operator. Earlier results, without optimal weighting, include Koul (1986) for minimum distance estimation with a continuum of moments.

[15] Monte-Carlo experiments not reported in this paper showed similar results when using the exponential density for π instead of the Gaussian density.

[16] See e.g. Heiss & Winschel (2008); Holtz (2011) for an introduction to sparse quadrature in economics and finance, and Owen (2003) for quasi-Monte-Carlo sampling.

2.3 Approximation and L^2-Smoothness Properties of the Mixture Sieve

This subsection provides more details on the approximation and L^p-smoothness properties of the mixture sieve. It also provides the necessary restrictions on the true density f_0 to be estimated. Gaussian mixtures can approximate any smooth univariate density but the rate of this approximation depends on both the smoothness and the tails of the density (see e.g. Kruijer et al., 2010). The tail densities parametrically model asymmetric fat tails in the density. This is useful in the second empirical example since a thin tail assumption may not hold for exchange rate data. The following lemma extends the approximation results of Kruijer et al. (2010) to a multivariate density with independent components and potentially fat tails.

Lemma 1 (Approximation Properties of the Gaussian and Tails Mixture)**.** *Suppose that the shocks $e = (e_{t,1}, \dots, e_{t,d_e})$ are independent with density $f = f_1 \times \cdots \times f_{d_e}$. Suppose that each marginal f_j can be decomposed into a smooth density $f_{j,S}$ and the two tails f_L, f_R of Definition 1:*

$$f_j = (1 - \omega_{j,1} - \omega_{j,2}) f_{j,S} + \omega_{j,1} f_L + \omega_{j,2} f_R.$$

Let each $f_{j,S}$ satisfy the assumptions of Kruijer et al. (2010):

i. Smoothness: $f_{j,S}$ is r-times continuously differentiable with bounded r-th derivative.

ii. Tails: $f_{j,S}$ has exponential tails, i.e. there exists $\bar{e}, M_f, a, b > 0$ such that:

$$f_{j,S}(e) \leq M_f e^{-a|e|^b}, \forall |e| \geq \bar{e}.$$

iii. Monotonicity in the Tails: $f_{j,S}$ is strictly positive and there exists $\underline{e} < \bar{e}$ such that $f_{j,S}$ is weakly decreasing on $(-\infty, \underline{e}]$ and weakly increasing on $[\bar{e}, \infty)$.

and $\|f_j\|_\infty \leq \overline{f}$ for all j. Then there exists a Gaussian and tails mixture $\Pi_k f = \Pi_k f_1 \times \cdots \times \Pi_k f_{d_e}$ satisfying the restrictions of Kruijer et al. (2010):

iv. Bandwidth: $\sigma_j \geq \underline{\sigma}_k = O(\frac{\log[k]^{2/b}}{k})$.

v. Location Parameter Bounds: $\mu_j \in [-\bar{\mu}_k, \bar{\mu}_k]$ with $\bar{\mu}_k = O\left(\log[k]^{1/b}\right)$

such that as $k \to \infty$:

$$\|f - \Pi_k f\|_{\mathcal{F}} = O\left(\frac{\log[k]^{2r/b}}{k^r}\right)$$

where $\|\cdot\|_{\mathcal{F}} = \|\cdot\|_{TV}$ or $\|\cdot\|_\infty$.

The space of true densities satisfying the assumptions will be denoted as $\mathcal{F}$ and $\mathcal{F}_k$ is the corresponding space of Gaussian and tails mixtures $\Pi_k f$.

Note that additional restrictions on f may be required for identification, such as mean zero, unit variance or symmetry. The assumption that the shocks are independent is not too strong for structural models where this, or a parametric factor structure is typically assumed. Note that under this assumption, there is no curse of dimensionality because the components f_j can be approximated separately. Also, the restriction $\|f_j\|_\infty \leq \overline{f}$ is only required for the approximation in supremum norm $\|\cdot\|_\infty$.

An important difficulty which arises in simulating from a nonparametric density is that draws are a very nonlinear transformation of the nonparametric density f. As a result, standard regularity conditions such as Hölder and L^p-smoothness are difficult to verify and may only hold under restrictive conditions. The following discusses these regularity conditions for the methods used in the previous literature and provides a L^p-smoothness result the mixture sieve (Lemma 2 below).

Bierens & Song (2012) use Inversion Sampling: they compute the CDF F_k from the nonparametric density and draw $F_k^{-1}(\nu_t^s), \nu_t^s \overset{iid}{\sim} \mathcal{U}_{[0,1]}$. Computing the CDF and its inverse to simulate is very computationally demanding. Also, while the CDF is linear in the density, its inverse is a highly non-linear transformation of the density. Hence, Hölder and L^p-smoothness results for the draws are much more challenging to prove without further restrictions.

Newey (2001) uses Importance Sampling for which Hölder conditions are easily verified but requires $S \to \infty$ for consistency alone. Furthermore, the choice of importance distribution is very important for the finite sample properties (the effective sample size) of the simulated moments. In practice, the importance distribution should give sufficient weight to regions for which the nonparametric density has more weight. Since the nonparametric density is unknown ex-ante, this is hard to achieve in practice.

Gallant & Tauchen (1993) use Accept/Reject (outside of an estimation setting): however, it is not practical for simulation-based estimation. Indeed, the required number of draws to generate an accepted draw depends on both the instrumental density and the target density $f_{\omega,\mu,\sigma}$. The latter varies with the parameters during the optimization. This also makes the L^p-smoothnes properties challenging to establish. In comparison, the following lemma shows that the required L^2-smoothness condition is satisfied by draws from a mixture sieve.

Lemma 2 (L^2-Smoothness of Simulated Mixture Sieves)**.** *Suppose that*

$$e_t^s = \sum_{j=1}^{k(n)} \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\omega_l, \sum_{l=0}^{j}\omega_l]} \left(\mu_j + \sigma_j Z_{t,j}^s\right), \quad \tilde{e}_t^s = \sum_{j=1}^{k(n)} \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\tilde{\omega}_l, \sum_{l=0}^{j}\tilde{\omega}_l]} \left(\tilde{\mu}_j + \tilde{\sigma}_j Z_{t,j}^s\right)$$

with $|\mu_j|$ *and* $|\tilde{\mu}_j| \leq \bar{\mu}_{k(n)}$, $|\sigma_j|$ *and* $|\tilde{\sigma}_j| \leq \bar{\sigma}$. *If* $\mathbb{E}(|Z_{t,j}^s|^2) \leq C_Z^2 < \infty$ *then there exists a finite constant* C *which only depends on* C_Z *such that:*

$$\left[\mathbb{E}\left(\sup_{\|f_{\omega,\mu,\sigma}-f_{\tilde{\omega},\tilde{\mu},\tilde{\sigma}}\|_m \leq \delta} \left|e_t^s - \tilde{e}_t^s\right|^2\right)\right]^{1/2} \leq C\left(1 + \bar{\mu}_{k(n)} + \bar{\sigma} + k(n)\right)\delta^{1/2}.$$

Lemma 2 is key in proving the L^2-smoothness conditions of the moments $\hat{\psi}_n^s$ required to establish the convergence rate of the objective function and stochastic equicontinuity results. Here, the L^p-smoothness constant depends on both the bound $\overline{\mu}_{k(n)}$ and the number of mixture components $k(n)$.[17] Kruijer et al. (2010) showed that both the total variation and supremum norms are bounded above by the pseudo-norm $\|\cdot\|_m$ on the mixture parameters (ω, μ, σ) up to a factor which depends on the bandwidth $\underline{\sigma}_{k(n)}$. As a result, the pseudo-norm $\|\cdot\|_m$ controls the distance between densities and the simulated draws as well. Furthermore, draws from the tail components are shown in the appendix to be L^2-smooth in their tail parameters ξ_L, ξ_R. Hence, draws from the Gaussian and tails mixture are L^2-smooth in both (ω, μ, σ) and ξ.

3 Asymptotic Properties of the Estimator

This section provides conditions under which the Sieve-SMM estimator in (6) is consistent. Its rate of convergence is derived and an asymptotic normality result for functionals of $\hat{\beta}_n$ is given.

3.1 Consistency

Consistency results are given under low-level conditions on the DGP using the Gaussian and tails mixture sieve with the CF.[18] First, the population objective Q_n is:

$$Q_n(\beta) = \int \left|\mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta)\right)\right|^2 \pi(\tau) d\tau. \tag{7}$$

The objective depends on n because (y_t^s, x_t) are not covariance stationary: the moments can depend on t. Under geometric ergodicity, it has a well-defined limit:[19]

$$Q_n(\beta) \overset{n\to\infty}{\to} Q(\beta) = \int \left|\lim_{n\to\infty} \mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta)\right)\right|^2 \pi(\tau) d\tau.$$

In the definition of the objective Q_n and its limit Q, the expectation is taken over both the data $(\mathbf{y}_t, \mathbf{x}_t)$ and the simulated samples $(\mathbf{y}_t^s, \mathbf{x}_t)$. The following assumption, provide a set of sufficient conditions on the true density f_0, the sieve space and a first set of conditions on the model (identification and time-series properties) to prove consistency.

[17] See e.g. Andrews (1994); Chen et al. (2003) for examples of L^p-smooth functions.

[18] Consistency results allowing for non-mixture sieves and other moments are given in Appendix C.1.

[19] Since the CF is bounded, the dominated convergence theorem can be used to prove the existence of the limit.

Assumption 1 (Sieve, Identification, Dependence)**.** *Suppose the following conditions hold:*

i. (Sieve Space) the true density f_0 and the mixture sieve space $\mathcal{F}_{k(n)}$ satisfy the assumptions of Lemma 1 with $k(n)^4 \log[k(n)]^4/n \to 0$ as $k(n)$ and $n \to \infty$. Θ is compact and $1 \leq \xi_L, \xi_R \leq \overline{\xi} < \infty$.

ii. (Identification) $\lim_{n\to\infty} \mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^s(\tau,\beta)\right) = 0, \pi$ a.s. $\Leftrightarrow \|\beta - \beta_0\|_{\mathcal{B}} = 0$ where π is the Gaussian density. For any $n, k \geq 1$ and for all $\varepsilon > 0$, $\inf_{\beta\in\mathcal{B}_k, \|\beta-\beta_0\|_{\mathcal{B}}\geq\varepsilon} Q_n(\beta)$ is strictly positive and weakly decreasing in both n and k.

iii. (Dependence) (y_t, x_t) is strictly stationary and α-mixing with exponential decay, the simulated $(y_t^s(\beta), x_t)$ are geometrically ergodic, uniformly in $\beta \in \mathcal{B}$.

Condition *i.* is stronger than the usual condition $k(n)/n \to 0$ in the sieve literature (see e.g Chen, 2007). The additional $\log[k(n)]$ term is due to the mixture being a non-linear sieve basis and the fourth power is due to the dependence. Indeed, the inequality in Lemma D15 implies that the variance is of order $k(n)^2 \log[k(n)]^2/\sqrt{n}$ instead of $\sqrt{k(n)\log[k(n)]/n}$ for iid data.

Condition *ii.* is the usual identification condition. It is assumed that the information from the joint distribution of $(\mathbf{y}_t, \mathbf{x}_t) = (y_t, \ldots, y_{t-L}, x_t, \ldots, x_{t-L})$ uniquely identifies $\beta = (\theta, f)$. Proving general global identification results is quite challenging in this setting and is left to future research. Local identification in the sense of Chen et al. (2014a) is also challenging to prove here because the dynamics imply that the distribution of (y_t^s, x_t, u_t^s) is a convolution of f with the distribution of $(y_{t-1}^s, x_t, u_{t-1}^s)$. Since the stationary distributions of (y_t^s, x_t, u_t^s) and $(y_{t-1}^s, x_t, u_{t-1}^s)$ are the same, the resulting distribution is the fixed point of its convolution with f. This makes derivatives with respect to f difficult to compute in many dynamic models. Note that the identification assumption does not exclude ill-posedness.[20] The space $\mathcal{F}$ is assumed to include the necessary restrictions (if any) for identification such as mean zero and unit variance. Global identification results for the stochastic volatility model in Example 1 are given in Appendix A.4.

Condition *iii.* is common in SMM estimation with dependent data (see e.g. Duffie & Singleton, 1993). In this setting, it implies two important features: the simulated (y_t^s, x_t) are α-mixing (Liebscher, 2005), and the initial condition bias is negligible: $Q_n(\beta_0) = O(1/n^2)$.[21]

Assumption 2 (Data Generating Process)**.** *y_t^s is simulated according to the dynamic model (1)-(2) where g_{obs} and g_{latent} satisfy the following Hölder conditions for some $\gamma \in (0, 1]$:*

y(i). $\|g_{obs}(y_1, x, \beta, u) - g_{obs}(y_2, x, \beta, u)\| \leq C_1(x, u)\|y_1 - y_2\|$ *with* $\mathbb{E}\left(C_1(x_t, u_t^s)^2 | y_{t-1}^s\right) \leq \bar{C}_1 < 1$.

y(ii). $\|g_{obs}(y, x, \beta_1, u) - g_{obs}(y, x, \beta_2, u)\| \leq C_2(y, x, u)\|\beta_1 - \beta_2\|_{\mathcal{B}}^{\gamma}$ *with* $\mathbb{E}\left(C(y_t^s, x_t, u_t^s)^2\right) \leq \bar{C}_2 < \infty$.

[20] See e.g. Carrasco et al. (2007b) and Horowitz (2014) for a review of ill-posedness in economics.

[21] See Proposition C4 in the supplemental material for the second result.

y(iii). $\|g_{obs}(y,x,\beta,u_1) - g_{obs}(y,x,\beta,u_2)\| \leq C_3(y,x)\|u_1 - u_2\|^\gamma$ *with* $\mathbb{E}\left(C_3(y_t^s, x_t)^2 | u_t^s\right) \leq \bar{C}_3 < \infty$.

u(i). $\|g_{latent}(u_1,\beta,e) - g_{latent}(u_2,\beta,e)\| \leq C_4(e)\|u_1 - u_2\|$ *with* $\mathbb{E}\left(C_4(e_t^s)^2\right) \leq \bar{C}_4 < 1$.

u(ii). $\|g_{latent}(u,\beta_1,e) - g_{latent}(u,\beta_2,e)\| \leq C_5(u,e)\|\beta_1 - \beta_2\|_{\mathcal{B}}^\gamma$ *with* $\mathbb{E}\left(C_5(u_{t-1}^s, e_t^s)^2\right) \leq \bar{C}_5 < \infty$.

u(iii). $\|g_{latent}(u,\beta,e_1) - g_{latent}(u,\beta,e_2)\| \leq C_6(u)\|e_1 - e_2\|$ *with* $\mathbb{E}\left(C_6(u_{t-1}^s)^2\right) \leq \bar{C}_6 < \infty$.

for any $(\beta_1,\beta_2) \in \mathcal{B}$, $(y_1,y_2) \in \mathbb{R}^{dim(y)}$, $(u_1,u_2) \in \mathbb{R}^{dim(u)}$ *and* $(e_1,e_2) \in \mathbb{R}^{dim(e)}$. *The norm* $\|\cdot\|_{\mathcal{B}}$ *is either the total variation or supremum norm.*

Conditions *y(ii), u(ii)* correspond to the usual Hölder conditions in GMM and M-estimation but placed on the DGP itself rather than the moments. Since the cosine and sine functions are Lipschitz, it implies that the moments are Hölder continuous as well.[22]

The decay conditions *y(i), u(i)* together with condition *y(iii)* ensure that the differences due to $\|\beta_1 - \beta_2\|_{\mathcal{B}}$ do not accumulate too much with the dynamics. As a result, keeping the shocks fixed, the Hölder condition applies to (y_t^s, u_t^s) as a whole. It also implies that the nonparametric approximation bias $\|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}$ does not accumulate too much. These conditions are similar to the L^2-Unit Circle condition which Duffie & Singleton (1993) suggest as an stronger alternative to geometric ergodicity in a uniform LLN and a CLT. The decay conditions play a more important role here since they are needed to control the nonparametric bias of the estimator.

Condition *u(iii)* ensures that the DGP preserves the L^2-smoothness properties derived for mixture draws in Lemma 2. This condition does not appear in the usual sieve literature which does not simulate from a nonparametric density. In the SMM literature, a Lipschitz or Hölder condition is usually given on the moments directly. Note that a condition analogous to *u(iii)* would also be required for parametric SMM estimation of a parametric distribution.

Assumption 2 does not impose that the DGP be smooth. This allows for kinks in g_{obs} or g_{latent} as in the sample selection model or the models of Deaton (1991) and Deaton & Laroque (1992). Assumption 2′ in Appendix B.2 extends Assumption 2 to allow for possible discontinuities in g_{obs}, g_{latent}. The following shows how to verify the conditions of Assumption 2 in Example 1 with χ_1^2 volatility shocks.[23]

[22] For any choice of moments that preserve identification and are Lipschitz, the main results will hold assuming $\|\tau\|_\infty\sqrt{\pi(\tau)}$ and $\int \sqrt{\pi(\tau)}d\tau$ are bounded. For both the Gaussian and the exponential density, these quantities turn out to be bounded. In general Lispchitz transformations preserve L^p-smoothness properties (see e.g. Andrews, 1994; van der Vaart & Wellner, 1996), here additional conditions on π are required to handle the continuum of moments with unbounded support.

[23] Some additional examples are given in Appendix C.4. They are not tied to the use of mixtures, and as a result, impose stronger restrictions on the density f such as bounded support.

Example 1 (Continued) (Stochastic Volatility). *If $|\rho_y| < 1$ then assumption y(i) is satisfied. Also:*

$$|\mu_{y,1} + \rho_{y,1} y_{t-1} - \mu_{y,2} - \rho_{y,2} y_{t-1}| \leq (|\mu_{y,1} - \mu_{y,2}| + |\rho_{y,1} - \rho_{y,2}|)(1 + |y_{t-1}|)$$

and thus condition y(ii) is satisfied assuming $\mathbb{E}(y_{t-1}^2)$ is bounded. Since f has mean zero and unit variance, $\mathbb{E}(y_{t-1}^2)$ is bounded if $|\mu_\sigma| \leq \bar{\mu}_\sigma < \infty$, $|\rho_\sigma| \leq \bar{\rho}_\sigma < 1$ and $\kappa_\sigma \leq \bar{\kappa}_\sigma < \infty$ for some $\bar{\mu}_\sigma, \bar{\rho}_\sigma, \bar{\kappa}_\sigma$. For condition y(iii), take $u_t = (\sigma_t^2, e_{t,1})$ and $\tilde{u}_t = (\tilde{\sigma}_t^2, \tilde{e}_{t,1})$:

$$|\sigma_t e_{t,1} - \tilde{\sigma}_t e_{t,1}| \leq |e_{t,1}| \sqrt{|\sigma_t^2 - \tilde{\sigma}_t^2|}, \quad |\sigma_t e_{t,1} - \sigma_t \tilde{e}_{t,1}| \leq \sigma_t |e_{t,1} - \tilde{e}_{t,1}|.$$

The first inequality is due to the Hölder continuity of the square-root function.[24] *σ_t and $\tilde{e}_{t,1}$ are independent, $\mathbb{E}(\sigma_t^2)$ is bounded above under the previous parameter bounds and $\mathbb{E}(e_{t,1}^2) = 1$ and so condition y(iii) holds term by term. If the volatility σ_t^2 is bounded below by a strictly positive constant for all paramater values then the Hölder continuity y(iii) can be strengthened to a Lipschitz continuity result. Given that σ_t^2 follows an AR(1) process, assumptions u(i), u(ii) and u(iii) are satisfied.*

The Hölder coefficient in conditions *y(ii), y(iii)* and *u(ii)* is assumed to be the same to simplify notation. If they were denoted γ_1, γ_2 and γ_3, in order of appearance, then the rate of convergence would depend on $\min(\gamma_1, \gamma_2\gamma_3)$ instead of γ^2. This could lead to sharper rates of convergence in section 3.2 and weaker condition for the stochastic equicontinuity result in section 3.3. As shown above, in Example 1 the Hölder coefficients are $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = 1/2$ when σ_t does not have a strictly positive lower bound.

Lemma 3 (Assumption 2/2′ implies L^2-Smoothness of the Moments). *Under either Assumption 2 or 2′, if the assumptions of Lemma 2 hold and π is the Gaussian density, then there exists $\overline{C} > 0$ such that for all $\delta > 0$, uniformly in $t \geq 1$, $(\beta_1, \beta_2) \in \mathcal{B}_{k(n)}$ and $\tau \in \mathbb{R}^{d_\tau}$:*

$$\mathbb{E}\left(\sup_{\|\beta_1 - \beta_2\|_m \leq \delta} \left| e^{i\tau'(\mathbf{y}_t^s(\beta_1), \mathbf{x}_t)} - e^{i\tau'(\mathbf{y}_t^s(\beta_2), \mathbf{x}_t)} \right|^2 \sqrt{\pi(\tau)} \right) \leq \overline{C} \max\left(\frac{\delta^{\gamma^2}}{\underline{\sigma}_{k(n)}^{2\gamma^2}}, [k(n) + \overline{\mu}_{k(n)} + \overline{\sigma}]^\gamma \delta^{\gamma^2/2} \right)$$

where $\|\beta\|_m = \|\theta\| + \|(\omega, \mu, \sigma)\|$ is the pseudo-norm on θ and the mixture parameters (ω, μ, σ) from Lemma 2. Also, since π is the Gaussian density the integral $\int \sqrt{\pi(\tau)} d\tau$ is finite.

Lemma 3 gives the first implication of Assumption 2. It shows that the moments $\hat{\psi}_t^s$ are L^2-smooth, uniformly in $t \geq 1$. The L^2-smoothness factor depends on the bounds of the sieve components. In the SMM and sieve literatures, the L^p-smoothness constant depends on neither k nor n by assumption. Here, drawing from the mixture distribution implies that the constant will increase

[24] For any two $x, y \geq 0$, $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x^2 - y^2|}$.

with the sample size n. The rate at which it increases is implied by the assumptions of Lemma 1.[25] Furthermore, because the index τ has unbounded support, the L^2-smoothness result involves the weights via $\sqrt{\pi}$. Without π, the L^2-smoothness result may not hold uniformly in $\tau \in \mathbb{R}^{d_\tau}$.

Lemma 4 (Nonparametric Approximation Bias)**.** *Suppose Assumptions 1 and 2 (or 2′) hold. Furthermore suppose that* $\mathbb{E}\left(\|y_t^s\|^2\right)$ *and* $\mathbb{E}\left(\|u_t^s\|^2\right)$ *are bounded for* $\beta = \beta_0$ *and* $\beta = \Pi_{k(n)}\beta_0$ *for all* $k(n) \geq 1$, $t \geq 1$ *then:*

$$Q_n(\Pi_{k(n)}\beta_0) = O\left(\max\left[\frac{\log[k(n)]^{4r/b+2}}{k(n)^{2r}}, \frac{\log[k(n)]^{4\gamma^2 r/b}}{k(n)^{2\gamma^2 r}}, \frac{1}{n^2}\right]\right) = O\left(\frac{\log[k(n)]^{4r/b+2}}{k(n)^{2\gamma^2 r}}\right)$$

where $\Pi_{k(n)}\beta_0$ *is the mixture sieve approximation of* β_0, γ *the Hölder coefficient in Assumption 2,* b *and* r *are the exponential tail index and the smoothness of the density* f_S *in Lemma 1.*

Lemma 4 gives the second implication of Assumption 2; it computes the value of the objective function Q_n at $\Pi_{k(n)}\beta_0$, which is directly related to the bias of the estimator $\hat{\beta}_n$. Two terms are particularly important for the rate of convergence: the smoothness of the true density r and the roughness of the DGP as measured by the Hölder coefficient $\gamma \in (0, 1]$. If r and γ are larger then the bias will be smaller. The rate in this lemma is different from the usual rate found in the sieve literature. Chen & Pouzo (2012) assume for instance that $Q_n(\Pi_{k(n)}\beta_0) \asymp \|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}^2$. In comparison, the rate derived here is:

$$Q_n(\Pi_{k(n)}\beta_0) \asymp \max\left(\|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}^2 \log\left(\|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}\right)^2, \|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{2\gamma^2}, 1/n^2\right)$$

with $\|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}} = O(\log[k(n)]^{2r/b}/k(n)^r)$ as given in Lemma 1. The $1/n^2$ term corresponds to the bias due to the nonstationarity, its order is implied by the geometric ergodicity condition and the boundedness of the moments. The log-bias term $\log\left(\|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}\right)$ is due to the dynamics: y_t^s depends on the full history $(e_t^s, \ldots, e_1^s)$ which are iid $\Pi_{k(n)}f_0$, so that the bias accumulates. The decay conditions *y(i), y(iii), u(i)* ensure that the resulting bias accumulation only inflates bias by a log term. The term $\|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{2\gamma^2}$ is due to the Hölder smoothness of the DGP. If the DGP is Lipschitz, i.e. $\gamma = 1$, and the model is static then the rate becomes $Q_n(\Pi_{k(n)}\beta_0) \asymp \|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}^2$, which is the rate assumed in Chen & Pouzo (2012).

Theorem 1 (Consistency)**.** *Suppose Assumptions 1 and 2 (or 2′) hold. Suppose that* $\beta \to Q_n(\beta)$ *is continuous on* $(\mathcal{B}_{k(n)}, \|\cdot\|_{\mathcal{B}})$ *and the numerical optimization and integration errors are such that* $\eta_n =$

[25] Under the assumption of Lemma 1: $\underline{\sigma}_{k(n)}^{-2\gamma^2} = O\left(k(n)^{2\gamma^2}/\log[(n)]^{4\gamma^2/b}\right)$ and $[k(n) + \overline{\mu}_{k(n)} + \overline{\sigma}]^\gamma = O\left(k(n)^\gamma\right)$. As a result, the maximum term is bounded above by $\max\left(k(n)^{2\gamma^2}, k(n)^\gamma\right)\delta^{\gamma^2/2}$ (up to a constant).

$o(1/n)$. *If for all* $\varepsilon > 0$ *the following holds:*

$$\max\left(\frac{log[k(n)]^{4r/b+2}}{k(n)^{2\gamma^2 r}}, \frac{k(n)^4 \log[k(n)]^4}{n}, \frac{1}{n^2}\right) = o\left(\inf_{\beta\in\mathcal{B}_{k(n)}, \|\beta-\beta_0\|_{\mathcal{B}}\geq\varepsilon} Q_n(\beta)\right) \tag{8}$$

where r *is the assumed smoothness of the smooth component* f_S *and* b *its exponential tail index. Then the Sieve-SMM estimator is consistent:*

$$\|\hat{\beta}_n - \beta_0\|_{\mathcal{B}} = o_p(1).$$

Theorem 1 is a consequence of the general consistency lemma in Chen & Pouzo (2012) reproduced as Lemma D12 in the appendix. They provide high level conditions which Assumption 2 together with Lemmas 3 and 4 verify for simulation-based estimation of static and dynamic models. Condition (8) in Theorem 1 allows for ill-posedness but requires the minimum to be well separated on the sieve space relative to the bias and the variance.

The variance term $k(n)^4 \log[k(n)]^4/n$ is derived using the inequality in Lemma D15 which is adapted from existing results of Andrews & Pollard (1994); Ben Hariz (2005). It is based on the moment inequalities for α-mixing sequences of Rio (2000) rather than coupling results (see e.g. Doukhan et al., 1995; Chen & Shen, 1998; Dedecker & Louhichi, 2002). This implies that the moments can be nonstationary, because of the initial condition, and depend on arbitrarily many lags as in Example 1 where y_t^s is a function of $e_t^s, \ldots, e_t^1$. It also allows for filtering procedures as in the first extension of the main results. The two main drawbacks of this inequality is that it requires uniformly bounded moments and implies a larger variance than, for instance, in the iid case. The boundedness restricts the class of moments used in Sieve-SMM and the larger variance implies a slower rate of convergence.

3.2 Rate of Convergence

Once the consistency of the estimator is established, the next step is to derive its rate of convergence. It is particularly important to derive rates that are as sharp as possible since a rate of a least $n^{-1/4}$ under the weak norm of Ai & Chen (2003) is required for the asymptotic normality results. This weak norm is introduced below for the continuum of complex valued moments. It is related to the objective function Q_n, and as such allows to derive the rate of convergence of $\hat{\beta}_n$.[26] Ultimately, the norm of interest in the strong norm $\|\cdot\|_{\mathcal{B}}$ which is generally not equivalent to the weak norm since the space is infinite dimensional. The two are related by the local measure of ill-posedness of Blundell et al. (2007) which allows to derive the rate of convergence in the strong norm, that is in either the total variation or the supremum norm.

[26] For a discussion see Ai & Chen (2003) and Chen (2007).

Assumption 3 (Weak Norm and Local Properties). *Let $\mathcal{B}_{osn} = \mathcal{B}_{k(n)} \cap \{\|\beta - \beta_0\|_{\mathcal{B}} \leq \varepsilon\}$ for $\varepsilon > 0$ small and for $(\beta_1, \beta_2) \in \mathcal{B}_{osn}$:*

$$\|\beta_1 - \beta_2\|_{weak} = \left[\int \left| \frac{d\mathbb{E}\left(\hat{\psi}_n^S(\tau, \beta_0)\right)}{d\beta} [\beta_1 - \beta_2] \right|^2 \pi(\tau) d\tau \right]^{1/2} \tag{9}$$

is the weak norm of $\beta_1 - \beta_2$. Suppose that there exists $\underline{C}_w > 0$ such that for all $\beta \in \mathcal{B}_{osn}$:

$$\underline{C}_w \|\beta - \beta_0\|_{weak}^2 \leq \int \left| \mathbb{E}\left(\hat{\psi}_n^S(\tau, \beta_0) - \hat{\psi}_n^S(\tau, \beta)\right) \right|^2 \pi(\tau) d\tau. \tag{10}$$

Assumption 3 adapts the weak norm of Ai & Chen (2003) to an objective with a continuum of complex-valued moments. Note that $\int |\mathbb{E}\left(\hat{\psi}_n^S(\tau, \beta_0) - \hat{\psi}_n^S(\tau, \beta)\right)|^2 \pi(\tau) d\tau = Q_n(\beta_0) + O_p(1/n^2)$ under geometric ergodicity. As a result, Assumption 3 implies that the weak norm is Lipschitz continuous with respect to $\sqrt{Q_n}$. Additional assumptions on the norm and the objective are usually required such as: $Q_n(\beta) \asymp \|\beta - \beta_0\|_{weak}^2$ and $Q_n(\beta) \leq C_{\mathcal{B}} \|\beta - \beta_0\|_{\mathcal{B}}$ (see e.g. Chen & Pouzo, 2015, Assumption 3.4). Instead of these assumptions, the results in this paper rely on Lemma 4 to derive the bias of the estimator. The resulting bias is larger than in the usual sieve literature.

