Incomplete: empirical results will be presented at seminar.

Marriage Matching, Risk Sharing and Family Labor Supplies: An Empirical Framework

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Abstract

The paper integrates marriage matching with the collective model of spousal labor supplies with full risk sharing. It derives observable implications of how marriage market conditions affect spousal labor supplies. In contrast to the sex ratio which is a partial measure, the model motivates a sufficient statistic for marriage market tightness. The framework also clarifies the identifying assumptions necessary to estimate causal effects of marriage market conditions on spousal labor supplies. The empirical section of the paper tests for marriage market effects on spousal labor supplies using data from a panel of US cities and states.

1 Introduction

Thirty years ago, Becker (1973; 1974; summarized in his 1991 book) introduced his landmark model of the marriage market. A cornerstone of that model is that resource transfers between spouses are used to clear the marriage market. The subsequent literature developed in two directions. First, researchers have found empirical evidence that is supportive of Becker's model, that a higher sex ratio (ratio of men to women) will result in more resource transfers from husbands to wives.[1] Second, Chiappori and his collaborators have developed a framework, the "collective model", for estimating household members' preferences when members may have divergent interests. A key feature of this framework is that it assumes efficient intrahousehold allocations. The intrahousehold allocation is what a social planner will choose if the planner's objective function is the weighted sum of household members' utilities where the weights reflect the bargaining power of each member. Researchers have also found empirical support for this model. An objective of current research is to integrate the collective model with a model of the marriage market.

Building on the above two strands of literature, this paper has does three things. First, it nests the collective model within the marriage market. Individuals choose who to marry or to remain unmarried. Endogenous spousal utility weights in the collective model are used to clear the marriage market. We show the existence of marriage market equilibrium.

Second, the model motivates a new empirical strategy for estimating the effects of changing marriage market conditions on intrahousehold allocations. Consider $\{i, j\}$ marriages where type i men marry type j women. Consider a data set with K type j wives who are married to type i husbands from from R different societies. A standard strategy is to regress wife k's labor supply, H_{ij}^{rk}, on the sex ratio, m_i^r/f_j^r, where m_i^r and f_j^r are the number of type i males and type j females in society r respectively:

$$H_{ij}^{rk} = \alpha_0 + \alpha_1 \ln \frac{m_i^r}{f_j^r} + u_{ij}^{rk},\ k = 1,.,K;\ r = 1,..,R \qquad (1)$$

u_{ij}^{rk} is the error term of the regression.

α_1 measures the elasticity of female labor supplies with respect to the sex ratio. If the demand for female labor is relatively low in society r, the sex ratio may respond and be high; thereby be negatively correlated with u_{ij}^{rk}. In this case, the OLS estimate of α_1 will not be consistent. Angrist addressed the endogeneity issue by instrumenting the current sex ratio with the sex ratio of the previous generation.

Our model delivers a different strategy to control for labor demand conditions. First, we propose a new measure of marriage market tightness in society r, $T_{ij}^r = \ln \mu_{i0}^r - \ln \mu_{0j}^r$, the log ratio of unmarried type i men to unmarried type

[1] E.g. Angrist 2002; Chiappori, Fortin, and Lacroix 2002; Francis 2005; Grossbard-Schechtman 1993; Seitz 2005, South and Trent.

j women.[2] An increase in T^r_{ij} increases the bargaining power of wives in $\{i,j\}$ marriages in society r.

Our empirical strategy for controlling for marriage market tightness is to match each society r marriage market with a comparison society r' which is constructed by matching the wage distributions between r and r'. The difference in marriage market tightness, $T^r_{ij} - T^{r'}_{ij}$, is used to explain the difference in mean spousal allocations in $\{i,j\}$ marriages between the two markets.

Let $\overline{H}^{r'}_{ij}$ be the average hours of work of wives in $\{i,j\}$ marriages in society r'. In our simplest setup, we regress:

$$H^{rk}_{ij} - \overline{H}^{r'}_{ij} = \beta_1(T^r_{ij} - T^{r'}_{ij}) + u^{rk}_{ij},\ k = 1,..,K;\ r = 1,..,R \tag{2}$$

Since the matching is based on equalizing wage distributions, the OLS estimate of β_1 should be a consistent estimate of the elasticity of market tightness on female labor supplies. Instead of T^r_{ij}, we can also use $\ln(m^r_i/f^r_j)$ in (2). Matching societies by wage distributions provide another way to control for labor demand conditions for researchers who want to estimate the effect of changing sex ratios on female labor supply.

Third, we estimate how differences in matched marriage markets tightness in the United States in 2000 affected mean intrahousehold allocations in $\{i,j\}$ marriages between those matched marriage markets.

2 The model

Consider a society in which there are I types of men, $i = 1,..,I$, and J types of women, $j = 1,..,J$. All type i men have the same preferences and ex-ante opportunities; and all type j women also have the same preferences and ex-ante opportunities. That is, the type of an individual is defined by his or her preferences and ex-ante opportunities.

Let m_i be the number of type i men and f_j be the number of type j women. M and F are the vectors of the numbers of each type of men and women respectively.

The model is a two period model. In the first period, individuals choose whether to marry and who to marry if they marry. An $\{i,j\}$ marriage is a marriage between a type i man and a type j woman. At the time of their marital choices, wages and non-labor income for each marital choice are random variables.

After their marital choices, and in the second period, intrahousehold allocations are chosen after wages and non-labor income for each household are realized. We consider a static model of marital, consumption and labor supply choices.[3]

[2] This measure is similar to the Beveridge curve measure of labor market tightness: ratio of the number of vacancies to number of unemployed.

[3] The extension to multi period married life is in Section ....

All men and unmarried women have positive hours of work. For notational simplicity, the theoretical model will assume that all married individuals also choose positive hours of work. As will become clear in the development, it is straightforward to extend the model to allow other kinds of marriages such as ones where the wife does not work, or cohabitation rather than marriage. In the empirical work, we will distinguish between marriages with working and non-working wives.

Let C_{ijgG} be the own consumption of wife G of type j matched to a type i husband g. K_{ijgG} is the amount of public good each of them consumes. H_{ijgG} is her labor supply.We normalize the total amount of time for each individual to 1. Her utility function is:

$$U_{ij}(C_{ijgG}, 1 - H_{ijgG}, K_{ijgG}, \varepsilon_{ijG}) = \widehat{Q}_{ij}(C_{ijgG}, 1 - H_{ijgG}, K_{ijgG}) + \Gamma_{ij} + \varepsilon_{ijG} \tag{3}$$

$\widehat{Q}_{ij}(.)$, her felicity function, depend on i, j which allows for differences in home production technologies across different marital matches. The invariant gain to an $\{i, j\}$ marriage for the woman, Γ_{ij}, shifts her utility according to the type of marriage and allows the model to fit the observed marriage matching patterns in the data. It may vary across different types of marriages and societies due to technological differences in different types of marriages, legal and cultural differences across societies. The important restriction is that Γ_{ij} does not affect her marginal utilities from consumption or labor supply.

