# REPUTATION WITH LONG RUN PLAYERS

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Abstract.

Previous work shows that reputation results may fail in repeated games with long-run players with equal discount factors. Attention is restricted to extensive-form stage games of perfect information. One and two-sided reputation results are provided for repeated games with two long-run players with equal discount factors where the first mover advantage is maximal. If one of the players is a Stackelberg type with positive probability, then that player receives the highest payoff, that is part of an individually rational payoff profile, in any perfect equilibria, as agents become patient. If both players are Stackelberg types with positive probability, then perfect equilibrium payoffs converge to a unique payoff vector; and the equilibrium play converges to the unique equilibrium of a continuous time war of attrition. All results generalize to simultaneous move stage games, if the stage game is a game of strictly conflicting interest.

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## 1. Introduction and Related Literature

This paper proves one and two-sided reputation results when two players with equal discount factors play a repeated game where the first mover advantage is maximal. The stage game, which is repeated in each period, is an extensive-form game of perfect information.

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A *Stackelberg strategy* is a player's optimal repeated game strategy, if the player could publicly commit to this strategy ex-ante; and a *Stackelberg type* is a commitment type that only plays the Stackelberg strategy. The *first mover advantage is maximal* for player 1 if the repeated game Stackelberg strategy delivers player 1 his highest payoff, that is part of an individually rational payoff profile, whenever player 2 best responds. Our first main result shows that if there is one-sided incomplete information and player 1 is a Stackelberg type with positive probability, then player 1 receives his highest possible payoff, in any equilibrium, as the discount factor converges to one, and the probability of being any other commitment type converges to zero. This *one-sided reputation result* extends to arbitrary probability distributions over other commitment types if these other types are uniformly learnable. Our second main result (*two-sided reputation result*) establishes that if there is incomplete information about both players' types and each player is a Stackelberg type with positive probability, then all equilibrium paths of play resemble the unique equilibrium of an appropriately defined continuous time war of attrition, as the time between repetitions of the stage game shrinks to zero. Also, all equilibrium payoffs converge to the unique equilibrium payoff of the war of attrition.

A one-sided reputation result was first established for finitely repeated games by Kreps and Wilson (1982) and Milgrom and Roberts (1982); and extended to infinitely repeated games by Fudenberg and Levine (1989). However, most reputation results in the literature are for repeated games where a long-run player, that is possibly a Stackelberg type, faces a sequence of short-run players (as in Fudenberg and Levine (1989, 1992)); or for repeated games where the player building the reputation is infinitely more patient than his rival and so the rival is essentially a short-run player, at the limit (for example, see Schmidt (1993b), Celantani, Fudenberg, Levine, and Pesendorfer (1996), Battigalli and Watson (1997) or Evans and Thomas (1997)). Also, previous research has shown that reputation results are fragile in infinitely repeated games where long-run players with equal discount factors play a simultaneous-move stage game. In particular, one-sided reputation results obtain only if the stage game is a *strictly conflicting interest* game (Cripps, Dekel, and Pesendorfer

(2005)), or if there is a strictly dominant action in the stage game (Chan (2000)).[1] For other simultaneous move games, such as the common interest game, a folk theorem by Cripps and Thomas (1997) shows that any individually rational and feasible payoff can be sustained in perfect equilibria of the infinitely repeated game, if the players are sufficiently patient (also see the analysis in Chan (2000)).

Almost all the recent work on reputation has focused on simultaneous move stage games. In sharp contrast, we restrict attention to extensive-form stage games of perfect information. The stage games we allow for include common interest games, the battle of the sexes, the chain store game as well as all strictly conflicting interest games (see section 2.1). For the class of games we consider, without incomplete information, the folk theorem of Fudenberg and Maskin (1986) applies, under a full dimensionality condition (see Wen (2002) or Mailath and Samuelson (2006)). Also, if the normal form representation of the extensive form game we consider is played simultaneously in each period, then under one-sided incomplete information, a folk theorem applies for a subset of the class of games we consider (see Cripps and Thomas (1997)). Consequently, our one-sided reputation result covers a significantly larger class of games than those covered by previous reputation results.

Our *two-sided reputation result* is motivated by the approach in Kreps and Wilson (1982) and is closely related to previous work by Abreu and Gul (2000) and Abreu and Pearce (2007). Abreu and Gul (2000) show that in a two player bargaining game, as the frequency of offers increases, the equilibria of the (two-sided) incomplete information game converges to the unique equilibrium of a continuous time war of attrition. Their two-sided reputation result builds on a one-sided reputation result for bargaining games due to Myerson (1991). Likewise, our two-sided reputation result builds on our one-sided reputation result that ensures that there is a unique equilibrium payoff in any continuation game with one-sided incomplete information.

[1]A game has strictly conflicting interests (Chan (2000)) if a best reply to the commitment action of player 1 yields the best feasible and individually rational payoff for player 1 and the minimax for player 2.

The only other two-sided reputation result for repeated games with long run players is by Abreu and Pearce (2007). In this paper, the authors allow for multiple types and elegantly show that the equilibrium payoff profile coincides with the Nash bargaining solution with endogenous threats for any specification of the stage-game. However, their paper studies a different economic environment than ours and is not directly comparable. Specifically, in Abreu and Pearce (2007) agents write binding contracts and commitment types announce their inflexible demands truthfully at the start of the repeated game. These enforceable contracts uniquely determine payoffs in the continuation game with one-sided incomplete information. In our paper, in contrast, continuation payoffs are unique as a consequence of our one-sided reputation result and no extra communication is assumed. Uniqueness in the one-sided incomplete information game is a key component for the two-sided reputation result. Without uniqueness, many equilibria can be generated in the game with two-sided incomplete information by leveraging the multiplicity in the continuation game with one-sided incomplete information.

The paper proceeds as follows: section 2 describes the model and discusses some examples that satisfy our assumptions; section 3 presents the main one-sided reputation result; and section 4 outlines the continuous time war of attrition and presents the two-sided reputation result as well as some comparative statics. All proofs that are not in the main text are in the appendix.

## 2. The Model

The stage game $\Gamma$ is a finite extensive-form game and the set of players in the game is $I = \{1, 2\}$.

**Assumption 1.** *The stage game $\Gamma$ is an extensive-form game of perfect information, that is, all information sets of $\Gamma$ are singletons.*

$\Gamma_N$ is the normal form of $\Gamma$. The finite set of pure stage game actions, $a_i$, for player $i$ in the game $\Gamma_N$ is denoted $A_i$ and the set of mixed stage game strategies $\alpha_i$ is denoted

$\mathcal{A}_i$. The payoff function of player $i$ for the game $\Gamma_N$ is $g_i : A_1 \times A_2 \to \mathbb{R}$. The minimax for player $i$, $\hat{g}_i = \min_{\alpha_j} \max_{\alpha_i} g_i(\alpha_i, \alpha_j)$. For games that satisfy Assumption 1 there exists $a_1^p \in A_1$ such that $g_2(a_1^p, a_2) \leq \hat{g}_2$ for all $a_2 \in A_2$.[2] The set of feasible payoffs $F = co\{g_1(a_1, a_2), g_2(a_1, a_2) : (a_1, a_2) \in A_1 \times A_2\}$; and the set of feasible and individually rational payoffs $G = F \cap \{(g_1, g_2) : g_1 \geq \hat{g}_1, g_2 \geq \hat{g}_2\}$. The highest payoff that is part of an individually rational payoff profile for player $i$, $\bar{g}_i = \max\{g_i : (g_i, g_j) \in G\}$. Also, let $M > \max_{g \in F} |g_i|$.

**Assumption 2** (Maximal First Mover Advantage for Player 1)**.** *The stage-game $\Gamma$ satisfies*

- $(i)$ *(Genericity) For any payoff profile $g \in G$ and $g' \in G$, if $g_1 = g_1' = \bar{g}_1$, then $g_2 = g_2'$, and either of the following*
- $(ii)$ *(Locally Non-Conflicting Interest) For payoff profile $g \in G$, if $g_1 = \bar{g}_1$, then $g_2 > \hat{g}_2$, or*
- $(iii)$ *(Strictly Conflicting Interest) There exists $(a_1^s, a_2^b) \in A_1 \times A_2$ such that $g_1(a_1^s, a_2^b) = \bar{g}_1$; and if $a_2 \in A_2$ is a best response to $a_1^s$, then $g_1(a_1^s, a_2) = \bar{g}_1$ and $g_2(a_1^s, a_2) = \hat{g}_2$.*

Assumption 2 is defined symmetrically for player 2. Item $(i)$ of Assumption 2, which is met generically by all extensive-form games, requires that the payoff profile where player $i$ obtains $\bar{g}_i$ is unique. For generic games, items $(ii)$ and $(iii)$ are mutually exclusive. Item $(ii)$ requires that the game have a common value component. In particular, in the payoff profile where player 1 receives his highest possible payoff player 2 receives a payoff that strictly exceeds her minimax value. In contrast, item $(iii)$ requires $\Gamma$ to be a game of strictly conflicting interest. A generic extensive-form game of perfect information $\Gamma$ does not satisfy Assumption 2 for player 1 only if, $(\bar{g}_1, \hat{g}_2) \in G$ and $\Gamma$ is not a strictly conflicting interest game. If a game satisfies $(i)$ and $(ii)$, then there exists $(a_1^s, a_2^b) \in A_1 \times A_2$ such that $g_1(a_1^s, a_2^b) = \bar{g}_1$.[3] Consequently, if $\Gamma$ is a generic locally non-conflicting interest game

[2]Consider the zero sum game obtained from $\Gamma_N$ where player 1's payoff is set equal to $-g_2(a_1, a_2)$. The minimax of this game is $(-\hat{g}_2, \hat{g}_2)$ by definition. Also, under Assumption 1, this game has a pure strategy Nash equilibrium, $(a_1^p, a_2) \in A_1 \times A_2$, by Zermello's lemma. Because the game is a zero sum game $g_2(a_1^p, a_2) = \hat{g}_2$.

[3]Note that $a_2^b$ need not be a best response to $a_1^s$.

for player 1 (satisfies ($i$) and ($ii$)), then there is a finite constant $\rho \geq 0$ such that

$$\left| \frac{g_2 - g_2(a_1^s, a_2^b)}{\bar{g}_1 - g_1} \right| \leq \rho \tag{1}$$

for any $(g_1, g_2) \in F$.[4] Also, if $\Gamma$ is a generic strictly conflicting interest game for player 1 (satisfies ($i$) and ($iii$)), then $g_2 - g_2(a_1^s, a_2^b) \leq \rho(\bar{g}_1 - g_1)$ for any $(g_1, g_2) \in F$.

In the repeated game $\Gamma^\infty$, the stage game $\Gamma$ is played in each of periods $t = 0, 1, 2, ....$ Players have perfect recall and can observe past outcomes. $H$ is the set of all possible histories for the stage game $\Gamma$ and $Y \subset H$ the set of all terminal histories of the stage game. $H_t \equiv Y^t$ denotes the set of partial histories at time $t$. A behavior strategy $\sigma_i \in \Sigma_i$ is a function $\sigma_i : \bigcup_{t=0}^{\infty} H_t \rightarrow \mathcal{A}_i$. A behavior strategy chooses a mixed stage game strategy given the partial history $h_t$. Players discount payoffs using their discount factor $\delta$. The players' continuation payoffs in the repeated game are given by the normalized discounted sum of the continuation stage-game payoffs

$$u_i(h_{-t}) = (1 - \delta) \sum_{k=t}^{\infty} \delta^{k-t} g((a_i, a_j)_k)$$

for history $h = \{h_t, h_{-t}\} = \{h_t, (a_1, a_2)_t, ...\}$.