Theorem 2 (Rate of Convergence). *Suppose that the assumptions for Theorem 1 hold and Assumption 3 also holds.The convergence rate in weak norm is:*

$$\|\hat{\beta}_n - \beta_0\|_{weak} = O_p\left(\max\left(\frac{\log[k(n)]^{r/b+1}}{k(n)^{\gamma^2 r}}, \frac{k(n)^2 \log[k(n)]^2}{\sqrt{n}} \right) \right). \tag{11}$$

The convergence rate in either the total variation or supremum norm $\|\cdot\|_{\mathcal{B}}$ is:

$$\|\hat{\beta}_n - \beta_0\|_{\mathcal{B}} = O_p\left(\frac{\log[k(n)]^{r/b}}{k(n)^r} + \tau_{\mathcal{B},n} \max\left(\frac{\log[k(n)]^{r/b+1}}{k(n)^{\gamma^2 r}}, \frac{k(n)^2 \log[k(n)]^2}{\sqrt{n}} \right) \right)$$

where $\tau_{\mathcal{B},n}$ is the local measure of ill-posedness of Blundell et al. (2007):

$$\tau_{\mathcal{B},n} = \sup_{\beta \in \mathcal{B}_{osn},\, \|\beta - \Pi_{k(n)}\beta_0\|_{weak} \neq 0} \frac{\|\beta - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}}{\|\beta - \Pi_{k(n)}\beta_0\|_{weak}}.$$

As usual in the (semi)-nonparametric estimation literature, the rate of convergence involves a bias/variance trade-off. As discussed before, the bias is larger than usual because of the dynamics and involves the Hölder smoothness γ of the DGP.

The variance term is of order $k(n)^2 \log[k(n)]^2/\sqrt{n}$ instead of $\sqrt{k(n)}/\sqrt{n}$ in the iid case or strictly stationary case with fixed number of lags in the moments. This is because the inequality in Lemma D15 is more conservative than the inequalities found in Theorem 2.14.2 of van der Vaart & Wellner (1996) for iid observations or the inequalities based on a coupling argument in

Doukhan et al. (1995); Chen & Shen (1998) for strictly stationary dependent data. However, in this simulation-based setting the dependence properties of y_t^s varies on θ over the parameter space Θ so that a coupling approach may not apply unless it only depends on finitely many lags of e_t and x_t. Determining whether this inequality can be sharpened in subject to future research.

The increased bias and variance imply a slower rate of convergence than usual. The optimal rate of convergence equates the bias and variance terms in equation (11). This is achieved (up to a log term) by picking $k(n) = O(n^{\frac{1}{2(2+\gamma^2 r)}})$. To illustrate, for a Lipschitz DGP $\gamma = 1$ and f_0 twice continuously differentiable $r = 2$ and $k(n) \asymp n^{1/8}$, the rate of convergence becomes:

$$\|\hat{\beta}_n - \beta_0\|_{weak} = O_p(n^{-1/4} \log(n)^{\max(2/b+1,2)}).$$

In comparison, if (y_t^s, x_t) were iid, keeping $\gamma = 1$ and $r = 2$, the variance term would be $\sqrt{k(n) \log[k(n)]/n}$ and the optimal $k(n) \asymp n^{1/5}$. The rate of convergence becomes:

$$\|\hat{\beta}_n - \beta_0\|_{weak} = O_p\left(n^{-2/5} \log(n)^{\max(2/b+1,2)}\right).$$

To achieve a rate faster than $n^{-1/4}$, as required for asymptotic normality, the smoothness of the true density f_0 must satisfy $r \geq 3/\gamma^2$ where γ is the Hölder coefficient in Assumption 2. In the Lipschitz case, $\gamma = 1$, at 3 derivatives are needed compared to 12 derivatives when $\gamma = 1/2$. In comparison, in the iid case 2 and 8 derivatives are needed for $\gamma = 1$ and $\gamma = 1/2$ respectively.

The following corollary shows that the number of simulated samples S can significantly reduce the sieve variance. This changes the bias-variance trade-off and improves the rate of convergence in the weak norm.

Corollary 1 (Number of Simulated Samples S and the Rate of Convergence)**.** *If a long sample $(y_1^s, \ldots, y_{nS}^s)$ can be simulated then the variance term becomes:*

$$\min\left(\frac{k(n)^2 \log[n]^2}{\sqrt{n \times S}}, \frac{1}{\sqrt{n}}\right).$$

As a result, for $S(n) \asymp k(n)^4 \log[k(n)]^4$ the rate of convergence in weak norm is:

$$\|\hat{\beta}_n - \beta_0\|_{weak} = O_p\left(\max\left(\frac{\log[k(n)]^{r/b+1}}{k(n)^{\gamma^2 r}}, \frac{1}{\sqrt{n}}\right)\right).$$

And the rate of convergence in either the total variation or the supremum norm is:

$$\|\hat{\beta}_n - \beta_0\|_{\mathcal{B}} = O_p\left(\frac{\log[k(n)]^{r/b}}{k(n)^r} + \tau_{\mathcal{B},n} \max\left(\frac{\log[k(n)]^{r/b+1}}{k(n)^{\gamma^2 r}}, \frac{1}{\sqrt{n}}\right)\right)$$

where $\tau_{\mathcal{B},n}$ is the local measure of ill-posedness in Theorem 2.

The assumption that a long sample can be simulated is called the ECA assumption in Kristensen & Salanié (2017); it is more commonly found in dynamic models than cross-sectional or panel data models. In the parametric SMM and Indirect Inference literature, S has an effect on the asymptotic variance whereas in the Sieve-SMM setting, Corollary 1 shows that increasing S with the sample size n can also improve the rate of convergence in the weak norm. Assuming undersmoothing so that the rate in weak norm is of order $1/\sqrt{n}$, the rate of convergence in the stronger norm $\|\cdot\|_{\mathcal{B}}$ becomes $O_p(k(n)^{-r} + \tau_{\mathcal{B},n}/\sqrt{n})$, up to a log term. This is faster than the rates of convergence found in the literature.

In practice, the number of simulated sample $S(n)$ required to achieve the rate in Corollary 1 can be very large. For $n = 1,000$, $\gamma = 1$ and $r = 2$, the optimal $k(n) \simeq 5$ and $S(n) = k(n)^4 \simeq 625$. The total number of simulated y_t^s required is $n \times S(n) = 625,000$. For iid data, the required number of simulations is $n \times S(n) = 5,000$. As a result, improving the rate of convergence of the estimator can be computationally costly since it involves increasing both the number of samples to simulate and the number of parameters to be estimate.

Remark 1 (An Illustration of the Local Measure of Ill-Posedness)**.** *The sieve measure of ill-posedness is generally difficult to compute. To illustrate a source of ill-posedness and its order of magnitude, consider the following basic static model:*

$$y_t^s = e_t^s \overset{iid}{\sim} f.$$

The only parameter to be estimated is the density f which can also be approximated with kernel density estimates. For this model the characteristic function is linear in f and as a consequence the weak norm for $f_1 - f_2$ is the weighted difference of the CFs ψ_{f_1}, ψ_{f_1} for f_1, f_2:

$$\|f_1 - f_2\|_{weak} = \left[\int |\psi_{f_1}(\tau) - \psi_{f_2}(\tau)|^2 \pi(\tau) d\tau\right]^{1/2}.$$

The weak norm is bounded above by 2 for any two densities f_1, f_2. However, the total variation and supremum distances are not bounded above: as a result the ratio between the weak norm and these stronger norms is unbounded. To illustrate, simplify the problem further and assume there is only one mixture component:

$$f_{1,k(n)}(e) = \underline{\sigma}_{k(n)}^{-1} \phi\left(\frac{e}{\underline{\sigma}_{k(n)}}\right), \quad f_{2,k(n)}(e) = \underline{\sigma}_{k(n)}^{-1} \phi\left(\frac{e - \mu_{k(n)}}{\underline{\sigma}_{k(n)}}\right).$$

As the bandwidth $\sigma_{k(n)} \to 0$, the two densities approach Dirac masses. Unless $\mu_{k(n)} \to 0$, the total variation and supremum distances between the two densities go to infinity while the distance in weak norm is bounded. The distance between f_1 and f_2 in weak, total-variation and supremum norm are given in Appendix A.3. For a well chosen sequence $\mu_{k(n)}$, the total variation and supremum distances are bounded

above and below while the weak norm goes to zero. The ratio provides the local measures of ill-posedness:

$$\tau_{TV,n} = O\left(\frac{k(n)}{\log[k(n)]^{2/b}}\right), \quad \tau_{\infty,n} = O\left(\frac{k(n)^2}{\log[k(n)]^{4/b}}\right).$$

Hence, this simple example suggests that Characteristic Function based Sieve-SMM estimation problems are at best mildly ill-posed.

3.3 Asymptotic Normality

This section derives asymptotic normality results for plug-in estimates $\phi(\hat{\beta}_n)$ where ϕ are smooth functionals of the parameters. As in Chen & Pouzo (2015), the main result finds a normalizing sequence $r_n \to \infty$ such that:

$$r_n \times \left(\phi\left(\hat{\beta}_n\right) - \phi\left(\beta_0\right)\right) \xrightarrow{d} \mathcal{N}\left(0, 1\right)$$

where $r_n = \sqrt{n}/\sigma_n^*$, for some sequence of standard errors $(\sigma_n^*)_{n\geq 1}$ which can go to infinity. If $\sigma_n^* \to \infty$, the plug-in estimates will converge at a slower than $\sqrt{n}$-rate. In addition, sufficient conditions for $\hat{\theta}_n$ to be root-n asymptotically normal, that is $\lim_{n\to\infty} \sigma_n^* < \infty$, are given in Appendix A.5 for the stochastic volatility model of Example 1.

To establish asymptotic normality results, stochastic equicontinuity results are required. However, the L^2-smoothness result only holds in the space of mixtures $\mathcal{B}_{k(n)}$ with the pseudo-norm $\|\cdot\|_m$ on the mixture parameters. This introduces two difficulties in deriving the results: a rate of convergence for the norm on the mixture components is required, and since $\beta_0 \notin \mathcal{B}_{k(n)}$ in general, the rate and the stochastic equicontinuity results need to be derived around a sequence of mixtures that are close enough to β_0 so that they extend to β_0. The following lemma provides the rate of convergence in the mixture norm.

Lemma 5 (Convergence Rate in Mixture Pseudo-Norm)**.** *Let $\delta_n = (k(n)\log[k(n)])^2/\sqrt{n}$ and $M_n = \log\log(n+1)$. Suppose the following undersmoothing assumptions hold:*

i. (Rate of Convergence) $\|\hat{\beta}_n - \beta_0\|_{weak} = O_p(\delta_n)$

ii. (Negligible Bias) $\|\Pi_{k(n)}\beta_0 - \beta_0\|_{weak} = o(\delta_n)$.

Furthermore, suppose that the population CF is smooth in β and satisfies:

iii. (Approximation Rate 1) Uniformly over $\beta \in \{\beta \in \mathcal{B}_{osn}, \|\beta - \beta_0\|_{weak} \leq M_n\delta_n\}$:

$$\int \left| \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0))}{d\beta}[\beta - \beta_0] - \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0))}{d\beta}[\beta - \beta_0] \right|^2 \pi(\tau)d\tau = O(\delta_n^2).$$

iv. (Approximation Rate 2) The approximating mixture $\Pi_{k(n)}\beta_0$ satisfies:

$$\int \left| \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0))}{d\beta} [\Pi_{k(n)}\beta_0 - \beta_0] \right|^2 \pi(\tau) d\tau = O(\delta_n^2).$$

Let $\underline{\lambda}_n$ be the smallest eigenvalue of the matrix

$$\int \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0))}{d(\theta, \omega, \mu, \sigma)}' \overline{\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0))}{d(\theta, \omega, \mu, \sigma)}} \pi(\tau) d\tau.$$

Suppose that $\underline{\lambda}_n > 0$ and $\delta_n \underline{\lambda}_n^{-1/2} = o(1)$ then the convergence rate in the mixture pseudo-norm is:

$$\|\hat{\beta}_n - \Pi_{k(n)}\beta_0\|_m = O_p\left(\delta_n \underline{\lambda}_n^{-1/2}\right)$$

where $\|\beta\|_m = \|(\theta, \omega, \mu, \sigma)\|$ is the pseudo-norm on θ and the mixture parameters (ω, μ, σ).

The rate of convergence in mixture norm $\|\cdot\|_m$ corresponds to the rate of convergence in the weak norm $\|\cdot\|_m$ times a measure of ill-posedness $\underline{\lambda}_n^{-1/2}$. Relations between the mixture norm and the strong norm $\|\cdot\|_{\mathcal{B}}$ imply that the local measure of ill-posedness in Theorem 2 can be computed using $\underline{\lambda}_n^{-1/2}$. Indeed, results in van der Vaart & Ghosal (2001); Kruijer et al. (2010) imply that $\|\beta - \Pi_{k(n)}\beta_0\|_{TV} \leq \underline{\sigma}_{k(n)}^{-1} \|\beta - \Pi_{k(n)}\beta_0\|_m$ and $\|\beta - \Pi_{k(n)}\beta_0\|_\infty \leq \underline{\sigma}_{k(n)}^{-2} \|\beta - \Pi_{k(n)}\beta_0\|_m$. These inequalities imply upper-bounds for ill-posedness in total variation and supremum norms:

$$\tau_{TV,n} \leq \underline{\lambda}_n^{-1/2} \underline{\sigma}_{k(n)}^{-1} \quad \text{and} \quad \tau_{\infty,n} \leq \underline{\lambda}_n^{-1/2} \underline{\sigma}_{k(n)}^{-2}.$$

The quantity $\underline{\lambda}_n^{-1/2}$ can be approximated numerically using sample estimates and $\underline{\sigma}_{k(n)}$ is the bandwidth in Lemma 1. As a result, even though the local measure of ill-posedness from Theorem 2 is generally not tractable, an upper bound can be computed using Lemma 5. Chen & Christensen (2017) shows how to achieve the optimal rate of convergence using plug-in estimates of the measure of ill-posedness in nonparametric instrumental variable regression, a similar approach should be applicable here using these bounds. This is left to future research.

Lemma 6 (Stochastic Equicontinuity Results)**.** *Let $\delta_{mn} = \delta_n \underline{\lambda}_n^{-1/2}$. Suppose that the assumptions of Lemma 5 hold and $(M_n \delta_{mn})^{\frac{\gamma^2}{2}} \max(\log[k(n)]^2, |\log[M_n \delta_{mn}]|^2) k(n)^2 = o(1)$, then a first stochastic equicontinuity result holds:*

$$\sup_{\|\beta - \Pi_{k(n)}\beta_0\|_m \leq M_n \delta_{mn}} \int \left| [\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] - \mathbb{E}[\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] \right|^2 \pi(\tau) d\tau = o_p(1/n).$$

Also, suppose that $\beta \to \int \mathbb{E}\left|\hat{\psi}_t^s(\tau, \beta_0) - \hat{\psi}_t^s(\tau, \beta)\right|^2 \pi(\tau) d\tau$ is continuous with respect to $\|\cdot\|_{\mathcal{B}}$ at $\beta = \beta_0$, uniformly in $t \geq 1$, then a second stochastic equicontinuity result holds:

$$\sup_{\|\beta - \Pi_{k(n)}\beta_0\|_m \leq M_n \delta_{mn}} \int \left| [\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \beta_0)] - \mathbb{E}[\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \beta_0)] \right|^2 \pi(\tau) d\tau = o_p(1/n).$$

Lemma 6 uses the rate of convergence in mixture norm to establish stochastic equicontinuity results. With these results, the moments $\hat{\psi}_n^s(\tau,\beta) - \hat{\psi}_n^s(\tau,\beta_0)$ can be substituted with a smoothed version under the integral of the objective function.

Remark 2 (Required Rate of Convergence)**.** *To achieve the rate of convergence required in Lemma 6, $k(n)$ must grow at a power of the sample size n, hence: $\log(n) \asymp \log[k(n)] \asymp |\log(\delta_{mn})|$. As a result, the condition on the rate of convergence in mixture norm $(M_n\delta_{mn})^{\frac{\gamma^2}{2}} \max(\log[k(n)]^2, |\log[M_n\delta_{mn}]|^2)k(n)^2 = o(1)$ in Lemma 6 can be simplified to:*

$$M_n\delta_n = o\left(\frac{\sqrt{\underline{\lambda}_n}}{[k(n)\log(n)]^{4/\gamma^2}}\right).$$

The following definition adapts the tools used in the sieve literature to establish asymptotic normality of smooth functionals (see e.g. Wong & Severini, 1991; Ai & Chen, 2003; Chen & Pouzo, 2015; Chen & Liao, 2015) to a continuum of complex valued moments.

Definition 2 (Sieve Representer, Sieve Score, Sieve Variance)**.** *Let $\beta_{0,n}$ be such that $\|\beta_{0,n} - \beta_0\|_{weak} = \inf_{\beta\in\mathcal{B}_{osn}} \|\beta - \beta_0\|_{weak}$, let $\overline{V}_{k(n)}$ be the closed span of $\mathcal{B}_{osn} - \{\beta_{0,n}\}$. The inner product $\langle\cdot,\cdot\rangle$ of $(v_1, v_2) \in \overline{V}_{k(n)}$ is defined as:*

$$\langle v_1, v_2\rangle = \frac{1}{2}\int \left[\psi_\beta(\tau, v_1)\overline{\psi_\beta(\tau, v_2)} + \overline{\psi_\beta(\tau, v_1)}\psi_\beta(\tau, v_2)\right]\pi(\tau)d\tau.$$

i. The Sieve Representer is the unique vector $v_n^* \in \overline{V}_{k(n)}$ such that $\forall v \in \overline{V}_{k(n)}$: $\langle v_n^*, v\rangle = \frac{d\phi(\beta_0)}{d\beta}[v]$.

ii. The Sieve Score S_n^* is:

$$\begin{aligned} S_n^* &= \frac{1}{2}\int \left[\psi_\beta(\tau, v_n^*)\overline{[\hat{\psi}_n^S(\tau,\beta_0) - \hat{\psi}_n(\tau)]} + \overline{\psi_\beta(\tau, v_n^*)}[\hat{\psi}_n^S(\tau,\beta_0) - \hat{\psi}_n(\tau)]\right]\pi(\tau)d\tau \\ &= \int Real\left(\psi_\beta(\tau, v_n^*)\overline{[\hat{\psi}_n^S(\tau,\beta_0) - \hat{\psi}_n(\tau)]}\right)\pi(\tau)d\tau. \end{aligned}$$

iii. The Sieve Long Run Variance σ_n^* is:

$$\sigma_n^{*2} = n\mathbb{E}\left(S_n^{*2}\right) = n\mathbb{E}\left(\left[\int Real\left(\psi_\beta(\tau, v_n^*)\overline{[\hat{\psi}_n^S(\tau,\beta_0) - \hat{\psi}_n(\tau)]}\right)\pi(\tau)d\tau\right]^2\right).$$

iv. The Scale Sieve Representer u_n^* is: $u_n^* = v_n^*/\sigma_n^*$.

Assumption 4 (Equivalence Condition)**.** *There exists $\underline{a} > 0$ such that for all $n \geq 1$: $\underline{a}\|v_n^*\|_{weak} \leq \sigma_n^*$. Furthermore, suppose that σ_n^* does not increase too fast: $\sigma_n^* = o(\sqrt{n})$.*

In Sieve-MD literature, Assumption 4 is implied by an eigenvalue condition on the conditional variance of the moments.[27] Because the moments are bounded and the data is geometrically ergodic, the long-run variance of the moments is bounded above uniformly in τ.[28] However, since τ has unbounded support, the eigenvalues of the variance may not have a strictly positive lower bound. Assumption 4 plays the role of the lower bound on the eigenvalues.[29]

Assumption 5 (Convergence Rate, Smoothness, Bias)**.** *$\mathcal{B}_{osn}$ is a convex neighborhood of β_0 where*

i. (Rate of Convergence) $M_n\delta_n = o(n^{-1/4})$ and $M_n\delta_n = o\left(\sqrt{\underline{\lambda}_n}/\left(k(n)\log(n)\right)^{4/\gamma^2}\right)$.

ii. (Smoothness) A linear expansion of ϕ is locally uniformly valid:

$$\sup_{\|\beta-\beta_0\|\leq M_n\delta_n} \frac{\sqrt{n}}{\sigma_n^*}\left|\phi(\beta) - \phi(\beta_0) - \frac{d\phi(\beta_0)}{d\beta}[\beta-\beta_0]\right| = o(1).$$

A linear expansion of the moments is locally uniformly valid:

$$\sup_{\|\beta-\beta_0\|_{weak}\leq M_n\delta_n} \left(\int \left|\mathbb{E}(\hat{\psi}_n^S(\tau,\beta)) - \mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0)) - \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0))}{d\beta}[\beta-\beta_0]\right|^2 \pi(\tau)d\tau\right)^{1/2} = O\left((M_n\delta_n)^2\right).$$

The second derivative is bounded:

$$\sup_{\|\beta-\beta_0\|_{weak}\leq M_n\delta_n} \left(\int \left|\frac{d^2\mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0))}{d\beta d\beta}[u_n^*,u_n^*]\right|^2 \pi(\tau)d\tau\right)^{1/2} = O(1).$$

iii. (Bias) The approximation bias is negligible:

$$\frac{\sqrt{n}}{\sigma_n^*}\frac{d\phi(\beta_0)}{d\beta}[\beta_{0,n}-\beta_0] = o(1).$$

Note that if $\mathcal{B}_{osn}$ is a convex neighborhood of β_0 then θ_0 is in the interior of Θ. Assumption 5 is standard in the literature. The first rate condition ensure the nonparametric component converges fast enough so that the central limit theorem dominates the asymptotic distribution (Newey, 1994; Chen et al., 2003), the second rate condition is required in Lemma 6. The smoothness and bias conditions can also be found in Ai & Chen (2003) and Chen & Pouzo (2015). The bias condition implies undersmoothing so that the variance term dominates asymptotically.

[27] See e.g. assumption 3.1(iv) in Chen & Pouzo (2015).

[28] This is shown in Appendix C.3.

[29] A discussion of this assumption is given in Appendix C.5

Theorem 3 (Asymptotic Normality)**.** *Suppose the assumptions of Theorems 1, 2 and lemmas 5, 6 hold as well as Assumptions 4 and 5, then as n goes to infinity:*

$$r_n \times \left(\phi(\hat{\beta}_n) - \phi(\beta_0)\right) \xrightarrow{d} \mathcal{N}(0,1)$$

where $r_n = \frac{\sqrt{n}}{\sigma_n^*} \to \infty$.

Theorem 3 shows that under the previous assumptions, inferences on $\phi(\beta_0)$ can be conducted using the confidence interval $[\phi(\hat{\beta}_n) \pm 1.96 \times \sigma_n^*/\sqrt{n}]$. The standard errors $\sigma_n^* > 0$ adjust automatically so that $r_n = \sqrt{n}/\sigma_n^*$ gives the correct rate of convergence. If $\lim_{n\to\infty} \sigma_n^* < \infty$, then $\phi(\hat{\beta}_n)$ is $\sqrt{n}-$convergent. A result for $\hat{\theta}_n$ is given in Proposition A1 in the Appendix for a smaller class of models that include the stochastic volatility model in Example 1.

As in Chen & Pouzo (2015) and Chen & Liao (2015), the sieve variance has a closed-form expression analogous to the parametric Delta-method formula. The notation is taken from Chen & Pouzo (2015), with sieve parameters $(\hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)$ the sieve variance can be estimated using:

$$\hat{\sigma}_n^{2*} = \frac{d\phi(\hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)}{d(\theta, \omega, \mu, \sigma)}' \hat{D}_n \hat{\mho}_n \hat{D}_n \frac{d\phi(\hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)}{d(\theta, \omega, \mu, \sigma)}$$

where

$$\hat{D}_n = \left(Real \left(\int \frac{d\hat{\psi}_n^S(\tau, \hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)}{d(\theta, \omega, \mu, \sigma)'} \overline{\frac{d\hat{\psi}_n^S(\tau, \hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)}{d(\theta, \omega, \mu, \sigma)}} \pi(\tau) d\tau \right) \right)^{-1}$$

$$\hat{\mho}_n = \int \hat{G}_n(\tau_1)' \hat{\Sigma}_n(\tau_1, \tau_2) \hat{G}_n(\tau_2) \pi(\tau_1) \pi(\tau_2) d\tau_1 d\tau_2.$$

$\hat{G}_n$ stacks the real and imaginary components of the gradient:

$$\hat{G}_n(\tau) = \begin{pmatrix} Real\left(\frac{d\hat{\psi}_n^S(\tau, \hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)}{d(\theta, \omega, \mu, \sigma)}\right) \\ Im\left(\frac{d\hat{\psi}_n^S(\tau, \hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)}{d(\theta, \omega, \mu, \sigma)}\right) \end{pmatrix}.$$

Let $Z_n^S(\tau, \beta) = \hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta)$ The covariance operator $\hat{\Sigma}_n$ approximates the population long-run covariance operator Σ_n:

$$\Sigma_n(\tau_1, \tau_2) = n\mathbb{E} \begin{pmatrix} Real\left(Z_n^S(\tau_1, \beta_0)\right) Real\left(Z_n^S(\tau_2, \beta_0)\right) & Real\left(Z_n^S(\tau_1, \beta_0)\right) Im\left(Z_n^S(\tau_2, \beta_0)\right) \\ Im\left(Z_n^S(\tau_1, \beta_0)\right) Im\left(Z_n^S(\tau_2, \beta_0)\right) & Im\left(Z_n^S(\tau_1, \beta_0)\right) Real\left(Z_n^S(\tau_2, \beta_0)\right) \end{pmatrix}.$$

Carrasco et al. (2007a) gives results for the Newey-West estimator of Σ_n. In practice, applying the block Bootstrap to the quantity

$$Real\left(\frac{d\hat{\psi}_n^S(\tau, \hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)}{d(\theta, \omega, \mu, \sigma)} \overline{\left(\hat{\psi}_n(\tau) - \hat{\psi}_n(\tau, \hat{\beta}_n)\right)}\right)$$

is more convenient than computing the large matrices $\hat{G}_n, \hat{\Sigma}_n$. $\hat{\beta}_n$ is held fixed across Bootstrap iterations so that the model is only estimated once. The Gaussian and uniform draws $Z_{j,t}^s$ and ν_t^s are re-drawn at each Bootstrap iteration.

4 Extensions

This section considers two extensions to the main results: the first covers auxiliary variables in the CF and the seconds allows for panel datasets with small T.

4.1 Using Auxiliary Variables

The first extension involves adding transformations of the data, such as using simple functions of y_t or a filtered volatility from an auxiliary GARCH model, to the CF $\hat{\psi}_n$. This approach can be useful in cases where (y_t, u_t) is Markovian but y_t alone is not, in which case functions of the full history $(y_t, \ldots, y_1)$ provide additional information about the unobserved u_t. It is used to estimate stochastic volatility models in sections 5 and 6. Other potential applications include filtering latent variables from an auxiliary linearized DSGE model to estimate a more complex, intractable non-linear DSGE model.

The auxiliary model consists of an auxiliary variable z_t^{aux} (the filtered GARCH volatility) and auxiliary parameters $\hat{\eta}_n^{aux}$ (the estimated GARCH parameters). The estimates $\hat{\eta}_n^{aux}$ are computed from the full sample $(y_1, \ldots, y_n, x_1, \ldots, x_n)$ and the auxiliary variables $z_t^{aux}, z_t^{s,aux}$ are computed using the full and simulated samples:[30]

$$z_t^{aux} = g_{t,aux}(y_t, \ldots, y_1, x_t, \ldots, x_1, \hat{\eta}_n^{aux}), \quad z_t^{s,aux} = g_{t,aux}(y_t^s, \ldots, y_1^s, x_t, \ldots, x_1, \hat{\eta}_n^{aux}).$$

The moment function $\hat{\psi}_n$ is now the joint CF of the lagged data $(\mathbf{y}_t, \mathbf{x}_t)$ and the auxiliary $\mathbf{z}_t^{aux}$:

$$\hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) = \sum_{t=1}^{n} e^{i\tau'(\mathbf{y}_t, \mathbf{x}_t, \mathbf{z}_t^{aux})}, \quad \hat{\psi}_n^s(\tau, \hat{\eta}_n^{aux}, \beta) = \sum_{t=1}^{n} e^{i\tau'(\mathbf{y}_t^s, \mathbf{x}_t, \mathbf{z}_t^{s,aux})}.$$

The following assumption provides sufficient conditions on the estimates $\hat{\eta}_n^{aux}$ and the filtering process $g_{t,aux}$ for the asymptotic properties in section 3 to also hold with auxiliary variables.

Assumption 6 (Auxiliary Variables)**.** *The estimates $\hat{\eta}_n^{aux}$ are such that:*

i. Compactness: with probability 1 $\hat{\eta}_n^{aux} \in E$ finite dimensional, convex and compact.

[30] Note that using the same estimates $\hat{\eta}_n^{aux}$ for filtering the data and the simulated samples avoids the complication of proving uniform convergence of the auxiliary parameters over the sieve space.

ii. Convergence: there exists a $\eta^{aux} \in E$ *such that:*

$$\sqrt{n}\,(\hat{\eta}_n^{aux} - \eta^{aux}) \xrightarrow{d} \mathcal{N}(0, V^{aux}).$$

iii. Lipschitz Continuity: for any two η_1^{aux}, η_2^{aux} *and for both* y_t^s *and* y_t*:*

$$\begin{aligned}&\|g_{t,aux}(y_t,\ldots,y_1,x_t,\ldots,x_1,\eta_1^{aux}) - z_{t,aux}(y_t,\ldots,y_1,x_t,\ldots,x_1,\eta_2^{aux})\| \\ &\leq C^{aux}(y_t,\ldots,y_1,x_t,\ldots,x_1) \times \|\eta_1^{aux} - \eta_2^{aux}\|\end{aligned}$$

with $\mathbb{E}(C^{aux}(y_t,\ldots,y_1,x_t,\ldots,x_1)^2) \leq \bar{C}^{aux} < \infty$ *and* $\mathbb{E}(C^{aux}(y_t^s,\ldots,y_1^s,x_t,\ldots,x_1)^2) \leq \bar{C}^{aux} < \infty$*. The average of the Lipschitz constants* $C_n^{aux} = \frac{1}{n}\sum_{t=1}^n C^{aux}(y_t,\ldots,y_1,x_t,\ldots,x_1)$ *is uniformly stochastically bounded, it is* $O_p(1)$*, for both the data and the simulated data.*

iv. Dependence: for all $\eta^{aux} \in E$*,* (y_t, x_t, z_t^{aux}) *is uniformly geometric ergodic.*

v. Moments: for all $\eta^{aux} \in E$*,* $\beta = \beta_0$ *and* $\beta = \Pi_{k(n)}\beta_0$*, the moments* $\mathbb{E}(\|z_t^{aux}\|^2)$ *and* $\mathbb{E}(\|z_t^{s,aux}\|^2)$ *exist and are bounded.*

vi. Summability: for any $(y_t,\ldots,y_1)$*,* $(\tilde{y}_t,\ldots,\tilde{y}_1)$*, any* $\eta^{aux} \in E$ *and for all* $t \geq 1$*:*

$$\|g_{t,aux}(y_t,\ldots,y_1,x_t,\ldots,x_1,\eta^{aux}) - z_{t,aux}(\tilde{y}_t,\ldots,\tilde{y}_1,x_t,\ldots,x_1,\eta^{aux})\| \leq \sum_{j=1}^{t}\rho_j\|y_j - \tilde{y}_j\|$$

with $\rho_j \geq 0$ *for all* $j \geq 1$ *and* $\sum_{j=1}^{\infty}\rho_j < \infty$*.*

vii. Central Limit Theorem for the Sieve Score:

$$\sqrt{n}Real\left(\int \psi_\beta(\tau, u_n^*, \eta^{aux})\overline{(\hat{\psi}_n(\tau,\hat{\eta}_n^{aux}) - \hat{\psi}_n^s(\tau,\hat{\eta}_n^{aux},\beta_0))}\pi(\tau)d\tau\right) \xrightarrow{d} \mathcal{N}(0,1)$$

The summability condition *iv.* is key in preserving the Hölder continuity and bias accumulation results of section 3 when using auxiliary variables in the CF. For auxiliary variables generated using the Kalman Filter or a GARCH model, this corresponds to a stability condition in the Kalman Filter or the GARCH volatility equations.