Finally, we assume ε_{ijG} is a type I extreme value random variable that is realized before marital decisions are made. Note that ε_{ijG} is independent of g. It does not depend on the specific identity of the type i male. The rationale for the extreme value assumption will be made clear later. The realizations of this random variable across different women of type j in the same society will produce different marital choices for different type j women in period one. If she chooses not to marry, then $i = 0$.

The specification of a representative man's problem is similar to that of women. Let c_{ijgG} be the own consumption of man g of type j matched to a type j woman G. K_{ijgG} is his public good consumption. Denote his labor supply by h_{ijgG}. If he chooses not to marry, then $j = 0$. The utility function for males is described by:

$$u_{ij}(c_{ijgG}, 1 - h_{ijgG}, K_{ijgG}, \varepsilon_{ijg}) = \widehat{q}_{ij}(c_{ijgG}, 1 - h_{ijgG}, K_{ijgG}) + \gamma_{ij} + \varepsilon_{ijg}, \tag{4}$$

$\widehat{q}_{ij}(.)$, his felicity function, depends on i, j will allow the model to fit observed labor supply behavior for different types of marriages. The invariant gain to an i, j marriage for the man, γ_{ij}, shifts his utility by i, j and allows the model to fit the observed marriage matching patterns in the data. It may vary across different types of marriages and societies due to technological differences in different types of marriages, legal and cultural differences across societies. The important restriction is that γ_{ij} does not affect his marginal utilities from consumption and labor supply.

Finally, we assume ε_{ijg} is a type I extreme value random variable that is realized before marital decisions are made. ε_{ijg} is independent of G. The realizations of this random variable across different men of type i in the same society will produce different marital choices for different type i men in period one.

2.1 The collective model with efficient risk sharing

We start first with intrahousehold allocation after the marriage decision has been made. Consider a particular husband g and his wife G in an $\{i, j\}$ marriage. Total non-labor family income is A_{ijgG} which is a random variable. The wage for the wife is also a random variable W_{ijgG}. The male's wage is another random variable w_{ijgG}. A_{ijgG}, W_{ijgG} and w_{ijgG} are realized in the second period, after the marriage decision.

The family budget constraint is:

$$c_{ijgG} + C_{ijgG} + K_{ijgG} \leq A_{ijgG} + W_{ijgG}(1 - H_{ijgG}) + w_{ijgG}(1 - h_{ijgG}) \quad (5)$$

Because wages and non-labor income, W_{ijgG}, w_{ijgG}, and A_{ijgG}, are random variables whose values are realized after marriage, in the second period, the spouses can share income risk in the first period.

The continuous joint distribution of A_{ijgG}, W_{ijgG} and w_{ijgG} with bounded support is characterized by the parameter vector Z. Z is known to individuals before their marriage decisions. Let $S_{ijgG} = \{W_{ijgG}, w_{ijgG}, A_{ijgG}\}$. Let $F(S_{ijgG}|Z)$ denote the cumulative multivariate wages and non-labor income distribution in the society.

Let $\mathbf{E}$ be the expectations operator. Following the collective model with full risk sharing, we pose the efficient risk sharing spousal arrangement as a planner solving the following problem:

$$\max_{\{C,c,H,h\}} \mathbf{E}(\widehat{Q}(C_{ijgG}, 1 - H_{ijgG}, K_{ijgG})|Z) + p_{ij}\mathbf{E}(\widehat{q}(c_{ijgG}, 1 - h_{ijgG}, K_{ijgG})|Z) \quad \text{(P1)}$$

subject to (5) for all S_{ijgG}

In problem (P1), the planner chooses family consumption and labor supplies to maximize the weighted sum of the wife's and the husband's expected felicities subject to their family budget constraint. $p_{ij} \in [0, 1]$ is the weight allocated to the husband's expected felicity. If $p_{ij} > 1$, the husband has more weight than the wife and vice versa. As in the collective model literature, p_{ij} depends on Z, marriage market conditions, and other factors affecting the gains to marriage in which the individuals live.

How p_{ij} is determined in the marriage market is a central focus of this paper. However the determination of p_{ij} is not a concern of the social planner in solving in problem (P1). The planner takes p_{ij} as exogenous. When the intrahousehold allocation is the solution to problem (P1), the intrahousehold allocation is efficient.

Let $C_{ijgG}(p_{ij}, S_{ijgG})$, $H_{ijgG}(p_{ij}, S_{ijgG})$, $c_{ijgG}(p_{ij}, S_{ijgG})$, $h_{ijgG}(p_{ij}, S_{ijgG})$, $K_{ijgG}(p_{ij}, S_{ijgG})$ be the optimal intrahousehold allocation when state S_{ijgG} is realized. Let $Q(p_{ij}, Z)$ and $q(p_{ij}, Z)$ be the expected indirect felicities of the wife and the husband respectively before the state S_{ijgG} is realized:

$$\mathbf{Q}_{ij}(p_{ij}, Z) = \mathbf{E}(Q_{ij}(C_{ijgG}(p_{ij}, S_{ijgG}), 1 - H_{ijgG}(p_{ij}, S_{ijgG}), K_{ijgG}(p_{ij}, S_{ijgG}))|Z)$$
$$\mathbf{q}_{ij}(p_{ij}, Z) = \mathbf{E}(q_{ij}(c_{ijgG}(p_{ij}, S_{ijgG}), 1 - h_{ijgG}(p_{ij}, S_{ijgG}), K_{ijgG}(p_{ij}, S_{ijgG}))|Z)$$

Assuming that own consumption, leisure and the public good are all normal goods, Appendix 1 shows that the solution to problem (P1) implies:

Proposition 1 *The wife's labor supply is increasing in* p_{ij} *whereas the husband's labor supply is decreasing in* p_{ij}*:*

$$\frac{\partial H_{ijgG}}{\partial p_{ij}} > 0 \; \forall S_{ijgG} \tag{6}$$

$$\frac{\partial h_{ijgG}}{\partial p_{ij}} < 0 \; \forall S_{ijgG} \tag{7}$$

And the expected felicity of the wife is decreasing in p_{ij} *whereas the expected felicity of the husband is increasing in* p_{ij} :

$$\frac{\partial \mathbf{Q}_{ij}(p_{ij}, Z)}{\partial p_{ij}} = -p_{ij} \frac{\partial \mathbf{q}_{ij}(p_{ij}|Z)}{\partial p_{ij}} < 0 \tag{8}$$