A Stackelberg strategy for player 1, denoted $\sigma_1(s)$ plays $a_1^s$, if the action profile in the previous period was $(a_1^s, a_2)$ and $g_1(a_1^s, a_2) = g_1(a_1^s, a_2^b) = \bar{g}_1$; and plays $a_1^p$, i.e., minimaxes player 2, for $n^p - 1$ periods and plays $a_1^s$ in the $n^p$th period, if the action profile in the previous period was $(a_1^s, a_2)$ and $g_1(a_1^s, a_2) < \bar{g}_1$. Also, in period zero, the Stackelberg strategy plays $a_1^s$. Intuitively, the Stackelberg strategy punishes player 2 for $n^p - 1$ periods if the opponent does not allow player 1 to get $\bar{g}_1$ in any period. The number of punishment periods $n_1^p - 1$ is the smallest integer such that

$$g_2(a_1^s, a_2) - g_2(a_1^s, a_2^b) < (n_1^p - 1)(g_2(a_1^s, a_2^b) - \hat{g}_2) \tag{2}$$

[4]If $\Gamma$ satisfies Assumption 2 ($i$) and ($ii$), then $\bar{g}_1 = \max\{g_1 : (g_1, g_2) \in F\}$. Consequently, the Lipschitz condition given in Equation (1) holds for all $g \in F$ and not just for $g \in G$.

for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < g_1(a_1^s, a_2^b)$. A Stackelberg type for player 2 is defined symmetrically. Note that if $\Gamma$ satisfies Assumption 2 for player 1, then $n_1^p \geq 1$ exists. Also observe that, for sufficiently high discount factor, whenever player 2 best responds to $\sigma_1(s)$, player 1's repeated game payoff is equal to $\bar{g}$. Consequently, if the stage game $\Gamma$ satisfies Assumption 2 for player 1, then player 1's *first mover advantage is maximal* in the repeated game.[5]

Let $\Omega$ denote the countable set of types and let $\mu = (\mu_1, \mu_2)$ denote a pair of probability measures over $\Omega$. Before time 0 nature selects player $i$ as a type $\omega$ with probability $\mu_i(\omega)$. $\Omega$ contains a normal type denoted $\omega_N$. The normal type maximizes expected normalized discounted utility. $\Omega$ also contains a Stackelberg type denoted $s$ that plays according to the Stackelberg strategy $\sigma_i(s)$. Let $\Omega_- = \Omega \backslash (\{\omega_N\} \cup \Omega_s)$. In words, $\Omega_-$ is the set of types other than the Stackelberg types and the normal type.[6]

Player $j$'s belief over player $i$'s types, $\mu_i : \bigcup_{t=0}^{\infty} H_t \to \Delta(\Omega)$ is a probability measure over $\Omega$ after each partial history $h_t$. A strategy profile $\sigma : \Omega_1 \times \Omega_2 \to \Sigma_1 \times \Sigma_2$ assigns a repeated game strategy to each type of each player. A normal player $i$'s expected continuation utility, following a partial history $h_t$, given that strategy profile $\sigma$ is used, is

$$U_i(\sigma|h_t) = \mu_j(\omega_N|h_t)\mathbb{E}_{(\sigma_i(\omega_N),\sigma_j(\omega_N))}[u_i(h_{-t})|h_t] + \mu_j(\Omega_s|h_t)\mathbb{E}_{(\sigma_i(\omega_N),\sigma_j(s))}[u_i(h_{-t})|h_t] + \sum_{\omega \in \Omega_-} \mu_j(\omega|h_t)\mathbb{E}_{(\sigma_i(\omega_N),\sigma_j(\omega))}[u_i(h_{-t})|h_t]$$

where $\mathbb{E}_{(\sigma_j,\sigma_i)}[u_i(h_{-t})|h_t]$ denotes the expectation over continuation histories $h_{-t}$ with respect to the probability measure generated by $(\sigma_i, \sigma_j)$ given that $h_t$ has occurred. Also, let $U_i(\sigma|h_t, \omega_j = \omega) = \mathbb{E}_{(\sigma_i(\omega_N),\sigma_j(\omega))}[u_i(h_{-t})|h_t]$.

[5]Suppose that the extensive-form stage-game of perfect information $\Gamma$ is generic. There exists a repeated game strategy $\sigma_1$, and a $\delta^* < 1$ such that, for all $\delta > \delta^*$ whenever player 2 best responds to $\sigma_1(s)$, player 1's repeated game payoff is equal to $\bar{g}$, if and only if, $\Gamma$ satisfies Assumption 2 for player 1.

[6]A few comments on notation: the analysis proceeds as if there is only one Stackelberg action $a_1^s$, only one punishment action $a_1^p$ and consequently a unique Stackelberg strategy $\sigma_1(s)$. This is without loss of generality, if there is more, name one arbitrarily as the Stackelberg action or the punishment action. Also, the subscript $i$ is suppressed in $\Omega_{s_i}$ and $\omega_{N_i}$ to avoid clutter. The expression $\mu_1(\Omega_s)$ should be interpreted as the probability that player 1 is a type in the set $\Omega_{s_1}$ where $s_1$ is the Stackelberg type for player 1. Likewise, $\mu_2(\Omega_s)$ denotes $\mu_2(\Omega_{s_2})$.

The repeated game where the initial probability over $\Omega$ is $\mu$ and the discount factor is $\delta$ is denoted $\Gamma^{\infty}(\mu, \delta)$. The analysis in the paper focuses on Bayes-Nash as well as perfect Bayesian equilibria of the game of incomplete information $\Gamma^{\infty}(\mu, \delta)$. In equilibrium, beliefs are obtained, where possible, using Bayes' rule given $\mu_i(\cdot|h_0) = \mu_i(\cdot)$ and conditioning on players' equilibrium strategies. If $\mu_2(\omega_N) = 1$ and $\mu_1(\Omega_s) > 0$, then belief $\mu_1(\cdot|h_t)$ is well defined after any history where player 1 has played according to $\sigma_1(s)$ in each period. Also, if $\mu_1(\Omega_s) > 0$ and $\mu_2(\Omega_s) > 0$, then beliefs are well defined after any history where both players have played according to $\sigma_i(s)$ in each period.

2.1. **Examples.** Theorem 1 and Theorem 2 provide one and two-sided reputation results for the following examples that satisfy Assumption 2. In these examples, if the repeated game is played under complete information, then the usual folk theorems apply and any individually rational payoff can be sustained in perfect equilibria for sufficiently high discount factors (see Wen (2002) or Mailath and Samuelson (2006) section 9.6). Also, the examples are not strictly conflicting interest games for player 1, so previous findings preclude reputation results, if the normal form representation of any of these games is played simultaneously and player 1 is building a reputation.

2.1.1. *Common interest games.* Consider the sequential-move common interest game depicted on the right in Figure 1. Assume that there is a (possibly small) probability that one of the two players is a Stackelbeg type that always plays the Stackelberg action (action $U$ at any information set for 1 and $L$ for 2). Theorem 1 implies that the player who is potentially a Stackelberg type can guarantee a payoff arbitrarily close to 1 in any perfect equilibrium of the repeated game, for sufficiently high discount factors.

2.1.2. *Battle of the sexes.* Theorem 1 and 2 provide one and two sided reputation results for the battle of the sexes game depicted in Figure 2. In particular, if each of the two players is a commitment type with probability $z_i > 0$, then Theorem 2 implies that the equilibrium path of play for this game resembles a war of attrition. During the "war" player 1 insists on playing $R$ while player 2 insists on playing $L$ and both receive per period payoff equal

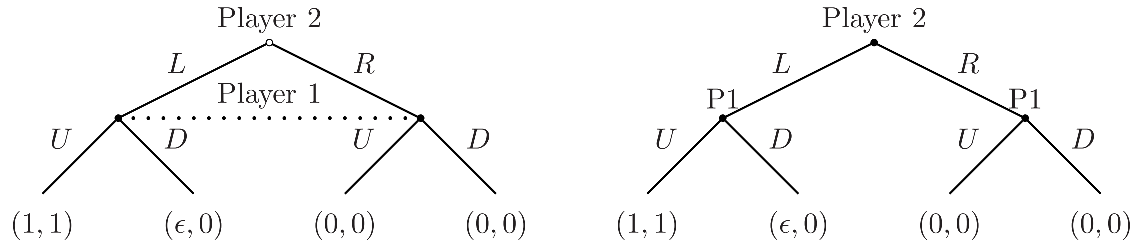


FIGURE 1. A simultaneous-move common interest game that does not satisfy Assumption 1 on the left and a sequential-move common interest game that satisfies Assumption 1 on the right ($\epsilon < 1$). For this game $n_1^p = n_2^p = 1$.

to 0. The "war" ends when one of the players reveals rationality by playing a best reply and accepting a continuation payoff equal to 1, while the opponent, who wins the war of attrition, receives continuation payoff equal to 2.

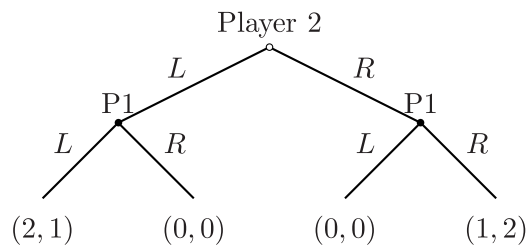


FIGURE 2. Battle of the sexes. In this game $a_1^s = R$, $a_2^s = L$. The minimax is 1 for player 1 and 0 for player 2. The game is a strictly conflicting interest game for player 2 and locally non-conflicting interest game for player 1. For this game $n_1^p = n_2^p = 1$.

2.1.3. *Stage game with a complex Stackelberg type.* In the previous two examples, the Stackelberg type was a simple type who played $a_i^s$ in each period. In the example depicted in Figure 3 the Stackelberg type of player 1 minimaxes player 2 by playing $R$ for two periods if player 2 plays $R$ against $L$.

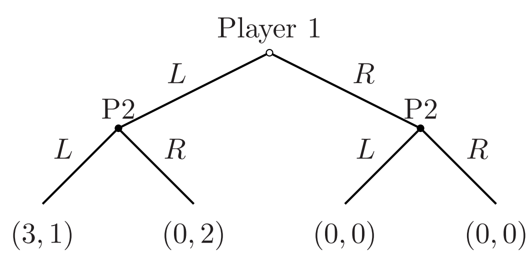


FIGURE 3. Stage-game with a complex Stackelberg type. For this game $n_1^p = 3$.

## 3. One-Sided Reputation

The central finding of this section, Theorem 1, establishes a one sided reputation result. The theorem maintains Assumption 1 and Assumption 2, and shows that if the probability that player 1 is a Stackelberg type is positive while the probability that player 2 is a commitment type is zero, then player 1 can guarantee a payoff close to $\bar{g}_1$, when the discount factor is sufficiently high, and the probability that player 1 is another commitment type is sufficiently low. Consequently, a generic extensive-form game of perfect information $\Gamma$ is not covered by Theorem 1 only if $(\hat{g}_1, \hat{g}_2) \in G$ and $\Gamma$ is a strictly conflicting interest game. This section also presents two corollaries to Theorem 1. Corollary 1 shows that the one-sided reputation result can also be established without Assumption 1 even under the weaker Bayes-Nash equilibrium concept, if the stage game is a strictly conflicting interest game, that is, satisfies Assumption 2 $(i)$ and $(iii)$. Corollary 2 extends the one-sided reputation result to the case where the probability that player 1 is another commitment type is arbitrary. This corollary maintains Assumptions 1 and 2 and, in addition, assumes that the commitment types in the support of $\mu_1$ are all uniformly learnable. Under these assumptions the corollary shows that player 1 can guarantee a payoff close to $\bar{g}_1$, if the discount factor is sufficiently high, by mimicking the Stackelberg type.

In order to provide some intuition for Theorem 1 and to establish the contrast with previous literature suppose, in contrast to Assumption 1, the extensive-form is as in Figure 1 (left). Also, suppose that player 1 is a Stackelberg type that plays $U$ in every period of the repeated game, with probability $z$. Cripps and Thomas (1997) have shown that there are many perfect equilibrium payoffs in this repeated game. In particular, they construct equilibria where players' payoffs are close to 0 when $z$ is close to 0 and $\delta$ is close to 1. In their construction, in the first $K$ periods player 2 plays $R$. As $\delta$ converges to 1, $K$ increases to ensure that the discounted payoffs converge to 0. To make $K$ large, Player 1's equilibrium strategy is chosen to be similar to the commitment type strategy; this ensures that player 1 builds a reputation very slowly. If this strategy exactly coincided with the commitment strategy, player 2 would not have the incentives to play $R$. Therefore this strategy is a mixed strategy that plays $D$ with small probability. To ensure that player 2 has an incentive to play $R$, she is punished when she plays $L$. Punishment entails a continuation payoff for player 2 that is close to 0, if player 2 plays $L$ and player 1 plays $D$ (thus revealing rationality). Player 1 is willing to mix between $U$ and $D$ in the first $K$ periods since player 2 only plays $R$ on the equilibrium path. Also, the punishment that follows $(D, L)$ is subgame perfect since, after such a history, the players are in a repeated game of complete information and any continuation payoff between 0 and 1 can be sustained in equilibrium, by a standard folk theorem.