Conditions *ii.* and *iii.* ensure that $\hat{\eta}_n^{aux}$ is well behaved and does not affect the rate of convergence. Condition *iv* implies that the inequality for the supremum of the empirical process still applies. Condition *vii.* assumes a CLT applies to the leading term in the expansion of $\phi(\hat{\beta}_n) - \phi(\beta_0)$. It could be shown by assuming an expansion of the form $\hat{\eta}_n^{aux} = \frac{1}{n}\sum_{t=1}^n \eta^{aux}(y_t, x_t) + o_p(1/\sqrt{n})$ and expanding $\hat{\psi}_n$, $\hat{\psi}_n^s$ around the probability limit η^{aux}. The following illustrates the Lipschitz and summability conditions for the SV with GARCH filtered volatility.

Example 1 (Continued) (Stochastic Volatility and GARCH(1,1) Filtered Volatility)**.** *For simplicity, assume there are only volatility dynamics:*

$$y_t = \sigma_t e_{t,1}$$

For simplicity, consider the absolute value GARCH(1,1) auxiliary model:[31]

$$y_t = \sigma_t^{aux} e_{t,1}, \quad \sigma_t^{aux} = \eta_1^{aux} + \eta_2^{aux}|y_t| + \eta_3^{aux}\sigma_{t-1}^{aux}.$$

The focus here is on the Lipschitz and summability conditions in the GARCH auxiliary model. First, to prove the Lipschitz condition, consider a sequence (y_t) and two sets of parameters $\eta^{aux}, \tilde{\eta}^{aux}$, by recursion:

$$\begin{aligned}|\sigma_t^{aux} - \tilde{\sigma}_t^{aux}| &= |\eta_1^{aux} - \tilde{\eta}_1^{aux} + (\eta_2^{aux} - \tilde{\eta}_2^{aux})|y_t| + (\eta_3^{aux} - \tilde{\eta}_3^{aux})\sigma_{t-1}^{aux} + \tilde{\eta}_3^{aux}(\sigma_{t-1}^{aux} - \tilde{\sigma}_{t-1}^{aux})| \\ &\leq \|\eta^{aux} - \tilde{\eta}^{aux}\| \times (\frac{1+\sigma_0^{aux}}{1-\overline{\eta}_3^{aux}} + [1+\overline{\eta}_2^{aux}][|y_t| + \cdots + (\overline{\eta}_3^{aux})^{t-1}|y_1|])\end{aligned}$$

$\overline{\eta}^{aux}$ are upper-bounds on the parameters. If $\mathbb{E}(|y_t|^2))$ and $\mathbb{E}(|y_t^s|^2))$ are finite and bounded and $0 \leq \overline{\eta}_3^{aux} < 1$ then the Lispchitz condition holds with:

$$\bar{C}^{aux} \leq \frac{1+\overline{\eta}_2^{aux}}{1-\overline{\eta}_3^{aux}}\left(1 + \sigma_0^{aux} + M_y\right)$$

where $\mathbb{E}(|y_t|^2)$ and $\mathbb{E}(|y_t^s|^2) \leq M_y$, for all $t \geq 1$ and $\beta \in \mathcal{B}$. Next, the proof for the summability is very similar, consider two time-series $y_t, \tilde{y}_t$ and a set of auxiliary parameters η^{aux}:

$$|\sigma_t^{aux} - \tilde{\sigma}_t^{aux}| \leq \overline{\eta}_2|y_t - \tilde{y}_t| + \overline{\eta}_3^{aux}|\sigma_{t-1}^{aux} - \tilde{\sigma}_{t-1}^{aux}|.$$

By a recursive argument, the inequality above becomes:

$$|\sigma_t^{aux} - \tilde{\sigma}_t^{aux}| \leq \overline{\eta}_2|y_t - \tilde{y}_t| + \overline{\eta}_3^{aux}\overline{\eta}_2|y_{t-1} - \tilde{y}_{t-1}| + \cdots + (\overline{\eta}_3^{aux})^{t-1}\overline{\eta}_2|y_1 - \tilde{y}_1| + (\overline{\eta}_3^{aux})^{t-1}|\sigma_0^{aux} - \tilde{\sigma}_0^{aux}|.$$

Suppose that σ_0^{aux} only depends on η^{aux} or is fixed, for instance equal to 0. Then the summability condition holds, if the upper-bound $\overline{\eta}_3^{aux} < 1$, with:

$$\rho_j = \overline{\eta}_2^{aux}(\overline{\eta}_3^{aux})^j, \quad \sum_{j=0}^{\infty}\rho_j = \frac{\overline{\eta}_2^{aux}}{1-\overline{\eta}_3^{aux}} < \infty.$$

The Lipschitz and summability conditions thus hold for the auxiliary GARCH model.

[31] The process is also known as the AVGARCH or TS-GARCH (see e.g. Bollerslev, 2010) and is a special case of the family GARCH model (see e.g. Hentschel, 1995). The method of proof is slightly more involved for a standard GARCH model, requiring for instance a lower bound on the volatility σ_t^{aux} together with finite and bounded fourth moments for y_t, y_t^s to prove the Lipschitz condition.

The following corollary shows that the results of section 3 also hold when addition auxiliary variables to the CF.

Corollary 2 (Asymptotic Properties using Auxiliary Variables)**.** *Suppose the assumptions for Theorems 1, 2 and 3 hold as well as Assumption 6, then the results of Theorems 1, 2 and 3 hold with auxiliary variables. The rate of convergence is unchanged.*

The proof of Corollary 2 is very similar to the proofs of the main results. Rather than repeating the full proofs, Appendix B.5 shows where the differences with and without the auxiliary variables are and explains why the main results are unchanged.

To compute standard errors, a block Bootstrap is applied to compute the variance term for the difference $\hat{\psi}_n(\cdot, \hat{\eta}_n^{aux}) - \hat{\psi}_n^S(\cdot, \beta_0, \hat{\eta}_n^{aux})$ in the sandwich formula for the standard errors. The unknown β_0 is replaced by $\hat{\beta}_n$ in practice.

4.2 Using Short Panels

The main theorems 1, 2 and 3 allow for either iid data or time-series. However, SMM estimation is also common in panel data settings where the time dimension T is small relative to the cross-sectional dimension n. The following provides a simple application of these results.

Example 2 (Dynamic Tobit Model)**.** *y_t follows a dynamic Tobit model:*

$$y_{j,t} = (x_{j,t}'\theta_1 + u_{j,t})\mathbb{1}_{x_{j,t}'\theta_1 + u_{j,t} \geq 0}$$
$$u_{j,t} = \rho u_{j,t-1} + e_{j,t}$$

where $|\rho| < 1$, $e_{j,t} \overset{iid}{\sim} f$, $\mathbb{E}(e_{j,t}) = 0$. The parameters to be estimated are $\theta = (\theta_1, \rho)$ and f.

An overview of the dynamic Tobit model is given in Arellano & Honoré (2001). Applications of the dynamic Tobit model include labor participation studies such as Li & Zheng (2008); Chang (2011). Li & Zheng (2008) find that estimates of ρ can be biased downwards under misspecification. This estimate matters for evaluating the probability of (re)-entering the labor market in the next period for instance.

Quantities of interest in the dynamic Tobit model includes the probability or re-entering the labor market $\mathbb{P}(y_{t+1} > 0 | x_{t+1}, \ldots, x_t, y_t = 0, y_{t-1}, \ldots, y_1)$ which depends on both the parameters θ and the distribution f. Marginal effects such as $\partial_{x_{t+1}} \mathbb{P}(y_{t+1} > 0 | x_{t+1}, \ldots, x_t, y_t = 0, y_{t-1}, \ldots, y_1)$ also depend on the true distribution f. As a result these quantities are sensitive to a particular choice of distribution f, this motivates a semi-nonparametric estimation approach for this model.

Other applications of simulation-based estimation in panel data settings include Gourinchas & Parker (2010) and Guvenen & Smith (2014) who consider the problem of consumption choices

with income uncertainty. For the simulation-based estimates, shocks to the income process are typically assumed to be Gaussian. Guvenen et al. (2015) use a very large and confidential panel data set from the U.S. Social Security Administration covering 1978 to 2013 to find that individual income shocks are display large negative skewness and excess kurtosis: the data strongly rejects Gaussian shocks.[32] They find that non-Gaussian income shocks help explain transitions between low and higher earnings states. Hence, a Sieve-SMM approach should also be of interest in the estimation of precautionary savings behavior under income uncertainty.

Because of the fixed T dimension, the initial condition (y_0, u_0) cannot be systematically handled using a large time dimension and geometric ergodicity argument as in the time-series case. Some additional restrictions on the DGP are given in the assumption below.

Assumption 7 (Data Generating Process for Panel Data)**.** *The data $(y_{j,t}, x_{j,t})$ with $j = 1, \ldots, n$, $t = 1, \ldots, T$ is generated by a DGP with only one source of dynamics either:*

$$\begin{aligned} y_{j,t} &= g_{obs}(x_{j,t}, \beta, u_{j,t}) \\ u_{j,t} &= g_{latent}(u_{j,t-1}, \beta, e_{j,t}) \end{aligned} \tag{12}$$

or

$$y_{j,t} = g_{obs}(y_{j,t-1}, x_{j,t}, \beta, e_{j,t}) \tag{13}$$

where $e_{j,t} \overset{iid}{\sim} f$ in both models. The observations are iid over the cross-sectional dimension j.

In situations where the DGPs in Assumption 7 are too restrictive, an alternative approach would be to estimate the distribution of $u_{j,1}$ conditional on $(y_{j,1}, x_{j,1})$. The methodology of Norets (2010) would apply to this particular estimation problem, the dimension of $(y_{j,1}, x_{j,1})$ should not be too large to avoid a curse of dimensionality. This is left to future research.

For the DGP in equation (12), geometric ergodicity applies to $u^s_{j,t}$ when simulating a longer history $u^s_{j,-m}, \ldots, u^s_{j,0}, \ldots, u^s_{j,1}, \ldots, u^s_{j,T}$ and letting the history increase with n, the cross-sectional dimension: $m/n \to c > 0$ as $n \to \infty$. For the DGP in equation (13), fixing $y^s_{j,1} = y_{j,1}$ ensures that $(y^s_{j,1}, \ldots, y^s_{j,T}, x_{j,1}, \ldots, x_{j,T})$ and $(y_{j,1}, \ldots, y_{j,T}, x_{j,1}, \ldots, x_{j,T})$ have the same distribution when $\beta = \beta_0$ (the DGP is assumed to be correctly specified).

The moments $\hat{\psi}_n, \hat{\psi}^s_n$ are the empirical CF of $(\mathbf{y}_t, \mathbf{x}_t)$ and $(\mathbf{y}^s_t, \mathbf{x}_t)$ respectively where $\mathbf{y}_t = (y_t, \ldots, y_{t-L})$ for $1 \leq L \leq T-1$; $\mathbf{y}_t, \mathbf{x}_t, \mathbf{y}^s_t$ are defined similarly. The identification Assumption 1 is assumed to hold for the choice of L.

[32] Also, Geweke & Keane (2000) estimate the distribution of individual income shocks using Bayesian estimates of a finite Gaussian mixture. They also find evidence of non-Gaussianity in the shocks. Arellano et al. (2017) use non-linear panel data methods to study the relation between incomes shocks and consumption. They provide evidence of persitence in earnings and conditional skewness.

The following lemma derives the initial condition bias for dynamic panel models with fixed T.

Lemma 7 (Impact of the Initial Condition)**.** *Suppose that Assumption 7 holds. If the DGP is given by (12) and $(y_{j,t}^s, u_{j,t}^s)$ with a long history for the latent variable $(u_{j,T}, \ldots, u_{j,0}, \ldots, u_{j,-m})$ where $m/n \to c > 0$ as $n \to \infty$. Suppose that $\mathbf{u}_{j,t}^s$ is geometrically ergodic in t and the integrals*

$$\int\int f(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}|\mathbf{u}_{j,t}^s)^2 f(\mathbf{u}_{j,t}^s) d\mathbf{y}_{j,t}^s d\mathbf{x}_{j,t}^s d\mathbf{u}_{j,t}^s, \quad \int\int f(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}|\mathbf{u}_{j,t}^s)^2 f^*(\mathbf{u}_{j,t}^s) d\mathbf{y}_{j,t}^s d\mathbf{x}_{j,t}^s d\mathbf{u}_{j,t}^s$$

are finite and bounded when $\beta = \beta_0$. Then, there exists a constant $\bar{\rho}_u \in (0,1)$ such that:

$$Q_n(\beta_0) = \int \left| \mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta_0) \right) \right|^2 \pi(\tau) d\tau = O\left(\bar{\rho}_u^m\right).$$

The effect of the initial condition is exponentially decreasing in m for DGP (12). If the DGP is given by (13) and the data is simulated with $y_{j,1}^s = y_{j,1}$ fixed then there is no initial condition effect:

$$Q_n(\beta_0) = \int \left| \mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta_0) \right) \right|^2 \pi(\tau) d\tau = 0$$

Simulating a long history $u_{j,T}^s, \ldots, u_{j,-m}^s$ implies that the impact of the initial condition $u_{j,m}^s = u_{-m}$ on the full simulated sample $y_{j,1}^s, \ldots, y_{j,T}^s$ delines exponentially fast in m. If m does not grow faster than n, that is $m/n \to c > 0$, than the dynamic bias accumulation is the same as in the time-series setting. In terms of bias, these m simulations play a similar role as the burn-in draws in MCMC estimation.

Corollary 3 (Asymptotic Properties for Short Panels)**.** *Suppose that Assumption 7 and Lemma 7 hold. For the DGP (12) in Assumption 7, assume that m is such that $\log[n]/m \to 0$ as $n \to \infty$. Suppose the assumptions for Theorems 1, 2 and 3 hold, then the resuls of Theorems 1, 2 and 3 hold. The rate of convergence in weak norm is the same as for iid data:*

$$\|\hat{\beta}_n - \beta_0\|_{weak} = O_p\left(\max\left(\frac{\log[k(n)]^{r/b+1}}{k(n)^{\gamma^2 r}}, \sqrt{\frac{k(n)\log[k(n)]}{n}} \right) \right).$$

The rate of convergence in total variance and supremum distance are:

$$\|\hat{\beta}_n - \beta_0\|_{\mathcal{B}} = O_p\left(\frac{\log[k(n)]^{r/b}}{k(n)^r} + \tau_{\mathcal{B},n} \max\left(\frac{\log[k(n)]^{r/b+1}}{k(n)^{\gamma^2 r}}, \sqrt{\frac{k(n)\log[k(n)]}{n}} \right) \right).$$

Remark 3. *For the DGP (13), the simulated history is finite and fixed so that the approximation bias is not inflated by the dynamics:*

$$\|\hat{\beta}_n - \beta_0\|_{weak} = O_p\left(\max\left(\frac{\log[k(n)]^{r/b}}{k(n)^{\gamma^2 r}}, \sqrt{\frac{k(n)\log[k(n)]}{n}} \right) \right).$$

As a result, the rate of convergence is the same as for static models.

The assumption that $\log[n]/m \to 0$ can be weakened to $m \to \infty$ and $\lim_{n\to\infty} \log[n]/m < -\log[\bar{\rho}_u]$. Heuristically, the requirement is $m \gg \log[n]$, for instance when $n = 1,000$ this implies $m \gg 7$: a short burn-in sample for $u_{j,t}$ is sufficient to reduce the impact of the initial condition. The following verifies some of the conditions in Assumption 2 for the Dynamic Tobit model.

Example 2 (Continued) (Dynamic Tobit)**.** *Since the function $x \to x\mathbb{1}_{x\geq 0}$ is Lipschitz the conditions y(i),y(ii) and y(iii) are satisfied as long as $\|\theta_1\|$ is bounded, $\mathbb{E}(\|x_t\|_2^2)$ is finite and $\mathbb{E}(u_t^2)$ is finite and bounded. The last variance is bounded if $|\rho| \leq \bar{\rho} < 1$ and $\mathbb{E}(e_t^2)$ is bounded above. The last condition is a restriction on the density f. Since $|\rho| \leq \bar{\rho} < 1$, condition u(i) is automatically satisfied. Together, $\mathbb{E}(u_t^2)$ bounded and linearity in ρ imply u(ii). Finally, linearity in e_t implies u(iii).*

5 Monte-Carlo Illustrations

This section illustrates the finite sample properties of the Sieve-SMM estimator. First, two very simple examples illustrate the estimator in the static and dynamic case against tractable estimators. Then, Monte-Carlo simulations are conducted for the stochastic volatility model Example 1 and Dynamic Tobit Example 2 for panel data.

For all Monte-Carlo simulations, the initial value for the mixture is a Gaussian density in the optimization routine. In most examples the Nelder & Mead (1965) algorithm in the NLopt package of Johnson (2014) was sufficient for optimization. In more difficult problems, such as the SV model with tail mixture components, the DIRECT global search algorithm of Jones et al. (1993) was applied to initialize the Nelder-Mead algorithm. The Monte-Carlo simulations were conducted using R[33] for all examples except for the AR(1) for which Matlab was used.

The Generalized Extreme Value (GEV) distribution is used in all Monte-Carlo examples. For the chosen parametrization, it displays negative skewness (-0.9) and excess kurtosis (3.9). It was also chosen because the approximation bias is larger for both kernel and mixture sieve estimates, and is thus more visible than alternative designs with smoother densities not reported here. This is useful when illustrating the increased bias due to the dynamics.

The Student t-distribution is also considered in the stochastic volatility design to illustrate the Sieve-SMM estimates with tail components. The density is smooth compared to the GEV. As a result, the bias is smaller and less visible.

[33]Some routines such as the computation of the CF and the simulation of mixtures were written in C++ and imported into R using Rcpp - see e.g. Eddelbuettel & Fran (2011a,b) for an introduction to Rcpp - and RcppArmadillo (Eddelbuettel & Sanderson, 2016) for linear algebra routines.

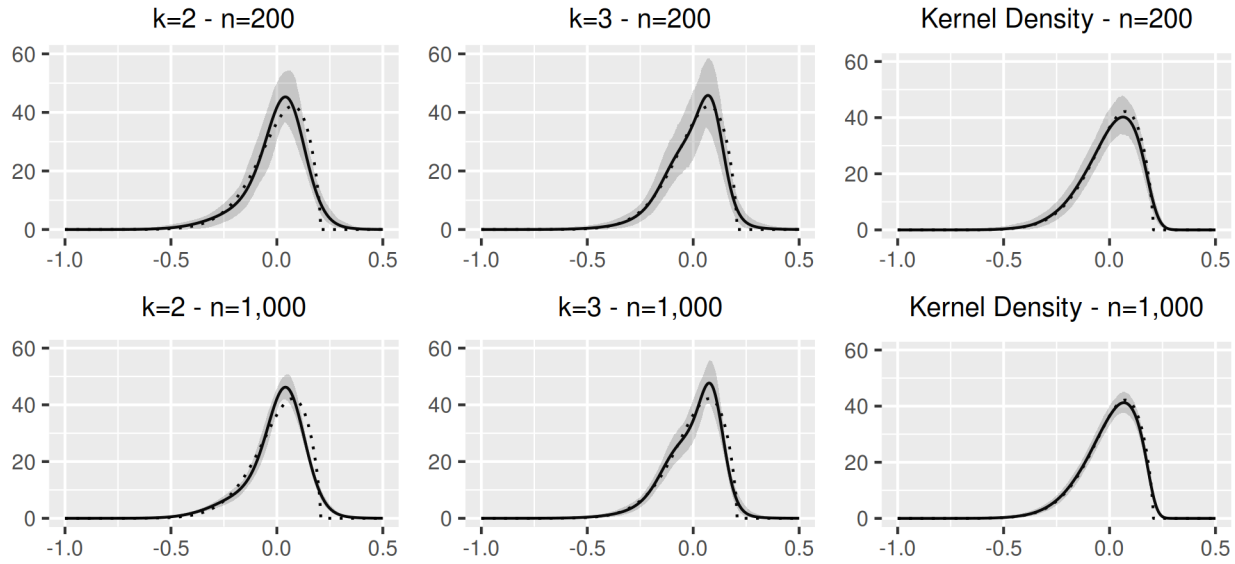
5.1 Basic Examples

The following basic tractable examples are used as benchmarks to understand the basic properties of the Sieve-SMM estimator in terms of bias and dynamic bias accumulation as well as the impact of dependence on the variance. As a benchmark, the estimates are compared to feasible kernel density and OLS estimates.

A Static Model

To illustrate Remark 1, the first example uses the static DGP: $y_t = e_t \overset{iid}{\sim} f$, the only parameter to be estimated is f and kernel density estimation is feasible. The true distribution f is the Generalized Extreme Value (GEV) distribution. It is a 3 parameter distribution which allows for asymmetry and displays excess kurtosis.[34] In a recent application, Ruge-Murcia (2017) uses the GEV distribution to model the third moment in inflation and productivity shocks in a small asset pricing model. The Sieve-SMM estimates $\hat{f}_n$ are compared to the feasible kernel density estimates $\hat{f}_{n,kde}$.

Figure 1: Static Model: Sieve-SMM vs. Kernel Density Estimates



Note: dotted line: true density, solid line: average estimate, bands: 95% pointwise interquantile range. Top panel $n = 200$ observation, bottom panel: $n = 1,000$ obervations. Left and middle: Sieve-SMM with $k = 2,3$ Gaussian mixture components respectively and $S = 1$. Right: kernel density estimates.

Figure 1 plots the density estimates for $k = 2,3$ with sample sizes $n = 200$ and $1,000$. The comparison between $k = 2$ and $k = 3$ illustrates the bias-variance trade-off: the bias is smaller

[34]The GEV distribution was first introduced by McFadden (1978) to unify the Gumbel, Fréchet and Weibull families.

for $k = 3$ but the variance of the estimates is larger compared to $k = 2$. Theorem 2 implies that when the sample size n increases, the number of mixture components k should increase as well to balance bias and variance. Here $k = 2$ appears to balance the bias and variance for $n = 200$ while $k \geq 3$ would be required for $n = 1,000$.

Autoregressive Dynamics

The second basic example considers an AR(1) model with an unknown distribution for the shocks:

$$y_t = \rho y_{t-1} + e_t,\ e_t \overset{iid}{\sim} (0,1).$$

The shocks are drawn from a GEV density as in the previous example. The empirical CFs are computed using one lagged observation:

$$\hat{\psi}_n(\tau) = \frac{1}{n}\sum_{t=1}^{n} e^{i\tau'(y_t, y_{t-1})}, \quad \hat{\psi}_n^s(\tau) = \frac{1}{n}\sum_{t=1}^{n} e^{i\tau'(y_t^s, y_{t-1}^s)}.$$

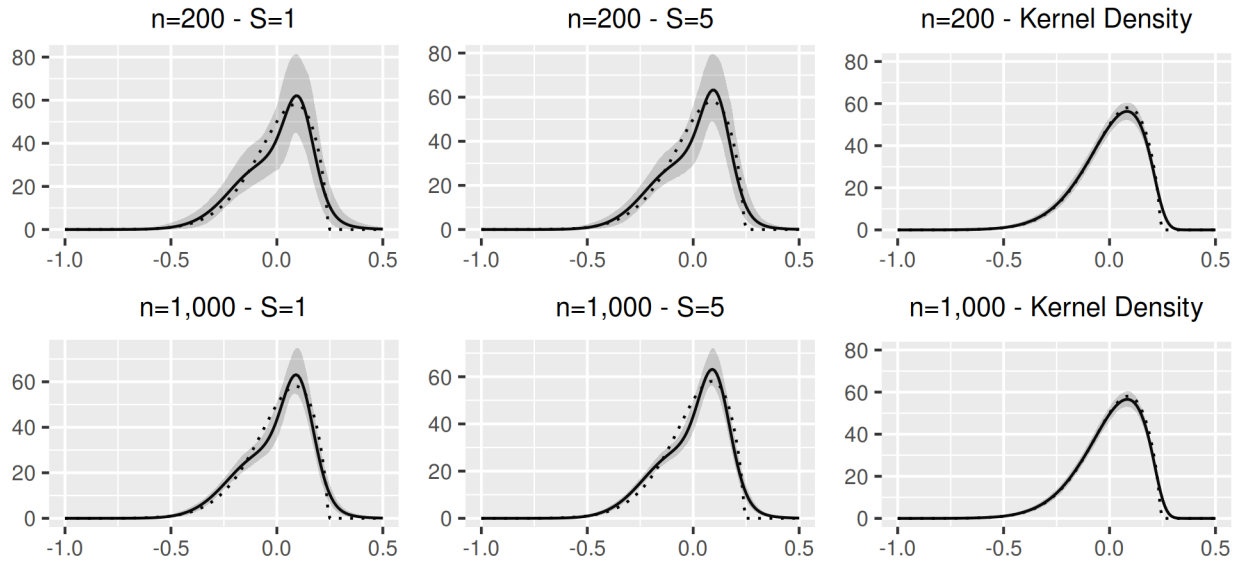
Knight & Yu (2002) note that additional lags do not improve the asymptotic properties of the estimator since y_t is Markovian of order 1.

This example illustrates Corollary 1 so the Monte-Carlo considers several choices of $S = 1, 5, 25$. Increasing S from 1 to 5 makes a notable difference on the variance of $\hat{f}_n$. Further increasing S has a much smaller effect on the variance of the estimates. Table 1 compares the Sieve-SMM with OLS estimates for $\rho = 0.95$ for $n = 200$ and $n = 1,000$, $S = 1, 5, 25$. In all cases, $k = 2$ mixture components are used.

Table 1: Autoregressive Dynamics: Sieve-SMM vs. OLS Estimates

Parameter: ρ		Sieve-SMM			OLS	True
		$S = 1$	$S = 5$	$S = 25$		
$n = 200$	Mean Estimate	0.942	0.934	0.933	0.927	0.95
	$\sqrt{n}\times$ Std. Deviation	(0.54)	(0.45)	(0.44)	(0.46)	-
$n = 1,000$	Mean Estimate	0.949	0.947	0.947	0.946	0.95
	$\sqrt{n}\times$ Std. Deviation	(0.47)	(0.38)	(0.37)	(0.34)	-

Figure 2: Autoregressive Dynamics: Sieve-SMM vs. Kernel Density Estimates



Note: dotted line: true density, solid line: average estimate, bands: 95% pointwise interquantile range. Top panel: $n = 200$, bottom panel: $n = 1,000$. Left and middle: Sieve-SMM with $S = 1, 5$ repsectively and $k = 2$. Right: infeasible kernel density estimates.

Figure 2 compares the Sieve-SMM estimates with kernel density assuming the shocks e_t are observed - this is an infeasible estimator. The top panel shows results for $n = 200$ and the bottom panel illustrates the larger sample size $n = 1,000$.

There are several features to note. First, as discussed in section 3.2, the bias is more pronounced under AR(1) dynamics than in the static case. The variance is larger with AR(1) dynamics compared to the static model. Second, as shown in Corollary 1 the number of simulated samples S shifts the bias/variance trade-off so that $k(n)$ can be larger.

5.2 Example 1: Stochastic Volatility

The stochastic volatility model of Example 1, illustrates the properties of the Sieve-SMM estimator for an intractable, non-linear state-space model. As a simplification, there are no mean dynamics:

$$y_t = \sigma_t e_{t,1}, \quad \log(\sigma_t) = \mu_\sigma + \rho_\sigma \log(\sigma_{t-1}) + \kappa_\sigma e_{t,2}$$

where $e_{t,2} \overset{iid}{\sim} \mathcal{N}(0,1)$ and $e_{t,1} \overset{iid}{\sim} f$ with mean zero and unit variance. Using an extension of the main results, a GARCH(1,1) auxiliary model is introduced:

$$y_t^{aux} = \sigma_t^{aux} e_t^{aux}, \quad (\sigma_t^{aux})^2 = \mu^{aux} + \alpha_1^{aux}[e_{t-1}^{aux}]^2 + \alpha_2^{aux}(\sigma_{t-1}^{aux})^2.$$

Using the data y_t, the parameters $\hat{\eta}_n^{aux} = (\mu_n^{aux}, \alpha_{1,n}^{aux}, \alpha_{2,n}^{aux})$ are estimated. The same $\hat{\eta}_n^{aux}$ is used to compute both filtered volatilities $\hat{\sigma}_t^{aux}, \hat{\sigma}_t^{s,aux}$. The empirical CFs uses both y and $\hat{\sigma}^{aux}$:[35]

$$\hat{\psi}_n(\tau) = \frac{1}{n}\sum_{t=1}^{n} e^{\tau'(y_t, y_{t-1}, \hat{\sigma}_t^{aux}, \log(\hat{\sigma}_{t-1}^{aux}))}, \quad \hat{\psi}_n^s(\tau, \beta) = \frac{1}{n}\sum_{t=1}^{n} e^{\tau'(y_t^s, y_{t-1}^s, \hat{\sigma}_t^{s,aux}, \log(\hat{\sigma}_{t-1}^{s,aux}))}.$$

The use of a GARCH model as an auxiliary model was suggested for indirect inference by Gouriéroux et al. (1993). Andersen et al. (1999) compare the EMM using ARCH, GARCH with the QML and GMM estimator using Monte-Carlo simulations. They find that EMM with GARCH(1,1) auxiliary model is more precise than GMM and QMLE in finite samples.