The final point in this section is to point out a well known implication of the efficient risk sharing model. A necessary condition for solving problem P1 is that given realized wages and non-labor income, i.e. S_{ijgG}, the planner solves problem P7:

$$\max_{C_{gG}, c_{gG}, L_{gG}, l_{gG}} \widehat{Q}_{ij}(C_{ijgG}, 1 - H_{ijgG}, K_{ijgG}) + p_{ij}\widehat{q}_{ij}(c_{ijgG}, 1 - h_{ijgG}, K_{ijgG}) \tag{P7}$$

subject to $c_{ijgG} + C_{ijgG} + K_{ijgG} \leq A_{ijgG} + W_{ijgG}(1 - H_{ijgG}) + w_{ijgG}(1 - h_{ijgG})$

Problem P7 is a unitary model of the family faced with wages W_{ijgG}, w_{ijgG}, and non-labor income A_{ijgG}. Thus we cannot reject a unitary model of the family for $\{i, j\}$ couples in the same society, by observing their spousal labor supplies behavior if they share risk efficiently.[4]

Problem P7 is also useful because it is a standard consumer choice problem. In particular, we know the spousal labor supplies functions must satisfy Slutsky symmetry, a testable restriction with spousal labor supply data when both spouses work. Although we do not test this implication in this paper, the rationale to pointing it out is to address later the question as to whether existing empirical rejections of Slutsky symmetry invalidate the efficient risk sharing model.

[4]This point is well known. Hayashi, Altonji and Kotlikoff, Lich Tyler, Mazzacco, Ogaki.

2.2 Marriage decision problems in the first period

In the first period, agents decide whether to marry and who to marry if they choose to marry. Consider a particular woman G of type j. Recall that she can choose between I types of men and whether or not to marry. She can choose between $I+1$ choices. Her expected utility in an $\{i,j\}$ marriage is:

$$\overline{V}(i,j,p_{ij},\varepsilon_{ijG}) = \mathbf{Q}_{ij}(p_{ij},Z) + \Gamma_{ij} + \varepsilon_{ijG} \tag{9}$$

Given the realizations of all the ε_{ijG}, she will choose the marital choice which maximizes her expected utility. Let $\underline{\varepsilon_{jG}} = [\varepsilon_{0jG},..,\varepsilon_{ijG},..,\varepsilon_{IjG}]$. The expected utility from her optimal choice will satisfy:

$$V^*(\underline{\varepsilon_{jG}}) = \max[\overline{V}(0,j,\varepsilon_{0jG}),..,\overline{V}(i,j,p_{ij},\varepsilon_{ijG}),..] \tag{10}$$

The problem facing men in the first stage is analogous to that of women. A man g of type i in an $\{i,j\}$ marriage, with ε_{ijg}, attains an expected utility of:

$$\overline{v}(i,j,p_{ij},\varepsilon_{ijg}) = \mathbf{q}_{ij}(p_{ij},Z) + \gamma_{ij} + \varepsilon_{ijg} \tag{11}$$

Given the realizations of all the ε_{ijg}, he will choose the marital choice which maximizes his expected utility. He can choose between $J+1$ choices. Let $\underline{\varepsilon_{ig}} = [\varepsilon_{i0g},..,\varepsilon_{ijg},..]$. The expected utility from his optimal choice will satisfy:

$$v^*(\underline{\varepsilon_{ig}}) = \max[\overline{v}(i,0,\varepsilon_{i0g}),..\overline{v}(i,j,p_{ij},\varepsilon_{ijg})..] \tag{12}$$

3 The Marriage Market

Our model of the marriage market follows CS. Assume that there are lots of men and women of each type, and each woman is solving (10) and each man is solving (12). Because ε_{ijG} are i.i.d. extreme value random variables, McFadden (1974) showed that for every type of woman j:

$$\frac{\overline{\mu}_{ij}}{f_j} = \frac{\exp(\Gamma_{ij} + \mathbf{Q}_{ij}(p_{ij},Z))}{\sum_{k=0}^{I}\exp(\Gamma_{kj} + \mathbf{Q}_{kj}(p_{kj},Z))}, \quad i = 0,1,..,J \tag{13}$$

where $\overline{\mu}_{ij}$ is the number of $\{i,j\}$ marriages demanded by j type females and $\overline{\mu}_{0j}$ is the number of type j females who choose to remain unmarried.

(13) implies:

$$\ln\overline{\mu}_{ij} - \ln\mu_{0j} = (\Gamma_{ij} - \Gamma_{0j}) + \mathbf{Q}_{ij}(p_{ij},Z) - \mathbf{Q}_{0j}(Z)\ , \quad i = 1,..,I \tag{14}$$

CS calls the left hand side of (14), $\ln\overline{\mu}_{ij} - \ln\mu_{0j}$, the net gains to a j type woman in an $\{i,j\}$ marriage relative to remaining umarried.

Similarly, for every type of man i,

$$\frac{\underline{\mu}_{ij}}{m_i} = \frac{\exp(\gamma_{ij} + \overline{q}_{ij}(p_{ij},Z))}{\sum_{k=0}^{J}\exp(\gamma_{ik} + \overline{q}_{ik}(p_{ik},Z))}, \quad j = 0,1,..,I \tag{15}$$

which implies:

$$\ln \underline{\mu}_{ij} - \ln \mu_{i0} = (\gamma_{ij} - \gamma_{i0}) + \overline{q}_{ij}(p_{ij}, Z) - \overline{q}_{i0}(Z), \quad j = 1, .., J, \tag{16}$$

where $\underline{\mu}_{ij}$ is the number of $\{i, j\}$ marriages supplied by j type males and $\underline{\mu}_{i0}$ is the number of type i males who choose to remain unmarried.

CS calls the left hand side of (16), $\overline{\mu}_{ij} - \mu_{i0}$, the net gains to a i type man in an $\{i, j\}$ marriage relative to remaining umarried.

Marriage market clearing requires the supply of wives to be equal to the demand for wives for each type of marriage:

$$\underline{\mu}_{ij} = \overline{\mu}_{ij} = \mu_{ij} \ \forall \ \{i > 0, \ j > 0\} \tag{17}$$

Imposing marriage market clearing (17), sum (14) and (16) to get:

$$\frac{\mu_{ij}}{\sqrt{\mu_{i0}\mu_{0j}}} = \frac{(\Gamma_{ij} - \Gamma_{0j}) + (\gamma_{ij} - \gamma_{i0}) + \mathbf{Q}_{ij}(p_{ij}, Z) + \mathbf{q}_{ij}(p_{ij}, Z) - \mathbf{Q}_{0j}(Z) - \mathbf{q}_{i0}(Z)}{2} \tag{18}$$

CS calls $\mu_{ij}(\mu_{i0}\mu_{0j})^{-\frac{1}{2}}$ the total gains to an (i, j) marriage relative to the couple remaining unmarried. Assuming transferable utilities, CS argued the total gains to marriage should be invariant across societies when preferences do not change across societies. As the rhs of (18) shows, even if preferences for consumption and leisure are invariant across societies, the total gains depend on the society that i and j live in. There are three reasons why the total gains in this model depend on the society. First, as the society changes, Z, the parameters which govern the distribution of wages and non-labor income change. Second, as p_{ij} changes due to changes in M and F or Z, the induced changes in spousal utilities do not cancel in (18). Third, legal and cultural differences in the marriage markets across societies will affect $(\Gamma_{ij} - \Gamma_{0j})$, $(\gamma_{ij} - \gamma_{i0})$.