Instead suppose that Assumption 1 is satisfied and player 1 moves after player 2, i.e., Figure 1 (right). When players move sequentially, the "follower" (player 1) observes the outcome of the behavior strategy used by his opponent. For the payoff of player 1 to be low, there should be many periods in which player 2 plays $R$. To give her an incentive to play $R$, player 1 must punish player 2 if she plays $L$. After any history where player 1 has not revealed rationality yet, punishing player 2 is also costly for player 1. Following a play of $L$ by player 2, in order for player 1 to punish player 2, he must be indifferent between $U$ and $D$. However, this is not possible since playing $U$ gives player 1 a payoff of 1 for the period and improves his reputation. On the other hand, a play of $D$ gives a payoff of zero for the period and moves the game into a punishment phase. Consequently, subgame perfection rules out player 1 punishing player 2 for playing $L$.

If the players move simultaneously in the stage game, then subgame perfection has no "bite" within the stage game. In the Cripps and Thomas (1997) construction because player 1 does not observe that player 2 has deviated and played $L$, and this never happens on the equilibrium path in the first $K$ periods, player 1 is willing to randomize between $U$ and $D$, and so, the required (off equilibrium-path) punishments can be sustained. Consequently, their construction avoids the logic outlined in the previous paragraph.

3.1. **The Main One-Sided Reputation Result.** Theorem 1 considers a repeated game $\Gamma^{\infty}(\mu,\delta)$ where $\mu_1(\Omega_s) > 0$ and $\mu_2(\omega_N) = 1$, that is, player 2 is known to be the normal type and player 1 is potentially a Stackelberg type. For the remainder of this section assume that $\mu_2(\omega_N) = 1$. Attention is restricted to perfect information stage games (Assumption 1) and to repeated games with maximal first mover advantage (Assumption 2). Within this class of games, the theorem demonstrates that a normal type for player 1 can secure a payoff arbitrarily close to $\bar{g}_1$ by mimicking the commitment type, in any equilibrium of the repeated game, for a sufficiently large discount factor ($\delta > \underline{\delta}$) and for sufficiently small probability mass on other commitment types ($\mu_1(\Omega_-) < \bar{\phi}$).

**Theorem 1.** *Posit Assumption 1, and Assumption 2 for player 1. For any $\underline{z}$ and $\gamma > 0$, there exists a $\underline{\delta} < 1$ and $\bar{\phi} > 0$ such that, for any $\delta \in (\underline{\delta}, 1)$, any $\mu$ with $\mu_1(\Omega_s) \geq \underline{z}$ and $\mu_1(\Omega_-) < \bar{\phi}$ and any perfect Bayesian equilibrium strategy profile $\sigma$ of $\Gamma^{\infty}(\mu,\delta)$, $U_1(\sigma) > \bar{g}_1 - \gamma$.*

The discussion that follows presents the various definitions and intermediate lemmas required for the proof Theorem 1. The proof of the Theorem is presented after all the intermediate results are established. First some preliminaries: Let $g_1(a_1^s, a_2^b) = \bar{g}_1$ for $(a_1^s, a_2^b) \in A_1 \times A_2$. Normalize payoffs, without loss of generality, such that:

(*i*) $\bar{g}_1 = 1$,
(*ii*) $g_1(a_1, a_2) \geq 0$ for all $a \in A$,
(*iii*) $g_2(a_1^s, a_2^b) = 0$,
(*iv*) There exists $l > 0$: $g_2(a_1^s, a_2) + (n_1^p - 1)g_2(a_1^p, a_2') \leq -2l$ for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < 1$ and $a_2' \in A_2$.

Condition (*iv*) implies that there exists a $\delta^* < 1$ such that, for all $\delta > \delta^*$,

$$g_2(a_1^s, a_2) + \sum_{k=1}^{n^p-1} \delta^k g_2(a_1^p, a_2') < -l$$

for any $a_2 \in A_2$ such that $g_1(a_1^s, a_2) < 1$ and $a_2' \in A_2$. For the remainder of the discussion we assume that $\delta > \delta^*$.

The main focus of analysis in the proof of Theorem 1 is player 2's resistance against a Stackelberg type. Intuitively, resistance is the expectation of the normalized discounted sum of the number of periods in which player 2 does not acquiesce to the demand of the Stackelberg type, in a particular equilibrium. Formally, the definition is as follows:

**Definition 1** (Resistance)**.** *Let $i(a) = 1$ if $a_1 = a_1^s$ and $g_1(a_1^s, a_2) < g_1(a_1^s, a_2^b)$, and $i(a) = 0$, otherwise. Let $i(\delta, h_{-t}) = (1-\delta)\sum_{k=t}^{\infty} \delta^{k-t} i(a_k)$. Player 2's continuation resistance, $R(\delta, \sigma_2|h_t) = \mathbb{E}_{(\sigma_1(s),\sigma_2)}[i(\delta, h_{-t})|h_t]$. Also, let $R(\delta, \sigma_2) = R(\delta, \sigma_2|h_0)$.*

The payoff to player 1 of using the Stackelberg strategy is at least $1 - n_1^p R(\delta, \sigma_2)$, by the definition of resistance and normalization $(i)$ and $(ii)$. Also, after any history $h_t$, the payoff to player 1 of using the Stackelberg strategy is at least $1 - n_1^p R(\delta, \sigma_2|h_t) - (1-\delta)n_1^p$.[7] This trivially implies the following lemma.

**Lemma 1.** *In any Bayes-Nash equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$, $U_1(\sigma) \geq 1 - n_1^p R(\delta, \sigma_2)$ and $U_1(\sigma|h_t) \geq 1 - n_1^p(R(\delta, \sigma_2|h_t) + (1-\delta))$ for any $h_t$ that has positive probability under $\sigma$. Also, in any perfect Bayesian equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$, $U_1(\sigma|h_t) \geq 1 - n_1^p(R(\delta, \sigma_2|h_t) + (1-\delta))$ for any $h_t$.*

The goal is to show that $R(\delta, \sigma_2)$ is bound by $C \max\{1-\delta, \phi\}$, for some constant $C$, in any equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$ where $\mu_1(\Omega_s) \geq \underline{z}$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$. Thus, if $\max\{1-\delta, \phi\}$ is close to zero, then $R(\delta, \sigma_2)$ is close to zero and $U_1(\sigma)$ is close to 1, in any equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$. The following definition introduces some reputation thresholds, denoted $z_n$ for a given resistance level $K^n \max\{1-\delta, \phi\}$.

[7]The extra term $(1-\delta)n^p$ appears since Player 1 may first have to endure a punishment stage.

**Definition 2** (Reputation Thresholds)**.** *Fix $\delta < 1$, $K > 1$ and $\phi \geq 0$. Let $\epsilon = \max\{1-\delta, \phi\}$. For each $n \geq 0$, let*

$$z_n = \sup\{z : \exists \text{ perfect Bayesian equilibrium } \sigma \text{ of } \Gamma^\infty(\mu, \delta), \quad \text{where } \mu_1(\Omega_s) = z \text{ and } \frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi, \text{ such that } R(\delta, \sigma_2) \geq K^n \epsilon\}. \tag{3}$$

*Also, define $q_n$ such that*

$$\frac{z_n}{1 - q_n} = z_{n-1}. \tag{4}$$

In words, $z_n$ is the highest reputation level of player 1 for which there exists an equilibrium of $\Gamma^\infty(\mu, \delta)$ in which player 2's resistance exceeds $K^n \epsilon$. The definition and $K^n > K^{n-1}$ implies that $z_n \leq z_{n-1}$. The $q_n$'s are real numbers that link the thresholds $z_n$. To interpret $q_n$, suppose that player 2 believes player 1 to be the Stackelberg type with probability $z_n$. Also, suppose that the total probability that any of player 1's types plays an action incompatible with $\sigma_1(s)$ at least once over the next $M$ periods is $q_n$. Consequently, if player 1 plays according to the Stackelberg strategy $\sigma_1(s)$ in each of the $M$ periods, then the posterior probability that player 2 places on player 1 being the Stackelberg type is $\frac{z_n}{1-q_n}$.

The development that follows will establish that $q_n \geq \underline{q} > 0$ for all $n$ such that $z_n \geq \underline{z}$, and all $\delta$ and $\phi$. If $q_n \geq \underline{q}$, then starting from $z_0 \leq 1$, there exists a $n^*$ such that $z_{n^*} \leq \underline{z}$. Since $z_{n^*} \leq \underline{z}$, if the initial reputation level is $\underline{z}$, then the maximal resistance of player 2 is at most $K^{n^*}\epsilon$, which is of the order of $\max\{1-\delta, \phi\}$. The following lemma formalizes this discussion.

**Lemma 2.** *Suppose that $q_n \geq \underline{q} > 0$ for all $\delta$, $\phi$ and all $n$ such that $z_n \geq \underline{z}$. There exists $n^*$ such that if $\max\{1-\delta, \phi\} < \frac{\gamma}{n_1^p K^{n^*}}$, then $U_1(\sigma) > 1 - \gamma$ for all perfect Bayesian equilibria $\sigma$ of $\Gamma^\infty(\mu, \delta)$ with $\mu_1(\Omega_s) \geq \underline{z}$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$.*

*Proof.* Let $n^*$ be the smallest the integer such that $(1-\underline{q})^{n^*} < \underline{z}$. Since $\underline{q} > 0$ such an integer exists. For all $\delta$ and $\phi$ such that $z_0 \geq \underline{z}$ we have $z_{n^*} < \underline{z}$. Consequently, by Definition 2,

$R(\delta, \sigma_2) < K^{n^*}\epsilon$ in any equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$ with $\mu_1(\Omega_s) \geq \underline{z}$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$. For any $\delta$ and $\phi$ where $z_0 < \underline{z}$, by Definition 2, $R(\delta, \sigma_2) < \epsilon < K^{n^*}\epsilon$ in any equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$ with $\mu_1(\Omega_s) \geq \underline{z}$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$. Consequently, by Lemma 1, $U_1(\sigma) > 1 - n_1^p K^{n^*}\epsilon$. So, if $\epsilon = \max\{1 - \delta, \phi\} < \frac{\gamma}{n_1^p K^{n^*}}$, then $U_1(\sigma) > 1 - \gamma$. □

In order to show that $q_n \geq \underline{q} > 0$ lower and upper bounds are established for player 2's payoffs. The argument hinges on the tension between player 2's magnitude of resistance and the speed at which player 1 builds a reputation. If player 2 resists the Stackelberg type of player 1, then player 2 must be doing so in anticipation that player 1 deviates from the Stackelberg strategy. Otherwise player 2 could do better by best responding to the Stackelberg strategy. The more player 2 resists player 1, the more player 2 must be expecting player 1 to deviate from the Stackelberg strategy. However, if player 1 is expected to deviate from the Stackelberg strategy with high probability, then the normal type of player 1 can build a reputation rapidly by imitating the Stackelberg type.

The upper bound for player 2's payoff is calculated for a reputation level $z$ close to the reputation threshold $z_n$ in an equilibrium where player 2's resistance is approximately equal to the maximal resistance possible given the reputation level. The following formally defines maximal resistance for player 2.[8]

**Definition 3** (Maximal Resistance)**.** *For any $\xi > 0$, let $z_\xi = z_n - \xi$ and*

$$K_\xi = \sup\{k : \exists \text{ perfect Bayesian equilibrium } \sigma \text{ of } \Gamma^\infty(\mu, \delta), \text{ where } \mu_1(\Omega_s) = z \text{ and } \frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi, \text{ such that } R(\delta, \sigma_2) \geq k\epsilon \text{ for some } z \in [z_\xi, z_n]\}. \tag{5}$$

*Also, define $q_\xi$ such that*

$$\frac{z_\xi}{1 - q_\xi} = z_n. \tag{6}$$

[8]This further definition is required since it is not guaranteed that when $\mu_1(\Omega_s) = z_n$, there exists a perfect equilibrium where resistance equals $K^n\epsilon$. However, by the definition of the threshold $z_n$, for $z$ close to $z_n$ there exists a perfect equilibrium where resistance is close to $K^n\epsilon$.

Observe that by the definition of $K_\xi$, there exists $z \in [z_\xi, z_n]$ and an equilibrium strategy profile $\sigma$ such that $R(\delta, \sigma_2) \geq (K_\xi - \xi)\epsilon$. Also, by the definition of $K_\xi$ and the definition of $z_n$, $K_\xi \geq K^n$. The definition of $z_n$ and $K_\xi \geq K^n$ implies that for any $z_n \geq z \geq z_\xi$, $R(\delta, \sigma_2) \leq K_\xi \epsilon$ in any perfect Bayesian equilibrium strategy profile $\sigma$ of $\Gamma^\infty(\mu, \delta)$ where $\mu_1(\Omega_s) = z$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$. The following lemma establishes an upper bound on Player 2's payoff in any equilibrium where the resistance is at least $(K_\xi - \xi)\epsilon$.