The parametrization is taken from Andersen et al. (1999): $\mu_\sigma = -0.736$, $\rho_\sigma = 0.90$, $\kappa_\sigma = 0.363$. Since Bayesian estimation is popular for SV models, the estimates are compared to a Gibbs sampling procedure, which assumes Gaussian shocks, using the R package *stochvol* of Kastner (2016). For Sieve-SMM estimation, the auxiliary GARCH filtered volatility estimates are computed using the R package *rugarch* of Ghalanos (2017).

The Monte-Carlo consists of 1,000 replications using $n = 1,000$ and $S = 2$. The distributions considered are the GEV and the Student t-distribution with 5 degrees of freedom. For the GEV density, $k = 4$ Gaussian mixture components are used and for the Student density, 4 Gaussian and 2 tail components are used.

Table 2: Stochastic Volatility: Sieve-SMM vs. Parametric Bayesian Estimates

Parameter		True	GEV		Student	
			Sieve-SMM	Bayesian	Sieve-SMM	Bayesian
$\frac{\mu_\sigma}{1-\rho_\sigma}$	Mean Estimate	-7.36	-7.28	-7.37	-7.29	-7.63
	Std. Deviation	-	(0.16)	(0.13)	(0.15)	(0.13)
ρ_σ	Mean Estimate	0.90	0.90	0.88	0.92	0.71
	Std. Deviation	-	(0.03)	(0.04)	(0.08)	(0.10)
κ_σ	Mean Estimate	0.36	0.40	0.40	0.29	0.74
	Std. Deviation	-	(0.05)	(0.06)	(0.06)	(0.12)

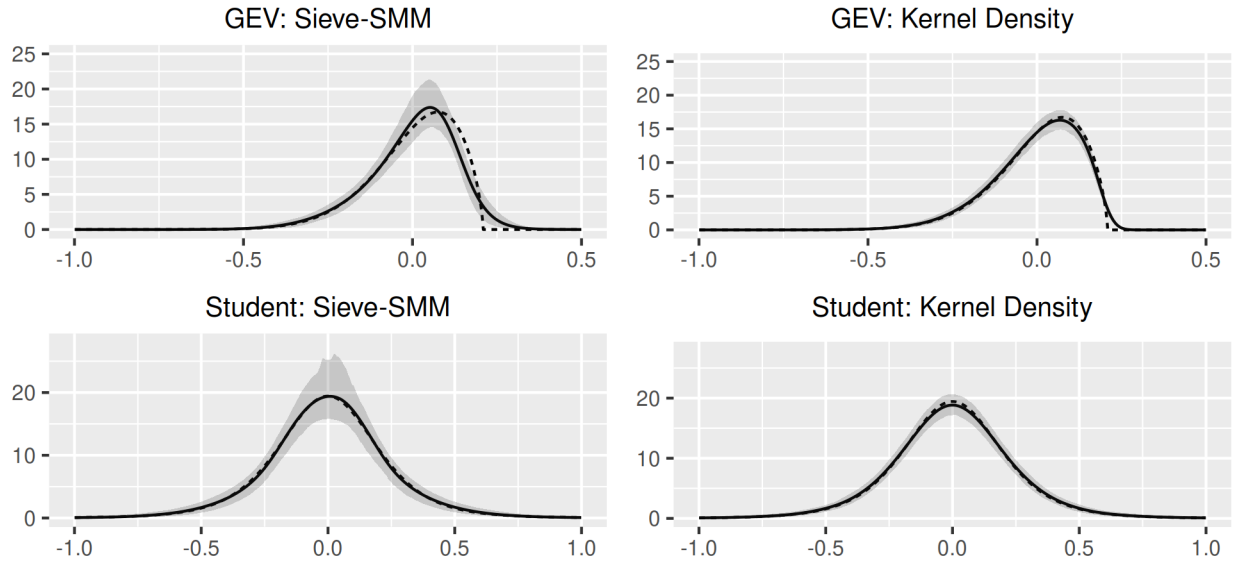
The standard deviations are comparable to the EMM with GARCH(1,1) generator found in Andersen et al. (1999). Results based only on the CF of $\mathbf{y}_t = (y_t, \ldots, y_{t-2})$ (not reported here) were

[35] The simulation results are similar whether $\hat{\sigma}^{aux}$ or $\log(\hat{\sigma}^{aux})$ is used in the CF.

more comparable to the GMM estimates reported in Andersen et al. (1999) - both for SMM and Sieve-SMM. Applying some transformations such as $\log(y_t^2)$ provided somewhat better results but information about potential asymmetries in f is lost. This motivated the first extension of the main result in section 4 to allow for auxiliary variables. Also not reported here, the bias and standard deviations of parametric estimates with f_0 are comparable to the GEV results.

Table 2 shows that the parametric Bayesian estimates and the SMM estimator are well behaved when the true density is Gaussian. For the GEV distribution, both the Sieve-SMM and the misspecified parametric Bayesian estimates are well behaved. However, under heavier tails, the Student t-distribution implies a significant amount of bias for the misspecified Bayesian estimates. The Sieve-SMM estimates are only slightly biased compared with the Bayesian estimates.

Figure 3: Stochastic Volatility: Sieve-SMM vs. Kernel Density Estimates



Note: dotted line: true density, solid line: average estimate, bands: 95% pointwise interquantile range. Top panel: estimates of a GEV density, bottom panel: estimates of a Student t-distribution with 5 *degrees of freedom.*

Figure 3 compares the density estimates with the infeasible kernel density estimates based on $e_{t,1}$ directly. The top panel shows the results for the GEV density and the bottom panel for the Student t-distribution. The Sieve-SMM is less precise than the infeasible estimator, as one would expect. As a comparison, the density is less precisely estimated than in the AR(1) case in figure 2. The two figures also illustrate bias reduction: the bias is larger for the AR(1) example which only uses $k = 2$ mixture components whereas the SV example uses $k = 4$.

The Monte-Carlo simulations for the stochastic volatility model highlight the lack of robust-

ness of the parametric Bayesian estimates to the tail behavior of the shocks. This is particularly important for the second empirical application where Sieve-SMM and Bayesian estimates differ a lot and there is evidence of fat tails and asymmetry in the shocks.

5.3 Example 2: Dynamic Tobit Model

The dynamic Tobit model in Example 2 illustrates the properties of the estimator in a non-linear dynamic panel data setting:

$$y_{j,t} = (\theta_1 + x'_{j,t}\theta_2 + u_{j,t})\mathbb{1}_{\theta_1 + x'_{j,t}\theta_2 + u_{j,t} \geq 0}$$
$$u_{j,t} = \rho u_{j,t-1} + e_{j,t}$$

with $j = 1, \ldots, n$ and $t = 1, \ldots, T$. The Monte-Carlo simulations consider a sample with $n = 200$, $T = 5$ for a total of $1,000$ observations. The burn-in sample for the latent variable $u_{j,t}$, described in section 4, is $m = 10$ which is about twice the log of n. The regressors x_t follow an AR(1) with Gaussian shocks. The AR process is calibrated so that x has mean 2, autocorrelation 0.3 and variance 2. The other parameters are chosen to be: $(\rho, \theta_1, \theta_2) = (0.8, -1.25, 1)$ and f is the GEV distribution as in the other examples. As a result, about 40% of the sample is censored. The numbers of simulated samples are $S = 1$ and $S = 5$. The moments used in the simulations are:

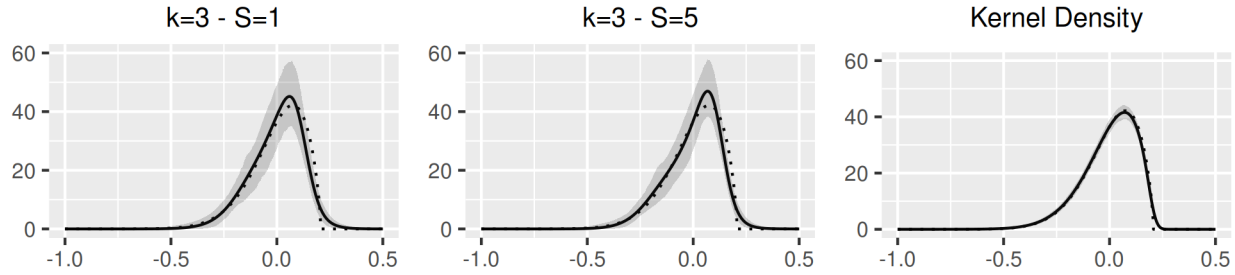
$$\hat{\psi}_n(\tau) = \frac{1}{nT}\sum_{t=2}^{T}\sum_{j=1}^{n} e^{i\tau'(y_t, y_{t-1}, x_t, x_{t-1})}, \; \hat{\psi}^s_n(\tau) = \frac{1}{nT}\sum_{t=2}^{T}\sum_{j=1}^{n} e^{i\tau'(y^s_t, y^s_{t-1}, x_t, x_{t-1})}.$$

Table 3: Dynamic Tobit: SMM vs. Sieve-SMM Estimates

Parameter		$S = 1$		$S = 5$		
		SMM	Sieve-SMM	SMM	Sieve-SMM	True
ρ	Mean	0.796	0.801	0.796	0.796	0.80
	Std. Deviation	(0.042)	(0.039)	(0.031)	(0.031)	-
θ_1	Mean	-1.259	-1.230	-1.250	-1.233	-1.25
	Std. Deviation	(0.234)	(0.200)	(0.178)	(0.169)	-
θ_2	Mean	1.002	1.002	1.000	0.997	1.00
	Std. Deviation	(0.059)	(0.052)	(0.045)	(0.043)	-

Table 3 compares the parametric SMM and the Sieve-SMM estimates. The numbers are comparable except for θ_1 which has a small bias for the Sieve-SMM estimates. Additional results for misspecified SMM estimates with simulated samples use Gaussian shocks instead of the true GEV distribution also show bias for θ_1, the average estimate is higher than -1.1. The other estimates were found to have negligible bias.[36]

Figure 4: Dynamic Tobit: Sieve-SMM vs. Kernel Density Estimates



Note: dotted line: true density, solid line: average estimate, bands: 95% pointwise interquantile range.

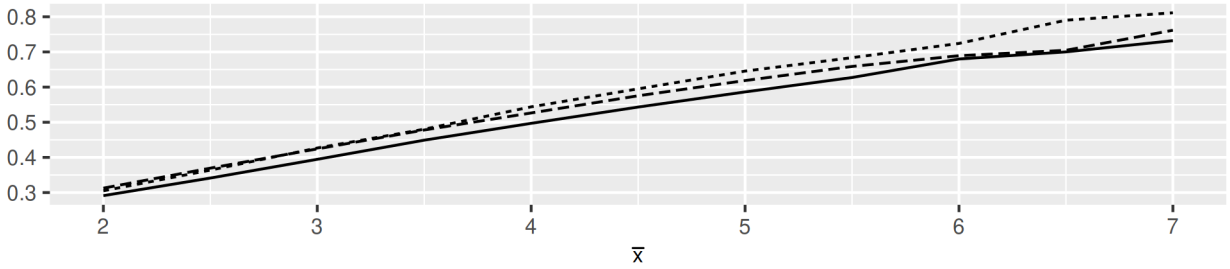
Figure 4 shows the Sieve-SMM estimates of the distribution of the shocks and the infeasible kernel density estimates of the unobserved e_t. Because of the censoring in the sample, note that the effective sample size for the Sieve-SMM estimates is smaller than for the kernel density estimates in this model. The left and middle plots show the sieve estimates when $S = 1, 5$; the right plot corresponds to the kernel density estimates.

Figure 5 illustrates the differences between SMM and Sieve-SMM for a counterfactual that involves the full density f. It shows the estimates of the probability of re-entering the market $\mathbb{P}(y_{j,5} > 0 | y_{j,4} = 0, x_5 = \cdots = x_1 = \bar{x})$ using the true value (θ_0, f_0), the SMM estimates $\hat{\theta}_n^{SMM}$ with Gaussian shocks and the Sieve-SMM estimates $(\hat{\theta}_n, \hat{f}_n)$. The true distribution is the GEV density which differs from the Gaussian density in the tails which implies a larger difference in the counterfactual when $\bar{x}$ is large, as shown in figure 5. For this particular counterfactual, the Sieve-SMM estimates are much closer to the true value for larger values of $\bar{x}$.

The Monte-Carlo simulations show the good finite sample behavior of the Sieve-SMM estimator with a non-smooth DGP. Indeed, the indicator function implies that the DGP is Lipschitz but not continuously differentiable. It also illustrates the extension to short panels in section 4.

[36] Li & Zheng (2008) consider an alternative design where ρ displays more significant bias.

Figure 5: Dynamic Tobit: SMM vs. Sieve-SMM Estimates of the Counterfactual



Note: Estimated counterfactual: $\mathbb{P}(y_{j,5} > 0 | y_{j,4} = 0, x_5 = \cdots = x_1 = \bar{x})$ *- solid line: true probability, dashed line: Sieve-SMM estimate, dotted line: SMM estimate with Gaussian shocks, 1 Monte-Carlo estimate for SMM, Sieve-SMM, probabilities computed using* 10^6 *Simulated Samples.*

6 Empirical Applications

This section considers two empirical examples of the Sieve-SMM estimator. The first example illustrates the importance of non-Gaussian shocks for welfare analysis and asset pricing using US monthly output data. The shocks are found to display both asymmetry and tails after controlling for time-varying volatility. As a result, the Sieve-SMM estimates imply welfare costs that are 25% greater than with the Gaussian SMM estimates. Furthermore, the effect of uncertainty on risk-free is nearly 3 times as large for the Sieve-SMM estimates compared to the Gaussian SMM estimates. The second one uses daily GBP/USD exchange rate data and highlights the bias and sensitivity implications of fat tails on parametric SV volatility estimates.

6.1 Welfare and Asset Pricing Implications of Non-Gaussian Shocks

The first example considers a simplified form of the DGP for output in the Long-Run Risks (LRR) model of Bansal & Yaron (2004). The data consists of monthly growth rate of US industrial production (IP), as a proxy for monthly consumption, from January 1960 to March 2017 for a total of 690 observations, from the FRED[37] database and downloaded via the R package Quandl.[38] IP is modeled using a stochastic volatility model with AR(1) mean dynamics:

$$\Delta c_t = \mu_c + \rho_c \Delta c_{t-1} + z_t e_{t,1}$$
$$\sigma_t^2 = \mu_\sigma + \rho_\sigma \sigma_{t-1}^2 + \kappa_\sigma [e_{t,2} - 1]$$

where $e_{t,2} \overset{iid}{\sim} \chi_1^2$ and $e_{t,1} \overset{iid}{\sim} f$ to be estimated assuming mean zero and unit variance. The stochastic volatility literature has mainly focused on the distribution of the shocks to the mean $e_{t,1}$ rather

[37] `https://fred.stlouisfed.org/`.
[38] `https://www.quandl.com/tools/r`

than the volatility[39] hence the volatility shocks are modelled parametrically in this application. Using the chi-squared distribution ensures that the volatility is non-negative. This DGP is a simplification of the one considered in Bansal & Yaron (2004). They assume that consumption is the sum of an AR(1) process and iid shocks with a common SV component. The DGP above only estimates the AR(1) component for simplicity given that the focus is of this example is on the shocks and the volatility rather than the mean dynamics. The volatility shocks are also assumed to be χ^2_1 rather than Gaussian to ensure non-negativity.

6.1.1 Empirical Estimates

The model is estimated using a Gaussian mixture and is compared with parametric SMM estimates. $S = 10$ simulated samples are used to perform the estimation. As in the Monte-Carlo an auxiliary GARCH(1,1) model is used. The empirical CF uses 2 lagged observations:

$$\hat{\psi}_n(\tau) = \frac{1}{n}\sum_{t=1}^{n} e^{i\tau'(\Delta c_t, \Delta c_{t-1}, \Delta c_{t-2}, \log(\hat{\sigma}_t^{aux}), \log(\hat{\sigma}_{t-1}^{aux}))}, \quad \hat{\psi}_n^s(\tau) = \frac{1}{n}\sum_{t=1}^{n} e^{i\tau'(\Delta c_t^s, \Delta c_{t-1}^s, \Delta c_{t-2}^s, \log(\hat{\sigma}_t^{s,aux}), \log(\hat{\sigma}_{t-1}^{s,aux}))}.$$

Table 4 shows the point estimates and the 95% confidence intervals for the parametric SMM, assuming Gaussian shocks, and the Sieve-SMM estimates using $k = 3$ mixture components. For reference, the OLS point estimate for ρ_c is 0.34 and the 95% confidence interval using HAC standard errors is $[0.23,\ 0.46]$ which is very similar to the SMM and Sieve-SMM estimates.[40]

Table 4: Industrial Production: Parametric and Sieve-SMM Estimates

		ρ_c	μ_σ	ρ_σ	κ_σ
SMM	Estimate	0.33	0.39	0.65	0.15
	95% CI	[0.22, 0.43]	[0.34, 0.45]	[0.22, 0.86]	[0.08, 0.26]
Sieve-SMM	Estimate	0.32	0.43	0.75	0.13
	95% CI	[0.20, 0.42]	[0.34, 0.55]	[0.35, 0.92]	[0.06, 0.29]

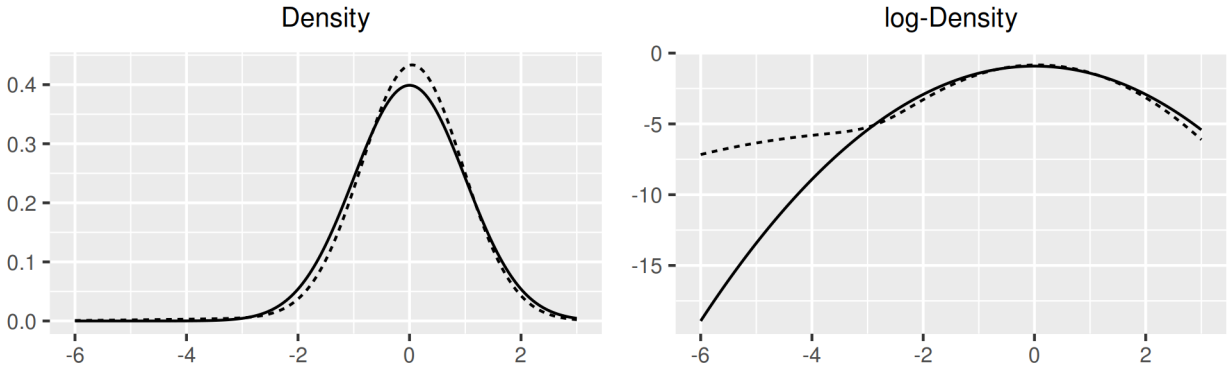
Figure 6 compares the densities estimated using the parametric SMM and Sieve-SMM. The log-density reveals a larger left tail for the sieve estimates and potential asymmetry: conditional on the volatility regime, large negative shocks are more likely than the Gaussian SV estimates

[39] See Fridman & Harris (1998); Mahieu & Schotman (1998); Liesenfeld & Jung (2000); Jacquier et al. (2004); Comte (2004); Jensen & Maheu (2010); Chiu et al. (2017) for instance.

[40] HAC standard errors are computed using the R package *sandwich* (Zeileis, 2004).

suggest. For instance, the log-difference at $e = -4$ is about 5 so that the ratio of densities is nearly 150 and the log-difference for $e = -5$ is roughly 10 so the density ratio is more than $20,000$.

Figure 6: Industrial Production: Sieve-SMM Density Estimate vs. Normal Density



Note: dotted line: Sieve-SMM density estimate, solid line: standard Normal density.

Table 5 shows that sieve estimated shocks have significant skewness and large kurtosis. It also shows the first four moments of the data compared to those implied by the estimates. Both sets of estimates match the first two moments similarly. The Sieve-SMM estimates provide a better fit for the skewness and kurtosis.

Table 5: Industrial Production: Moments of Δc_t, Δc_t^s and e_t^s

		Mean	Std Dev	Skewness	Kurtosis
Data	y_t	0.21	0.75	-0.92	7.56
SMM	y_t^s	0.25	0.66	0.06	4.39
Sieve-SMM	y_t^s	0.24	0.67	-0.35	6.65
SMM	e_t^s	0.00	1.00	0.00	3.00
Sieve-SMM	e_t^s	0.00	1.00	-0.75	7.74

Altogether, these results suggest significant non-Gaussian features in the shocks with both negative skewness and excess kurtosis. The welfare implications and the impact on the risk-free rate are now discussed.

6.1.2 Welfare Implications

The first implication considered here is the welfare effect of the fluctuations implied by each set of estimates. The approach considered here is based on the simple calculation approach of Lucas (1991, 2003).[41] The main advantage of this approach is that it does not require a full economic model: only a statistical model for output and a utility function are needed. To set the framework, a brief overview of his setting is now given. Lucas (1991) considers a setting where consumption is iid log-normal with constant growth rate $C_t = e^{\mu t + \sigma e_t}$ where $e_t \overset{iid}{\sim} \mathcal{N}(0,1)$ and has a certainty equivalent $C_t^\star = e^{\mu t + \sigma^2/2}$.

For a given level of risk-aversion $\gamma \geq 0$ and time preference $e^{-a} \in (0,1)$, he defines the welfare cost of business cycle fluctuations as the proportion λ by which the C_ts increase to achieve the same lifetime utility as under $C_t^\star$. This implies the following equation:

$$(1+\lambda)^{1-\gamma} \sum_{t \geq 0} e^{-at} \mathbb{E}_0 \left(\frac{C_t^{1-\gamma} - 1}{1-\gamma} \right) = \sum_{t \geq 0} e^{-at} \frac{C_t^{\star 1-\gamma} - 1}{1-\gamma}.$$

The estimates for the cost of business cycle fluctuations depends only on γ and σ in the Gaussian case: $\log(1+\lambda) = \gamma \frac{\sigma^2}{2}$. Lucas estimates this cost to be very small in the US.

Combining the SMM and Sieve-SMM with Monte-Carlo simulations[42], the welfare cost of business cycle fluctuations is now computed under Gaussian and mixture SV dynamics. Table 6 compares the two welfare costs for different levels of risk aversion with the baseline iid Gaussian case of Lucas.[43] For the full range of risk aversion considered here the welfare cost is estimated to be above 1% of monthly consumption. As a comparison Lucas (1991) estimates the welfare cost to be very small, a fraction of a percent, while Krusell et al. (2009) estimates it to be around 1%.[44] Both SV models imply much larger costs for business cycle fluctuations compared to the iid results: for $\gamma = 4$ and an annual income of \$55,000 the estimated welfare cost is \$990, \$800 and \$7 for Sieve-SMM, SMM and Gaussian iid estimates respectively. The Sieve-SMM estimates imply a welfare cost that is nearly \$200, or 25%, higher than the parametric SMM welfare estimates. This difference is quite large highlighting the non-negligible role of asymmetry in welfare.

[41] A number of alternative methods to estimate the welfare effect of business cycle fluctuations exist in the literature using, to cite only a few, models with heterogeneous agents (Krusell & Smith, Jr., 1999; Krusell et al., 2009), asset pricing models (Alvarez & Jermann, 2004; Barro, 2006a) and RBC models (Cho et al., 2015).

[42] Expectations are taken over 1,000 Monte-Carlo samples for an horizon of 5,000 months or about 420 years.

[43] The iid case is calibrated to match the mean and standard deviation of monthly IP growth. The monthly time preference parameter is chosen to match a quarterly rate of 0.99.

[44] Additional calculations and results under an AR(1) process and using linearized DSGE models are also given in Reis (2009).

Table 6: Welfare Cost of Business Cycle Fluctuations λ (%)

Risk Aversion γ	2	4	6	10
Gaussian iid	0.01	0.01	0.02	0.03
SMM	1.32	1.46	1.53	1.65
Sieve-SMM	1.54	1.80	1.93	2.12

6.1.3 Implications for the risk-free rate

The second implication considers the effect of uncertainty on the risk-free rate. As discussed in the introduction, the Euler equation implies that the risk-free rate r_t satisfies: $e^{-r_t} = e^{-a}\mathbb{E}_t\left((C_{t+1}/C_t)^{-\gamma}\right)$ where e^{-a} and γ are the time preference and risk aversion parameters. To explain the low-level of the risk-free rate observed in the data (Weil, 1989) a number of resolutions have been proposed including the long-run risks model of Bansal & Yaron (2004), which involves stochastic volatility and a recursive utility, and the rare disasters literature which involves very low frequency, high impact shocks and a power utility (Rietz, 1988; Barro, 2006b). This empirical application considers a simple power utility together with the higher frequency of shocks (monthly) over a recent period (since 1960) to achieve a similar result.

Given the AR(1) mean dynamics and volatility process postulated for IP growth, the risk-free rate can be written as:

$$r_t = a + \underbrace{\gamma\mu_c + \gamma\rho_c\Delta c_t}_{\text{Predictable Component}} \underbrace{- \log\left(\int e^{-\gamma e_{t+1,1}\sqrt{\mu_\sigma+\rho_\sigma\sigma_t^2+\kappa_\sigma[e_{t+1,2}-1]}} f(e_{t+1,1}) f_{\chi_1^2}(e_{t+1,2}) de_{t+1,1} de_{t+1,2}\right)}_{\text{Effect of uncertainty}}$$

where $f_{\chi_1^2}$ is the density of a χ_1^2 distribution.

Other than time preference a, there are two components in the risk-free rate: a predictable component $\gamma\mu_c + \gamma\rho_c\Delta c_t$ and another factor which only depends on the distribution of the shocks, it is the effect of uncertainty. In the second term, the integral over $e_{t+1,1}$ is the moment generating function of $e_{t+1,1}$ evaluated at $-\gamma\sqrt{\mu_\sigma + \rho_\sigma\sigma_t^2 + \kappa_\sigma[e_{t+1,2}-1]}$ and has closed-form when the distribution is either a Gaussian or a Gaussian mixture:

$$\begin{aligned}
&\int e^{-\gamma e_{t+1,1}\sqrt{\mu_\sigma+\rho_\sigma\sigma_t^2+\kappa_\sigma[e_{t+1,2}-1]}} f(e_{t+1,1}) f_{\chi_1^2}(e_{t+1,2}) de_{t+1,1} de_{t+1,2} \\
&= \sum_{j=1}^{k} \omega_j \int e^{-\gamma\mu_j\sqrt{\mu_\sigma+\rho_\sigma\sigma_t^2+\kappa_\sigma[e_{t+1,2}-1]} + \frac{\gamma^2}{2}\sigma_j^2(\mu_\sigma+\rho_\sigma\sigma_t^2+\kappa_\sigma[e_{t+1,2}-1])} f_{\chi_1^2}(e_{t+1,2}) de_{t+1,2}.
\end{aligned}$$

The integral over $e_{t+1,2}$ is computed using Gaussian quadrature. Using this formula, table 7 computes the effect of uncertainty on the risk-free rate over a range of values for risk aversion γ for a Gaussian AR(1) model as well as the parametric SMM and Sieve-SMM SV estimates. The effect of uncertainty is estimated to be nearly 3 times as large under the Sieve-SMM estimates compared to the Gaussian SMM estimates. Given that the risk free-rate is predicted to be much lower with the

Table 7: Effect of uncertainty on the risk-free rate (% annualized)

Risk aversion γ	2	4	6	10
Gaussian AR(1)	-0.12	-0.24	-0.35	-0.59
SMM	-0.09	-0.37	-0.84	-2.34
Sieve-SMM	-0.25	-1.02	-2.32	-6.59

Sieve-SMM estimates, the results suggest that the non-Gaussian features in the shocks matter for precautionary savings. Altogether, the results suggest that the choice of distribution f matters in computing both welfare effects and the risk-free rate.

6.2 GBP/USD Exchange Rate Data

The second example highlights the effect of fat tails and outliers on SV estimates for GBP/USD exchange rate data. The results highlight the presence of heavy tails even after controlling for time-varying volatility. Similar findings were also documented with parametric methods (see e.g. Fridman & Harris, 1998; Liesenfeld & Jung, 2000). This paper also finds significant asymmetry in the distribution of the shocks. Furthermore, comparing the estimates with common Bayesian estimates shows that parametric estimates severely underestimate the persistence of the volatility. Mahieu & Schotman (1998) also consider a mixture approximation for the distribution of the shocks in a SV model, using quasi-MLE for weekly exchange rate data. However, they do not provide asymptotic theory for their estimator and quasi-MLE does not estimate asymmetries in the density which turns out to be significant in this setting.

The data consists of a long series of daily exchange rate data between the British Pound and the US Dollar (GBP/USD) downloaded using the R package Quandl. The data begins in January 2000 and ends in December 2016 for a total of $5,447$ observations. The exchange rate is modeled using a log-normal stochastic volatility model with no mean dynamics:

$$y_t = \mu_y + \sigma_t e_{t,1}, \quad \log(\sigma_t) = \rho_\sigma \log(\sigma_{t-1}) + \kappa_\sigma e_{t,2}$$

where $e_{t,2} \overset{iid}{\sim} \mathcal{N}(0,1)$ and $e_{t,1} \overset{iid}{\sim} f$ to be estimated assuming mean zero and unrestricted variance. This allows to model extreme events associated with volatility clustering, when σ_t is large, as well as more isolated extreme events, represented by the tails of f. For this empirical application, μ_σ is set to 0 and f is only constrained to have unit variance. This illustrates the type of flexibility allowed when using mixtures for estimation. The data y_t consists of the daily log-growth rate of the GBP/USD exchange rate:

$$y_t = 100 \times \log\left(\frac{GBP/USD_t}{GBP/USD_{t-1}}\right).$$

Sieve-SMM estimates are compared to a common Gibbs sampling Bayesian estimate using the R package *stochvol* (Kastner, 2016). Two sets of Sieve-SMM estimates are computed: the first uses a Gaussian mixture with $k = 5$ components and the second a Gaussian and tails mixture with $k = 5$ components: 3 Gaussians and 2 tails. The two Sieve-SMM estimators have the same number of parameters to be estimated.

Table 8 shows the posterior mean and 95% credible interval for the Bayesian estimates as well as the point estimates and te 95% confidence interval for two Sieve-SMM estimators. The Bayesian estimate for the persistence of volatility ρ_z is much smaller than the SMM and Sieve-SMM estimates: it is outside their 95% confidence intervals. This reflects the bias issues discussed in the Monte-Carlo when f has large tails. As a robustness check, the estimates for the Sieve-SMM are similar when removing observations after the United Kingdom European Union membership referendum, that is between June 23rd and December 31st 2016: $(\hat{\rho}_n, \hat{\sigma}_z) = (0.96, 0.23)$ for the Gaussian mixture and $(0.97, 0.20)$ for the Gaussian and tails mixture. The Bayesian estimates are also of the same order of magniture $(0.26, 1.27)$. The density estimates $\hat{f}_n$ are also very similar when removing these observations.