There are feasibility constraints that the stocks of married and single agents of each gender and type cannot exceed the aggregate stocks of agents of each gender in the society:

$$f_j = \mu_{0j} + \sum_i \mu_{ij} \tag{19}$$

$$m_i = \mu_{i0} + \sum_j \mu_{ij} \tag{20}$$

We can now define a rational expectations equilibrium. There are two parts to the equilibrium, corresponding to the two stages at which decisions are made by the agents. The first corresponds to decisions made in the marriage market; the second to the intra-household allocation. In equilibrium, agents make marital status decisions optimally, the sharing rules clear each marriage market, and conditional on the sharing rules, agents choose consumption and labor supply optimally. Formally:

Definition 2 *A rational expectations equilibrium consists of a distribution of males and females across individual type, marital status, and type of marriage* $\{\hat{\mu}_{0j}, \hat{\mu}_{i0}, \hat{\mu}_{ij}\}$, *a set of decision rules for marriage* $\{\widehat{V}^*(\underline{\varepsilon_{jG}}), \widehat{v}^*(\underline{\varepsilon_{ig}})\}$, *a set of decision rules for spousal consumption and leisure* $\{\hat{C}_{ijgG}, \hat{c}_{ijgG}, \hat{L}_{ijgG}, \hat{l}_{ijgG}, \widehat{P}_{ijgG}\}$, *and a set of shadow prices* $\{\widehat{p}_{ij}\}$ *such that:*

1. *The decision rules* $\{\widehat{V}^*(\underline{\varepsilon_{jG}}), \widehat{v}^*(\underline{\varepsilon_{ig}})\}$ *solve (10) and (12);*

2. *All marriage markets clear implying (17), (19), (20) hold;*

3. *For an* $\{i, j\}$ *marriage, the decision rules* $\{\hat{C}_{ijgG}, \hat{c}_{ijgG}, \hat{L}_{ijgG}, \hat{l}_{ijgG}, \widehat{P}_{ijgG}\}$ *solve (P1).*

Theorem 3 *A rational expectations equilibrium exists.*

Sketch of proof: We have already demonstrated (1) and (3). So what needs to be done is to show that there is a set of shadow prices, $\{\widehat{p}_{ij}\}$ which clears the marriage market. Let $\underline{p}$ be the vector of shadow prices for society x. For every marriage market $\{i, j, \pi\}$ excluding $i = 0$ or $j = 0$, define the excess demand function for marriages by men:

$$E_{ij}(\underline{p}) = \underline{\mu}_{ij}(\underline{p}) - \overline{\mu}_{ij}(\underline{p}) \tag{21}$$

The demand and supply functions (13) and (15), for every marriage market $\{i, j\}$, satisfy the weak gross substitute property. So the excess demand functions also satisfy the weak gross substitute property. Mas-Colell, Winston and Green (1995: p. 646, exercise 17.F.16^C) provide a proof of existence of market equilibrium when the excess demand functions satisfy the weak gross substitute property. For convenience, we reproduce their proof in our context in Appendix 2.[5]

Remark: In monogamous marriage markets, where different types of spouses are substitutes for each other (since an individual can at most marry one type), the weak gross substitute property is generic. Thus existence of marriage market equilibrium is more general than our specific random utility model for spousal choice, which we use for empirical convenience. Kelso and Crawford (1982) were the first to use the gross substitute property to demonstrate existence in matching models.

4 Marriage market identification with approximately equal bargaining power

Consider the marriage market $\{i, j\}$. As derived in Section 3, marriage market clearing, equation (18), implies:

[5] The proof does not rely on Walras Law or that excess demand is homogenous of degree zero in $\underline{p}$, both of which our model does not satisfy.

$$\frac{\mu_{ij}}{\sqrt{\mu_{i0}\mu_{0j}}} = \frac{(\Gamma_{ij} - \Gamma_{0j}) + (\gamma_{ij} - \gamma_{i0}) + \mathbf{Q}_{ij}(p_{ij}, Z) + \mathbf{q}_{ij}(p_{ij}, Z) - \mathbf{Q}_{0j}(Z) - \mathbf{q}_{i0}(Z)}{2} \tag{22}$$

p_{ij} is the equilibrium shadow price which clears the marriage market. $p_{ij} = 1$ is an important benchmark where the particular marriage market clears with equal bargaining power between the spouses.

Using a first order Taylor series expansion around $p_{ij} = 1$,

$$\mathbf{q}_{ij}(p_{ij}, Z) \simeq \mathbf{q}_{ij}(1, Z) + (p_{ij} - 1)\frac{\partial \mathbf{q}_{ij}(p_{ij}, Z)}{\partial p_{ij}}|_{p_{ij}=1} \tag{23}$$

$$\mathbf{Q}_{ij}(p_{ij}, Z) \simeq \mathbf{Q}_{ij}(1, Z) + (p_{ij} - 1)\frac{\partial \mathbf{Q}_{ij}(p_{ij}, Z)}{\partial p_{ij}}|_{p_{ij}=1} \tag{24}$$

$$= \mathbf{Q}_{ij}(1, Z) - (p_{ij} - 1)\frac{\partial \mathbf{q}_{ij}(p_{ij}, Z)}{\partial p_{ij}}|_{p_{ij}=1} \tag{25}$$

The last line, (25), obtains because of (8), an implication of efficient spousal risk sharing.

Using (23) and (25), (22) becomes:

$$\frac{\mu_{ij}}{\sqrt{\mu_{i0}\mu_{0j}}} = \frac{(\Gamma_{ij} - \Gamma_{0j}) + (\gamma_{ij} - \gamma_{i0}) + \mathbf{Q}_{ij}(1, Z) + \mathbf{q}_{ij}(1, Z) - \mathbf{Q}_{0j}(Z) - \mathbf{q}_{i0}(Z)}{2} \tag{26}$$

The right hand side of (26) is independent of p_{ij}, the equilibrium shadow price. It only depends on i, j, the types of spouses involved as well as the type of marriage that they are engaged in.

Because μ_{ij}, μ_{i0}, μ_{0j} are observed, we can estimate the total gains to marriage, $\mu_{ij}(\mu_{i0}\mu_{0j})^{-\frac{1}{2}}$. If two different societies have different $Z's$ and or $\Gamma_{ij}, \Gamma_{0j}, \gamma_{ij}, \gamma_{i0}$, they will have different total gains.

Equation (26) is familiar from CS where it was derived under the hypothesis of transferable utilities without post marital uncertainty.