**Lemma 3.** *Posit Assumption 1, and Assumption 2 for player 1. Pick any $z_n \geq z \geq z_\xi$ and perfect Bayesian equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$ with $\mu_1(\Omega_s) = z$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$, such that $R(\delta, \sigma_2) \geq (K_\xi - \xi)\epsilon$. For the chosen equilibrium profile $\sigma$,*

(UB)
$$U_2(\sigma) \leq \rho\epsilon n_1^p(q_\xi K_\xi + (q_n + q_\xi)K^n + K^{n-1} + \frac{5(1-\delta)}{\epsilon}) - z(K_\xi - \xi)\epsilon l + (1 - \delta + \phi)M.$$

*Proof.* Assumption 1 implies that there exists $a_1^p \in A_1$ such that $g_2(a_1^p, a_2) \leq \hat{g}_2$ for any $a_2 \in A_2$. This is the only use of Assumption 1 in the proof of this lemma. Consequently, Assumption 1 is redundant for generic strictly conflicting interest games for this lemma.

Pick $z_n \geq z \geq z_\xi$ and fix a perfect Bayesian equilibrium $\sigma = (\sigma_1, \sigma_2) = (\{\sigma_1(\omega)\}_{\omega\in\Omega}, \sigma_2)$ of the game $\Gamma^\infty(\mu, \delta)$ with $\mu_1(\Omega_s) = z$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$, such that $R(\delta, \sigma_2) \geq (K_\xi - \xi)\epsilon$. Let $\sigma_2^*$ denote a pure repeated game strategy for player 2 in the support of player 2's equilibrium strategy $\sigma_2$. For any strategy $\sigma_2^*$ in the support of $\sigma_2$ perfect equilibrium implies that $U_2(\sigma_1, \sigma_2^*|h_t) = U_2(\sigma_1, \sigma_2|h_t)$ for any $h_t$. Let

$$T = \min\{T : \Pr_{(\sigma_1, \sigma_2^*)}\{\exists t \leq T : (a_1)_t \neq \sigma_1(s|h_t)\} > q_\xi\}, \tag{7}$$

where $(a_1)_t \neq \sigma_1(s|h_t)$ denotes the event that player 1 plays an action that differs in *outcome* from the action played by the Stackelberg strategy given $h_t$ and the probability $\Pr_{(\sigma_1,\sigma_2^*)}$ is calculated assuming that player 2 uses pure strategy $\sigma_2^*$, player 1's types play according to profile $\sigma_1$ and the measure over player 1's types is given by $\mu$. In words, $T$ is the first period $t$ such that, the total probability with which player 1 is expected to deviate from the

Stackelberg strategy $\sigma_1(s|h_t)$ at least once, in any period $t \leq T$, exceeds $q_\xi$. By definition, for any $T' < T$, $\Pr_{(\sigma_1,\sigma_2^*)}\{\exists t \leq T' : (a_1)_t \neq \sigma_1(s|h_t)\} \leq q_\xi$. Also, let

$$T_n = \min\{T : \Pr_{(\sigma_1,\sigma_2^*)}\{\exists t \leq T : (a_1)_t \neq \sigma_1(s|h_t)\} > q_n + q_\xi\}. \tag{8}$$

The definition implies that $T_n \geq T$. Also, trivially, $\frac{z_\xi}{1-q_n-q_\xi} > \frac{z_\xi}{(1-q_n)(1-q_\xi)}$.

Suppose $\mu_1(\Omega_s) = z$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$. If player 1 has played according to $\sigma_1(s)$ in any history $h_t$ that is consistent with $\sigma_2^*$, then for any $t < T$, $\mu_1(\Omega_s|h_t) \geq z$ and $\frac{\mu_1(\Omega_-|h_t)}{\mu_1(\Omega_s|h_t)} \leq \phi$; for any $t \geq T$, $\mu_1(\Omega_s|h_t) \geq z_n$ and $\frac{\mu_1(\Omega_-|h_t)}{\mu_1(\Omega_s|h_t)} \leq \phi$; for $t \geq T_n$, $\mu_1(\Omega_s|h_t) \geq z_{n-1}$ and $\frac{\mu_1(\Omega_-|h_t)}{\mu_1(\Omega_s|h_t)} \leq \phi$.

For some period $t < T$, suppose that player 1 has always played an action compatible with $\sigma_1(s)$ in history $h_t$. After history $h_t$, $\mu_1(\Omega_s|h_t) \geq z$ and $\frac{\mu_1(\Omega_-|h_t)}{\mu_1(\Omega_s|h_t)} \leq \phi$. So, player 2's resistance after $h_t$ is at most $K_\xi\epsilon$, by Definition 3. So, by Lemma 1, $U_1(\sigma|h_t) \geq 1-n_1^p(K_\xi\epsilon+(1-\delta))$. Note that, $(U_1(\sigma|h_t), U_2(\sigma|h_t, \omega = \omega_N)) \in F$, so player 2's continuation payoff after history $h_t$ conditional on player 1 being the normal type, $U_2(\sigma|h_t, \omega = \omega_N)$, is at most $\rho(1 - U_1(\sigma|h_t))$, by Assumption 2 and Equation (1). Also, player 2's payoff in periods 0 through $t-1$ is at most $(1-\delta)M$ since player 1 has always played an action compatible with $\sigma_1(s)$ in history $h_t$. Consequently, player 2's repeated game payoff, if she is facing a normal type of player 1, and player 1 deviates from the Stackelberg strategy for the first time in period $t < T$ is as follows:

$$U_2(\sigma, h_t|\omega = \omega_N) \leq (1-\delta)M + \rho(1 - U_1(\sigma|h_t)) \leq (1-\delta)M + \rho n_1^p(K_\xi\epsilon + (1-\delta)), \tag{9}$$

where

$$U_2(\sigma, h_t|\omega = \omega_N) = \sum_{k=0}^{t-1}(1-\delta)\delta^k g_2((a_1,a_2)_k) + \delta^t U_2(\sigma|h_t, \omega = \omega_N),$$

and $\{(a_1,a_2)_0, ..., (a_1,a_2)_{t-1}\} = h_t$.

Suppose in $h_T$ player 1 has not deviated from the Stackelberg strategy, then player 1's equilibrium continuation payoff $U_1(\sigma|h_T)$ must be at least as large as $\delta^{n_1^p}(1 - n_1^p(K^n\epsilon + (1-\delta))$. This is because player 1 can mimic $\sigma_1(s)$ for the next $n_1^p$, receive at least zero in these periods by normalization $(ii)$, increase his reputation to at least $z_n$ and thereby guarantee a continuation payoff of at least $1 - n_1^p(K^n\epsilon + (1-\delta))$ by Lemma 1. This implies that player 2's continuation payoff is at most $\rho(1-(1-n_1^p(K^n\epsilon + 2(1-\delta)))$. So, if player 2 is facing the normal type of player 1, and player 1 deviates from the Stackelberg strategy in period $t = T$, then player 2's repeated game payoff

$$U_2(\sigma, h_t|\omega = \omega_N) \leq (1-\delta)M + \rho n_1^p(K^n\epsilon + 2(1-\delta)). \tag{10}$$

For any period, $T < t < T_n$, suppose in $h_t$ player 1 has not deviated from the Stackelberg strategy. If player 2 is facing the normal type of player 1, and player 1 deviates from the Stackelberg strategy in period $T < t < T_n$, then player 2's repeated game payoff $U_2(\sigma, h_t|\omega = \omega_N) \leq (1-\delta)M + \rho n_1^p(K^n\epsilon + (1-\delta))$.

Suppose in history $h_{T_n}$ player 1 has not deviated from the Stackelberg strategy. In period $T_n$, if player 1 plays according to $\sigma_1^s$, then his reputation will exceed $z_{n-1}$ in the next period. Consequently, by the same reasoning as in period $T$, if player 2 is facing the normal type of player 1, and player 1 deviates from the Stackelberg strategy in period $t = T_n$, then player 2's repeated game payoff

$$U_2(\sigma, h_t|\omega = \omega_N) \leq (1-\delta)M + \rho n_1^p(K^{n-1}\epsilon + 2(1-\delta)). \tag{11}$$

For any period, $t > T_n$, suppose in $h_t$ player 1 has not deviated from the Stackelberg strategy. If player 2 is facing the normal type of player 1, and player 1 deviates from the Stackelberg strategy in period $t > T_n$, then player 2's repeated game payoff, $U_2(\sigma, h_t|\omega = \omega_N) \leq (1-\delta)M + \rho n_1^p(K^{n-1}\epsilon + 1 - \delta)$.

Player 2 can get at most $M$ against any other commitment type and this happens with probability $\phi z \leq \phi$. Since player 2's resistance is $(K_\xi - \xi)\epsilon$ in the equilibrium under consideration, she loses $(K_\xi - \xi)\epsilon l$ against the Stackelberg type, and this happens with probability $z$. The probability that player 1 is a normal type and takes action $a_1 \neq \sigma_1(s|h_t)$ for the first time in any period $t < T$ is at most $q_\xi$; and an upper-bound on player 2's repeated game payoff, conditional on this event, is given by Equation (9). The probability that player 1 is a normal type and takes action $a_1 \neq \sigma_1(s|h_t)$ for the first time in any period $T \leq t < T_n$ is at most $q_\xi + q_n$; and an upper-bound on player 2's payoff is given by Equation (10). Finally, the probability that player 1 is a normal type and takes action $a_1 \neq \sigma_1(s|h_t)$ for the first time in any period $T_n \leq t$ is at most $1 - z < 1$; and an upper-bound on player 2's payoff is given by Equation (11). Consequently,

$$U_2(\sigma) \leq q_\xi \rho n_1^p K_\xi \epsilon + (q_\xi + q_n)\rho n_1^p K^n \epsilon + \rho n_1^p K^{n-1}\epsilon - z(K_\xi - \xi)\epsilon l + 5\rho n_1^p(1-\delta) + (1-\delta+\phi)M$$

delivering the required inequality. Observe that if $T = \infty$, then the bound is still valid. □

Although the previous lemma was stated for perfect Bayesian equilibria, since all considered histories were on an equilibrium path, perfection was not needed for the result. In contrast, in the following lemma, which establishes a lower bound for player 2's equilibrium payoffs, both perfection and Assumption 1 are crucial. In order to bound payoffs in a particular equilibrium, the lemma considers an alternative strategy for player 2 that plays $a_1^b$, as long as Player 1 plays according to the Stackelberg strategy, and reverts back to playing according to the equilibrium strategy once Player 1 deviates from the Stackelberg strategy. The argument then finds a lower bound for player 1's payoff, using Lemma 1, and converts this into a lower bound for player 2. Since the alternative strategy considered for player 2 may generate a history that has zero probability on the equilibrium path, the argument for player 1's lower bound hinges on both perfection and perfect information (Assumption 1).

**Lemma 4.** *Posit Assumption 1 and Assumption 2 for player 1. Suppose that $z_n \geq z \geq z_\xi$ and that $\mu_1(\Omega_s) = z$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$. In any perfect Bayesian equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$,*

$$U_2(\sigma) \geq -\rho\epsilon n_1^p\left(q_\xi K_\xi + (q_\xi + q_n)K^n + K^{n-1} + \frac{6(1-\delta)}{\epsilon}\right) - \phi M. \tag{LB}$$

*Proof.* Fix a perfect Bayesian equilibrium $\sigma$ of $\Gamma^\infty(\mu, \delta)$ where $z_n \geq z \geq z_\xi$, $\mu_1(\Omega_s) = z$ and $\frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)} \leq \phi$. If $\Gamma$ is a generic game of strictly conflicting interest for player 1 (Assumption

2 ($i$) and ($iii$)), then $U_2(\sigma) \geq \hat{g}_2 = g_2(a_1^s, a_2^b) = 0$ in any Bayes-Nash equilibrium which exceeds the right-hand side of Equation (LB).

Posit Assumption 1 and Assumption 2 ($i$) and ($ii$) for player 1. We calculate the payoff of player 2 if she deviates and uses the following alternative repeated game strategy $\sigma_2^*$. Suppose that player 2 always plays $a_2^b$, a pure action, if player 1 has played the Stackelberg strategy $\sigma_1(s)$ in every prior node of the repeated game and plays according to the equilibrium strategy $\sigma_2$ if player 1 has deviated from the Stackelberg strategy $\sigma_1(s)$ in a prior node of the repeated game. Using this strategy player 2 will receive payoff equal to zero in any period where player 1 plays $a_1(s)$. Let strategy profile $\sigma^* = (\sigma_1, \sigma_2^*)$.