Figure 7 compares the density $\hat{f}_n$ of $e_{t,1}$ for the Bayesian and Sieve-SMM estimates. The log-density $\log[\hat{f}_n]$ is also computed as it higlights the differences in the tails. The Bayesian assumes Gaussian shocks, so the log-density is quadratic, the density declines faster in the tails compared to the other two estimates. For the mixture with tail components, the density decays much slower than for both the Bayesian and Gaussian mixture estimates.

Table 9 compares the first four moments in the data to those implied by the estimates.[45] The Bayesian estimates fit the fourth moment of the full dataset best. Note that for time series data, estimates of kurtosis can be very unprecise (Bai & Ng, 2005). Hence a robustness check can be important: when removing the observation corresponding to United Kingdom European Union

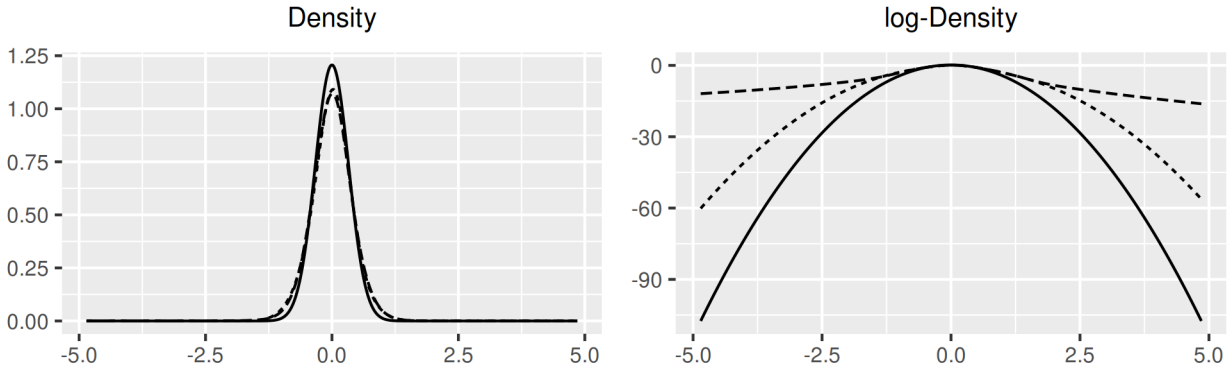
[45]The moments for the Bayesian and Sieve-SMM estimates are computed using numerical simulations.

Table 8: Exchange Rate: Bayesian and Sieve-SMM Estimates

		ρ_z	σ_z
Bayesian	Estimate	0.24	1.31
	95% CI	[0.16, 0.34]	[1.21, 1.41]
Sieve-SMM	Estimate	0.96	0.22
	95% CI	[0.59, 0.99]	[0.06, 0.83]
Sieve-SMM Tails	Estimate	0.97	0.19
	95% CI	[0.62, 0.99]	[0.05, 0.79]

Note: CI is the credible interval for the Bayesian and the confidence interval for the frequentist estimates.

Figure 7: Exchange Rate: Density and log-Density Estimates



Note: solid line: Gaussian density, dotted line: Gaussian mixture, dashed: Gaussian and tails mixture.

membership referendum on June 23rd 2016 which is the largest variation in the sample,[46] the kurtosis drops to about 10. Furthermore, when removing all observations between June 23rd and December 31st 2016, the kurtosis declines further to about 9. As discussed above, the point estimates remain similar when removing these observations. The Sieve-SMM estimates match the fourth moment of the restricted sample more closely but the Gaussian mixture fits the third moment poorly. The Gaussian and tails mixture fits all four moments of the restricted sample best. It also has the lowest value for the sample objective function. The Gaussian and tails mixture is thus the preferred specifications for this dataset.

In terms of forecasting, there are three main implications. First, the Bayesian estimates severely

[46]It is associated with a depreciation of the the GBP of more than 8 log percentage points. This is much larger than typical daily fluctuations.

Table 9: Exchange Rate: Moments of y_t, y_t^s and e_t^s

		Mean	Std Dev	Skewness	Kurtosis
Data	y_t	0.00	0.49	-1.15	21.05
Data*	y_t	0.00	0.47	-0.32	8.92
Bayesian	y_t^s	0.00	0.52	0.00	18.47
Sieve-SMM	y_t^s	0.00	0.85	0.10	5.88
Sieve-SMM tails	y_t^s	0.00	0.45	-0.28	7.74
Bayesian	e_t^s	0.00	1.00	0.00	3.00
Sieve-SMM	e_t^s	0.00	1.00	-0.06	3.68
Sieve-SMM tails	e_t^s	0.00	1.00	-0.17	4.83

Note: Data corresponds to the full sample: January 1st 2000-December 31st 2016. Data is a restricted sample: January 1st 2000-June 22nd 2016. Sieve-SMM: Gaussian mixture, Sieve-SMM tails: mixture with tail components.*

underestimate the persistence of the volatility: as a result, forecasts would underestimate the persistence of a high volatility episode. Second, $\hat{f}_n$ displays a significant amount of tails: a non-negligible amount of large shocks are isolated rather than associated with high volatility regimes. Third, there is evidence of asymmetry in $\hat{f}_n$: large depreciations in the GBP relative to the USD are historically more likely than large appreciations.

7 Conclusion

Simulation-based estimation is a powerful approach to estimate intractable models. This paper extends the existing parametric literature to a semi-nonparametric setting using a Sieve-SMM estimator. General asymptotic results are given using the mixture sieve for the distribution of the shocks and the empirical characteristic function as a moment function. On the theoretical side, this paper provides new and more general results for static models and allows for a new class of dynamics in the Sieve-GMM literature. Monte-Carlo simulations illustrate the range of applications of the method and its finite sample properties. Extensions to a larger class of moments and short panels are given.

Two empirical applications highlight the importance of the density in the shocks in practice. The first one shows asymmetry and tail behavior in output shocks. Welfare estimates suggest that

the cost of business cycle fluctuations are larger under these non-Gaussian shocks. The risk-free rate is also significantly lower, reflecting the greater downside risks in the estimated distribution and the additional precautionary savings it implies.

The second empirical example highlights the effect of misspecification on volatility estimates. Sieve-SMM estimation applied to daily GBP/USD exchange rate data reveals significant tail behavior and asymmetry, even after controlling for the time-varying volatility. The parametric Bayesian estimates are not robust to misspecification and large rare events.

Going forward, a number of extensions to this paper's results should be of interest. On the theoretical side, extending the inequality in this paper to unbounded moments would allow for more general Sieve-GMM settings as in Chen et al. (2013). The results could also be extended to a generalization of Indirect Inference with both infinite dimensional moments and parameters. The mixture sieve can be extended to accommodate heteroskedasticiy as in Norets (2010) or multivariate densities without the independence assumption as in De Jonge & Van Zanten (2010). On the empirical side, the results in this paper suggest that the distribution of the shocks is important in estimating welfare effects in DSGE models or risk-premia in asset pricing models. Also, using the results in this paper, the Sieve-SMM can be applied to estimate cross-sectional heterogeneity in short panels where fixed effects cannot be differenced out.

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Appendix A Background Material

A.1 The Characteristic Function and Some of its Properties

The joint characteristic function (CF) of $(\mathbf{y}_t, \mathbf{x}_t)$ is defined as

$$\psi : \tau \to \mathbb{E}\left(e^{i\tau'(\mathbf{y}_t, \mathbf{x}_t)}\right) = \mathbb{E}\left(\cos(\tau'(\mathbf{y}_t, \mathbf{x}_t)) + i\sin(\tau'(\mathbf{y}_t, \mathbf{x}_t))\right).$$

An important result for the CF is that the mapping between distribution and CF is bijective: two CFs are equal if, and only if they come from the same distribution $f_1 = f_2 \Leftrightarrow \psi_{f_1} = \psi_{f_2}$. The characteristic function has several other attractive features:

i. Existence: The CF is well defined for any probability distribution: it can be computed even if no moment of (y_t, x_t) exist.

ii. Boundedness: The CF is bounded $|\psi(\tau)| \leq 1$ for any distribution. As a result, the objective function $\hat{Q}_n^S$ is always well defined assuming the density π is integrable.

iii. Continuity in f: The CF is continuous in the distribution $f_n \to f_0$ implies $\psi_{f_n} \to \psi_{f_0}$.

iv. Continuity in τ: The CF is continuous in τ.

The continuity properties are very useful when the data y_t does not have a continuous density, e.g. discrete, but the density of the shocks f is continuous as in Example 2. For instance, the data generated by:

$$y_t = \mathbb{1}_{x_t'\theta + e_t \geq 0}$$

is discrete but its conditional characteristic function is continuous in both f and θ:

$$\mathbb{E}\left(e^{i\tau_y y_t}|x_t\right) = 1 - F(x_t'\theta) + F(x_t'\theta)e^{i\tau_y},$$

where F is the CDF of $e_t \sim f$. As a result, the joint CF is also continuous:

$$\mathbb{E}\left(e^{i\tau(y_t, x_t)}\right) = \mathbb{E}\left(e^{i\tau_x x_t}[1 - F(x_t'\theta) + F(x_t'\theta)e^{i\tau_y}]\right).$$

The empirical CDF however is not continuous. As a result, a population objective Q based on the CF is continuous but the one based on a CDF is not.

A.2 Computing the Sample Objective Function $\hat{Q}_n^S$

This section discusses the numerical implementation of the Sieve-SMM estimator. First, several transformations are used to normalize the weights ω and impose restrictions such as mean zero

$\sum_j \omega_j \mu_j = 0$ and unit variance $\sum_j \omega_j(\mu_j^2 + \sigma_j^2) = 1$ without requiring constrained optimization. For the weights, take $k-1$ unconstrained parameters $\tilde{\omega}$ and apply the transformation:

$$\omega_1 = \frac{1}{1 + \sum_{\ell=1}^{k-1} e^{\tilde{\omega}_\ell}}, \quad \omega_j = \frac{e^{\tilde{\omega}_{j-1}}}{1 + \sum_{\ell=1}^{k-1} e^{\tilde{\omega}_\ell}} \text{ for } j = 2, \ldots, k.$$

The resulting $\omega_1, \ldots, \omega_k$ are positive and sum to one. To impose a mean zero restriction take $\mu_2, \ldots, \mu_k$ unconstrained and compute:

$$\mu_1 = -\frac{\sum_{j=2}^{k} \omega_j \mu_j}{\omega_1}$$

The mixture has mean zero by construction. In practice, it is assumed that $\sigma_j \geq \underline{\sigma}_k$. Take unconstrained $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k$ and compute:

$$\sigma_j = \underline{\sigma}_k + e^{\tilde{\sigma}_j}.$$

The resulting σ_j are greater or equal than the lower bound $\underline{\sigma}_k \geq 0$. To impose unit variance, restrict $\tilde{\sigma}_1 = 0$ and then divide μ, σ by $\sqrt{\sum_j \omega_j(\mu_j^2 + \sigma_j^2)}$: standardized this way, the mixture has unit variance.

Once the parameters ω, μ, σ are appropriately transformed and normalized, the mixture draws e_t^s can be simulated, and then y_t^s itself is simulated. Numerical integration is used to approximate the sample objective function $\hat{Q}_n^S$. For an integration grid $\tau_1, \ldots, \tau_m$ with weights $\pi_1, \ldots, \pi_m$ compute the vectors:

$$\hat{\psi}_n = (\hat{\psi}_n(\tau_1), \ldots, \hat{\psi}_n(\tau_m))', \quad \hat{\psi}_n^S = (\hat{\psi}_n^S(\tau_1), \ldots, \hat{\psi}_n^S(\tau_m))'$$

and the objective:

$$\hat{Q}_n^S(\beta) = (\hat{\psi}_n - \hat{\psi}_n^S)' \text{diag}(\pi_1, \ldots, \pi_m)(\hat{\psi}_n - \hat{\psi}_n^S).$$

In practice, the objective function is computed the same as for a parametric SMM estimator. If a linear operator B is used to weight the moments, then the finite matrix approximation B_m is computed on $\tau_1, \ldots, \tau_m$ and the objective becomes $(\hat{\psi}_n - \hat{\psi}_n^S)' B' \text{diag}(\pi_1, \ldots, \pi_m)(\hat{\psi}_n - \hat{\psi}_n^S)'$; a detailed overview on computing the objective function with a linear operator B, using quadrature, is given in the appendix of Carrasco & Kotchoni (2016).

A.3 Local Measure of Ill-Posedness

The following provides the derivations for Remark 1. Recall that the simple model consists of:

$$f_{1,k(n)}(e) = \underline{\sigma}_{k(n)}^{-1} \phi\left(\frac{e}{\underline{\sigma}_{k(n)}}\right), \quad f_{2,k(n)}(e) = \underline{\sigma}_{k(n)}^{-1} \phi\left(\frac{e - \mu_{k(n)}}{\underline{\sigma}_{k(n)}}\right).$$

The only difference between the two densities is the location parameter $\mu_{k(n)}$ in $f_{2,k(n)}$. The total variance, weak and supremum distances between $f_{1,k(n)}$ and $f_{1,k(n)}$ are given below:

i. **Distance in the Weak Norm**

The distance between f_1 and f_2 in the weak norm is:

$$\|f_1 - f_2\|_{weak}^2 = 2 \int e^{-\underline{\sigma}_{k(n)}^2 \tau^2} \sin(\tau \mu_{k(n)})^2 \pi(\tau) d\tau.$$

When $\mu_{k(n)} \to 0$, $\sin(\tau \mu_{k(n)})^2 \to 0$ as well. By the dominated convergence theorem this implies that $\|f_{1,k(n)} - f_{2,k(n)}\|_{weak} \to 0$ as $\mu_{k(n)} \to 0$ regardless of the sequence $\underline{\sigma}_{k(n)} > 0$. The rate at which the distance in weak norm goes to zero when $\mu_{k(n)} \to 0$ can be approximated using the power series for the sine function $\|f_1 - f_2\|_{weak} = |\mu_{k(n)}| \sqrt{2 \int e^{-\underline{\sigma}_{k(n)}^2 \tau^2} \tau^2 \pi(\tau) d\tau} + o(|\mu_{k(n)}|)$. For $\mu_{k(n)} \to 0$, the distance in weak norm declines linearly in $\mu_{k(n)}$. For a specific choice of sequence $(\mu_{k(n)})$ the total variation and supremum distances can be shown to be bounded below. As a result, the ratio with the distance in weak norm is proportional to $|\mu_{k(n)}|^{-1} \to +\infty$.

ii. **Total Variation Distance**

The total variation distance between $f_{1,k(n)}$ and $f_{2,k(n)}$ is bounded below and above by[47]:

$$1 - e^{-\frac{\mu_{k(n)}^2}{8\underline{\sigma}_{k(n)}}} \leq \|f_1 - f_2\|_{TV} \leq \sqrt{2} \left(1 - e^{-\frac{\mu_{k(n)}^2}{8\underline{\sigma}_{k(n)}}} \right)^{1/2}.$$

For any $\varepsilon > 0$, one can pick $\mu_{k(n)} = \pm \underline{\sigma}_{k(n)} \sqrt{-8 \log(1 - \varepsilon^2)}$ so that $\|f_{1,k(n)} - f_{2,k(n)}\|_{TV} \in [\varepsilon^2/2, \varepsilon]$. However, for the same choice of $\mu_{k(n)}$, the paragraph above implies that $\|f_{1,k(n)} - f_{2,k(n)}\|_{weak} \to 0$ as $\underline{\sigma}_{k(n)} \to 0$. The ratio goes to infinity when $\underline{\sigma}_{k(n)} \to 0$:

$$\frac{\|f_{1,k(n)} - f_{2,k(n)}\|_{TV}}{\|f_{1,k(n)} - f_{2,k(n)}\|_{weak}} \geq \underline{\sigma}_{k(n)}^{-1} \frac{1}{\sqrt{2}\varepsilon \sqrt{-8 \log(1 - \varepsilon^2)}}$$

iii. **Distance in the Supremum Norm**

Using the intermediate value theorem the supremum distance can be computed as:

$$\begin{aligned} \|f_{1,k(n)} - f_{2,k(n)}\|_\infty &= \sup_{e \in \mathbb{R}} \frac{1}{\underline{\sigma}_{k(n)}} \left| \phi\left(\frac{e}{\underline{\sigma}_{k(n)}} \right) - \phi\left(\frac{e - \mu_{k(n)}}{\underline{\sigma}_{k(n)}} \right) \right| \\ &= \sup_{\tilde{e} \in \mathbb{R}} \frac{|\mu_{k(n)}|}{\underline{\sigma}_{k(n)}^2} \left| \phi'\left(\frac{\tilde{e}}{\underline{\sigma}_{k(n)}} \right) \right| = \frac{|\mu_{k(n)}|}{\underline{\sigma}_{k(n)}^2} \|\phi'\|_\infty \end{aligned}$$

[47] The bounds make use of the relationship between the Hellinger distance $H(f_1, f_2)$: $H(f_1, f_2)^2 \leq \|f_1 - f_2\|_{TV} \leq \sqrt{2} H(f_1, f_2)$. The Hellinger distance between two univariate Gaussian densities is available in closed-form: $H(f, g)^2 = 1 - \sqrt{\frac{2\sigma_f \sigma_g}{\sigma_f^2 + \sigma_g^2}} e^{-\frac{1}{4} \frac{(\mu_f - \mu_g)^2}{(\sigma_f^2 + \sigma_g^2)}}$.

For any $\varepsilon > 0$, pick $\mu_k = \pm\varepsilon\underline{\sigma}^2_{k(n)}/\|\phi'\|_\infty$ then the distance is supremum norm is fixed, $\|f_{1,k(n)} - f_{2,k(n)}\|_\infty = \varepsilon$, for any strictly positive sequence $\underline{\sigma}_{k(n)} \to 0$. However, the distance in weak norm goes to zero, again the ratio goes to infinity when $\underline{\sigma}_{k(n)} \to 0$:

$$\frac{\|f_{1,k(n)} - f_{2,k(n)}\|_\infty}{\|f_{1,k(n)} - f_{2,k(n)}\|_{weak}} \geq \underline{\sigma}^{-2}_{k(n)}\varepsilon\|\phi'\|_\infty$$

The degree of ill-posedness depends on the bandwidth $\underline{\sigma}_{k(n)}$ in both cases. In order to achieve the approximation rate in Lemma 1, the bandwidth $\underline{\sigma}_{k(n)}$ must be $O(\log[k(n)]^{2/b}/k(n))$. As a result the local measures of ill-posedness for the total variation and supremum distances are:

$$\tau_{TV,n} = O\left(\frac{k(n)}{\log[k(n)]^{2/b}}\right), \quad \tau_{\infty,n} = O\left(\frac{k(n)^2}{\log[k(n)]^{4/b}}\right).$$

A.4 Identification in the Stochastic Volatility Model

This section provides an identification result for the SV model in the first empirical application:

$$\begin{aligned} y_t &= \mu_y + \rho_y y_{t-1} + \sigma_t e_{t,1}, \quad e_{t,1} \overset{iid}{\sim} f \\ \sigma_t^2 &= \mu_\sigma + \rho_\sigma \sigma_{t-1}^2 + \kappa_\sigma e_{t,2} \end{aligned}$$

with the restriction $e_{t,1} \sim (0,1)$, $|\rho_y|, |\rho_\sigma| < 1$ and $e_{t,2}$ follows a known distribution standardized to have mean zero and unit variance.[48] Suppose the CF $\hat{\psi}_n$ includes y_t and two lagged observations (y_{t-1}, y_{t-2}) and that the moment generating functions of (y_t, y_{t-1}, y_{t-2}) and $e_{t,1}$ are analytic so that all the moments are finite and characterise the density. Suppose that for two sets of parameters β_1, β_2 we have: $Q(\beta_1) = Q(\beta_2) = 0$. This implies that π almost surely:

$$\mathbb{E}(\hat{\psi}^s_n(\tau, \beta_1)) = \mathbb{E}(\hat{\psi}^s_n(\tau, \beta_2)), \quad \forall \tau \in \mathbb{R}^3. \tag{A.14}$$

Using the notation $\tau = (\tau_1, \tau_2, \tau_3)$ this implies that for any integers $\ell_1, \ell_2, \ell_3 \geq 0$:

$$\begin{aligned} i^{\ell_1+\ell_2+\ell_3}\mathbb{E}_{\beta_1}(y_t^{\ell_1} y_{t-1}^{\ell_2} y_{t-2}^{\ell_3}) &= \frac{d^{\ell_1+\ell_2+\ell_3}\mathbb{E}(\hat{\psi}^s_n(\tau, \beta_1))}{d\tau_1^{\ell_1} d\tau_2^{\ell_2} d\tau_3^{\ell_3}}\Big|_{\tau=0} \\ &= \frac{d^{\ell_1+\ell_2+\ell_3}\mathbb{E}(\hat{\psi}^s_n(\tau, \beta_2))}{d\tau_1^{\ell_1} d\tau_2^{\ell_2} d\tau_3^{\ell_3}}\Big|_{\tau=0} = i^{\ell_1+\ell_2+\ell_3}\mathbb{E}_{\beta_2}(y_t^{\ell_1} y_{t-1}^{\ell_2} y_{t-2}^{\ell_3}) \end{aligned}$$

In particular for $\ell_1 = 1, \ell_2 = 0, \ell_3 = 0$, it implies $\mu_{y,1} = \mu_{y,2}$ so that the mean is identified. Then, taking $\ell_1 = 2, \ell_2 = 0, \ell_3 = 0$ implies that $\mathbb{E}_{\beta_1}(\sigma_t^2)/(1-\rho_{y,1}^2) = \mathbb{E}_{\beta_2}(\sigma_t^2)/(1-\rho_{y,2}^2)$. For $\ell_1 = \ell_2 = 1, \ell_3 = 0$ it implies $\rho_{y,1}\mathbb{E}_{\beta_1}(\sigma_t^2)/(1-\rho_{y,1}^2) = \rho_{y,2}\mathbb{E}_{\beta_2}(\sigma_t^2)/(1-\rho_{y,2}^2)$ which, given the

[48] This assumption makes the derivations easier in terms of notation.

result above implies $\rho_{y,1} = \rho_{y,2}$ and then $\mathbb{E}_{\beta_1}(\sigma_t^2) = \mathbb{E}_{\beta_2}(\sigma_t^2)$. The latter implies $\mu_{\sigma,1}/(1-\rho_{\sigma,1}) = \mu_{\sigma,2}/(1-\rho_{\sigma,2})$. Taking $\ell_1 = 2, \ell_2 = 2, \ell_3 = 0$ and $\ell_1 = 2, \ell_2 = 0, \ell_3 = 0$ implies two additional moment conditions (after de-meaning):[49] $\rho_{\sigma,1}\kappa_{\sigma,1}^2/(1-\rho_{\sigma,1}^2) = \rho_{\sigma,2}\kappa_{\sigma,2}^2/(1-\rho_{\sigma,2}^2)$ and $\rho_{\sigma,1}^2\kappa_{\sigma,1}^2/(1-\rho_{\sigma,1}^2) = \rho_{\sigma,2}^2\kappa_{\sigma,2}^2/(1-\rho_{\sigma,2}^2)$. If $\rho_{\sigma,1}, \rho_{\sigma,2} \neq 0$ this imples $\rho_{\sigma,1} = \rho_{\sigma,2}$ and $\kappa_{\sigma,1}, \kappa_{\sigma,2}$ and also $\mu_{\sigma,1} = \mu_{\sigma_2}$.

Overall if $\rho_\sigma \neq 0$, then condition (A.14) implies $\theta_1 = \theta_2$, the parametric component is identified. Since θ is identified, all the moments of σ_t are known. After recentering, this implies that for all $\ell_1 \geq 3$ if $\mathbb{E}_\theta(\sigma_t^{\ell_1}) \neq 0$:

$$\mathbb{E}_{f_1}(e_{t,1}^{\ell_1}) = \mathbb{E}_{f_1}(e_{t,2}^{\ell_1}). \tag{A.15}$$

If σ_t is non-negative, which is implied by e.g. $e_{t,2} \sim \chi_1^2$ and parameter constraints, then all moments are stictly positive so that (A.15) holds. Since the moment generating function is analytic and the first two moments are fixed, (A.15) implies $f_1 = f_2$. Altogether, if $\rho_\sigma \neq 0$ and $\sigma_t > 0$ then the joint CF of (y_t, y_{t-1}, y_{t-2}) identifies β.

A.5 Additional Results on Asymptotic Normality

The following provides two additional results on the root-n asymptotic normality of $\hat{\theta}_n$. A positive result is given in Proposition A1 and a negative result is given in Remark A4. The results apply to DGPs of the form:[50]

$$y_t = g_{obs}(y_{t-1}, \theta, u_t)$$
$$u_t = g_{latent}(u_{t-1}, \theta, e_t)$$

where g_{obs}, g_{latent} are smooth in θ. In this class of models, the data depends on f only through e_t. Examples 1 and 2 satisfy this restriction but dynamic programming models typically don't. The smoothness restriction holds in Example 1 but not Example 2.

Proposition A1 (Sufficient Conditions for Asymptotic Normality of $\hat{\theta}_n$)**.** *If $\mathbb{E}_{\theta_0,f}(\mathbf{y}_t^s)$ and $\mathbb{V}_{\theta_0,f}(\mathbf{y}_t^s)$ do not depend on f then $\hat{\theta}_n$ is root-n asymptotically normal if:*

$$\mathbb{E}_{\theta_0,f_0}\left(\frac{d\mathbf{y}_t^s}{d\theta'}\left[\begin{pmatrix} 1 & \mathbf{y}_t^{s\prime} \end{pmatrix} \otimes I_{d_\mathbf{y}}\right]\right)$$

has rank greater or equal than d_θ when $t \to \infty$.

[49] Since μ_y, ρ_y are identified, it is possible to compute $\mathbb{E}([y_t - \mu_y - \rho_y y_{t-1}]^2[y_{t-1} - \mu_y - \rho_y y_{t-2}]^2) = \mathbb{E}(\sigma_t^2\sigma_{t-1}^2)$ from the information given by the CF.

[50] The regressors x_t are omitted here to simplify notation in the proposition and the proof, results with x_t can be derived in a similar way as in this section.

Proposition A1 provides some sufficient conditions for models where the mean and the variance of y_t^s do not vary with f, this holds for Example 1 but not Example 2. This condition requires that $\mathbf{y}_t^s$ varies sufficiently with θ on average to affect the draws. The proof of the proposition is given at the end of this subsection.

Example 1 (Continued) (Stochastic Volatility)**.** *Recall the DGP for th stochastic volatility model:*

$$y_t = \sum_{j=0}^{t} \rho_y^j \sigma_{t-j} e_{t-j,1} \quad \sigma_t^2 = \sum_{j=0}^{t} \rho_\sigma^j (\mu_\sigma + \kappa_\sigma e_{t-j,2}).$$

It is assumed that the initial condition is $y_0 = \sigma_0 = 0$ in the following. To reduce the number of derivatives to compute, suppose $\mu_\sigma, \kappa_\sigma$ are known and $e_{t-j,2}$ is normalized so that it has mean zero and unit variance. During the estimation $e_{t,1}$ is also restricted to have mean zero, unit variance which implies that the mean of y_t^s and its variance do not depent on f. First, compute the derivatives of y_t^s with respect to ρ_y, ρ_σ:

$$\begin{aligned}
\frac{dy_t^s}{d\rho_y} &= \sum_{j=1}^{\infty} j\rho_y^{j-1} \sigma_{t-j} e_{t-j,1} \\
\frac{dy_t^s}{d\rho_\sigma} &= 0.5 \sum_{j=0}^{\infty} \rho_y^j \frac{d\sigma_{t-j}^2}{d\rho_\sigma} e_{t-j,1}/\sigma_{t-j} \quad \textit{where} \quad \frac{d\sigma_{t-j}^2}{d\rho_\sigma} = \sum_{\ell=1}^{t-j} \ell \rho_\sigma^{\ell-1} (\mu_\sigma + \kappa_\sigma e_{\ell,2}).
\end{aligned}$$

Both derivatives have mean zero, the derivatives of the lags are zero as well. Hence, $\mathbb{E}\left(\frac{d\mathbf{y}_t^s}{d\theta'}\mathbf{y}_t^s\right)$ must have rank greater than 2 for Proposition A1 to apply. Now, compute a first set of expectations:

$$\begin{aligned}
\mathbb{E}(\frac{dy_t^s}{d\rho_y} y_t^s) &= \sum_{j=1}^{t} j\rho_y^{2j-1} \mathbb{E}(\sigma_{t-j}^2) \\
\mathbb{E}(\frac{dy_t^s}{d\rho_y} y_{t-1}^s) &= \sum_{j=0}^{t-1} (j+1)\rho_y^{2j} \mathbb{E}(\sigma_{t-j-1}^2) \\
\mathbb{E}(\frac{dy_t^s}{d\rho_y} y_{t-2}^s) &= \sum_{j=0}^{t-2} (j+2)\rho_y^{2j+1} \mathbb{E}(\sigma_{t-j-2}^2) \\
\mathbb{E}(\frac{dy_{t-1}^s}{d\rho_y} y_t^s) &= \sum_{j=1}^{t-1} j\rho_y^{2j} \mathbb{E}(\sigma_{t-j-1}) \\
\mathbb{E}(\frac{dy_{t-2}^s}{d\rho_y} y_t^s) &= \sum_{j=1}^{t-2} j\rho_y^{2j+1} \mathbb{E}(\sigma_{t-j-2}).
\end{aligned}$$

The remaining expectation for ρ_y can be deduced from the expectations above. Since $\mathbb{E}(\frac{dy_t^s}{d\rho_y} y_{t-1}^s) > 0$, these expectations are not all equal to zero as long as $\mathbb{E}(\sigma_t^2) > 0$. If ρ_σ was known then the rank condition would

hold. For the second set of expectations:

$$\mathbb{E}(\frac{dy_t^s}{d\rho_\sigma}y_t^s) = \sum_{j=0}^{t}\rho_y^j\mathbb{E}(\frac{d\sigma_{t-j}^2}{d\rho_\sigma}) = \sum_{j=0}^{t}\rho_y^j\sum_{\ell=1}^{t-j}\ell\rho_\sigma^{2\ell-1}\mu_\sigma$$

$$\mathbb{E}(\frac{dy_t^s}{d\rho_\sigma}y_{t-1}^s) = \sum_{j=1}^{t}\rho_y^{j+1}\mathbb{E}(\frac{d\sigma_{t-j}^2}{d\rho_\sigma}) = \sum_{j=1}^{t}\rho_y^{j+1}\sum_{\ell=1}^{t-j}\ell\rho_\sigma^{2\ell-1}\mu_\sigma$$

$$\mathbb{E}(\frac{dy_t^s}{d\rho_\sigma}y_{t-2}^s) = \sum_{j=2}^{t}\rho_y^{j+2}\mathbb{E}(\frac{d\sigma_{t-j}^2}{d\rho_\sigma}) = \sum_{j=1}^{t}\rho_y^{j+1}\sum_{\ell=1}^{t-j}\ell\rho_\sigma^{2\ell-1}\mu_\sigma.$$

The remaining derivatives can be computed similarly. The calculations above imply that the matrix is full rank only if $\rho_\sigma \neq 0$ and $\mu_\sigma \neq 0$ since all the expectations above are zero when either $\rho_\sigma = 0$ or $\mu_\sigma = 0$.