That $\mu_{ij}(\mu_{i0}\mu_{0j})^{-\frac{1}{2}}$ measures the total gains to a $\{i, j\}$ type marriage in an efficient spousal risk sharing marriage market model is important because it shows that transferable utilities is not necessary to obtain equation (26). As discussed in CS, $\mu_{ij}(\mu_{i0}\mu_{0j})^{-\frac{1}{2}}$ is an intuitive measure of total gains because it says that the more $\{i, j\}$ marriages there are relative to the geometric average of the unmarrieds, the larger is the total gains to that type of marriage.

As discussed in CS, (26) does not have any overidentifying assumption. There is no way to test the marriage matching model using (26). (26) is derived under the assumption that bargaining power between the spouses are approximately equal.

Finally, we have assumed that individuals can freely choose their hours of work. If workers are rationed in their hours of work, the labor supplies models proposed in this paper are misspecified. In particular $\mathbf{q}_{ij}$ and $\mathbf{Q}_{ij}$ will be misspecified. But as (26) shows, we can still identify marriage market parameters because $\mathbf{q}_{ij}$ and $\mathbf{Q}_{ij}$ do not need to be separately identified.

5 Multi-markets restrictions

Let the equilibrium shadow prices be $\{p_{ij}(\Gamma, \gamma, Z, M, F)\}$. The equilibrium quasi supply by women for $\{i, j\}$ marriages satisfies:

$$\ln \frac{\mu_{ij}}{\mu_{0j}} = (\Gamma_{ij} - \Gamma_{0j}) + \mathbf{Q}_{ij}(p_{ij}(\Gamma, \gamma, Z, M, F), Z) - \mathbf{Q}_{0j}(Z) \tag{27}$$

Then we have the following comparative static:

$$\frac{\partial \ln \frac{\mu_{ij}}{\mu_{0j}}}{\partial \omega} = \frac{\partial(\Gamma_{ij} - \Gamma_{0j})}{\partial \omega} + \frac{\partial \mathbf{Q}_{ij}}{\partial p_{ij}} \frac{\partial p_{ij}}{\partial \omega} + \frac{\partial(\mathbf{Q}_{ij} - \mathbf{Q}_{0j})}{\partial Z} \frac{\partial Z}{\partial \omega} \tag{28}$$

(28) decomposed the change in net gains to marriage for wives into three components. The first component is the change utility due to the change in net invariant gains for wives. The second component is the utility change from the change in relative bargaining power. The third component is the utility change from the changes in wage and non-labor income distributions.

The equilibrium quasi demand by men for $\{i, j\}$ marriages satisfy:

$$\ln \frac{\mu_{ij}}{\mu_{i0}} = (\gamma_{ij} - \gamma_{i0}) + \mathbf{q}_{ij}(p_{ij}(\Gamma, \gamma, Z, M, F), Z) - \mathbf{q}_{i0}(Z) \tag{29}$$

Then we have the following comparative static for a scalar parameter $\omega \in \{\Gamma, \gamma, Z, M, F\}$:

$$\frac{\partial \ln \frac{\mu_{ij}}{\mu_{i0}}}{\partial \omega} = \frac{\partial(\gamma_{ij} - \gamma_{i0})}{\partial \omega} + \frac{\partial \mathbf{q}_{ij}}{\partial p_{ij}} \frac{\partial p_{ij}}{\partial \omega} + \frac{\partial(\mathbf{q}_{ij} - \mathbf{q}_{i0})}{\partial Z} \frac{\partial Z}{\partial \omega} \tag{30}$$

The interpretation of (30) is similar to that given for wives.

The difference in net spousal gains, T_{ij}, is equal to the log of the ratio of the number of unmarried type i men to unmarried type j women:

$$T_{ij} = \ln \frac{\mu_{ij}}{\mu_{0j}} - \ln \frac{\mu_{ij}}{\mu_{i0}} = \ln \frac{\mu_{i0}}{\mu_{0j}} \tag{31}$$

T_{ij} is a measure of marriage market tightness or the net spousal gain of the wife relative to her husband. (31) says marriage market tightness increases when the number of unmarried type i men increases relative to the number of unmarried type j women.

Using (8), (30) and (28),

$$\frac{\partial T_{ij}}{\partial \omega} = \frac{\partial((\Gamma_{ij} - \Gamma_{0j}) - (\gamma_{ij} - \gamma_{0j}))}{\partial \omega} \tag{32}$$
$$+ (\frac{\partial(\mathbf{Q}_{ij} - \mathbf{Q}_{0j}) - (\mathbf{q}_{ij} - \mathbf{q}_{i0})}{\partial Z})\frac{\partial Z}{\partial \omega} + (1 + p_{ij})\frac{\partial \mathbf{Q}_{ij}}{\partial p_{ij}}\frac{\partial p_{ij}}{\partial \omega}$$

(32) says that the change in the difference in net spousal gains is equal to three terms. The first term is the change in the relative spousal invariant gains. The second term is the change in the difference in spousal utilities from a change in the wages and non-labor income distributions. The third term is proportional to the change in the wife's utility from a change in her husband's relative bargaining power, p_{ij}. Since $(1+p_{ij}) > 0$ and $\frac{\partial Q_{ij}}{\partial p_{ij}} < 0$, if p_{ij} increases, the wife's net gain will fail relative to her husband and vice versa.

The empirical content of (32) is as follows. Consider two societies, r and r'. Let

$$\overline{X}_{ij}^{rr'} = \frac{1}{2}(X_{ij}^{r} + X_{ij}^{r'}) \tag{33}$$

be the average of X_{ij}^r and $X_{ij}^{r'}$.

Using (32), (33) becomes:

$$T_{ij}^r - \overline{T}_{ij}^{rr'} = ((\Gamma_{ij}^r - \Gamma_{0j}^r) - (\gamma_{ij}^r - \gamma_{0j}^r)) - ((\overline{\Gamma}_{ij}^{rr'} - \overline{\Gamma}_{0j}^{rr'}) - (\overline{\gamma}_{ij}^{rr'} - \overline{\gamma}_{0j}^{rr'})) \tag{34}$$
$$+ (1 + \overline{p}_{ij}^{rr'})\frac{\partial \mathbf{Q}_{ij}}{\partial p_{ij}}|_{\overline{p}_{ij}^{rr'}}(p_{ij}^r - \overline{p}_{ij}^{rr'}) + \frac{\partial((\mathbf{Q}_{ij} - \mathbf{Q}_{0j}) - (\mathbf{q}_{ij} - \mathbf{q}_{i0}))}{\partial Z}|_{\overline{Z}^{rr'}}(Z^r - \overline{Z}^{rr'})$$

Given base society r and $\{i, j\}$, choose the partner city, r', such that the cumulative wage and non-labor income distributions, $F(S_{ijgG}|Z^r)$ and $F(S_{ijgG}|Z^{r'})$, are the same in both cities. In other words, between the two cities r and r', $Z^r - Z^{r'} = 0$.