Suppose that, $T$ and $T_n$ are defined as in Lemma 3, given that player 2 uses strategy $\sigma_2^*$. If player 1 has played according to $\sigma_1(s)$ in any history $h_t$ compatible with $\sigma_2^*$, then for any $t < T$, $\mu_1(\Omega_s|h_t) \geq z$ and $\frac{\mu_1(\Omega_-|h_t)}{\mu_1(\Omega_s|h_t)} \leq \phi$; for any $t \geq T$, $\mu_1(\Omega_s|h_t) \geq z_n$ and $\frac{\mu_1(\Omega_-|h_t)}{\mu_1(\Omega_s|h_t)} \leq \phi$; for $t \geq T_n$, $\mu_1(\Omega_s|h_t) \geq z_{n-1}$ and $\frac{\mu_1(\Omega_-|h_t)}{\mu_1(\Omega_s|h_t)} \leq \phi$.

For period $t < T$, suppose that player 1 has always played an action compatible with $\sigma_1(s)$ and player 2 has played $a_2^b$ in history $h_t$ and player 1 deviates from the Stackelberg strategy in period t. At any information set where player 1 deviates from the Stackelberg strategy in period $t$, he can instead play according to $\sigma_1(s)$ for $n_1^p$ periods, get at least zero in these periods, and receive $1 - n_1^p(K_\xi \epsilon + (1-\delta))$ as a continuation payoff from period $n^p$ onwards by Lemma 1. Consequently, $\mathbb{E}_{\sigma^*}[(1-\delta)g_1(a_1,a_2) + \delta U_1(\sigma|h_t, a_1, a_2)|h_t, a_1 \neq a_1^s] \geq \delta^{n_1^p}(1 - n_1^p(K_\xi \epsilon + (1-\delta)))$ where the expectation is taken with respect to $(a_1, a_2)$ using repeated game strategy profile $\sigma^*$ conditioning on player 1 deviating from the Stackelberg strategy in period $t$.[9] Perfection is required for this inequality because the history $h_t$ is not necessarily on the equilibrium path. Perfect information (Assumption 1) is also required here since Player 2 may have played $a_2^b$ in period $t$ and this may have probability zero on the equilibrium path.[10] If player 1 is the normal type and deviates from the Stackelberg

[9] $(a_1)_t \neq a_1^s$ denotes the event that player 1 plays an action that differs in *outcome* from the action played by the Stackelberg strategy, i.e., $a_1^s$.

[10] Observe that in the Common Interest game example discussed at the beginning of this section, without the perfect information assumption, this bound on Player 1's payoffs is not valid. This is because in the first K periods, Player 1 does not expect to see action $L$ on the equilibrium path. So, the continuation payoff

strategy for the first time in period $t$, then player 2's continuation payoff

$$U_2(\sigma^*|h_t, a_1 \neq a_1^s, \omega = \omega_N) = \mathbb{E}_{\sigma^*}[(1-\delta)g_2(a_1, a_2) + \delta U_2(\sigma|h_t, a_1, a_2)|h_t, a_1 \neq a_1^s]$$
$$\geq -\rho(1 - U_1(\sigma^*|h_t, a_1 \neq a_1^s))$$

because $(\mathbb{E}_{\sigma^*}[(1-\delta)g_1(a_1, a_2) + \delta U_1(\sigma|h_t, a_1, a_2)|h_t, a_1 \neq a_1^s], U_2(\sigma^*|h_t, a_1 \neq a_1^s, \omega = \omega_N)) \in F$ and Equation (1). Player 2's payoff for periods 0 through $t-1$ is at least zero since player 1's action in each of these periods is $a_1^s$ in history $h_t$ and player 2 plays $a_2^b$. Consequently, if player 2 is facing a normal type of player 1, and player 1 deviates from the Stackelberg strategy for the first time in period $t < T$, then her repeated game payoff $U_2(\sigma^*, h_t|\omega = \omega_N) \geq -\rho(1 - U_1(\sigma|h_t)) \geq -\rho n_1^p(K_\xi \epsilon + 2(1-\delta))$.

For any period, $T \leq t < T_n$, suppose in $h_t$ player 1 has not deviated from the Stackelberg strategy and deviates from the Stackelberg strategy in period $t$, then player 2's repeated game payoff $U_2(\sigma^*, h_t|\omega = \omega_N) \geq -\rho n_1^p(K^n \epsilon + 2(1-\delta))$. For any period, $t \geq T_n$, suppose in $h_t$ player 1 has not deviated from the Stackelberg strategy and deviates from the Stackelberg strategy in period $t$, then player 2's repeated game payoff $U_2(\sigma^*, h_t|\omega = \omega_N) \geq -\rho n_1^p(K^{n-1}\epsilon + 2(1-\delta))$.

Player 2 can get at least $-M$ against any other commitment type with probability at most $\phi$, gets zero against the Stackelberg type with probability at most $z$. Following an identical reasoning as in Lemma 3 for the events that player 1 is a normal type and deviates from the Stackelberg type for the first time at time $t < T$ or time $T \leq t < T_n$ or time $t \geq T_n$ implies that

$$U_2(\sigma) \geq U_2(\sigma^*) \geq -\rho n_1^p K_\xi \epsilon q_\xi - \rho n_1^p K^n \epsilon(q_\xi + q_1) - \rho n_1^p K^{n-1}\epsilon - 6\rho n_1^p(1-\delta) - \phi M$$

delivering the required inequality. Observe that if $T = \infty$ the bound is still valid □

after $(D, L)$ can be arbitrarily chosen since Player 1 has revealed rationality. With perfect information, in contrast, player 1 knows that player 2 has played L, and so the continuation payoff associated to revealing rationality must be greater than always playing $U$.

**Lemma 5.** *Let* $\underline{q} = \underline{z}(\frac{l}{2\rho n_1^p} - \frac{7}{\underline{z}K} - \frac{2M}{\underline{z}\rho n_1^p K})$ *and pick* $K$ *such that* $\underline{q} > 0$*. If* $z_n \geq \underline{z}$*, then* $q_n \geq \underline{q}$*, for all* $\delta$ *and* $\phi$*.*

*Proof.* Combining the lower bound for $U_2(\sigma)$, given by Equation (LB) established in Lemma 4, and the upper bound for $U_2(\sigma)$, given by Equation (UB) established in Lemma 3, and simplifying by canceling $\epsilon$ delivers

$$z(K_\xi - \xi)l \leq 2\rho n_1^p(q_\xi K_\xi + (q_\xi + q_n)K^n + K^{n-1} + \frac{6(1-\delta)}{\epsilon}) + \frac{2(1-\delta+\phi)M}{\epsilon}.$$

Taking $\xi \to 0$ implies that $z \to z_n$, $q_\xi \to 0$. Also, $K_\xi \geq K^n$ for each $\xi$ implies that $\lim_{\xi\to 0}(K_\xi - \xi) = \lim_{\xi \to 0} K_\xi \geq K^n$. Consequently,

$$z_n K^n l \leq 2\rho n_1^p(q_n K^n + K^{n-1} + \frac{6(1-\delta)}{\epsilon}) + \frac{2(1-\delta+\phi)M}{\epsilon}.$$

Rearranging, $q_n \geq \frac{z_n l}{2\rho n_1^p} - \frac{1}{K} - \frac{6(1-\delta)}{\epsilon K^n} - \frac{(1-\delta+\phi)M}{\epsilon \rho n_1^p K^n}$. Recall that $\epsilon = \max\{1-\delta, \phi\}$ and $z_n \geq \underline{z}$ so

$$q_n \geq \underline{z}(\frac{l}{2\rho n_1^p} - \frac{7}{\underline{z}K} - \frac{2M}{\underline{z}\rho n_1^p K}) = \underline{q} > 0$$

delivering the required inequality. □

Given Lemma 5, Lemma 2 can be applied to complete the proof of Theorem 1.

*Proof of Theorem 1.* Pick $1-\underline{\delta} < \frac{\gamma}{K^{n^*} n_1^p}$ and pick $\bar{\phi} < \frac{\gamma}{K^{n^*} n_1^p}\underline{z}$. By Lemma 5 if $z_n \geq \underline{z}$, then, for all $\delta$, $\phi$, $q_n \geq \underline{q}$. Consequently, by Lemma 2, $\max\{1-\delta, \frac{\mu_1(\Omega_-)}{\mu_1(\Omega_s)}\} < \frac{\gamma}{K^{n^*} n_1^p}$ implies that $U_1(\sigma) > 1-\gamma$ for all perfect Bayesian equilibria $\sigma$ of $\Gamma^\infty(\mu,\delta)$ with $\mu_1(\Omega_s) \geq \underline{z}$, $\mu_1(\Omega_-) < \bar{\phi}$ and $\delta > \underline{\delta}$. □

As first demonstrated by Cripps, Dekel, and Pesendorfer (2005), it is possible to obtain reputation results for simultaneous-move strictly conflicting interest stage games, i.e., games that do not satisfy Assumption 1. The following corollary to Theorem 1 maintains Assumption 2 $(i)$ and $(iii)$, and shows that Player 1 can guarantee a payoff arbitrarily close to $\bar{g}_1$ even without Assumption 1. Consequently, this corollary provides an alternative argument for Cripps, Dekel, and Pesendorfer (2005).

**Corollary 1.** *Posit Assumption 2 (i) and (iii) for player 1. For any* $\underline{z}$ *and* $\gamma > 0$*, there exists a* $\underline{\delta} < 1$ *and* $\bar{\phi} > 0$ *such that, for any* $\delta \in (\underline{\delta}, 1)$*, any* $\mu$ *with* $\mu_1(\Omega_s) \geq \underline{z}$ *and* $\mu_1(\Omega_-) < \bar{\phi}$ *and any Bayes-Nash equilibrium strategy profile* $\sigma$ *for the repeated game* $\Gamma^\infty(\mu, \delta)$*,* $U_1(\sigma) > \bar{g}_1 - \gamma$*.*

*Proof.* Redefine $z_n$ and $q_n$ in Definition 2 and $z_\xi$, $K_\xi$ and $q_\xi$ in Definition 3 using Bayes-Nash equilibrium instead of perfect Bayesian Equilibrium. The upper-bound given by Equation (UB) established in Lemma 3 remains valid for Bayes-Nash equilibria. This is because all the arguments were constructed on an equilibrium path without any appeals to perfection. Also, $U_2(\sigma) \geq \hat{g}_2 = 0$ in any Bayes-Nash equilibrium, by Lemma 4. Consequently, Lemma 2, which also holds in Bayes-Nash equilibria, implies the result. □

3.2. **Uniformly Learnable Types.** The analysis in this subsection restricts the set of commitment types to learnable types, and shows that player 1 can guarantee payoff close to $\bar{g}_1$, for arbitrary probability distributions over commitment types, if the discount factor is sufficiently large. The intuition for the result (Corollary 2) is as follows: if the other commitment types are uniform learnable, then player 1 can repeatedly play the Stackelberg action and ensure that player 2's posterior belief that player 1 is a type in $\Omega_-$ is arbitrarily small in finitely many periods (Lemma 7). However, if player 2's posterior belief that player 1 is a type in $\Omega_-$ is small, then Theorem 1 implies that player 1's payoff is close to $\bar{g}_1 = 1$ for sufficiently large discount factors.

If a uniformly learnable type is not the Stackelberg type, then that type must reveal itself not to be the Stackelberg type, at a rate that is bounded away from zero, uniformly in histories, by the definition given below. The restriction to uniformly learnable types rules out perverse types that may punish player 2 for learning. For example, consider a type that always plays according to the Stackelberg strategy, if player 2 plays an action $a_2$ in period where $a_1^s$ is played such that $g_1(a_1^s, a_2) < g_1(a_1^s, a_2^b)$; and minimaxes player 2 forever, if player 2 plays an action $a_2$ in a period where $a_1^s$ is played such that $g_1(a_1^s, a_2) = g_1(a_1^s, a_2^b)$. This perverse type is not uniformly learnable because after any history where player 1 has played the Stackelberg strategy, and player 2 has played an action different than $a_2^b$, the perverse type never reveals and so the revelation rate is not bounded away from zero. The following is the formal definition of uniformly learnable types.