Remark A4 ($\hat{\theta}_n$ is generally not an adaptive estimator of θ_0)**.** *For the estimator $\hat{\theta}_n$ to be adaptive*[51] *an orthogonality condition is required, namely:*

$$\frac{d^2Q(\beta_0)}{d\theta df}[f - f_0] = 0, \text{ for all } f \in \mathcal{F}_{osn}.$$

For the CF, this amounts to:

$$\lim_{n\to\infty}\int Real\left(\frac{d\mathbb{E}(\hat{\psi}_n^s(\tau,\beta_0))}{d\theta}\overline{\frac{d\mathbb{E}(\hat{\psi}_n^s(\tau,\beta_0))}{df}[f-f_0]}\pi(\tau)d\tau\right) = 0.$$

Given the restrictions on the DGP and using the notation in the proof of Proposition A1, it implies:

$$\lim_{t\to\infty}\int Real\left(i\tau'\frac{dg_t(\theta_0,e_1)}{d\theta}e^{i\tau'[g_t(\theta_0,e_1)-g_t(\theta_0,e_2)]}f_0(e_1)\Delta f(e_2)\pi(\tau)d\tau de_1 de_2\right) = 0.$$

After some simplification, the orthogonality condition can be re-written as:

$$\lim_{t\to\infty}\int \tau'\frac{dg_t(\theta_0,e_1)}{d\theta}\sin\left(\tau'[g_t(\theta_0,e_1)-g_t(\theta,e_2)]\right)f_0(e_1)\Delta f(e_2)\pi(\tau)d\tau de_1 de_2 = 0.$$

This function is even in τ so that it does not average out over τ in general when π is chosen to be the Gaussian or the exponential density with mean-zero. Hence, the orthogonality condition holds if the integral of $\frac{dg_t(\theta_0,e_1)}{d\theta}\sin\left(\tau'[g_t(\theta_0,e_1)-g_t(\theta,e_2)]\right)f_0(e_1)\Delta f(e_2)$ over e_1 and e_2 is zero. This is the case if $g_t(\theta_0,e_1)$ is separable in e_1 and f_0, f are symmetric densities which is quite restrictive.

Proof of Proposition A1. Chen & Pouzo (2015), pages 1031-1033 and their Remark A.1, implies that $\hat{\theta}_n$ is root-n asymptotically normal if:

$$\lim_{n\to\infty}\inf_{v\in\overline{V},v_\theta\neq 0}\frac{1}{\|v_\theta\|_1^2}\int\left|\frac{d\mathbb{E}(\hat{\psi}_n^s(\tau,\beta_0))}{d\theta}v_\theta + \frac{d\mathbb{E}(\hat{\psi}_n^s(\tau,\beta_0))}{df}[v_f]\right|^2\pi(\tau)d\tau > 0.$$

[51] If the estimator is adaptive then $\hat{\theta}_n$ is root-n asymptotically normal and its asymptotic variance does not depend on $\hat{f}_n$, i.e. it has the same asymptotic variance as the CF based parametric SMM estimator with f_0 known.

By definition of $\overline{V}$ the vector $v = (v_\theta, v_f)$ has the form $v_\theta \in \mathbb{R}^{d_\theta}$ and $v_f = \sum_{j=0}^{\infty} a_j[f_j - f_0]$ for a sequence $(a_1, a_2, \dots)$ in $\mathbb{R}$ and $(f_1, f_2, \dots)$ such that $(\theta_j, f_j) \in \mathcal{B}_{osn}$ for some θ_j. To prove the result, we can proceed by contradiction suppose that for some non-zero v_θ and a v_f:

$$\int \left| \frac{d\mathbb{E}(\hat{\psi}_n^s(\tau, \beta_0))}{d\theta} v_\theta + \frac{d\mathbb{E}(\hat{\psi}_n^s(\tau, \beta_0))}{df}[v_f] \right|^2 \pi(\tau) d\tau = 0. \tag{A.16}$$

This implies that $\frac{d\mathbb{E}(\hat{\psi}_n^s(\tau,\beta_0))}{d\theta} v_\theta + \frac{d\mathbb{E}(\hat{\psi}_n^s(\tau,\beta_0))}{df}[v_f] = 0$ for all τ (π almost surely). This implies that the following holds:

$$\frac{d\mathbb{E}(\hat{\psi}_n^s(0, \beta_0))}{d\theta} v_\theta + \frac{d\mathbb{E}(\hat{\psi}_n^s(0, \beta_0))}{df}[v_f] = 0 \tag{A.17}$$

$$\frac{d^2\mathbb{E}(\hat{\psi}_n^s(\tau, \beta_0))}{d\theta d\tau}\Big|_{\tau=0} v_\theta + \frac{d^2\mathbb{E}(\hat{\psi}_n^s(\tau, \beta_0))}{df d\tau}[v_f]\Big|_{\tau=0} = 0 \tag{A.18}$$

$$\frac{d^3\mathbb{E}(\hat{\psi}_n^s(\tau, \beta_0))}{d\theta d\tau d\tau_\ell}\Big|_{\tau=0} v_\theta + \frac{d^3\mathbb{E}(\hat{\psi}_n^s(\tau, \beta_0))}{df d\tau d\tau_\ell}\Big|_{\tau=0}[v_f] = 0 \tag{A.19}$$

for all $\ell = 1, \dots, d_{\mathbf{y}}$. To simplify notation the following will be used: $f(e) = f(e_1) \times \cdots \times f(e_t)$ and $\Delta f_j(e) = [f_k(e_1) - f_0(e_1)]f_0(e_2) \times \cdots \times f_0(e_t) + f_0(e_1)[f_j(e_2) - f_0(e_2)]f_0(e_3) \times \cdots \times f_0(e_t) + \cdots + f_0(e_1) \dots f_0(e_{t-1})[f_j(e_t) - f_0(e_t)]$ and $\mathbf{y}_t^s = g_t(\theta, e_t^s, \dots, e_1^s)$ (the dependence on x is removed for simplicity). The first order derivatives can be written as:

$$\frac{d\mathbb{E}(\hat{\psi}_t^s(\tau, \beta_0))}{d\theta} = \int i\tau' \frac{dg_t(\theta_0, e)}{d\theta} e^{i\tau' g_t(\theta_0, e)} f_0(e) de, \quad \frac{d\mathbb{E}(\hat{\psi}_t^s(\tau, \beta_0))}{df}[v_f] = \sum_{j=0}^{\infty} a_j \int e^{i\tau' g_t(\theta_0, e)} \Delta f_j(e) de$$

For $\tau = 0$ this yields $\frac{d\mathbb{E}(\hat{\psi}_t^s(0,\beta_0))}{d\theta} = 0$ and $\frac{d\mathbb{E}(\hat{\psi}_t^s(0,\beta_0))}{df}[v_f] = 0$, so equality (A.17) holds automatically. Taking derivatives and setting $\tau = 0$ again implies:

$$\frac{d^2\mathbb{E}(\hat{\psi}_t^s(\tau, \beta_0))}{d\theta d\tau}\Big|_{\tau=0} = i \int \frac{dg_t(\theta_0, e)}{d\theta'} f_0(e) de$$
$$\frac{d^2\mathbb{E}(\hat{\psi}_t^s(\tau, \beta_0))}{df d\tau}[v_f]\Big|_{\tau=0} = i \sum_{j=0}^{\infty} a_j \int g_t(\theta_0, e) \Delta f_j(e) de$$

If $\mathbb{E}(\mathbf{y}_t^s)$ does not depend on f then $\int g_t(\theta_0, e)\Delta f_j(e) de = 0$ for all j and $\frac{d^2\mathbb{E}(\hat{\psi}_t^s(\tau,\beta_0))}{df d\tau}[v_f]\Big|_{\tau=0} = 0$ holds automatically. This implies that condition (A.18) becomes:

$$\mathbb{E}\left(\frac{d\mathbf{y}_t^s}{d\theta}\right) v_\theta = 0 \tag{A.20}$$

If $\mathbb{E}\left(\frac{d\mathbf{y}_t^s}{d\theta}\right)$ has rank greater or equal than d_θ then condition (A.20) holds only if $v_\theta \neq 0$; this is a contradiction. If the rank is less than d_θ, then taking derivatives with respect to τ again yields $\frac{d^3\mathbb{E}(\hat{\psi}_n^s(0,\beta_0))}{df d\tau d\tau'}\Big|_{\tau=0}[v_f] = -\sum_{j=0}^{\infty} a_j \int g_t(\theta, e) g_t(\theta, e)' \Delta f_j(e) de = 0$ assuming $\mathbb{E}(\mathbf{y}_t^s \mathbf{y}_t^{s\prime})$ does not depend

on f. Computing the other derivatives imply that condition (A.19) becomes $-v_\theta' \int \frac{dg(\theta_0)}{d\theta'} g(\theta_0, e) f_0(e) de$ i.e.:

$$v_\theta' \mathbb{E}\left(\frac{d\mathbf{y}_t^s}{d\theta'} \mathbf{y}_{t,\ell}^s \right) = 0 \text{ for all } \ell = 1, \ldots, d_{\mathbf{y}}. \tag{A.21}$$

Then, stacking conditions (A.20)-(A.21) together implies:

$$v_\theta' \mathbb{E}\left(\frac{d\mathbf{y}_t^s}{d\theta'} \left[\begin{pmatrix} 1 & \mathbf{y}_t^{s\prime} \end{pmatrix} \otimes I_{d_{\mathbf{y}}} \right] \right) = 0. \tag{A.22}$$

If the matrix has rank greater or equal to d_θ then it implies $v_\theta = 0$ which is a contradiction. Hence (A.16) holds only if $v_\theta = 0$ which proves the result. □

Appendix B Proofs for the Main Results

The proofs for the main results allow for a bounded linear operator B, as in Carrasco & Florens (2000), to weight the moments. In the appendices, the operator is assumed to be fixed:

$$\hat{Q}_n^S(\beta) = \int \left| B\hat{\psi}_n(\tau) - B\hat{\psi}_n^S(\tau, \beta) \right|^2 \pi(\tau) d\tau.$$

Since B is bounded linear there exists a $M_B > 0$ such that for any two CFs:

$$\int \left| B\hat{\psi}_n(\tau) - B\hat{\psi}_n^S(\tau, \beta) \right|^2 \pi(\tau) d\tau \leq M_B^2 \int \left| \hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta) \right|^2 \pi(\tau) d\tau.$$

As a result, the rate of convergence for the objective function with the weighting B is the same as the rate of convergence without.[52]

B.1 Properties of the Mixture Sieve

Lemma B8 (Kruijer et al. (2010))**.** *Suppose that f is a continuous univariate density satisfying:*

i. Smoothness: f is r-times continuously differentiable with bounded r-th derivative.

ii. Tails: f has exponential tails, i.e. there exists $\bar{e}, M_{f_1}, a, b > 0$ such that:

$$f_1(e) \leq M_{f_1} e^{-a|e|^b}, \ \forall |e| \geq \bar{e}.$$

iii. Monotonicity in the Tails: f is strictly positive and there exists $\underline{e} < \bar{e}$ such that f_S is weakly decreasing on $(-\infty, \underline{e}]$ and weakly increasing on $[\bar{e}, \infty)$.

[52] For results on estimating the optimal B see Carrasco & Florens (2000); Carrasco et al. (2007a). Using their method would lead to $M_{\hat{B}} \to \infty$ as $n \to \infty$ resulting in a slower rate of convergence for $\hat{\beta}_n$. Further investigation of this effect and the resulting rate of convergence are left to future research.

Let $\mathcal{F}_k$ be the sieve space consisting of Gaussian mixtures with the following restrictions:

iv. Bandwidth: $\sigma_j \geq \underline{\sigma}_k = O(\frac{\log[k(n)]^{2/b}}{k})$.

v. Location Parameter Bounds: $\mu_j \in [-\bar{\mu}_k, \bar{\mu}_k]$.

vi. Growth Rate of Bounds: $\bar{\mu}_k = O\left(\log[k]^{1/b}\right)$.

Then there exists $\Pi_k f \in \mathcal{F}_k$, a mixture sieve approximation of f, such that as $k \to \infty$:

$$\|f - \Pi_k f\|_{\mathcal{F}} = O\left(\frac{\log[k(n)]^{2r/b}}{k(n)^r}\right)$$

where $\|\cdot\|_{\mathcal{F}} = \|\cdot\|_{TV}$ or $\|\cdot\|_\infty$.

Proof of Lemma 2. :
The difference between e_t^s and $\tilde{e}_t^s$ can be split into two terms:

$$\sum_{j=1}^{k(n)} \left(\mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\omega_l, \sum_{l=0}^{j}\omega_l]} - \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\tilde{\omega}_l, \sum_{l=0}^{j}\tilde{\omega}_l]}\right)\left(\mu_j + \sigma_j Z_{t,j}^s\right) \tag{B.23}$$

$$\sum_{j=1}^{k(n)} \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\tilde{\omega}_l, \sum_{l=0}^{j}\tilde{\omega}_l]}\left(\mu_j - \tilde{\mu}_j + [\sigma_j - \tilde{\sigma}_j]Z_{t,j}^s\right). \tag{B.24}$$

To bound the term (B.23) in expectation, combine the fact that $|\mu_j| \leq \bar{\mu}_{k(n)}$, $|\sigma_j| \leq \bar{\sigma}$ and ν_t^s and $Z_{t,j}^s$ are independent so that:

$$\left[\mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2 \leq \delta}\left|\sum_{j=1}^{k(n)}\left(\mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\omega_l, \sum_{l=0}^{j}\omega_l]} - \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\tilde{\omega}_l, \sum_{l=0}^{j}\tilde{\omega}_l]}\right)\left(\mu_j + \sigma_j Z_{t,j}^s\right)\right|^2\right)\right]^{1/2}$$
$$\leq \sum_{j=1}^{k(n)}\left[\mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2 \leq \delta}\left|\mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\omega_l, \sum_{l=0}^{j}\omega_l]} - \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\tilde{\omega}_l, \sum_{l=0}^{j}\tilde{\omega}_l]}\right|^2\right)\right]^{1/2}\left(\bar{\mu}_{k(n)} + \bar{\sigma}\mathbb{E}\left(|Z_{t,j}^s|^2\right)^{1/2}\right).$$

The last term is bounded above by $\bar{\mu} + \bar{\sigma}C_Z$. Next, note that $\mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\omega_l, \sum_{l=0}^{j}\omega_l]} - \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\tilde{\omega}_l, \sum_{l=0}^{j}\tilde{\omega}_l]} \in \{0, 1\}$ so that:

$$\mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2 \leq \delta}\left|\mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\omega_l, \sum_{l=0}^{j}\omega_l]} - \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\tilde{\omega}_l, \sum_{l=0}^{j}\tilde{\omega}_l]}\right|^2\right)$$
$$= \mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2 \leq \delta}\left|\mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\omega_l, \sum_{l=0}^{j}\omega_l]} - \mathbb{1}_{\nu_t^s \in [\sum_{l=0}^{j-1}\tilde{\omega}_l, \sum_{l=0}^{j}\tilde{\omega}_l]}\right|\right).$$

Also, for any j: $|\sum_{l=0}^{j}\tilde{\omega}_l - \sum_{l=0}^{j}\omega_l| \leq \sum_{l=0}^{j}|\tilde{\omega}_l - \sum_{l=0}^{j}\omega_l| \leq \left(\sum_{l=0}^{j}|\tilde{\omega}_l - \omega_l|^2\right)^{1/2} \leq \|\tilde{\omega}-\omega\|_2 \leq \delta$. Following a similar approach to Chen et al. (2003):

$$\begin{aligned}
&\left[\mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2\leq\delta}\left|\mathbb{1}_{\nu_t^s\in[\sum_{l=0}^{j-1}\omega_l,\sum_{l=0}^{j}\omega_l]} - \mathbb{1}_{\nu_t^s\in[\sum_{l=0}^{j-1}\tilde{\omega}_l,\sum_{l=0}^{j}\tilde{\omega}_l]}\right|\right)\right]^{1/2} \\
&\leq \left[\mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2\leq\delta}\left|\mathbb{1}_{\nu_t^s\in[(\sum_{l=0}^{j-1}\tilde{\omega}_l)-\delta,(\sum_{l=0}^{j}\tilde{\omega}_l)+\delta]} - \mathbb{1}_{\nu_t^s\in[\sum_{l=0}^{j-1}\tilde{\omega}_l,\sum_{l=0}^{j}\tilde{\omega}_l]}\right|\right)\right]^{1/2} \\
&= \left[\left([(\sum_{l=0}^{j}\tilde{\omega}_l)+\delta] - [(\sum_{l=0}^{j-1}\tilde{\omega}_l)-\delta] - [(\sum_{l=0}^{j}\tilde{\omega}_l) - (\sum_{l=0}^{j-1}\tilde{\omega}_l)]\right)\right]^{1/2} = \sqrt{2\delta}.
\end{aligned}$$

Overall the term (B.23) is bounded above by $\sqrt{2}(1+C_Z)\left(\bar{\mu}_{k(n)}+\bar{\sigma}+k(n)\right)\sqrt{\delta}$. The term (B.24) can be bounded above by using the simple fact that $0 \leq \mathbb{1}_{\nu_t^s\in[\sum_{l=0}^{j-1}\tilde{\omega}_l,\sum_{l=0}^{j}\tilde{\omega}_l]} \leq 1$ and:

$$\begin{aligned}
&\left[\mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2\leq\delta}\left|\sum_{j=1}^{k(n)}\mathbb{1}_{\nu_t^s\in[\sum_{l=0}^{j-1}\tilde{\omega}_l,\sum_{l=0}^{j}\tilde{\omega}_l]}\left(\mu_j-\tilde{\mu}_j+[\sigma_j-\tilde{\sigma}_j]Z_{t,j}^s\right)\right|^2\right)\right]^{1/2} \\
&\leq \sum_{j=1}^{k(n)}\left[\mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2\leq\delta}\left|(\mu_j-\tilde{\mu}_j)+[\sigma_j-\tilde{\sigma}_j]Z_{t,j}^s\right|^2\right)\right]^{1/2} \\
&\leq \sum_{j=1}^{k(n)}\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2\leq\delta}\left(|\mu_j-\tilde{\mu}_j|+|\sigma_j-\tilde{\sigma}_j|C_Z\right) \\
&\leq (1+C_Z)\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2\leq\delta}\left(\sum_{j=1}^{k(n)}|\mu_j-\tilde{\mu}_j|^2+|\sigma_j-\tilde{\sigma}_j|^2\right)^{1/2} \leq (1+C_Z)\delta.
\end{aligned}$$

Without loss of generality assume that $\delta \leq 1$ so that:

$$\left[\mathbb{E}\left(\sup_{\|(\omega,\mu,\sigma)-(\tilde{\omega},\tilde{\mu},\tilde{\sigma})\|_2\leq\delta}\left|e_t^s-\tilde{e}_t^s\right|^2\right)\right]^{1/2} \leq 2\sqrt{2}(1+C_Z)\left(1+\bar{\mu}_{k(n)}+\bar{\sigma}+k(n)\right)\delta^{1/2}.$$

which concludes the proof. □

Lemma B9 (Properties of the Tails Distributions)**.** *Let $\bar{\xi} \geq \xi_1, \xi_2 \geq \underline{\xi} > 0$. Let $\nu_{t,1}^s$ and $\nu_{t,2}^s$ be uniform $\mathcal{U}_{[0,1]}$ draws and:*

$$e_{t,1}^s = -\left(\frac{1}{\nu_{t,1}^s}-1\right)^{\frac{1}{2+\xi_1}}, \quad e_{t,2}^s = \left(\frac{1}{1-\nu_{t,2}^s}-1\right)^{\frac{1}{2+\xi_2}}.$$

The densities of $e_{t,1}^s, e_{t,2}^s$ satisfy $f_{e_{t,1}^s}(e) \sim e^{-3-\xi_1}$ as $e \to -\infty$, $f_{e_{t,2}^s}(e) \sim e^{-3-\xi_2}$ as $e \to +\infty$. There exists a finite C bounding the second moments $\mathbb{E}\left(|e_{t,1}^s|^2\right) \leq C < \infty$ and $\mathbb{E}\left(|e_{t,2}^s|^2\right) \leq C < \infty$. Furthermore, the

draws $y^s_{t,1}$ and $y^s_{t,2}$ are L^2-smooth in ξ_1 and ξ_2 respectively:

$$\left[\mathbb{E}\left(\sup_{|\xi_1-\tilde{\xi}_1|\leq\delta}|e^s_{t,1}(\xi_1)-e^s_{t,1}(\tilde{\xi}_1)|^2\right)\right]^{1/2}\leq C\delta,\quad \left[\mathbb{E}\left(\sup_{|\xi_2-\tilde{\xi}_2|\leq\delta}|e^s_{t,2}(\xi_2)-e^s_{t,2}(\tilde{\xi}_2)|^2\right)\right]^{1/2}\leq C\delta$$

Where the constant C only depends on $\underline{\xi}$ and $\bar{\xi}$.

Proof of Lemma B9. :
To reduce notation, the t and s subscripts will be dropped in the following. The proof is similar for both e_1 and e_2 so the proof is only given for e_1.

First, the densities of e_1 and e_2 are derived, the first two results follow. Noting that the draws are defined using quantile functions, inverting the formula yields: $\nu_1 = \frac{1}{1-e_1^{2+\xi_1}}$. This is a proper CDF on $(-\infty, 0]$ since $e_1 \to \frac{1}{1-e_1^{2+\xi_1}}$ is increasing and has limits 0 at $-\infty$ and 1 at 0. Its derivative is the density function: $(2+\xi_1)\frac{e_1^{1+\xi_1}}{(1-e_1^{2+\xi_1})^2}$ which is continuous on $(-\infty, 0]$ and has an asymptote at $-\infty$: $(2+\xi_1)\frac{e_1^{1+\xi_1}}{(1-e_1^{2+\xi_1})^2}\times e_1^{3+\xi_1} \to (2+\xi_1)$ as $e_1 \to -\infty$. Since $\xi_1 \in [\underline{\xi}, \bar{\xi}]$ with $0 < \underline{\xi}$ then $\mathbb{E}|e_1|^2 \leq C < \infty$ for some finite $C > 0$. Similar results hold for e_2 which has density $(2+\xi_2)\frac{e_2^{1+\xi_2}}{(1+e_2^{2+\xi_2})^2}$ on $[0, +\infty)$.

Second, $\xi_1 \to e_1(\xi_1)$ is shown to be L^2-smooth. Let $|\xi_1 - \tilde{\xi}_1| \leq \delta$, using the mean value theorem, for each ν_1 there exists an intermediate value $\check{\xi}_1 \in [\xi_1, \tilde{\xi}_1]$ such that:

$$\left(\frac{1}{\nu_1}-1\right)^{\frac{1}{2+\xi_1}}-\left(\frac{1}{\nu_1}-1\right)^{\frac{1}{2+\tilde{\xi}_1}}=\frac{1}{2+\check{\xi}_1}\log(\frac{1}{\nu_1}-1)\left(\frac{1}{\nu_1}-1\right)^{\frac{1}{2+\check{\xi}_1}}(\xi_1-\tilde{\xi}_1).$$

The first part is bounded above by $1/(2+\underline{\xi})$, the second part is bounded above by:

$$\log(\frac{1}{\nu_1}+1)\left(\frac{1}{\nu_1}+1\right)^{\frac{1}{2+\underline{\xi}}}$$

and the last term is bounded above, in absolute value, by δ.

Finally, in order to conclude the proof, the following integral needs to be finite:

$$\int_0^1 \log(\frac{1}{\nu_1}+1)\left(\frac{1}{\nu_1}+1\right)^{\frac{2}{2+\underline{\xi}}}d\nu_1.$$

By a change of variables, it can be re-written as:

$$\int_2^\infty \log(\nu)\nu^{\frac{2}{2+\underline{\xi}}-2}d\nu.$$

Since $\frac{2}{2+\underline{\xi}} - 2 < -1$, the integral is finite and thus:

$$\left[\mathbb{E}\left(\sup_{|\xi_1-\tilde{\xi}_1|\leq\delta}|e^s_{t,1}(\xi_1)-e^s_{t,1}(\tilde{\xi}_1)|^2\right)\right]^{1/2} \leq \frac{\delta}{2+\underline{\xi}}\sqrt{\int_2^\infty \log(v)v^{\frac{2}{2+\underline{\xi}}-2}dv}.$$

□

Proof of Lemma 1. The proof proceeds by recursion. Denote $\pi_{k(n)}f_j \in \mathcal{B}B_{k(n)}$ the mixture approximation of f_j from Lemma B8. For $d_e = 1$, Lemma B8 implies

$$\|f_1 - \Pi_{k(n)}f_1\|_{TV} = O(\frac{\log[k(n)]^{r/b}}{k(n)^r}), \quad \|f_1 - \Pi_{k(n)}f_1\|_\infty = O(\frac{\log[k(n)]^{r/b}}{k(n)^r}).$$

Suppose the result holds for $f_1 \times \cdots \times f_{d_e}$. Let $f = f_1 \times \cdots \times f_{d_e} \times f_{d_e+1}$; let:

$$\begin{aligned} d_{t+1} &= f_1 \times \cdots \times f_{d_e} \times f_{d_e+1} - \Pi_{k(n)}f_1 \times \cdots \times \Pi_{k(n)}f_{d_e} \times \Pi_{k(n)}f_{d_e+1} \\ d_t &= f_1 \times \cdots \times f_{d_e} - \Pi_{k(n)}f_1 \times \cdots \times \Pi_{k(n)}f_{d_e}. \end{aligned}$$

The difference can be re-written as a recursion:

$$d_{t+1} = d_t f_{d_e+1} + \Pi_{k(n)}f_1 \times \cdots \times \Pi_{k(n)}f_{d_e}\left(f_{d_e+1} - \Pi_{k(n)}f_{d_e+1}\right).$$

Since $\int f_{d_e+1} = \int \Pi_{k(n)}f_1 \times \cdots \times \Pi_{k(n)}f_{d_e} = 1$, the total variation distance is:

$$\|d_{t+1}\|_{TV} \leq \|d_t\|_{TV} + \|f_{d_e+1} - \Pi_{k(n)}f_{d_e+1}\|_{TV} = O\left(\frac{\log[k(n)]^{r/b}}{k(n)^r}\right).$$

And the supremum distance is:

$$\begin{aligned} &\|d_{t+1}\|_\infty \leq \|d_t\|_\infty\|f_{d_e+1}\|_\infty + \|\Pi_{k(n)}f_1 \times \cdots \times \Pi_{k(n)}f_{d_e}\|_\infty\|f_{d_e+1} - \Pi_{k(n)}f_{d_e+1}\|_\infty \\ &\leq \|d_t\|_\infty\left(\|f_{d_e+1}\|_\infty + \|f_1 \times \cdots \times f_{d_e}\|_\infty\|f_{d_e+1} - \Pi_{k(n)}f_{d_e+1}\|_\infty\right) = O\left(\frac{\log[k(n)]^{r/b}}{k(n)^r}\right). \end{aligned}$$

□

Definition B3 (Pseudo-Norm $\|\cdot\|_m$ on $\mathcal{B}_{k(n)}$)**.** *Let $\beta_1, \beta_2 \in \mathcal{B}_{k(n)}$ where $\beta_l = (\theta_l, f_l), l = 1, 2$ with $f_j = f_{1,j} \times \ldots f_{d_e,j}$, each $f_{l,j}$ as in definition 1. The pseudo-norm $\|\cdot\|_m$ is the ℓ^2 norm on $(\theta, \omega, \mu, \sigma, \xi)$, the associated distance is:*

$$\|\beta_1 - \beta_2\|_m = \|(\theta_1, \omega_1, \mu_1, \sigma_1, \xi_1) - (\theta_2, \omega_2, \mu_2, \sigma_2, \xi_2)\|_2$$

using the vector notation $\omega_1 = (\omega_{1,1}, \ldots, \omega_{1,k(n)+2}, \ldots, \omega_{d_e,1}, \ldots, \omega_{d_e,k(n)+2})$ for $\theta, \omega, \mu, \sigma, \xi$.

Remark B5. *Using lemma 6 in Kruijer et al. (2010), for any two mixtures f_1, f_2 in $\mathcal{B}_{k(n)}$:*

$$\|f_1 - f_2\|_\infty \leq C_\infty \frac{\|f_1 - f_2\|_m}{\underline{\sigma}^2_{k(n)}}, \quad \|f_1 - f_2\|_{TV} \leq C_{TV} \frac{\|f_1 - f_2\|_m}{\underline{\sigma}_{k(n)}}$$

for some constants $C_\infty, C_{TV} > 0$. The result extends to $d_e > 1$, for instance when $d_e = 2$:

$$f_1^1 f_1^2 - f_2^1 f_2^2 = f_1^1(f_1^2 - f_2^2) + (f_1^2 - f_2^1) f_2^2$$

In total variation distance the difference becomes:

$$\begin{aligned}\|f_1^1 f_1^2 - f_2^1 f_2^2\|_{TV} \leq &= \|f_1^2 - f_2^2\|_{TV} + \|f_1^2 - f_2^1\|_{TV} \\ &\leq C_{TV} \frac{\|f_1^2 - f_2^2\|_m + \|f_1^2 - f_2^1\|_m}{\underline{\sigma}_{k(n)}} \leq C_{TV,2} \frac{\|f_1 - f_2\|_m}{\underline{\sigma}_{k(n)}}.\end{aligned}$$

A recursive argument yields the result for arbitrary $d_e > 1$. In supremum distance a similar result holds assuming $\|f_1^j\|_\infty, \|f_2^j\|_\infty$, with $j = 1, 2$, are bounded above by a constant.