(34) reduces to:

$$T_{ij}^r - \overline{T}_{ij}^{rr'} = ((\Gamma_{ij}^r - \Gamma_{0j}^r) - (\gamma_{ij}^r - \gamma_{0j}^r)) - ((\overline{\Gamma}_{ij}^{rr'} - \overline{\Gamma}_{0j}^{rr'}) - (\overline{\gamma}_{ij}^{rr'} - \overline{\gamma}_{0j}^{rr'})) \tag{35}$$
$$+ (1 + \overline{p}_{ij}^{rr'})\frac{\partial \mathbf{Q}_{ij}}{\partial p_{ij}}|_{\overline{p}_{ij}^{rr'}}(p_{ij}^r - \overline{p}_{ij}^{rr'}) \tag{36}$$

Let H_{ij}^{rk} be the hours of work of wife k in an $\{i, j\}$ marriage in society r. Now consider the OLS regression with $K \times L \times R$ observations, where there are K women, L types of marriages, $ij = 1, .., L$ and R pairs of matched cities, $\{r, r'|r = 1, ..R\}$:

$$H_{ij}^{rk} - \overline{H}_{ij}^{rr'} = \alpha_0 + \alpha_1(T_{ij}^r - \overline{T}_{ij}^{rr'}) + u_{ij}^r, \; k = 1, .., K^r; \; ij = 1, .., L; \; r = 1, ..R \tag{37}$$

u_{ij}^r is the error term of the regression. Since $(1 + p_{ij})\frac{\partial \mathbf{Q}_{ij}}{\partial p_{ij}} < 0$, $(T_{ij}^r - \overline{T}_{ij}^{rr'})$ is negatively related to $(p_{ij}^r - \overline{p}_{ij}^{rr'})$ from (35). α_1 estimates the elasticity of mean hours of work of the wives with respect to marriage market tightness. We know from (6) that the wife's labor supply is positively correlated with p^r_{ij} and thus negatively correlated with marriage market tightness, $T^r_{ij} = \ln \frac{\mu^r_{i0}}{\mu^r_{0j}}$. So the estimate of α_1 should be negative.

If husbands obtain more invariant gains relative to wives in one society than the other, the bargaining power of the spouses across the two societies will systematically be different. So in general, $((\Gamma^r_{ij} - \Gamma^r_{0j}) - (\gamma^r_{ij} - \gamma^r_{0j})) - ((\overline{\Gamma}^{rr'}_{ij} - \overline{\Gamma}^{rr'}_{0j}) - (\overline{\gamma}^{rr'}_{ij} - \overline{\gamma}^{rr'}_{0j}))$, the difference in relative spousal invariant gains, will be correlated with $(p^r_{ij} - \overline{p}^{rr'}_{ij})$. Thus the OLS estimate of α_1 in (37) is consistent only if $((\Gamma^r_{ij} - \Gamma^r_{0j}) - (\gamma^r_{ij} - \gamma^r_{0j})) - ((\overline{\Gamma}^{rr'}_{ij} - \overline{\Gamma}^{rr'}_{0j}) - (\overline{\gamma}^{rr'}_{ij} - \overline{\gamma}^{rr'}_{0j})) = 0$. Our first empirical test assumes that $((\Gamma^r_{ij} - \Gamma^r_{0j}) - (\gamma^r_{ij} - \gamma^r_{0j})) - ((\overline{\Gamma}^{rr'}_{ij} - \overline{\Gamma}^{rr'}_{0j}) - (\overline{\gamma}^{rr'}_{ij} - \overline{\gamma}^{rr'}_{0j})) = 0$ across all pairs of societies.

The assumption that $((\Gamma^r_{ij} - \Gamma^r_{0j}) - (\gamma^r_{ij} - \gamma^r_{0j})) - ((\overline{\Gamma}^{rr'}_{ij} - \overline{\Gamma}^{rr'}_{0j}) - (\overline{\gamma}^{rr'}_{ij} - \overline{\gamma}^{rr'}_{0j})) = 0$ across all pairs of societies is very strong. In some pairs of societies, the difference in difference in invariant spousal gains may be positive and negative in others. It is also likely that these differences persists across different types of marriages for a given pair of societies. Our second test exploits this persistence and uses the difference in differences methodology. Consider two types of marital matches, $\{i, j\}$ and $\{i', j'\}$, where $i, j \neq i', j'$. Here we assume that:

$$((\Gamma_{ij} - \Gamma_{0j}) - (\gamma_{ij} - \gamma_{0j})) = ((\Gamma_{i'j'} - \Gamma_{0j'}) - (\gamma_{i'j'} - \gamma_{0j'})) \tag{38}$$

In other words, relative spousal invariant gains is the same for marriage matches in society r. Now consider the OLS regression:

$$H^{rk}_{ij} - \overline{H}^{rr'}_{ij} = \alpha^r_0 + \alpha_1(T^r_{ij} - \overline{T}^{rr'}_{ij}) + u^r_{ij},\ k = 1,..,K^r;\ ij = 1,..,L;\ r = 1,..R \tag{39}$$

α^r_0 and $\alpha^{r'}_0$ are fixed effects for societies, r and r' respectively. So if (38) is valid, the OLS estimate of α_1 in (39) is consistent.

Our third empirical test assumes that $((\Gamma^r_{ij} - \Gamma^r_{0j}) - (\gamma^r_{ij} - \gamma^r_{0j})) - ((\overline{\Gamma}^{rr'}_{ij} - \overline{\Gamma}^{rr'}_{0j}) - (\overline{\gamma}^{rr'}_{ij} - \overline{\gamma}^{rr'}_{0j}))$ is uncorrelated with $(\ln \frac{m^r_i}{f^r_j} - \ln \frac{\overline{m}^{rr'}_i}{\overline{f}^{rr'}_j})$, the deviation in the sex ratios. In other words, individuals do not move across matched societies in response to deviations in relative spousal invariant gains. In this case, we instrument $(T^r_{ij} - \overline{T}^{rr'}_{ij})$ with $(\ln \frac{m^r_i}{f^r_j} - \ln \frac{\overline{m}^{rr'}_i}{\overline{f}^{rr'}_j})$ in (37) and (39). Note that we do not assume that the deviations in sex ratios are uncorrelated with the wage distributions in the two cities. Because we have matched societies by their wage distributions, differences in labor supply effects due to differences in wage distributions are directly controlled for. Thus any causal effect of differences in sex ratios on labor supplies for matched cities are through its effect on marriage market tightness.

Our use of sex ratios as instruments stands the traditional analysis based on the OLS regression of female labor supplies on sex ratio on its head. In the traditional analysis, the exogeneity of the sex ratio to labor demand conditions is a necessary condition for consistency of the OLS estimate of α_1. In our setup, it is fine that the sex ratio respond to labor demand conditions. Because we control for labor demand conditions directly through matching by wage distributions, variations in labor demand conditions are important in generating variations in sex ratios to tease out marriage market effects.