**Definition 4** (Uniformly Learnable Types)**.** *A type $\omega$ is uniformly learnable with respect to $s$ if there exists $\varepsilon_\omega > 0$ such that, after any history $h_l$ where $\sigma_1(s|h_l)_l = a_1^s$, either $\Pr_{\sigma_1(\omega)}((a_1)_l \neq a_1^s|h_l) > \varepsilon_\omega$; or there is an $h_t = \{h_l, (a_1^s, a_2)_l, ..., (a_1, a_2)_{t-1}\}$, where $l < t \leq l + n_1^p - 1$, $(a_2)_l \neq a_2^b$ and $(a_1)_k = a_1^p$ for $l < k < t$, such that $\Pr_{\sigma_1(\omega)}((a_1)_t \neq a_1^p|h_t) > \varepsilon_\omega$; or $\Pr_{\sigma_1(\omega)}((a_1)_t \neq \sigma_1(s|h_t)_t|h_t) = 0$ for all $t \geq l$.*

After any history, a uniformly learnable type deviates from the Stackelberg strategy with probability $\varepsilon_\omega$ either during the phase where $a_1^s$ is played or during the $n_1^p - 1$ period punishment phase that potentially follows; or always plays according to the Stackelberg strategy. Lemma 7, established in the Appendix, shows under the uniformly learnable types assumption there exists a period $T$ such that if player 1 repeatedly plays according to $\sigma_1(s)$ in history $h_T$, then the probability that player 1 is a type that is different than the Stackelberg type is small, with high probability. Applying the lemma delivers the following corollary to Theorem 1.

**Corollary 2.** *Posit Assumption 1 and Assumption 2 for player 1. Assume that each $\omega \in \Omega_-$ is uniformly learnable. For any $\underline{z}$ and $\gamma > 0$, there exists a $\underline{\delta} < 1$ such that, for any $\delta \in (\underline{\delta}, 1)$ and any perfect equilibrium strategy profile $\sigma$ for the repeated game $\Gamma^\infty(\mu, \delta)$ with $\mu_1(\Omega_s) \geq \underline{z}$, $U_1(\sigma) > \bar{g}_1 - \gamma$.*

*Proof.* Pick $\phi$ such that $n_1^p K^{n^*}\phi + \phi < \gamma$ where $K$ and $n^*$ are defined as in Theorem 1. By Lemma 7 there exists $T$ such that $\frac{\mu_1(\Omega_-(h_T)|h_T)}{\mu_1(\Omega_s(h_T)|h_T)} < \phi$ with probability $1-\phi$ if player 1 played according to $\sigma_1(s)$ in history $h_T$ and there were $N$ periods in which $(a_1^s, a_2)$ where $a_2 \neq a_2^b$ was played. Consequently, by Theorem 1, $U_1(\sigma|h_T) > 1 - n_1^p(K^{n^*} \max\{1-\delta, \phi\} + (1-\delta))$, with probability $1 - \phi$. Suppose that player 1 plays according to $\sigma_1(s)$ and let $\tau$ denote the first (random) time such that in history $h_\tau$ there are $N$ periods in which $(a_1^s, a_2)$ where

$a_2 \neq a_2^b$ was played. If player 1 uses strategy $\sigma_1(s)$, then before time $\tau$ there are $N$ periods in which $(a_1^s, a_2 \neq a_2^b)$ is played, after time $\tau$, Theorem 1 and Lemma 7 imply that, with probability $1-\phi$, there are at most $K^{n^*} \max\{1-\delta, \phi\}$ periods in which $(a_1^s, a_2 \neq a_2^b)$ is played. Because strategy $\sigma_1^s$ is available to player 1,

$$\begin{aligned} U_1(\sigma) &\geq (1-\delta^\tau) - n_1^p N(1-\delta) + \delta^\tau(1-\phi)(1 - n_1^p(K^{n^*} \max\{1-\delta, \phi\} + (1-\delta)) \\ &\geq 1 - n_1^p(K^{n^*} \max\{1-\delta, \phi\} + (N+1)(1-\delta)) - \phi \end{aligned}$$

So, if $\delta > \underline{\delta} = \max\{\frac{\gamma - n_1^p K^{n^*}\phi - \phi}{(N+1)n_1^p}, 1-\phi\}$, then $U_1(\sigma) > 1-\gamma$. □

3.3. **Examples with no Reputation Effects.** In the non-generic version of the common interest game outlined in Figure 4 (left) suppose that player 2 is building a reputation and $\mu_2(\Omega_s) < \epsilon$. Suppose that the Stackelberg type of player 2 always plays $L$. Consider the following equilibrium construction: on the equilibrium path player 1 plays $R$ and the normal type of player 2 plays $L$ for K periods. After period $K$ they play $(L, L)$. Choose $K$ such that both players receive 1/2. Notice that no reputation is built on the equilibrium path. Also, suppose that the normal type of player 2 always plays $R$ if player 1 deviates from $R$ on the equilibrium path. Once player 2 plays $R$ she is known to be the normal type and the stage-game equilibrium $(L, R)$ is played forever. Consequently, player 1 receives $\epsilon$ if he deviates from the equilibrium strategy which is less than 1/2.[11] In the game depicted in Figure 4 (right) suppose that player 1 is building a reputation. Player 2 always playing $R$ is a dominant action and so is a best response to any type of player 1.

## 4. Two-Sided Reputation and a War of Attrition

The main finding presented in this section, Theorem 2, establishes a two-sided reputation result. Throughout the discussion assume that $\mu_1(\Omega_s) = z_1 > 0$, $\mu_2(\Omega_s) = z_2 > 0$ and $\mu_i(\Omega_-) = 0$, that is, for both players, the probability of being the Stackelberg type is positive and the probability of being another commitment type is zero. Set $\delta = e^{-r\Delta}$.

[11] If player 2's Stackelberg type always plays $R$, then the normal type reveals by playing $L$ and then switches to the equilibrium where player 1 receives $\epsilon$.

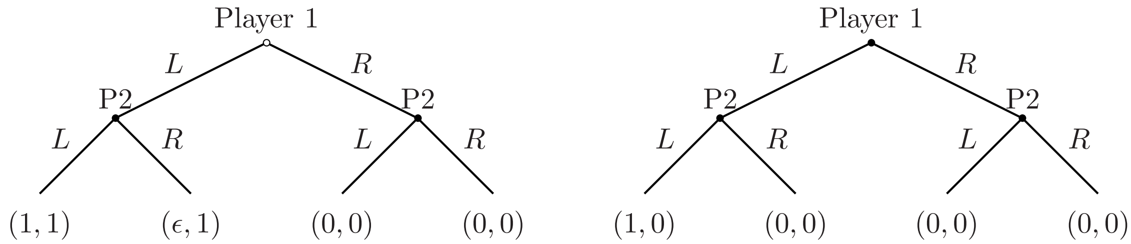


FIGURE 4. On the left is a non-generic common interest game. This game satisfies Assumption 2 (*i*) and (*iii*) or (*iv*) but does not satisfy (*ii*) (genericity). On the right is a conflicting interest game. This game satisfies Assumption 2 (*i*) and (*ii*) but does not satisfy (*iii*) or (*iv*).

The focus of analysis is on sequences of repeated games $\Gamma^\infty(\mu, \Delta_n)$ parameterized by the time $\Delta_n$ between repetitions of the stage game. Theorem 2 maintains Assumption 1 and Assumption 2 for both players and demonstrated that as $\Delta_n \to 0$, any sequence of perfect equilibrium payoff profiles $\{(U_1(\sigma_n), U_2(\sigma_n))\}$ of the repeated game $\Gamma^\infty(\mu, \Delta_n)$ converges to a unique limit. This unique limit is the unique equilibrium payoff profile of a continuous time war of attrition which is outlined below. Corollary 3 shows that the perfect information assumption (Assumption 1) can be discarded if the stage game is a generic game of strictly conflicting interest (Assumption 2 (*i*) and (*iii*)).

In what follows we *assume that* $\sigma_1(s)$ *is not a best response to* $\sigma_2(s)$.[12] This assumption is maintained out of convenience in order to focus attention on the interesting cases. If $\sigma_1(s)$ is a best response to $\sigma_2(s)$, then Theorem 1 immediately implies that $(U_1(\sigma_n), U_2(\sigma_n))$ converges to the unique payoff profile $(\bar{g}_1, g_2)$ for any sequence of perfect equilibria $\sigma_n$ of $\Gamma^\infty(\mu, \Delta_n)$.[13] For the remainder of the discussion normalize payoffs, without loss of generality such that,

(*i*) $\bar{g} = \bar{g}_1 = \bar{g}_2$

(*ii*) $g_1(a_1^b, a_2^s) = g_2(a_1^s, a_2^b) = 0$

[12]For games that satisfy Assumption 2, $\sigma_1(s)$ is a best response to $\sigma_2(s)$ if and only if $\sigma_2(s)$ is a best response to $\sigma_1(s)$.

[13]Observe that given the definition of resistance, the lower bound for player 1's payoff in Lemma 1 is valid. This is because the Stackelberg type of Player 2 best responds to the Stackelberg strategy of Player 1. Also, the Equations (LB) and (UB) are unchanged. So, the theorem implies that $R(\delta, \sigma_2)$ converges to zero. So, $(U_1(\sigma_n), U_2(\sigma_n)) \to (\bar{g}_1, g_2)$.

The Stackelberg types under consideration are $n_i^p$ state automatons. To deal with this we introduce the following device: Let $n^p$ denote the smallest common multiple of $n_1^p$ and $n_2^p$; and let

$$l_1 = -\frac{1}{n^p}\sum_{t=0}^{n^p-1} g_1(\sigma_1(s)_t, \sigma_2(s)_t).$$

where $(\sigma_1(s)_t, \sigma_2(s)_t)$ is the action profile in period $t$ if the players use strategy profile $(\sigma_1(s), \sigma_2(s))$ starting from period 0. Also, define $l_2$ symmetrically. So $l_i$ is the average loss incurred from playing the Stackelberg strategy against the Stackelberg strategy.[14] Note that for $n^p > 1$, the difference between using the time average instead of $l_i$ is bounded as follows:

$$|\sum_{t=0}^{n^p-1} e^{-r\Delta t} l_i - \sum_{t=0}^{n^p-1} e^{-r\Delta t} g_1(\sigma_1(s)_t, \sigma_2(s)_t)| \leq M n^p (1 - e^{-r\Delta(n^p-1)}). \tag{12}$$

Consequently, the error from using $l_i$ as the cost of employing the Stackelberg strategy against the Stackelberg strategy converges to zero as $\Delta \to 0$.

4.1. **The War of Attrition.** The War of Attrition is played over continuous time by the two players. At time zero, both players simultaneously choose either to concede or insist. If both players choose to insist, then the continuous time war ensues. The game continues until one of the two players concedes. Each player can concede at any time $t \in [0, \infty]$. If player $i$ concedes at time $t \in [0, \infty]$ and player $j$ continues to play insist through time $t$, then player $i$'s payoff is $-l_i\left(1 - e^{-rt}\right)$ and player $j$'s payoff is $e^{-rt}\bar{g} - l_j\left(1 - e^{-rt}\right)$. If both players concede concurrently at time $t$, then they receive payoff $e^{-rt} g_i(t) - l_i\left(1 - e^{-rt}\right)$ and $e^{-rt} g_j(t)$ $-l_j\left(1 - e^{-rt}\right)$ where $(g_i(t), g_j(t)) \in G$, and consequently, $-\rho(\bar{g} - g_j(t)) \leq g_i(t) \leq \rho(\bar{g} - g_j(t))$. Before the game begins at time 0, nature chooses a type for each player independently. A player is chosen as either a Stackelberg type that never concedes, with probability $z_i > 0$, or a normal type, with probability $1 - z_i$.

This War of Attrition is closely related to the repeated game $\Gamma^\infty(\mu, \Delta)$ for $\Delta \approx 0$: Insisting corresponds to playing according to the Stackelberg strategy in each period, and

[14] To be exact $l_1 = -\frac{1}{n^p} g_1(a_1^s, a_2^s) + (\frac{n^p}{n_1^p} - 1) g_1(a_1^s, a_2^p) + (\frac{n^p}{n_2^p} - 1) g_1(a_1^p, a_2^s) + (n_p + 1 - \frac{n^p}{n_1^p} - \frac{n^p}{n_2^p}) g_1(a_1^p, a_2^p)$.

conceding corresponds deviating from the Stackelberg strategy. Players incur cost $l_i$ from insisting on the Stackelberg strategy against the Stackelberg strategy. They insist on the Stackelberg strategy hoping that their rival will deviate. If one of the players deviates from the Stackelberg strategy and the other does not, then the player that deviated is known to be the rational type with certainty. After such a history, Theorem 1 implies that the player known as normal receives a payoff of zero and the rival receives payoff $\bar{g}$. This corresponds exactly to the payoffs when one of the players conceding at time $t$ in the War of Attrition. Both players incur the cost $l_i$ for $t$ units of time, i.e., $l_i\left(1-e^{-rt}\right)$; the conceding player receives continuation payoff of zero; and the player that wins receives continuation payoff of $\bar{g}$, i.e., $e^{-rt}\bar{g}$. If both players reveal rationality concurrently in period $t$, that is, if both players play concede in the War of Attrition in period $t$, then Theorem 1 puts no restrictions on continuation payoffs. So, agents receive an arbitrary payoff from the set of individually rational and feasible repeated game payoffs.