B.2 Consistency

Assumption 2′ (Data Generating Process - L^2-Smoothness)**.** *y_t^s is simulated according to the dynamic model (1)-(2) where g_{obs} and g_{latent} satisfy the following L^2-smoothness conditions for some $\gamma \in (0, 1]$ and any $\delta \in (0, 1)$:*

$y(i)'$. For some $0 \leq \bar{C}_1 < 1$:

$$\begin{aligned}&\left[\mathbb{E}\left(\sup_{\|\beta_1 - \beta_2\|_{\mathcal{B}} \leq \delta} \|g_{obs}(y_t^s(\beta_1), x_t, \beta_1, u_t^s(\beta_1)) - g_{obs}(y_t^s(\beta_2), x_t, \beta_1, u_t^s(\beta_1))\|^2 \Big| y_t^s(\beta_1), y_t^s(\beta_2)\right)\right]^{1/2} \\ &\leq \bar{C}_1 \|y_t^s(\beta_1) - y_t^s(\beta_2)\|\end{aligned}$$

$y(ii)'$. For some $0 \leq \bar{C}_2 < \infty$:

$$\left[\mathbb{E}\left(\sup_{\|\beta_1 - \beta_2\|_{\mathcal{B}} \leq \delta} \|g_{obs}(y_t^s(\beta_1), x_t, \beta_1, u_t^s(\beta_1)) - g_{obs}(y_t^s(\beta_1), x_t, \beta_2, u_t^s(\beta_1))\|^2\right)\right]^{1/2} \leq \bar{C}_2 \delta^\gamma$$

$y(iii)'$. For some $0 \leq \bar{C}_3 < \infty$:

$$\begin{aligned}&\left[\mathbb{E}\left(\sup_{\|\beta_1 - \beta_2\|_{\mathcal{B}} \leq \delta} \|g_{obs}(y_t^s(\beta_1), x_t, \beta_1, u_t^s(\beta_1)) - g_{obs}(y_t^s(\beta_1), x_t, \beta_1, u_t^s(\beta_2))\|^2 \Big| u_t^s(\beta_1), u_t^s(\beta_2)\right)\right]^{1/2} \\ &\leq \bar{C}_3 \|u_t^s(\beta_1) - u_t^s(\beta_2)\|^\gamma\end{aligned}$$

$u(i)'$. *For some* $0 \leq \bar{C}_4 < 1$

$$\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_{\mathcal{B}}\leq\delta} \|g_{latent}(u_{t-1}^s(\beta_1),\beta,e_t^s(\beta_1)) - g_{latent}(u_{t-1}^s(\beta_2),\beta,e_t^s(\beta_1))\|^2\right)\right]^{1/2}$$
$$\leq \bar{C}_4\|u_{t-1}^s(\beta_1) - u_{t-1}^s(\beta_2)\|$$

$u(ii)'$. *For some* $0 \leq \bar{C}_5 < \infty$*:*

$$\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_{\mathcal{B}}\leq\delta} \|g_{latent}(u_{t-1}^s(\beta_1),\beta_1,e_t^s(\beta_1)) - g_{latent}(u_{t-1}^s(\beta_1),\beta_2,e_t^s(\beta_1))\|^2\right) \leq \bar{C}_5\delta^\gamma$$

$u(iii)'$. *For some* $0 \leq \bar{C}_5 < \infty$*:*

$$\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_{\mathcal{B}}\leq\delta} \|g_{latent}(u_{t-1}^s(\beta_1),\beta_1,e_t^s(\beta_1)) - g_{latent}(u_{t-1}^s(\beta_1),\beta_1,e_t^s(\beta_2))\|^2 \Big| e_t^s(\beta_1), e_t^s(\beta_2)\right)$$
$$\leq \bar{C}_6\|e_1 - e_2\|$$

for $\|\beta_1 - \beta_2\|_{\mathcal{B}} = \|\theta_1 - \theta_2\| + \|f_1 - f_2\|_\infty$ *or* $\|\theta_1 - \theta_2\| + \|f_1 - f_2\|_{TV}$.

Proof of Lemma 3: First note that the cosine and sine functions are uniformly Lispchitz on the real line with Lipschitz coefficient 1. This implies for any two $(\mathbf{y}_1, \mathbf{y}_2, \mathbf{x})$ and any $\tau \in \mathbb{R}^{d_\tau}$:

$$|\cos(\tau'(\mathbf{y}_1,\mathbf{x})) - \cos(\tau'(\mathbf{y}_2,\mathbf{x}))| \leq |\tau'(\mathbf{y}_1 - \mathbf{y}_2, 0)| \leq \|\tau\|_\infty\|\mathbf{y}_1 - \mathbf{y}_2\|$$
$$|\sin(\tau'(\mathbf{y}_1,\mathbf{x})) - \sin(\tau'(\mathbf{y}_2,\mathbf{x}))| \leq |\tau'(\mathbf{y}_1 - \mathbf{y}_2, 0)| \leq \|\tau\|_\infty\|\mathbf{y}_1 - \mathbf{y}_2\|.$$

As a result, the moment function is also Lipschitz in $\mathbf{y}, \mathbf{x}$:

$$|e^{i\tau'(\mathbf{y}_1,\mathbf{x})} - e^{i\tau'(\mathbf{y}_2,\mathbf{x})}|\pi(\tau)^{\frac{1}{4}} \leq 2\|\tau\|_\infty\pi(\tau)^{\frac{1}{4}}\|\mathbf{y}_1 - \mathbf{y}_2\|.$$

Since π is chosen to be the Gaussian density, it satisfies $\sup_\tau \|\tau\|_\infty\phi(\tau)^{\frac{1}{4}} \leq C_\pi < \infty$ and $\phi(\tau)^{\frac{1}{2}} \propto \phi(\tau/\sqrt{2})$ which has finite integral.

The Lispschitz properties of the moments combined with the conditions properties of π imply that the L^2-smoothness of the moments is implied by the L^2-smoothness of the simulated data itself. As a result, the remainder of the proof establishes the L^2-smoothness of $\mathbf{y}_t^s$.

First note that since $\mathbf{y}_t = (y_t, \ldots, y_{t-L})$:

$$\|\mathbf{y}_t(\beta_1) - \mathbf{y}_t(\beta_2)\| \leq \sum_{j=1}^{L} \|y_{t-j}(\beta_1) - y_{t-j}(\beta_2)\|.$$

To bound the term in $\mathbf{y}$ above, it suffices to bound the expression for each term y_t with arbitrary $t \geq 1$. Assumptions 2, 2′ imply that, for some $\gamma \in (0, 1]$:

$$\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|y_t(\beta_1) - y_t(\beta_2)\|^2\right)\right]^{1/2} \leq \overline{C}_1 \left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|y_{t-1}(\beta_1) - y_{t-1}(\beta_2)\|^2\right)\right]^{1/2} + \overline{C}_2 \frac{\delta^\gamma}{\underline{\sigma}^{2\gamma}_{k(n)}}$$
$$+ \overline{C}_3 \left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|u_t(\beta_1) - u_t(\beta_2)\|^2\right)\right]^{\gamma/2}.$$

The term $\frac{\delta^\gamma}{\underline{\sigma}^{2\gamma}_{k(n)}}$ comes from the fact that $\|\beta_1 - \beta_2\|_\infty \leq \frac{\|\beta_1-\beta_2\|_m}{\underline{\sigma}^2_{k(n)}}$ and $\|\beta_1 - \beta_2\|_{TV} \leq \frac{\|\beta_1-\beta_2\|_m}{\underline{\sigma}_{k(n)}}$ on $\mathcal{B}_{k(n)}$. Without loss of generality, suppose that $\underline{\sigma}_{k(n)} \leq 1$.[53] Applying this inequality recursively, and using the fact that y^s_0, u^s_0 are the same regardless of β, yields:

$$\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|y_t(\beta_1) - y_t(\beta_2)\|^2\right)\right]^{1/2} \leq \frac{\overline{C}_2}{1-\overline{C}_1} \frac{\delta^\gamma}{\underline{\sigma}^{2\gamma}_{k(n)}} + \overline{C}_3 \sum_{l=0}^{t-1} \overline{C}_1^l \left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|u_{t-l}(\beta_1) - u_{t-l}(\beta_2)\|^2\right)\right]^{\gamma/2}.$$

Using Lemmas 2 and B9 and the same approach as above:

$$\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|u_t(\beta_1) - u_t(\beta_2)\|^2\right)\right]^{1/2} \leq \overline{C}_4 \left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|u_{t-1}(\beta_1) - u_{t-1}(\beta_2)\|^2\right)\right]^{1/2} + \overline{C}_5 \frac{\delta^\gamma}{\underline{\sigma}^{2\gamma}_{k(n)}}$$
$$+ \overline{C}_6 C \left(k(n) + \bar{\mu}_{k(n)} + \bar{\sigma}\right) \delta^{\gamma/2}.$$

Again, applying this inequality recursively yields:

$$\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|u_t(\beta_1) - u_t(\beta_2)\|^2\right)\right]^{1/2} \leq \frac{\overline{C}_5}{1-\overline{C}_4} \frac{\delta^\gamma}{\underline{\sigma}^{2\gamma}_{k(n)}} + \frac{\overline{C}_6}{1-\overline{C}_4} C \left(k(n) + \bar{\mu}_{k(n)} + \bar{\sigma}\right) \delta^{\gamma/2}.$$

Putting everything together:

$$\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|y_t(\beta_1) - y_t(\beta_2)\|^2\right)\right]^{1/2}$$
$$\leq \frac{\overline{C}_2}{1-\overline{C}_1} \frac{\delta^\gamma}{\underline{\sigma}^{2\gamma}_{k(n)}} + \frac{\overline{C}_3}{1-\overline{C}_1} \left(\frac{\overline{C}_5}{1-\overline{C}_4} \frac{\delta^\gamma}{\underline{\sigma}^{2\gamma}_{k(n)}} + \frac{\overline{C}_6}{1-\overline{C}_4} C \left(k(n) + \bar{\mu}_{k(n)} + \bar{\sigma}\right) \delta^{\gamma/2}\right)^\gamma.$$

Without loss of generality, suppose that $\delta \leq 1$. Then, for some positive constant $\overline{C}$:

$$\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m} \|y_t(\beta_1) - y_t(\beta_2)\|^2\right)\right]^{1/2} \leq \overline{C} \max\left(\frac{\delta^{\gamma^2}}{\underline{\sigma}^{2\gamma^2}_{k(n)}}, [k(n) + \overline{\mu}_{k(n)} + \overline{\sigma}]^\gamma \delta^{\gamma^2/2}\right).$$

□

[53] Recall that by assumption $\underline{\sigma}_{k(n)} = O(\frac{log[k(n)]^{2/b}}{k(n)})$ goes to zero.

Lemma B10 (Covering Numbers). *Under the L^2-smoothness of the DGP (as in Lemma 3), the bracketing number satisfies for $x \in (0,1)$ and some $\overline{C}$:*

$$N_{[]}(x, \Psi_{k(n)}(\tau), \|\cdot\|_{L^2}) \leq (3[k(n)+2]+d_\theta)\left(2\max(\bar{\mu}_{k(n)}, \underline{\sigma})\overline{C}^{2/\gamma^2}\frac{\left(k(n)+\overline{\mu}_{k(n)}+\overline{\sigma}\right)^{2/\gamma}+\underline{\sigma}^4_{k(n)}}{x^{2/\gamma^2}}+1\right)^{3[k(n)+2]+d_\theta}.$$

For $\tau \in \mathbb{R}^{d_\tau}$, let $\Psi_{k(n)}(\tau)$ be the set of functions $\Psi_{k(n)}(\tau) = \left\{\beta \to e^{i\tau'(\mathbf{y}_t(\beta),\mathbf{x}_t)}\pi(\tau)^{1/2},\ \beta \in \mathcal{B}_{k(n)}\right\}$. The bracketing entropy of each set $\Psi_{k(n)}(\tau)$ satisfies for some $\tilde{C}$:

$$\log\left(N_{[]}(x, \Psi_{k(n)}(\tau), \|\cdot\|_{L^2}\right) \leq \tilde{C}k(n)\log[k(n)])|\log\delta|.$$

Using the above, for some $\tilde{C}_2 < \infty$:

$$\int_0^1 \log^2\left(N_{[]}(x, \Psi_k(n), \|\cdot\|_{L^2}\right)dx \leq \tilde{C}_2 k(n)^2\log[k(n)]^2.$$

Proof of Lemma B10: Since $\mathcal{B}_{k(n)}$ is contained in a ball of radius $\max(\overline{\mu}_{k(n)}, \overline{\sigma}, \|\theta\|_\infty)$ in $\mathbb{R}^{3[k(n)+2]+d_\theta}$ under $\|\cdot\|_m$, the covering number for $\mathcal{B}_{k(n)}$ can be computed under the $\|\cdot\|_m$ norm using a result from Kolmogorov & Tikhomirov (1959).[54] As a result, the covering number $N(x, \mathcal{B}_{k(n)}, \|\cdot\|_m)$ satisfies:

$$N(x, \mathcal{B}_{k(n)}, \|\cdot\|_m) \leq 2\,(3[k(n)+2]+d_\theta)\left(\frac{2\max(\bar{\mu}_{k(n)}, \bar{\sigma})}{x}+1\right)^{3[k(n)+2]+d_\theta}.$$

The rest follows from Lemma 3 and Appendix C. □

Proof of Theorem 1: If the assumptions of Corollary C3 hold then the result of Theorem 1 holds as well. The following relates the previous lemmas and assumptions to the required assumption for the corollary.

Assumption 1 implies Assumptions C8 and C9. Furthermore, by Lemmas 3 and B10, Assumptions 1 with 2 (or 2′) imply Assumption C11 with $\sqrt{C_n/n} = O(\frac{k(n)^2\log^2[k(n)]}{\sqrt{n}})$ using the norm $\|\cdot\|_m$. The order of $Q_n(\Pi_{k(n)}\beta_0)$ is given in Lemma 4. This implies that all the assumptions for Corollary C3 so that the estimator is consistent if $\sqrt{C_n/n} = o(1)$ which concludes the proof. □

[54] See also Fenton & Gallant (1996) for an application of this result for the sieve estimation of a density and Coppejans (2001) for a CDF.

B.3 Rate of Convergence

Proof of Lemma 4: First, using the assumption that B is a bounded linear operator:

$$Q_n(\Pi_{k(n)}\beta_0) \leq M_B^2 \int \left|\mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)\right)\right|^2 \pi(\tau)d\tau$$
$$\leq 3M_B^2 \left(\int \left|\mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta_0)\right)\right|^2 \pi(\tau)d\tau + \int \left|\mathbb{E}\left(\hat{\psi}_n^S(\tau, \beta_0) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)\right)\right|^2 \pi(\tau)d\tau\right)$$

Each term can be bounded above individually. Re-write the first term in terms of distribution:

$$\left|\mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta_0)\right)\right| = \left|\frac{1}{n}\sum_{t=1}^n \int e^{i\tau'(\mathbf{y}_t, \mathbf{x_t})}[f_t^*(\mathbf{y}_t, \mathbf{x_t}) - f_t(\mathbf{y}_t, \mathbf{x_t})]d\mathbf{y}_t d\mathbf{x}_t\right|$$

where f_t is the distribution of $(\mathbf{y}_t(\beta_0), \mathbf{x}_t)$ and f_t the stationary distribution of $(\mathbf{y}_t(\beta_0), \mathbf{x}_t)$. Using the geometric ergodicity assumption, for all τ:

$$\left|\frac{1}{n}\sum_{t=1}^n \int e^{i\tau'(\mathbf{y}_t, \mathbf{x_t})}[f_t^*(\mathbf{y}_t, \mathbf{x_t}) - f_t(\mathbf{y}_t, \mathbf{x_t})]d\mathbf{y}_t d\mathbf{x}_t\right| \leq \frac{1}{n}\sum_{t=1}^n \int \left|f_t^*(\mathbf{y}_t, \mathbf{x_t}) - f_t(\mathbf{y}_t, \mathbf{x_t})\right| d\mathbf{y}_t d\mathbf{x}_t$$
$$= \frac{2}{n}\sum_{t=1}^n \|f_t^* - f_t\|_{TV} \leq \frac{2C_\rho}{n}\sum_{t=1}^n \rho^t \leq \frac{2C_\rho}{(1-\rho)n}$$

for some $\rho \in (0,1)$ and $C_\rho > 0$. This yields a first bound:

$$\int \left|\mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^S(\tau, \beta_0)\right)\right|^2 \pi(\tau)d\tau \leq \frac{4C_\rho^2}{(1-\rho)^2}\frac{1}{n^2} = O\left(\frac{1}{n^2}\right).$$

The mixture norm $\|\cdot\|_m$ is not needed here to bound the second term since it involves population CFs. Some changes to the proof of Lemma 3 allows to find bounds in terms of $\|\cdot\|_{\mathcal{B}}$ and $\|\cdot\|_{TV}$ for which Lemma 1 gives the approximation rates.

To bound the second term, re-write the simulated data as:

$$y_t^s = g_{obs,t}(x_t, \ldots, x_1, \beta, e_t^s, \ldots, e_1^s), \quad u_t^s = g_{latent,t}(\beta, e_t^s, \ldots, e_1^s)$$

with $\beta = (\theta, f)$ and $e_t^s \sim f$. Under Assumption 2 or 2′, using the same sequence of shocks (e_t^s):

$$\mathbb{E}\left(\left\|g_{obs,t}(x_t, \ldots, x_1, \beta_0, e_t^s, \ldots, e_1^s) - g_{obs,t}(x_t, \ldots, x_1, \Pi_{k(n)}\beta_0, e_t^s, \ldots, e_1^s)\right\|\right) \leq \overline{C}\|\Pi_{k(n)}f_0 - f_0\|_{\mathcal{B}}^\gamma.$$

This is similar to the proof of Lemma 3, first re-write the difference as:

$$\mathbb{E}\Big(\Big\|g_{obs}(g_{obs,t-1}(x_{t-1}, \ldots, x_1, \beta_0, e_{t-1}^s, \ldots, e_1^s), x_t, \beta_0, g_{latent}(g_{latent,t-1}(\beta_0, e_{t-1}^s, \ldots, e_1^s), \beta_0, e_t^s))$$
$$- g_{obs}(g_{obs,t-1}(x_{t-1}, \ldots, x_1, \Pi_{k(n)}\beta_0, e_{t-1}^s, \ldots, e_1^s), x_t, \Pi_{k(n)}\beta_0, g_{latent}(g_{latent,t-1}(\Pi_{k(n)}\beta_0, e_{t-1}^s, \ldots, e_1^s), \Pi_{k(n)}\beta_0, e_t^s)\Big\|\Big).$$

Using Assumptions 2-2′, there is a recursive relationship:

$$\begin{aligned}
&\mathbb{E}\Big(\Big\|g_{obs}(g_{obs,t-1}(x_{t-1},\ldots,x_1,\beta_0,e_{t-1}^s,\ldots,e_1^s),x_t,\beta_0,g_{latent}(g_{latent,t-1}(\beta_0,e_{t-1}^s,\ldots,e_1^s),\beta_0,e_t^s))\\
&-g_{obs}(g_{obs,t-1}(x_{t-1},\ldots,x_1,\Pi_{k(n)}\beta_0,e_{t-1}^s,\ldots,e_1^s),x_t,\Pi_{k(n)}\beta_0,g_{latent}(g_{latent,t-1}(\Pi_{k(n)}\beta_0,e_{t-1}^s,\ldots,e_1^s),\Pi_{k(n)}\beta_0,e_t^s)\Big\|\Big)\\
&\leq\Big[\mathbb{E}\Big(\Big\|g_{obs}(g_{obs,t-1}(x_{t-1},\ldots,x_1,\beta_0,e_{t-1}^s,\ldots,e_1^s),x_t,\beta_0,g_{latent}(g_{latent,t-1}(\beta_0,e_{t-1}^s,\ldots,e_1^s),\beta_0,e_t^s))\\
&-g_{obs}(g_{obs,t-1}(x_{t-1},\ldots,x_1,\Pi_{k(n)}\beta_0,e_{t-1}^s,\ldots,e_1^s),x_t,\Pi_{k(n)}\beta_0,g_{latent}(g_{latent,t-1}(\Pi_{k(n)}\beta_0,e_{t-1}^s,\ldots,e_1^s),\Pi_{k(n)}\beta_0,e_t^s)\Big\|^2\Big)\Big]^{1/2}\\
&\leq\overline{C}_1\Big[\mathbb{E}\Big(\Big\|g_{obs,t-1}(x_{t-1},\ldots,x_1,\beta_0,e_{t-1}^s,\ldots,e_1^s)-g_{obs,t-1}(x_{t-1},\ldots,x_1,\Pi_{k(n)}\beta_0,e_{t-1}^s,\ldots,e_1^s)\Big\|^2\Big)\Big]^{1/2}\\
&+\overline{C_2}\|\beta_0-\Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{\gamma}+\overline{C}_3\Big[\mathbb{E}\Big(\Big\|g_{latent,t}(\beta_0,e_t^s,\ldots,e_1^s)-g_{latent,t}(\Pi_{k(n)}\beta_0,e_t^s,\ldots,e_1^s)\Big\|^2\Big)\Big]^{\gamma/2}.
\end{aligned}$$

The last term also has a recursive structure:

$$\begin{aligned}
&\Big[\mathbb{E}\Big(\Big\|g_{latent,t}(\beta_0,e_t^s,\ldots,e_1^s)-g_{latent,t}(\Pi_{k(n)}\beta_0,e_t^s,\ldots,e_1^s)\Big\|^2\Big)\Big]^{1/2}\\
&\leq\overline{C}_4\left[\mathbb{E}\Big(\Big\|g_{latent,t-1}(\beta_0,e_{t-1}^s,\ldots,e_1^s)-g_{latent,t-1}(\Pi_{k(n)}\beta_0,e_{t-1}^s,\ldots,e_1^s)\Big\|^2\Big)\right]^{1/2}+\overline{C}_5\|\beta_0-\Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{\gamma}.
\end{aligned}$$

Together these inequalities imply:

$$\begin{aligned}
&\mathbb{E}\Big(\Big\|g_{obs}(g_{obs,t-1}(x_{t-1},\ldots,x_1,\beta_0,e_{t-1}^s,\ldots,e_1^s),x_t,\beta_0,g_{latent}(g_{latent,t-1}(\beta_0,e_{t-1}^s,\ldots,e_1^s),\beta_0,e_t^s))\\
&-g_{obs}(g_{obs,t-1}(x_{t-1},\ldots,x_1,\Pi_{k(n)}\beta_0,e_{t-1}^s,\ldots,e_1^s),x_t,\Pi_{k(n)}\beta_0,g_{latent}(g_{latent,t-1}(\Pi_{k(n)}\beta_0,e_{t-1}^s,\ldots,e_1^s),\Pi_{k(n)}\beta_0,e_t^s)\Big\|\Big)\\
&\leq\frac{1}{1-\overline{C}_1}\left(\overline{C}_2\|\beta_0-\Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{\gamma}+\overline{C}_3\frac{\overline{C}_5^{\gamma}}{(1-\overline{C}_4)^{\gamma}}\|\beta_0-\Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{\gamma^2}\right).
\end{aligned}$$

Recall that $\|\tau\|_\infty\sqrt{\pi(\tau)}$ is bounded above and $\pi(\tau)^{1/4}$ is integrable so that:

$$\begin{aligned}
&\int\left|\mathbb{E}\left(e^{i\tau'(\mathbf{y}_t(\beta_0,x_t,\ldots,x_1))}-e^{i\tau'(\mathbf{y}_t(\Pi_{k(n)}\beta_0,x_t,\ldots,x_1))}\right)\right|^2\pi(\tau)d\tau\\
&\leq\frac{1}{1-\overline{C}_1}\left(\overline{C}_2\|\beta_0-\Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{\gamma}+\overline{C}_3\frac{\overline{C}_5^{\gamma}}{(1-\overline{C}_4)^{\gamma}}\|\beta_0-\Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{\gamma^2}\right)\sup_{\tau}[\|\tau\|_\infty\sqrt{\pi(\tau)}]\int\pi(\tau)^{1/4}d\tau.
\end{aligned}$$

To conclude the proof, the difference due to e_t^s needs to be bounded. In order to do so, it suffice to bound the following integral:

$$\int e^{i\tau'(\mathbf{y}_t(y_0,u_0,x_t,\ldots,x_1,\beta_0,e_t^s,\ldots,e_1^s),\mathbf{x}_t)}\left(f_0(e_t^s)\times\cdots\times f_0(e_1^s)-\Pi_{k(n)}f_0(e_t^s)\times\cdots\times\Pi_{k(n)}f_0(e_1^s)\right)f_{\mathbf{x}}(\mathbf{x}_t)de_t^s\ldots de_1^sd\mathbf{x}_t.$$

A direct bound on this integral yields a term of order of $t\|f_0-\Pi_{k(n)}f_0\|_{TV}$ which increases too fast with t to generate useful rates. Rather than using a direct bound, consider Assumptions 2-2′. The time-series y_t^s can be approximated by another time-series term which only depends on a fixed and

finite $(e_t^s, \ldots, e_{t-m}^s)$ for a given integer $m \geq 1$. Making m grow with n at an appropriate rate allows to balance the bias $m\|f_0 - \Pi_{k(n)} f_0\|_{TV}$ (computed from a direct bound) and the approximation due to $m < t$.

The m-approximation rate of y_t is now derived. Let $\beta = (\theta, f) \in \mathcal{B}$, $e_t^s, \ldots, e_1^s \sim f$ and $\tilde{y}_t^s$ such that $\tilde{y}_{t-m}^s = 0, \tilde{u}_{t-m}^s = 0$ and then $\tilde{y}_j^s = g_{obs}(\tilde{y}_{j-1}^s, x_j, \beta, \tilde{u}_j^s), \tilde{u}_j^s = g_{latent}(\tilde{u}_{j-1}^s, \beta, e_j^s)$ for $t - m + 1 \leq j \leq t$. Each observation t is approximated by its own time-series. For observation $t - m$, by construction:

$$\mathbb{E}\left(\left\|y_{t-m}^s - \tilde{y}_{t-m}^s\right\|\right) = \mathbb{E}\left(\left\|y_{t-m}^s\right\|\right) \leq \left[\mathbb{E}\left(\left\|y_{t-m}^s\right\|^2\right)\right]^{1/2}$$
$$\mathbb{E}\left(\left\|u_{t-m}^s - \tilde{u}_{t-m}^s\right\|\right) = \mathbb{E}\left(\left\|u_{t-m}^s\right\|\right) \leq \left[\mathbb{E}\left(\left\|u_{t-m}^s\right\|^2\right)\right]^{1/2}$$

Then, for any $t \geq \tilde{t} \geq t - m$:

$$\mathbb{E}\left(\left\|u_{\tilde{t}}^s - \tilde{u}_{\tilde{t}}^s\right\|\right) \leq \overline{C}_4 \left[\mathbb{E}\left(\left\|u_{\tilde{t}-1}^s - \tilde{u}_{\tilde{t}-1}^s\right\|^2\right)\right]^{1/2}$$
$$\mathbb{E}\left(\left\|y_{\tilde{t}}^s - \tilde{y}_{\tilde{t}}^s\right\|\right) \leq \overline{C}_3 \overline{C}_4^{\gamma} \left[\mathbb{E}\left(\left\|u_{\tilde{t}-1}^s - \tilde{u}_{\tilde{t}-1}^s\right\|^2\right)\right]^{\gamma/2} + \overline{C}_1 \left[\mathbb{E}\left(\left\|y_{\tilde{t}-1}^s - \tilde{y}_{\tilde{t}-1}^s\right\|^2\right)\right]^{1/2}.$$

The previous two results and a recursion arguments leads to the following inequality:

$$\mathbb{E}\left(\left\|u_t^s - \tilde{u}_t^s\right\|\right) \leq \overline{C}_4^m \left[\mathbb{E}\left(\left\|u_{t-m}^s\right\|^2\right)\right]^{1/2} \tag{B.25}$$
$$\mathbb{E}\left(\left\|y_t^s - \tilde{y}_t^s\right\|\right) \leq \overline{C}_3 \overline{C}_4^{\gamma m} \left[\mathbb{E}\left(\left\|u_{t-m}^s\right\|^2\right)\right]^{\gamma/2} + \overline{C}_1^m \left[\mathbb{E}\left(\left\|y_{t-m}^s\right\|^2\right)\right]^{1/2}. \tag{B.26}$$

For $\beta = \beta_0, \Pi_{k(n)}\beta_0$ since the expectations are finite and bounded by assumption, $\mathbb{E}\left(\left\|y_t^s - \tilde{y}_t^s\right\|\right) \leq \overline{C} \max(\overline{C}_1, \overline{C}_4)^{\gamma m}$ with $0 \leq \max(\overline{C}_1, \overline{C}_4) < 1$ and some $\overline{C} > 0$. For the first observations $t \leq m$ the data is unchanged, $y_t^s = \tilde{y}_t^s$, so that the bound still holds. The integral can be split and bounded:

$$\begin{aligned}
&\left| \int e^{i\tau'(\mathbf{y}_t(y_0,u_0,x_t,\ldots,x_1,\beta_0,e_t^s,\ldots,e_1^s),\mathbf{x}_t)} \left(f_0(e_t^s) \times \cdots \times f_0(e_1^s) - \Pi_{k(n)} f_0(e_t^s) \times \cdots \times \Pi_{k(n)} f_0(e_1^s)\right) f_{\mathbf{x}}(\mathbf{x}_t) de_t^s \ldots de_1^s d\mathbf{x}_t \right| \\
&\leq \left| \mathbb{E}\left([\hat{\psi}_n^S(\tau, \beta_0) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] - [\tilde{\psi}_n^S(\tau, \beta_0) - \tilde{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] \right) \right| \\
&+ \int \left| \left(f_0(e_t^s) \times \cdots \times f_0(e_{t-m+1}^s) - \Pi_{k(n)} f_0(e_t^s) \times \cdots \times \Pi_{k(n)} f_0(e_{t-m+1}^s)\right) f_{\mathbf{x}}(\mathbf{x}_t) de_t^s \ldots de_{t-m+1}^s d\mathbf{x}_t \right| \\
&\leq 4\overline{C} \max(\overline{C}_1, \overline{C}_4)^{\gamma m} + 2m\|\Pi_{k(n)} f_0 - f_0\|_{TV}.
\end{aligned}$$

The last inequality is due to the cosine, and sine function being uniformly Lipschitz continuous and equations (B.25)-(B.26). Recall that $\|\Pi_{k(n)} f_0 - f_0\|_{TV} = O(\frac{\log[k(n)]^{2r/b}}{k(n)^r})$. To balance the two terms, choose:

$$m = -\frac{r}{\gamma \log \max(\overline{C}_1, \overline{C}_4)} \log[k(n)] > 0$$

so that $\max(\overline{C}_1, \overline{C}_4)^{\gamma m} = k(n)^{-r}$ and

$$\overline{C}\max(\overline{C}_1, \overline{C}_4)^{\gamma m} + 2m\|\Pi_{k(n)}f_0 - f_0\|_{TV} = O\left(\frac{\log[k(n)]^{2r/b+1}}{k(n)^r}\right).$$