Our fourth empirical investigation approximates:

$$1 + \overline{p}_{ij}^{rr'} \simeq 1 + p + \Delta p_{ij} + \Delta p^{rr'}$$

In other words, there is a component of the bargaining power of husbands that is common to $\{i, j\}$ marriages in all societies. In this case, (35) becomes:

$$\begin{aligned} T_{ij}^r - \overline{T}_{ij}^{rr'} &= ((\Gamma_{ij}^r - \Gamma_{0j}^r) - (\gamma_{ij}^r - \gamma_{0j}^r)) - ((\overline{\Gamma}_{ij}^{rr'} - \overline{\Gamma}_{0j}^{rr'}) - (\overline{\gamma}_{ij}^{rr'} - \overline{\gamma}_{0j}^{rr'})) \quad (40) \\ &+ (1 + p + \Delta p_{ij} + \Delta p^{rr'}) \frac{\partial \mathbf{Q}_{ij}}{\partial p_{ij}} |_{\overline{p}_{ij}^{rr'}} \quad (41) \end{aligned}$$

Now consider the following OLS regression:

$$H_{ij}^{rk} - \overline{H}_{ij}^{rr'} = \alpha_0^r + (\alpha_1 + \alpha_1^{rr'} + \alpha_{1ij})(T_{ij}^r - \overline{T}_{ij}^{rr'}) + u_{ij}^r, \ k = 1, .., K^r; \ ij = 1, .., L; \ r = 1, ..R \quad (42)$$

(42) expands (39) to include interaction terms with $(T_{ij}^r - \overline{T}_{ij}^{rr'})$. Of interest are the coefficients on the interaction terms α_{1ij}. From the estimates of α_{1ij}, we can get a sense of the bargaining power of the husbands as the type of marriage changes. Consider marriages with low total gains such as that between younger men and older women. If we hypothesize that wives in these marriages have to provide more bargaining power to their husbands, then α_{1ij} for these marriages should be larger than for marriages between older men and younger women.

6 One period marriage without uncertainty

Most of literature on the collective model deals with a static model of intrahousehold allocations without uncertainty. That is, wages and non-labor income are known as of the time the individuals enter into the marriage. Our marriage matching framework can accommodate this case.

Let observed wages, non-labor income and labor supplies be equal to true wages, non-labor income and labor supplies plus measurement error:

$$W_{ij} = W_{ij} + \varepsilon_{ijgG}^{W\pi} \quad (43)$$

$$w_{ij} = w_{ij} + \varepsilon_{ijgG}^{w\pi} \quad (44)$$

$$A_{ij} = A_{ij} + \varepsilon_{ijgG}^{A\pi} \quad (45)$$

$$H_{ij} = H_{ij} + \varepsilon_{ijgG}^{L\pi} \quad (46)$$

$$h_{ij} = h_{ij} + \varepsilon_{ijgG}^{l\pi} \quad (47)$$

$\varepsilon^{W\pi}_{ijgG}$, $\varepsilon^{w\pi}_{ijgG}$, $\varepsilon^{L\pi}_{ijgG}$, $\varepsilon^{l\pi}_{ijgG}$and $\varepsilon^{A\pi}_{ijgG}$ are measurment errors which are uncorrelated with the true values. Marriages are still identified by $\{i, j, \pi\}$. Thus we can still use p_{ij}, the bargaining weight of the husband to clear the marriage market. Given p_{ij}, instead of problem P1, the planner will now solve:

$$\max_{\{C,c,L,l\}} \widehat{Q}(C_{ij}, 1 - H_{ij}, K_{ij}) + p_{ij})\widehat{q}(c_{ij}, 1 - h_{ij}, K_{ij}) \tag{P1a}$$
$$\text{subject to } C_{ij} + c_{ij} + K_{ij} \leq A_{ij} + W_{ij}(1 - H_{ij}) + w_{ij}(1 - h_{ij}) \;\forall\, S_{ij}$$

(8), appropriately reinterpreted, continues to hold which is what is critical for marriage market clearing. Thus as long as we can identify the type of an individual and the types of marriages that the individual can enter into, i.e. $\{i, j\}$, the empirical tests that we develop in this paper remain valid.

Thus our empirical results should be interpreted with care. *Even if our empirical results is consistent with our model predictions, they do not shed light on whether there is efficient risk sharing within the family or not.*

It is also convenient at this point to discuss empirical tests of the static collective model using spousal labor supplies such as CFL. In their paper, they estimate restricted spousal labor supplies models where the restrictions are derived from a static collective model. They instrument spousal wages with education, father's education, age, city size, religion. Different values of these instruments define different types of individuals in different regions. There is no instrument which captures the transitory component of wages.[6] Our interpretation of their empirical results is that they provide evidence of efficient bargaining between different types of spouses. Their empirical results are not informative about whether there is efficient risk sharing with the household as we suppose, or whether there is not as they supposed. In order to empirically distinguish between whether there is efficient risk sharing or not, one would need an instrument for transitory wage shocks when one estimates spousal labor supplies equations.

Our static formulation of the collective model in this section is also close to Del Bocca and Flinn's formulation. Instead of competitive marriage market clearing as we use in this paper, they use two different household allocation models and the deferred acceptance algorithm to construct a marriage market equilibrium. The difference in equilibrium constructions may not be significant in large marriage markets.[7]. What is empirically significant between their paper and ours is that they impose the restriction that the invariant gains to marriage and utilities from consumption and labor supply are the same for all types of marriages. This restriction imposes strong restrictions on marriage matching

[6] Although age changes for an individual over time, the changes are deterministic. Also, the previous section shows that our model extends to multi-period marriages.

[7] Dagsvik () has shown that when individuals' preferences over different spouses are characterized by McFadden's random utility model, using a non-transferable utility deferred acceptance algorithm to construct a large marriage market equilibrium results in a marriage matching function that is closely related to that discussed in this paper (See CS for further discussion).

patterns and spousal labor supplies in a single marriage market. We use the exactly opposite assumption where we do not impose any structure on invariant gains and utilities from consumption and labor supply across different types of marriages. Thus we do not impose any marriage matching and spousal labor supplies pattern in a single marriage market. The behavioral difference between our two models can be illustrated as follows. For a large class of household allocation models, if all marriages have the same invariant gains and utilities from consumption and labor supplies, a man with a low wage who wants to marry a woman with a high wage will have to work many more hours. This is due to the fact that the low wage man cannot make himself more attractive in other ways to the high wage woman. But if invariant gains and utilities from consumption and labor supplies are different for different types of marriages, a man with a low wage who wants to marry a woman with a high wage may do more house work and work more or less hours in the labor market.[8] We have broken the link between marriage matching and labor supplies in a single marriage market. The "true" model is likely in between our two formulations.[9]

7 Empirical results

(To be added)

8 Literature review

(To be added)

9 Appendis 2: Proof of existence of equilibrium

In the proof, we need:

$$E_{ij}(\underline{p}) > 0 \text{ as } \underline{p} \to \infty \qquad \text{(Condition A1)}$$

$$E_{ij}(\underline{p}) < 0 \text{ as } \underline{p} \to 0 \qquad \text{(Condition A2)}$$

That is, the utility functions q and Q must be such that as $\underline{p}$ approaches 0, men will not want to marry. And as $\underline{p}$ approaches ∞, women will not want to marry.