The War of Attrition outlined above differs from the game analyzed in Abreu and Gul (2000). In the War of Attrition presented here, the payoffs that the players receive, if they concede concurrently, depend on $t$ and are potentially non-stationary. In contrast, in Abreu and Gul (2000) concurrent concessions involve stationary payoffs. Nevertheless, the argument below (for condition $(i)$) shows that the non-stationarity of payoffs does not introduce any new complications and Abreu and Gul (2000) 's analysis applies without alteration. In particular, the unique equilibrium of the War of Attrition satisfies three conditions: $(i)$ at most one agent concedes with positive probability at time zero, $(ii)$ after time zero each player concedes with constant hazard rate $\lambda_i$, $(iii)$ the normal types finish conceding at some finite time $T$. Consequently, at time $T$ the posterior probability that an agent faces a Stackelberg type equals one.

In order to provide a rationale for condition $(i)$ suppose that both players were to concede with positive probability at time $t$. If they concede concurrently, then player 1's payoff is $g_1(t)$ and player 2's payoff is $g_2(t)$. By Assumption 2, $g_i(t) \leq \rho(\bar{g} - g_j(t))$. Consequently, for one of the two players $g_i(t) < \bar{g}$. But for this player $i$ waiting to see whether player $j$

quits at time $t$ and then quitting immediately afterwards does strictly better than quitting at time $t$. Consequently, both players cannot concede with positive probability at any time $t$; and in particular, at most one of the players can concede with positive probability at time zero.

For some insight into condition $(ii)$ note that player $i$ cannot concede with certainty at any time. If player $i$ were to concede with certainty at time $t$, then by not conceding player $i$ would ensure that player $j$ believes that player $i$ is the Stackelberg type with probability one. But this would induce player $j$ to concede immediately improving $i$'s payoff. Consequently, condition $(ii)$ implies that the hazard rate $\lambda_i$ must leave $j$ indifferent between conceding immediately and waiting for an additional $\triangle$ units of time and then conceding. Conceding immediately guarantees player $j$ zero. By waiting for $\triangle$ units of time player $j$ incurs cost $l_j(1-e^{-r\triangle})$, but receives $\bar{g}$ if $i$ quits which happens with probability $\triangle\ \lambda_i$. Consequently, $0 = \lim_{\triangle\to 0} \triangle\ \lambda_i \bar{g} - l_j(1-e^{-r\triangle})$ and so $\lambda_i = \lim_{\triangle\to 0} \frac{(1-e^{-r\triangle})l_j}{\bar{g}\triangle} = \frac{rl_j}{\bar{g}}$.

Once one of the players' normal type has finished conceding and the player is known as the Stackelberg type with certainty, then the normal type of the other player should also concede immediately. Because a Stackelberg type never concedes, the normal player has no incentive to insist. Consequently, condition $(iii)$ holds and both players complete conceding by the same finite time $T$. Conditions $(i)$ through $(iii)$ imply the following lemma:

**Lemma 6.** *Let $F_i(t)$ denote the cumulative probability that player $i$ concedes by time $t$, that is, $F_i(t)$ is $1-z_i$ multiplied by the probability that the normal type of player $i$ quits by time $t$. Let $\lambda_i = \frac{rl_j}{\bar{g}}$, $T_i = \frac{-\ln z_i}{\lambda_i}$, $T = \min\{T_1, T_2\}$ and $c_i \in [0,1]$, then*

$$F_i(t) = 1 - c_i e^{-\lambda_i t} \text{ for all } t \le T < \infty, \text{ and } F_i(T) = 1 - z_i,$$

*where $1-c_i$ is the probability that player $i$ concedes at time 0, and $1-c_i > 0$ if and only if $T_i > T_j$. The unique sequential equilibrium of the War of Attrition is $(F_1, F_2)$. Also, the unique equilibrium payoff vector for the War of Attrition is $((1-c_2)\bar{g}, (1-c_1)\bar{g})$.*

*Proof.* Observe if $F_1$ jumps at time $t$, then $F_2$ does not jump at time $t$. This follows from the argument provided for condition $(i)$. The rest of the argument in Abreu and Gul (2000) applies verbatim. Thus $F_1$ and $F_2$ comprise the unique equilibrium for the War of Attrition.

Suppose that player 1 is the player that concedes with positive probability at time zero. Since player 1 concedes with positive probability at time zero, he is indifferent between conceding immediately and receiving a payoff equal to zero and continuing. Consequently, player 1's equilibrium payoff must equal zero. Player 2 is also indifferent between quitting and conceding at any time after time zero. This implies that player 2's expected payoff at time $t > 0$, conditional on neither player conceding by time $t$, is equal to zero. Consequently, player 2's equilibrium payoff at the start of the game must equal $(1 - c_1)\bar{g}$. □

4.2. **The Main Two Sided Reputation Result.** Let $G_1^n(t)$ denote the cumulative probability that player 1 reveals that he is rational by time $t$, if he is playing against the Stackelberg type of player 2, in an equilibrium $\sigma$ of the repeated game $\Gamma^\infty(\mu, \Delta^n)$. Theorem 2 demonstrates that the distributions $G_i^n$ have a limit and proves that this limiting distribution solves the War of Attrition and is thus equal to $F_i$. The Theorem then proceeds to show that convergence of $G_i^n$ to $F_i$ implies that the equilibrium payoffs in the repeated game also converge to the unique equilibrium payoff for the War of Attrition.

**Theorem 2.** *Suppose that $\sigma_1(s)$ is not a best response to $\sigma_2(s)$. Also, $\Gamma$ satisfies Assumption 1 and Assumption 2 for both players. For any $z = (z_1 > 0, z_2 > 0)$ and any $\epsilon > 0$, there exists a $\Delta^* > 0$ such that, for any $\Delta < \Delta^*$, any equilibrium strategy profile $\sigma$ for the repeated game $\Gamma^\infty(\mu, \Delta)$ with $\mu_i(\Omega_s) = z_i$ and $\mu_i(\Omega_-) = 0$, $|U_1(\sigma) - (1 - c_2)\bar{g}| < \epsilon$ and $|U_2(\sigma) - (1 - c_1)\bar{g}| < \epsilon$.*

4.2.1. *Comparative Statics.* The unique limit equilibrium payoff for the repeated game is $(1-c_2)\bar{g}$ and $(1-c_1)\bar{g}$, by Theorem 2. Suppose that $1-c_2 = 0$ hence $T_1 > T_2$. Since player 2 never concedes at time zero, player 1's payoff is equal to zero, and player 2's payoff is equal to $(1 - c_1)\bar{g}$. Hence, player 1 is the weaker player. The identity of the agent who quits with positive probability at time zero, player 1 in this case, is determined by the quitting time if the agent was not expected to quit at time zero ($T_1$). The time zero quitting probability $c_1$ ensures that $F_1(T_2) = 1 - z_1$. Consequently, player 2's payoff is decreasing in $\lambda_1$ and $z_1$; and increasing in $\lambda_2$ and $z_2$. The concession rate of player 2, in turn, is increasing in player 1's cost of resisting the Stackelberg action, $l_1$. Thus, player 2's payoff is increasing in $l_1$ and $z_2$; and decreasing in $l_2$ and $z_1$. Also note that because $c_1 > 0$, player 2's payoff is strictly less than $\bar{g}$ and so the limit equilibrium payoff is inefficient.

4.2.2. *Proof of Theorem 2.* Suppose in partial history $h_t$ player $i$ has played according to $\sigma_i^s$ and player $j$ has deviated from $\sigma_j^s$, then $\mu_i(\Omega_s|h_t) > z_i$ and $\mu_j(\Omega_s|h_t) = 0$. Consequently, Theorem 1 implies that $U_i(\sigma|h_t) \geq \bar{g} - K(\Delta)$ where $K(\Delta) = (1 - e^{-r\Delta})n^p K^{n^*} + n^p(1 - e^{-r\Delta})M$ and $K$ and $n^*$ are independent of $\Delta$. Also, let $M(\Delta) = 2Mn^p(1 - e^{-r\Delta n^p})$ which converges to zero when $\Delta$ goes to zero. Take $\Delta$ sufficiently small so that $\rho K(\Delta) + 2M(\Delta) < l_i$ and $\bar{g} - K(\Delta) > 0$.

*Step* 1. Let the set $R_i(n)$ denote all pure repeated game strategies $\sigma_i$ such that: if $\sigma_i$ is played against the opponents Stackelberg strategy $\sigma_j(s)$, then in all periods $l < n$ the strategy picks an action compatible with the Stackelberg strategy $\sigma_i(s)$; in period $n$, the strategy picks an action which generated an outcome incompatible with $\sigma_i(s)$ at some decision node.

Observe that the sets $R_i(n)$ are disjoint and their union $\bigcup_n R_i(n)$ gives all pure repeated game strategies excluding the set of strategies $N_i$ that never deviate from $\sigma_i(s)$, if played against $\sigma_j(s)$. Let

$$G_i(t) = (1 - z_i) \sum_{\Delta n \leq t} \sigma_i(R_i(n)),$$

where, for mixed repeated game strategy $\sigma_i$, $\sigma_i(R)$ denotes the probability that a pure strategy in the set $R$ is played. $G_i(t)$ is the probability that player $i$ will reveal rationality by playing an action incompatible with the Stackelberg type by time $t$. Step 4 given below shows that for all equilibria $\sigma$ of the repeated game $\Gamma^\infty(\mu, \Delta)$, there exists a time $T$ such that $G_i(T) = 1 - z_i$, that is, every normal player eventually reveals rationality, if faced with a sufficiently long history of play compatible with the Stackelberg type. Consequently, for any equilibrium $\sigma$, $\sigma_i(N_i) = 0$.

If $\sigma_i(R_i(n)) > 0$, then let $U_i(\sigma|R_i(n))$ denote the expected repeated game payoff for player $i$, under mixed strategy profile $\sigma$, conditional on player $i$ having picked a strategy in the set $R_i(n)$. If $\sigma_i(R_i(n)) = 0$, then let $U_i(\sigma|R_i(n)) = \sup_{\sigma_i' \in R_i(n)} U_i(\sigma_i', \sigma_j)$. Also, for any real number $t$, let $U_i(\sigma|R_i(t)) = U_i(\sigma|R_i(\bar{n}))$ where $\bar{n} = \max_n\{\Delta n \leq t\}$. Observe that for any equilibrium mixed strategy profile $\sigma$ ,

$$U_i(\sigma) = \sum_{n=0}^{\infty} \sigma_i(R_i(n))U_i(\sigma|R_i(n)) + \sigma_i(N_i)U_i(\sigma|N_i) = \frac{1}{1-z_i}\int_{t=0}^{\infty} U_i(\sigma|R_i(t))dG_i(t).$$

*Step* 2. Define

$$\begin{aligned}\bar{U}_i(t,k) &= e^{-r\min\{t,k\}}\bar{g} - l_i(1-e^{-r\min\{t,k\}}) && \text{if } t \geq k \\ &= -l_i(1-e^{-r\min\{t,k\}}) && \text{if } t < k.\end{aligned}$$

In this definition $t, k \in \mathbb{R}_+$. For any equilibrium profile $\sigma^n$ for $\Gamma^\infty(\mu, \Delta^n)$

$$U_i(\sigma^n) \leq \frac{1}{1-z_i}\int_t\int_k \bar{U}_i(t,k)dG_j^n(k)dG_i^n(t) + \rho K(\Delta) + M(\Delta).$$

*Proof of Step 2.* Fix an equilibrium strategy profile $\sigma$. Pick $t$ such that $\Delta n = t$ for some $n$. We bound $U_i(\sigma|R_i(t))$. If $j$ does not reveal rationality in any period $k \leq t$, then player $i$ will reveal rationality. Consequently, the continuation utility, by Theorem 1, for player $j$ will be at least $\bar{g} - K(\Delta)$. This implies that player $i$'s continuation utility after period $t$ is at most $\rho K(\Delta)$. Also, player $i$ will incur $-(1-e^{-rt})l_i$ since both players will play according to the Stackelberg action up to period $t$. This event occurs with probability $1 - G_j(t)$. If player $j$ reveals rationality at any time $\Delta m = k \leq t$, then player 1 will receive payoff at most $\bar{g}$ from that period onwards and will incur $-(1-e^{-rk})l_i$ up to time $k$. Consequently,

$$U_i(\sigma|R_i(t)) \leq \int_{\{t\geq k\}} (e^{-rk}\bar{g} - l_i(1-e^{-rk}))dG_j(k) + (1-G_j(t))(\rho K(\Delta) - l_i(1-e^{-rt})) + M(\Delta)$$

$$U_i(\sigma|R_i(t)) \leq \int \bar{U}_i(t,k)dG_j(k) + \rho K(\Delta) + M(\Delta)$$

where the factor $M(\Delta)$ corrects for revelations that occur during punishment phases as well as the inaccuracy of using $l_i$ as the cost of resisting the Stackelberg strategy. Hence,

$$U_i(\sigma) \leq \frac{1}{1-z_i}\int\int \bar{U}_i(t,k)dG_j(k)dG_i(t) + \rho K(\Delta) + M(\Delta)$$

□

*Step* 3 (The proof is in the Appendix). Define

$$\begin{aligned} \underline{U}_i(t,k) &= e^{-r\min\{t,k\}}\bar{g} - l_i(1-e^{-r\min\{t,k\}}) && \text{if } t > k \\ &= -l_i(1-e^{-r\min\{t,k\}}) && \text{if } t \leq k \end{aligned}$$

In this definition $t, k \in \mathbb{R}_+$. For any equilibrium profile $\sigma^n$ for the repeated game $\Gamma^\infty(z, \Delta^n)$

$$U_i(\sigma^n) \geq \frac{1}{1-z_i}\int\int \underline{U}_i(t,k)dG_j^n(k)dG_i^n(t) - K(\Delta)\rho - M(\Delta)$$

*Step* 4 (The proof is in the Appendix). There exists a $T$ such that $G_i^n(T) = 1 - z_i$.