Combining all the bounds above yields:

$$Q_n(\Pi_{k(n)}\beta_0) = O\left(\max\left[\frac{\log[k(n)]^{4r/b+2}}{k(n)^{2r}}, \frac{\log[k(n)]^{4\gamma^2 r/b}}{k(n)^{2\gamma^2 r}}, \frac{1}{n^2}\right]\right)$$

where $\|\cdot\|_{\mathcal{B}} = \|\cdot\|_\infty$ or $\|\cdot\|_{TV}$ so that $\|\beta_0 - \Pi_{k(n)}\beta_0\|_{\mathcal{B}}^{\gamma^2} = O(\frac{\log[k(n)]^{4\gamma^2 r/b}}{k(n)^{2\gamma^2 r}})$. The term due to the non-stationarity is of order $1/n^2 = o\left(\max\left[\frac{\log[k(n)]^{4r/b+2}}{k(n)^{2r}}, \frac{\log[k(n)]^{4\gamma^2 r/b}}{k(n)^{2\gamma^2 r}}\right]\right)$ so it can be ignored. This concludes the proof. □

Proof of Theorem 2: The theorem is a corollary of Theorem C5 with a mixture sieve. Lemma 4 gives an explicit derivation of $\sqrt{Q_n(\Pi_{k(n)}\beta_0)}$ in this setting. □

B.4 Asymptotic Normality

Remark B6. *Note that for each τ the matrix $B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0))}{d(\theta,\omega,\mu,\sigma)}'\overline{B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0))}{d(\theta,\omega,\mu,\sigma)}}$ is singular - the requirement is that the average, over τ, of this matrix is invertible. Lemma 5 states that $\hat{\beta}_n$ and the approximation $\Pi_{k(n)}\beta_0$ have a representation that are at a distance $\delta_n\underline{\lambda}_n^{-1/2}$ of each other in $\|\cdot\|_m$ norm.*

Proof of Lemma 5: Using the simple inequality $1/2|a|^2 \leq |a-b|^2 + |b|^2$ for any $a, b \in \mathbb{R}$:

$$\begin{aligned}
0 &\leq 1/2\int\left|B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0))}{d\beta}[\hat{\beta}_n - \Pi_{k(n)}\beta_0]\right|^2\pi(\tau)d\tau \\
&\leq \int\left|B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0))}{d\beta}[\hat{\beta}_n - \beta_0]\right|^2\pi(\tau)d\tau \\
&+ \int\left|B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0))}{d\beta}[\hat{\beta}_n - \beta_0] - B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0))}{d\beta}[\hat{\beta}_n - \Pi_{k(n)}\beta_0]\right|^2\pi(\tau)d\tau \\
&\leq \int\left|B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0))}{d\beta}[\hat{\beta}_n - \beta_0]\right|^2\pi(\tau)d\tau + \int\left|B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0))}{d\beta}[\Pi_{k(n)}\beta_0 - \beta_0]\right|^2\pi(\tau)d\tau \\
&+ \int\left|B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0))}{d\beta}[\hat{\beta}_n - \beta_0] - B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\beta_0))}{d\beta}[\hat{\beta}_n - \Pi_{k(n)}\beta_0]\right|^2\pi(\tau)d\tau.
\end{aligned}$$

By assumption the term on the left is $O_p(\delta_n^2)$, by assumption ii. the middle term is $O_p(\delta_n^2)$ and assumption i. implies that the term on the right is also $O_p(\delta_n^2)$. It follows that:

$$\int\left|B\frac{d\mathbb{E}(\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0))}{d\beta}[\hat{\beta}_n - \Pi_{k(n)}\beta_0]\right|^2\pi(\tau)d\tau = O_p(\delta_n^2). \tag{B.27}$$

Now note that both $\hat{\beta}_n$ and $\Pi_{k(n)}\beta_0$ belong to the finite dimensional space $\mathcal{B}_{k(n)}$ parameterized by $(\theta, \omega, \mu, \sigma)$. To save space, $\hat{\beta}_n$ will be represented by $\hat{\varphi}_n = (\hat{\theta}_n, \hat{\omega}_n, \hat{\mu}_n, \hat{\sigma}_n)$ and $\Pi_{k(n)}\beta_0$ by $\varphi_{k(n)} = (\theta_{k(n)}, \omega_{k(n)}, \mu_{k(n)}, \sigma_{k(n)})$. Using this notation, equation (B.27) becomes:

$$\int \left| B \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0))}{d\beta} [\hat{\beta}_n - \Pi_{k(n)}\beta_0] \right|^2 \pi(\tau) d\tau = \int \left| B \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0))}{d(\theta, \omega, \mu, \sigma)} [\hat{\varphi}_n - \varphi_{k(n)}] \right|^2 \pi(\tau) d\tau$$
$$= trace \left([\hat{\varphi}_n - \varphi_{k(n)}]' \int B \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0))}{d(\theta, \omega, \mu, \sigma)}' \overline{B \frac{d\mathbb{E}(\hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0))}{d(\theta, \omega, \mu, \sigma)}} \pi(\tau) d\tau [\hat{\varphi}_n - \varphi_{k(n)}] \right)$$
$$\geq \underline{\lambda}_n \|\hat{\varphi}_n - \varphi_{k(n)}\|^2 = \underline{\lambda}_n \|\hat{\beta}_n - \Pi_{k(n)}\beta_0\|_m^2.$$

It follows that $0 \leq \underline{\lambda}_n \|\hat{\beta}_n - \Pi_{k(n)}\beta_0\|_m^2 \leq O_p(\delta_n^2)$ so that the rate of convergence in mixture norm is:

$$\|\hat{\beta}_n - \Pi_{k(n)}\beta_0\|_m = O_p\left(\delta_n \underline{\lambda}_n^{-1/2}\right).$$

□

Lemma B11 (Stochastic Equicontinuity). *Let $M_n = \log\log(n+1)$ and $\delta_{mn} = \delta_n / \sqrt{\underline{\lambda}_n}$. Suppose that the assumptions of Lemma 5 and Assumption C11 hold then for any $\eta > 0$, uniformly over $\beta \in \mathcal{B}_{k(n)}$:*

$$\left[\mathbb{E} \left(\sup_{\|\beta - \Pi_{k(n)}\beta_0\|_m \leq M_n \delta_{mn}} \left| [\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] - \mathbb{E}[\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] \right|^2 \pi(\tau)^{\frac{2}{2+\eta}} \right) \right]^{1/2}$$
$$\leq C \frac{(M_n \delta_{mn})^{\frac{\gamma^2}{2}}}{\sqrt{n}} \int_0^1 \left(x^{-\vartheta/2} \sqrt{\log N([x M_n \delta_{mn}]^{\frac{2}{\gamma^2}}, \mathcal{B}_{k(n)}, \|\cdot\|_m)} + \log^2 N([x M_n \delta_{mn}]^{\frac{2}{\gamma^2}}, \mathcal{B}_{k(n)}, \|\cdot\|_m) \right) dx$$

For the mixture sieve the integral is a $O(k(n)\log[k(n)] + k(n)|\log(M_n\delta_{mn})|)$ so that:

$$\left[\mathbb{E} \left(\int \sup_{\|\beta - \Pi_{k(n)}\beta_0\|_m \leq M_n \delta_{mn}} \left| [\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] - \mathbb{E}[\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] \right|^2 \pi(\tau) d\tau \right) \right]^{1/2}$$
$$= O\left((M_n \delta_{mn})^{\frac{\gamma^2}{2}} \max(\log[k(n)]^2, |\log[M_n \delta_{mn}]|^2) \frac{k(n)^2}{\sqrt{n}} \right)$$

Now suppose that $(M_n \delta_{mn})^{\frac{\gamma^2}{2}} \max(\log[k(n)]^2, |\log[M_n\delta_{mn}]|^2) k(n)^2 = o(1)$. The first stochastic equicontinuity result is:

$$\left[\mathbb{E} \left(\int \sup_{\|\beta - \Pi_{k(n)}\beta_0\|_m \leq M_n \delta_{mn}} \left| [\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] - \mathbb{E}[\hat{\psi}_n^S(\tau, \beta) - \hat{\psi}_n^S(\tau, \Pi_{k(n)}\beta_0)] \right|^2 \pi(\tau) d\tau \right) \right]^{1/2} = o(1/\sqrt{n}).$$

Also, suppose that $\beta \to \int \mathbb{E}\left|\hat{\psi}_t^s(\tau,\beta_0) - \hat{\psi}_t^s(\tau,\beta)\right|^2 \pi(\tau)d\tau$ *is continuous at* $\beta = \beta_0$ *under the norm* $\|\cdot\|_{\mathcal{B}}$*, uniformly in* $t \geq 1$*. Then, the second stochastic equicontinuity result is:*

$$\left[\mathbb{E}\left(\int \sup_{\|\beta - \Pi_{k(n)}\beta_0\|_m \leq M_n\delta_{mn}} \left|[\hat{\psi}_n^S(\tau,\beta) - \hat{\psi}_n^S(\tau,\beta_0)] - \mathbb{E}[\hat{\psi}_n^S(\tau,\beta) - \hat{\psi}_n^S(\tau,\beta_0)]\right|^2 \pi(\tau)d\tau\right)\right]^{1/2} = o(1/\sqrt{n}).$$

Proof of Lemma B11. This proof relies on the results in Lemma 3 together with Lemma D16. First, Lemma 3 implies that, after simplifying the bounds, for some $C > 0$:

$$\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_m \leq \delta, \|\beta_j - \Pi_{k(n)}\beta_0\|_m \leq M_n\delta_{m,n}, j=1,2} \left|\hat{\psi}_t^s(\tau,\beta_1) - \hat{\psi}_t^s(\tau,\beta_2)\right|^2\right)\right]^{1/2} \frac{\sqrt{\pi(\tau)}}{(M_n\delta_{m,n})^{\gamma^2/2}}$$
$$\leq Ck(n)^{2\gamma^2}\left(\frac{\delta}{M_n\delta_{m,n}}\right)^{\gamma^2/2}.$$

Next, apply the inequality of Lemma D15 to generate the bound:

$$\left[\mathbb{E}\left(\sup_{\|\beta - \Pi_{k(n)}\beta_0\|_m \leq M_n\delta_{m,n}} \left|[\hat{\psi}_n^S(\tau,\beta) - \hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)] - \mathbb{E}[\hat{\psi}_n^S(\tau,\beta) - \hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)]\right|^2\right)\right]^{1/2} \sqrt{\pi(\tau)}$$
$$\leq \overline{C}\frac{(M_n\delta_{m,n})^{\gamma^2/2}}{\sqrt{n}} \int_0^1 \left(x^{-\vartheta/2}\sqrt{\log N(\left[\frac{xM_n\delta_{mn}}{k(n)^{2\gamma^2}}\right]^{\frac{2}{\gamma^2}}, \mathcal{B}_{k(n)}, \|\cdot\|_m) + \log^2 N(\left[\frac{xM_n\delta_{mn}}{k(n)^{2\gamma^2}}\right]^{\frac{2}{\gamma^2}}, \mathcal{B}_{k(n)}, \|\cdot\|_m)}\right) dx$$

for some $\overline{C} > 0, \vartheta \in (0,1)$. Since $\int \sqrt{\pi(\tau)}d\tau < \infty$, the term on the left-hand side ca be squared and multiplied by $\sqrt{\pi(\tau)}$. Then, taking the integral:

$$\left[\mathbb{E}\left(\int \sup_{\|\beta - \Pi_{k(n)}\beta_0\|_m \leq M_n\delta_{m,n}} \left|[\hat{\psi}_n^S(\tau,\beta) - \hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)] - \mathbb{E}[\hat{\psi}_n^S(\tau,\beta) - \hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)]\right|^2 \pi(\tau)d\tau\right)\right]^{1/2}$$
$$\leq \overline{C}_\pi\frac{(M_n\delta_{m,n})^{\gamma^2/2}}{\sqrt{n}} \int_0^1 \left(x^{-\vartheta/2}\sqrt{\log N(\left[\frac{xM_n\delta_{mn}}{k(n)^{2\gamma^2}}\right]^{\frac{2}{\gamma^2}}, \mathcal{B}_{k(n)}, \|\cdot\|_m) + \log^2 N(\left[\frac{xM_n\delta_{mn}}{k(n)^{2\gamma^2}}\right]^{\frac{2}{\gamma^2}}, \mathcal{B}_{k(n)}, \|\cdot\|_m)}\right) dx$$

where $\overline{C}_\pi = \overline{C}\int \sqrt{\pi(\tau)}d\tau$. The integral on the right-hand side is a $O(k(n)^2 \max(\log[k(n)]^2, \log[M_n\delta_{m,n}]^2))$.

To prove the final statement, notation will be shortened using $\Delta\hat{\psi}_t^s(\tau,\beta) = \hat{\psi}_t^s(\tau,\beta_0) - \hat{\psi}_t^s(\tau,\beta)$.

Note that, by applying Davydov (1968)'s inequality:

$$
\begin{aligned}
&n\mathbb{E}\left|\Delta\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)-\mathbb{E}[\Delta\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)]\right|^2\\
&\leq \frac{1}{n}\sum_{t=1}^{n}\mathbb{E}\left|\Delta\hat{\psi}_t^s(\tau,\Pi_{k(n)}\beta_0)-\mathbb{E}[\Delta\hat{\psi}_t^s(\tau,\Pi_{k(n)}\beta_0)]\right|^2\\
&+\frac{24}{n}\sum_{m=1}^{n}(n-m)\alpha(m)^{1/3}\max_{1\leq t\leq n}\left(\mathbb{E}\left|\Delta\hat{\psi}_t^s(\tau,\Pi_{k(n)}\beta_0)-\mathbb{E}[\Delta\hat{\psi}_t^s(\tau,\Pi_{k(n)}\beta_0)]\right|^6\right)^{2/3}\\
&\leq \left(1+24\sum_{m\geq 1}\alpha(m)^{1/3}\right)\max_{1\leq t\leq n}\left(\mathbb{E}\left|\Delta\hat{\psi}_t^s(\tau,\Pi_{k(n)}\beta_0)-\mathbb{E}[\Delta\hat{\psi}_t^s(\tau,\Pi_{k(n)}\beta_0)]\right|^6\right)^{2/3}\\
&\leq 4^{8/3}\left(1+24\sum_{m\geq 1}\alpha(m)^{1/3}\right)\max_{1\leq t\leq n}\left(\mathbb{E}\left|\Delta\hat{\psi}_t^s(\tau,\Pi_{k(n)}\beta_0)-\mathbb{E}[\Delta\hat{\psi}_t^s(\tau,\Pi_{k(n)}\beta_0)]\right|^2\right)^{2/3}.
\end{aligned}
$$

The last inequality is due to $|\Delta\hat{\psi}_t^s(\tau,\beta)|\leq 2$. By the continuity assumption the last term is a $o(1)$ when $\|\beta_0-\Pi_{k(n)}\|_{\mathcal{B}}\to 0$. As a result:

$$
\int\mathbb{E}\left|\Delta\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)-\mathbb{E}[\Delta\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)]\right|^2\pi(\tau)d\tau=o(1/n).
$$

To conclude the proof, apply a triangular inequality and the results above:

$$
\begin{aligned}
&\left[\mathbb{E}\left(\int\sup_{\|\beta-\Pi_{k(n)}\beta_0\|_m\leq M_n\delta_{mn}}\left|[\hat{\psi}_n^S(\tau,\beta)-\hat{\psi}_n^S(\tau,\beta_0)]-\mathbb{E}[\hat{\psi}_n^S(\tau,\beta)-\hat{\psi}_n^S(\tau,\beta_0)]\right|^2\pi(\tau)d\tau\right)\right]^{1/2}\\
&\leq\left[\mathbb{E}\left(\int\sup_{\|\beta-\Pi_{k(n)}\beta_0\|_m\leq M_n\delta_{mn}}\left|[\hat{\psi}_n^S(\tau,\beta)-\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)]-\mathbb{E}[\hat{\psi}_n^S(\tau,\beta)-\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)]\right|^2\pi(\tau)d\tau\right)\right]^{1/2}\\
&+\left(\int\mathbb{E}\left|\Delta\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)-\mathbb{E}[\Delta\hat{\psi}_n^S(\tau,\Pi_{k(n)}\beta_0)]\right|^2\pi(\tau)d\tau\right)^{1/2}=o(1/\sqrt{n}).
\end{aligned}
$$

□

Remark B7. *Note that* $\delta_n=\frac{k(n)^2\log[k(n)]^2}{\sqrt{n}}=o(1)$ *by assumption so that* $\log[\delta_n]^2=O(\log(n)^2)$. *Furthermore, it is assumed that* $\delta_n=o\left(\sqrt{\underline{\lambda}_n}\right)$ *and* $\delta_{m,n}=o(1)$, *so that* $\max(\log[k(n)]^2,\log[M_n\delta_{m,n}]^2))$ *is dominated by a* $O(\log(n))$. *The condition* $k(n)^2\max(\log[k(n)]^2,\log[M_n\delta_{m,n}]^2)$ *can thus be re-written as:*

$$
(M_n\delta_{mn})^{\frac{\gamma^2}{2}}[k(n)\log(n)]^2=o(1)
$$

which is equivalent to:

$$
\delta_n=o\left(\frac{\sqrt{\underline{\lambda}_n}}{M_n[k(n)\log(n)]^{\frac{4}{\gamma^2}})}\right).
$$

Furthermore, since $\delta_n = \frac{k(n)^2 \log[k(n)]^2}{\sqrt{n}}$*, this condition can be re-written in terms of* $k(n)$*:*

$$k(n) = o\left(\left(\frac{\sqrt{\underline{\lambda}_n}}{M_n \log(n)^{\frac{4}{\gamma^2}}}\right)^{\frac{1}{2+4/\gamma^2}} n^{\frac{1}{2(2+4/\gamma^2)}}\right).$$

Proof of Theorem 3: Theorem 3 mostly follows from Theorem C6 with two differences: the rate of convergence and the stochastic equicontinuity results in mixture norm. Lemmas 5 and B11 provide these results for the mixture sieve. Hence, given these results, Theorem 3 is a corollary of Theorem C6. □

B.5 Extension 1: Using Auxiliary Variables

Proof of Corollary 2: Since the proof of Corollary 2 is very similar to the main proofs, only the differences in the steps are highlighted.

i. **Consistency**: The objective function with auxiliary variables is:

$$Q_n(\beta) = \int \left| \mathbb{E}\left(\hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) - \hat{\psi}_n^s(\tau, \hat{\eta}_n^{aux}, \beta)\right) \right|^2 \pi(\tau) d\tau.$$

To derive its rate of convergence consider:

$$\begin{aligned}\int \left| \hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) - \mathbb{E}\left(\hat{\psi}_n(\tau, \hat{\eta}_n^{aux})\right) \right|^2 \pi(\tau) d\tau \leq\ & 9 \int \left| \hat{\psi}_n(\tau, \eta^{aux}) - \mathbb{E}\left(\hat{\psi}_n(\tau, \eta^{aux})\right) \right|^2 \pi(\tau) d\tau \\ & + 9 \int \left| \hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) - \hat{\psi}_n(\tau, \eta^{aux}) \right|^2 \pi(\tau) d\tau \\ & + 9 \int \left| \mathbb{E}\left(\hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) - \hat{\psi}_n(\tau, \eta^{aux})\right) \right|^2 \pi(\tau) d\tau.\end{aligned}$$

The first term is $O_p(1/n)$. By the Lipschitz condition, the second term satisfies:

$$\begin{aligned}\int \left| \hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) - \hat{\psi}_n(\tau, \eta^{aux}) \right|^2 \pi(\tau) d\tau &\leq \|\hat{\eta}_n^{aux} - \eta^{aux}\|^2 |C_n^{aux}|^2 \int \|\tau\|_\infty \pi(\tau) d\tau \\ &= O_p(1/n) O_p(1).\end{aligned}$$

C_n^{aux} is an average of the Lipschitz constants in the assumptions. The third term can be bounded using the Lipschitz assumption and the Cauchy-Schwarz inequality:

$$\begin{aligned}\int \left| \hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) - \hat{\psi}_n(\tau, \eta^{aux}) \right|^2 \pi(\tau) d\tau &\leq \mathbb{E}\|\hat{\eta}_n^{aux} - \eta^{aux}\|^2 \mathbb{E}|C_n^{aux}|^2 \int \|\tau\|_\infty \pi(\tau) d\tau \\ &= O_p(1/n^2) O_p(1).\end{aligned}$$

Altogether, these inequalities imply:

$$\int \left| \hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) - \mathbb{E}\left(\hat{\psi}_n(\tau, \hat{\eta}_n^{aux})\right) \right|^2 \pi(\tau) d\tau = O_p(1/n^2).$$

The L^2-smoothness result still holds given the summability condition:

$$\left[\mathbb{E}\Big(\sup_{\|\beta_1-\beta_2\|_{\mathcal{B}}\leq\delta,\eta\in E}\|g_{aux}(y_t^s(\beta_1),\ldots,y_1^s(\beta_1),x_t,\ldots,x_1;\eta)-g_{aux}(y_t^s(\beta_2),\ldots,y_1^s(\beta_2),x_t,\ldots,x_1;\eta)\|^2\Big)\right]^{1/2}$$
$$\leq \sum_{j=1}^{t}\rho_j\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_{\mathcal{B}}\leq\delta,\eta\in E}\|y_j^s(\beta_1)-y_j^s(\beta_2)\|^2\right)\right]^{1/2}$$
$$\leq \left(\sum_{j=1}^{\infty}\rho_j\right)\sup_{t\geq 1}\left[\mathbb{E}\left(\sup_{\|\beta_1-\beta_2\|_{\mathcal{B}}\leq\delta,\eta\in E}\|y_t^s(\beta_1)-y_t^s(\beta_2)\|^2\right)\right]^{1/2}$$
$$\leq \bar{C}\left(\sum_{j=1}^{\infty}\rho_j\right)\max\left(\frac{\delta^{\gamma^2}}{\underline{\sigma}_{k(n)}^{2\gamma^2}},[k(n)+\overline{\mu}_{k(n)}+\overline{\sigma}]^{\gamma}\delta^{\gamma^2/2}\right)$$

The last inequality is a consequence of Lemma 3.

$$\int\left|\hat{\psi}_n^s(\tau,\hat{\eta}_n^{aux})-\mathbb{E}\left(\hat{\psi}_n^s(\tau,\hat{\eta}_n^{aux})\right)\right|^2\pi(\tau)d\tau \leq 9\int\left|\hat{\psi}_n^s(\tau,\eta^{aux})-\mathbb{E}\left(\hat{\psi}_n^s(\tau,\eta^{aux})\right)\right|^2\pi(\tau)d\tau$$
$$+9\int\left|\hat{\psi}_n^s(\tau,\hat{\eta}_n^{aux})-\hat{\psi}_n^s(\tau,\eta^{aux})\right|^2\pi(\tau)d\tau$$
$$+9\int\left|\mathbb{E}\left(\hat{\psi}_n^s(\tau,\hat{\eta}_n^{aux})-\hat{\psi}_n^s(\tau,\eta^{aux})\right)\right|^2\pi(\tau)d\tau.$$

The first term is a $O_p(\delta_n^2)$ given the L^2-smoothness above and the main results. The last two terms are $O_p(1/n^2)$ as in the calculations above.

Together, these results imply that the rate of convergence for the objective function is $O_p(\delta_n^2)$ as before. As a result, given that the other assumptions hold, the estimator is consistent.

ii. **Rate of Convergence**: The variance term is still $O_p(\delta)$ as discussed above. The only term remaining to discuss if the bias accumulation term.

Recall that the first part of the bias term involves changing f in g_{obs}, g_{latent} while keeping the shocks e_t^s unchanged. Using the same method of proof as for the L^2-smoothness it can be shown that the first bias term is only inflated by $\sum_{j=1}^{\infty}\rho_j<\infty$: a finite factor.

The second part involves changing the shocks keeping g_{obs}, g_{latent} unaffected. An alternative simulated sequence $\tilde{y}_t^s$ where part of the history is changed $\tilde{y}_{t-j}^s=\tilde{u}_{t-j}^s=0$ for $j\geq m$. For a well chosen sequence m, the difference between y_t^s and $\tilde{y}_t^s$ declines exponentially in m. Here $\tilde{z}_t^s$ only depends on a finite number of shocks since $\tilde{y}_{t-m}^s=\cdots=\tilde{y}_1^s=0$. The difference between z_t^s and $\tilde{z}_t^s$ becomes:

$$\mathbb{E}\left(\|z_t^s-\tilde{z}_t^s\|\right)\leq\sum_{j=1}^{t}\rho_j\mathbb{E}\left(\|y_j^s-\tilde{y}_j^s\|\right)\leq\left(\sum_{j=1}^{\infty}\rho_j\right)\bar{C}\max(\bar{C}_1,\bar{C}_4)^{\gamma m}$$

where the last inequality comes from Lemma 4. To apply this lemma, the bounded moment condition $v.$ is required.

Overall, the bias term is unchanged. As a result, the rate of convergence is the same as in the main proofs.

iii. **Asymptotic Normality**: The L^2-smoothness result was shown above to be unchanged. As a result, stochastic equicontinuity can be proved the same way as before. The Lipschitz condition also implies stochastic equicontinuity in η^{aux} using the same approach as for the rate of convergence of the objective. The asymptotic expansion can be proved the same way as in the main results where $\hat{\psi}_n(\tau)$ and $\hat{\psi}_n^s(\tau, \beta_0)$ are replaced with $\hat{\psi}_n(\tau, \hat{\eta}_n^{aux})$ and $\hat{\psi}_n^s(\tau, \hat{\eta}_n^{aux}, \beta_0)$. Eventually, the expansion implies:

$$\frac{\sqrt{n}}{\sigma_n^*}\left(\phi(\hat{\beta}_n) - \phi(\beta_0)\right) = \sqrt{n} Real\left(\int \psi_\beta(\tau, u_n^*, \eta^{aux})\overline{\left(\hat{\psi}_n(\tau, \hat{\eta}_n^{aux}) - \hat{\psi}_n^s(\tau, \hat{\eta}_n^{aux}, \beta_0)\right)}\pi(\tau)d\tau\right) + o_p(1)$$

where the term on the right is asymptotically normal by assumption.

□

B.6 Extension 2: Using Short Panels

Proof of Lemma 7. The second part of the lemma is implied by Remark **??**.

For the first part of Lemma 7, using the notation for the proof of Proposition C4: f is the distribution for the simulated $\mathbf{y}_{j,t}^s$ and $\mathbf{u}_{j,t}^s$ and f^* is the stationary distribution. Note that $f(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}|\mathbf{u}_{j,t}^s) = f^*(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}|\mathbf{u}_{j,t}^s)$ for $\beta = \beta_0$ and $\|f_{\mathbf{u}} - f_{\mathbf{u}}^*\|_{TV} \leq C_u \bar{\rho}_u^m$ for some $C_u > 0$ and $\bar{\rho}_u \in (0,1)$.

$$\begin{aligned}
\sqrt{Q_n(\beta_0)} &\leq M_B \left(\int \left|\mathbb{E}\left(\hat{\psi}_n(\tau) - \hat{\psi}_n^s(\tau, \beta_0)\right)\right|^2 \pi(\tau)d\tau\right)^{1/2} \\
&= M_B \left(\int \left|\frac{1}{n}\sum_{j=1}^n \int e^{i\tau'(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t})}\left(f(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}) - f^*(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t})\right) d\mathbf{y}_{j,t}^s d\mathbf{x}_{j,t}\right|^2 \pi(\tau)d\tau\right)^{1/2} \\
&= M_B \left(\int \left|\frac{1}{n}\sum_{j=1}^n \int e^{i\tau'(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t})} f^*(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}|\mathbf{u}_{j,t}^s)\left(f(\mathbf{u}_{j,t}^s) - f^*(\mathbf{u}_{j,t}^s)\right) d\mathbf{y}_{j,t}^s d\mathbf{x}_{j,t} d\mathbf{u}_{j,t}^s\right|^2 \pi(\tau)d\tau\right)^{1/2} \\
&\leq M_B \int f^*(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}|\mathbf{u}_{j,t}^s)\left|f(\mathbf{u}_{j,t}^s) - f^*(\mathbf{u}_{j,t}^s)\right| d\mathbf{y}_{j,t}^s d\mathbf{x}_{j,t} d\mathbf{u}_{j,t}^s.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality implies:

$$\begin{aligned}
&\int f^*(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}|\mathbf{u}_{j,t}^s)\left|f(\mathbf{u}_{j,t}^s) - f^*(\mathbf{u}_{j,t}^s)\right| d\mathbf{y}_{j,t}^s d\mathbf{x}_{j,t} d\mathbf{u}_{j,t}^s \\
&\leq \left(\int f^*(\mathbf{y}_{j,t}^s, \mathbf{x}_{j,t}|\mathbf{u}_{j,t}^s)^2\left|f(\mathbf{u}_{j,t}^s) - f^*(\mathbf{u}_{j,t}^s)\right| d\mathbf{y}_{j,t}^s d\mathbf{x}_{j,t} d\mathbf{u}_{j,t}^s\right)^{1/2}\left(\int \left|f(\mathbf{u}_{j,t}^s) - f^*(\mathbf{u}_{j,t}^s)\right| d\mathbf{u}_{j,t}^s\right)^{1/2}.
\end{aligned}$$

By assumption the first term is finite and bounded while the second term is a $O(\bar{\rho}_u^{m/2})$. Taking squares on both sides on the inequality concludes the proof. □

Proof of Corollary 3: As discussed in section 4 asymptotic are conducted over the cross-sectional dimension n for the moments:

$$\hat{\psi}_j(\tau) = \frac{1}{T}\sum_{t=1}^{T} e^{i\tau'(\mathbf{y}_{j,t},\mathbf{x}_{j,t})}, \quad \hat{\psi}_j^s(\tau) = \frac{1}{T}\sum_{t=1}^{T} e^{i\tau'(\mathbf{y}_{j,t}^s,\mathbf{x}_{j,t})}$$

which are iid under the stated assumptions. The bias can accumulate dynamically for DGP (12), as in the time-series case, but it accumulates with m instead of sample size. Assumption 2 or $2'$ ensure that the bias does not accumulate too much when $m \to \infty$. Lemma 7 shows how the assumed DGPs handle the initial condition problem in the panel setting. Note that:

$$n\bar{\rho}_u^m = e^{\log[n]+m\log[\bar{\rho}_u]} = e^{m(\log[n]/m+\log[\bar{\rho}_u])} \to 0$$

as $m, n \to \infty$ if $\lim_{m,n\to\infty} \log[n]/m < -\log[\bar{\rho}_u] > 0$. Given, this result and the dynamic bias accumulation the results for the iid case apply with an inflation bias term for DGP (12). □