Let $\beta_{ij} = (1 + p_{ij})^{-1}$ where $\beta_{ij} \in [0, 1]$ is the utility weight of the wife in an $\{i, j\}$ marriage and $(1 - \beta_{ij})$ is the utility weight of the husband.

[8] As Becker long pointed out, the low wage man may not work in the labor market at all.

[9] A more general model, with type invariant utilities from consumption and leisure may be formulated by adding explicit household production and estimated with time use data. It will also have to deal explicitly with children. This is beyond the scope of this paper.

We know:

$$\frac{\partial \underline{\mu}_{ij}}{\partial p_{ij}} > 0 \tag{48}$$

$$\frac{\partial \underline{\mu}_{ij}}{\partial p_{ik}} < 0,\ k \neq j \tag{49}$$

$$\frac{\partial \underline{\mu}_{kl}(\beta)}{\partial p_{ij}} = 0;\ k \neq i, l \neq j \tag{50}$$

$$\frac{\partial \overline{\mu}_{ij}}{\partial p_{ij}} < 0 \tag{51}$$

$$\frac{\partial \overline{\mu}_{ij}}{\partial p_{kj}} > 0,\ k \neq i \tag{52}$$

$$\frac{\partial \overline{\mu}_{kl}(\beta)}{\partial p_{ij}} = 0;\ k \neq i, l \neq j \tag{53}$$

Let β be a matrix with typical element β_{ij} and the IxJ matrix function $E(\beta)$ be:

$$E(\beta) = \underline{\mu}(\beta) - \overline{\mu}(\beta) \tag{54}$$

An element of $E(\beta)$, $E_{ij}(\beta)$, is the excess demand for j type wives by i type men given β.

An equilibrium exists if there is a β^* such that $E(\beta^*) = 0$.

Assume that there exists a function $f(\beta) = \alpha E(\beta) + \beta$, $\alpha > 0$ which maps $[0,1]^{I*J} \to [0,1]^{I*J}$ and is non-decreasing in β. Tarsky's fixed point theorem says if a function $f(\beta)$ maps $[0,k]^N \to [0,k]^N$, $k > 0$, and is non-decreasing in β, there exists $\beta^* \in [0,k]^N$ such that $\beta^* = f(\beta^*)$. Let $f(\beta) = \alpha E(\beta) + \beta$, $k = 1$ and $N = I * J$, and apply Tarsky's theorem to get $\beta^* = \alpha E(\beta^*) + \beta^* \Rightarrow E(\beta^*) = 0$.

Thus the proof of existence reduces to showing $f(\beta)$ which has the required properties.

We know from (48) to (53) that:

$$\frac{\partial E_{ij}(\beta)}{\partial \beta_{ij}} < 0 \tag{55}$$

$$\frac{\partial E_{ik}(\beta)}{\partial \beta_{ij}} > 0 \tag{56}$$

$$\frac{\partial E_{kj}(\beta)}{\partial \beta_{ij}} > 0 \tag{57}$$

$$\frac{\partial E_{kl}(\beta)}{\partial \beta_{ij}} = 0;\ k \neq i, l \neq j \tag{58}$$

(55) to (58) imply that $E(\beta)$ satisfies the Weak Gross Substitutability (WGS) assumption.

We now show that the WGS property of $E(\beta)$ implies that we can construct $f(\beta)$, such that $f(\beta)$ maps $[0,1]^{I*J} \to [0,1]^{I*J}$ and is non-decreasing in β. The proof follows the solution to exercise 17.F.16^C of Mas-Colell, Whinston and Green given in their solution manual. (N.B. Unlike them, we do not start with Gross Substitution, we begin from WGS, but it turns out to be sufficient for Tarsky's conditions)

For notational convenience, now onwards we'll treat the matrix function $E(\beta)$, as a vector function.

Let $N = I * J$ and 1_N be a $N \times 1$ vector of ones. $E(\beta) : [0,1]^N \rightarrow R^N$ is continuously differentiable and satisfies $E(0_N) >> 0_N$ and $E(1_N) << 0_N$ (Conditions A1 and A2).

For every $\beta \in [0,1]^N$ and any n, if $\beta_n = 0$, then $E_n(\beta) > 0$.

For every $\beta \in [0,1]^N$ and any n, if $\beta_n = 1$, then $E_n(\beta) < 0$.

If $\beta = \{0_N, 1_N\}$, the facts follow from Conditions A1 and A2. Otherwise, they are due to Conditions A1 and A2, and (55) to (58), i.e. WGS.

For each n, define $C_n = \{\beta \in [0,1]^N : E_n(\beta) \geq 0\}$ and $D_n = \{\beta \in [0,1]^N : E_n(\beta) \leq 0\}$.

Then $C_n \subset \{\beta \in [0,1]^N : \beta_n < 1\}$ and $D_n \subset \{\beta \in [0,1]^N : \beta_n > 0\}$.

Then by continuity, the following two minima, $_{ij}((1-\beta_n)/E_n(\beta) : \beta \in C_n)$ and $_{ij}(-\beta_n/E_n(\beta) : \beta \in D_n)$, exist and are positive. Let $\underline{\beta}_n > 0$ be smaller than those two minima. Then, for all $\alpha \in (0, \underline{\beta}_n)$ and any $\beta \in [0,1]^N$, we have $0 \leq \alpha E_n(\beta) + \beta_n \leq 1$.

For each n, define $L_n =_{ij} \{|\partial E_n(\beta)/\partial \beta_n| : \beta \in [0,1]^N\}$. Then, for all $\alpha \in (0, 1/L_n)$,

$$\frac{\partial(\alpha E_n(\beta) + \beta_n)}{\partial \beta_n} = \alpha \frac{\partial E_n(\beta)}{\partial \beta_n} + 1 \geq -\alpha L_n + 1 > 0$$
$$\frac{\partial(\alpha E_n(\beta) + \beta_n)}{\partial \beta_m} = \alpha \frac{\partial E_n(\beta)}{\partial \beta_m} \geq 0; n \neq m, \text{ follows from (55) to (58).}$$

Now let $K = _{ij}\{\underline{\beta}_1, .., \underline{\beta}_N, 1/L_1, .., 1/L_N\}$, choose $\alpha \in (0, K)$, then $f(\beta) = \alpha E(\beta) + \beta \in [0,1]^N$ and $\partial f(\beta)/\partial \beta_n \geq 0$ for every $\beta \in [0,1]^N$, and any n. Hence Tarsky's conditions are satisfied.