*Step* 5. There exists a subsequence $\{n_k\} \subset \{n\}$ such that $(G_1^{n_k}(t), G_2^{n_k}(t)) \to (\hat{G}_1(t), \hat{G}_2(t))$.

*Proof of Step 5.* Since $G_1^n$ and $G_2^n$ are distribution functions, by Helly's theorem they have a (possibly) subsequential limit $\hat{G}_1(t), \hat{G}_2(t)$. Also, since the support of the $G_1^n$ and $G_2^n$'s is uniformly bounded by the previous lemma, the limiting functions $\hat{G}_1(t), \hat{G}_2(t)$ are also distribution functions. □

*Step* 6 (The proof is in the Appendix). The distribution functions $\hat{G}_1(t)$ and $\hat{G}_2(t)$ do not have any common points of discontinuity.

*Step* 7 (The proof is in the Appendix). If $(G_1^n(t), G_2^n(t)) \to (\hat{G}_1(t), \hat{G}_2(t))$, then

$$\lim U_i(\sigma^n)(1-z_i) = \int\int \bar{U}_i(t,k)d\hat{G}_j(k)d\hat{G}_i(t) = \int\int \underline{U}_i(t,k)d\hat{G}_j(k)d\hat{G}_i(t).$$

*Step* 8 (The proof is in the Appendix). The distribution functions $(\hat{G}_1(t), \hat{G}_2(t))$ solve the War of Attrition and consequently $(\hat{G}_1(t), \hat{G}_2(t)) = (F_1(t), F_2(t))$.

*Step* 9. Observe that $\lim U_i(\sigma^n) = \frac{1}{1-z_i}\int\int \bar{U}_i(t,k)dF_j(k)dF_i(t)$. However, $\frac{1}{1-z_i}\int\int \bar{U}_i(t,k)dF_j(k)dF_i(t)$ is just the expected utility of player $i$ from playing the War of Attrition. Consequently, $\frac{1}{1-z_i}\int\int \bar{U}_i(t,k)dF_j(k)dF_i(t) = (1-c_j)\bar{g}$ thus completing the argument.

Assumption 1 and Assumption 2 were used for two purposes in the previous Theorem. First, to bound the continuation payoff of the two agents. Second, to show that the two agents never concede concurrently, at the limit. Both of these can be achieved if we drop Assumption 1, but assume Assumption 2 ($i$) and ($iii$). The bound on continuation payoffs continue to hold due to Corollary 1. Also, to prove that the two players never concede concurrently, only the bound of continuation payoffs and the Lipschitz given by Equation 1, which also holds under Assumption 2 ($i$) and ($iii$), is required.

**Corollary 3.** *Suppose that $\Gamma_N$ satisfies Assumption 2 ($i$) and ($iii$) for both players. For any $z = (z_1 > 0, z_2 > 0)$ and any $\epsilon > 0$, there exists a $\Delta^*$ such that, for any $\Delta < \Delta^*$, any equilibrium strategy profile $\sigma$ for the repeated game $\Gamma^\infty(\mu, \Delta)$ with $\mu_i(\Omega_s) = z_i$ and $\mu_i(\Omega_-) = 0$, $|U_1(\sigma) - (1-c_2)\bar{g}| < \epsilon$ and $|U_2(\sigma) - (1-c_1)\bar{g}| < \epsilon$.*

## Appendix A. Omitted Proofs

### A.1. **Omitted Steps in the Proof of Corollary 2.**

**Lemma 7** (Uniform Learning)**.** *Suppose player 1 has played according to $\sigma_1(s)$ in history $h_t$, and let $\Omega_s(h_t) \supset \Omega_s$ denote the set of types that behave identical to a Stackelberg type given $h_t$ and let $\Omega_-(h_t) \subset \Omega_-$ denote the set of commitment types not in $\Omega_s(h_t)$. Assume* *that $\mu_1(\Omega_s) = \underline{z} > 0$ and all $\omega \in \Omega_-$ are uniformly learnable. For any $\phi > 0$, there exists a $T$ such that, $\Pr_{(\sigma_1(s),\sigma_2)}\{h : \frac{\mu_1(\Omega_-(h_T)|h_T)}{\mu_1(\Omega_s(h_T)|h_T)} < \phi\} > 1-\phi$, for any strategy $\sigma_2$ of player 2.*

*Proof.* We show that for each finite subset $W \subset \Omega_-$ and any $\varepsilon > 0$, there exists a $T$ such that, $\Pr_{(\sigma_1(s),\sigma_2)}\{h : \mu_1(W \cap \Omega_-(h_T)|h_T) < \varepsilon\} > 1-\varepsilon$, for any strategy $\sigma_2$ of player 2. Proving this is sufficient for the result since $W$ can be picked such that $\mu_1(W)$ is arbitrarily close to $\mu_1(\Omega_-)$.

Step 1. Let $L_t^\omega(h) = \frac{\Pr_{\sigma(\omega)}((a_1)_t=\sigma_1(s|h_t)|h_t)}{\Pr_{\sigma(s)}((a_1)_t=\sigma_1(s|h_t)|h_t)} L_{t-1}^\omega(h)$ and $L_0^\omega(h) = \frac{\mu_1(\omega)}{\mu_1(\Omega_s)}$. By Fudenberg and Levine (1992) Lemma 4.2, $L_t^\omega(h) = \frac{\mu_1(\omega|h_t)}{\mu_1(\Omega_s|h_t)}$ and $(L_t^\omega, H_t)$ is a supermartingale, under $\Pr_{(\sigma_1(s),\sigma_2)}$. Observe $\Pr_{(\sigma_1(s),\sigma_2)}((a_1)_t = \sigma_1(s|h_t)|h_t) = 1$ for $\Pr_{(\sigma_1(s),\sigma_2)}$ $-a.e.$ history. Let $L^\omega(K,\varepsilon)$ denote the set of histories such that either $L_T^\omega(h) < \varepsilon$ or $1-\Pr_{\sigma(\omega)}((a_1)_t = \sigma_1^s|h_t) < \varepsilon$ in all but $K$ periods for any $T > K$. Fudenberg and Levine (1992) Theorem 4.1 implies that there exists a $K_\omega$ independent of $\sigma_2$ such that $\Pr_{(\sigma_1(s),\sigma_2)}\{L^\omega(K_\omega,\varepsilon)\} > 1-\varepsilon$.

Step 2. Let $\xi = \min_{\omega\in W} \varepsilon_\omega$ where $\varepsilon_\omega$ is the uniform probability, implied by uniformly learnable types. For each $\omega \in W$, after any history $h_T$ where there has been $N$ periods where player 2 has played an action different than $a_2^b$ against $a_1^s$, either $\Pr_\omega((a_1)_t \neq \sigma_1(s|h_t)|h_t) > \xi$ at least $N$ times or $\omega \in \Omega_s(h_T)$.

Step 3. Pick $\frac{\varepsilon}{|W|} < \xi$. Pick $N$ such that $N > K_\omega$ and $\Pr_{(\sigma_1(s),\sigma_2)}\{L^\omega(K_\omega, \frac{\varepsilon}{|W|})\} > 1 - \frac{\varepsilon}{|W|}$ for all $\omega \in W$. Consequently, $\Pr_{(\sigma_1(s),\sigma_2)}\{\cap_{\omega\in W} L^\omega(K_\omega, \frac{\varepsilon}{|W|})\} > 1-\varepsilon$. For any $h \in \cap_{\omega\in W} L^\omega(K_\omega, \frac{\varepsilon}{2|W|}))$ where there has been $N$ periods where player 2 has played an action different than $a_2^b$ against $a_1^s$ in $h_T$, by Step 2, either $L_T^\omega(h) < \frac{\varepsilon}{|W|}$ or $|\Pr_{\sigma(\omega)}((a_1)_t = \sigma_1(s|h_t)|h_t) - 1| < \frac{\varepsilon}{|W|} < \xi$ all but $T$ times. However, by the definition of $\xi$ in Step 2, either $L_T^\omega(h) < \frac{\varepsilon}{2|W|}$ or for any $\omega$ with $L_T^\omega(h) > \frac{\varepsilon}{2|W|}$ by Step 2 $\omega \in \Omega_s(h_T)$. That is, either $L_T^\omega(h) < \frac{\varepsilon}{2|W|}$ or $\omega \in \Omega_s(h_T)$. So all $\omega \in W$ with $\mu_1(\omega|h_T) > \frac{\varepsilon}{2|W|}$ are in $\Omega_s(h_T)$. Hence $\mu_1(W \cap \Omega_-(h_T)|h_T) < \varepsilon$ for any $h \in \cap_{\omega\in W} L^\omega(K_\omega, \frac{\varepsilon}{2|W|})$ delivering the result. □

A.2. **Omitted Steps in the Proof of Theorem 2.**

*Proof of Step 3. Statement.* Let

$$\underline{U}_i(t,k) = e^{-r\min\{t,k\}}\bar{g} - l_i(1-e^{-r\min\{t,k\}}) \qquad \text{if } t > k$$
$$= -l_i(1-e^{-r\min\{t,k\}}) \qquad \text{if } t \leq k$$

For any equilibrium profile $\sigma^n$ for $\Gamma^\infty(\mu, \Delta^n)$

$$U_i^n(\sigma^n) \geq \frac{1}{1-z_i}\int\int \underline{U}_i(t,k)dG_j^n(k)dG_i^n(t) - K(\Delta)(1+\rho) - 2M(\Delta)$$

*proof.* Fix an equilibrium strategy $\sigma$ and suppose that $j$ behaves according to $\sigma_j$. Fix an equilibrium strategy profile $\sigma$. Pick $t$ such that $\Delta n = t$ for some $n$. We bound $U_i(\sigma|R_i(t))$. If $j$ reveals rationality in any period $\Delta m = k < t$, then player $i$ incurs $-l_i$ up to that time, and receives continuation payoff $\bar{g} - K(\Delta)$. This exceeds $-(1-e^{-rk})l_i + e^{-rk}(\bar{g} - K(\Delta))$. If player $i$ reveals rationality first in period $t$, then she receives as a continuation $-\rho K(\Delta) \leq 0$. If player $j$ reveals first in period $t$, then player $i$ receives in continuation $\bar{g} - K(\Delta) > 0$. Consequently,

$$U_i(\sigma|R_i(t)) \geq \int_{k<t}(e^{-rk}\bar{g} - (1-e^{-rk})l_i)dG_j(k) - (1-G_j(t^-))(1-e^{-rt})l_i - (1+\rho)K(\Delta) - M(\Delta)$$

where $1 - G_j(t^-)$ denotes the probability that player $j$ reveals at a time $k \geq t$. This implies that

$$U_i(\sigma|R_i(t)) \geq \int \underline{U}_i(t,k)dG_j(k) - (1+\rho)K(\Delta) - M(\Delta)$$

Hence,

$$U_i(\sigma) \geq \frac{1}{1-z_i}\int\int \underline{U}_i(t,k)dG_j(k)dG_i(t) - (1+\rho)K(\Delta) - M(\Delta)$$

proving the result. □

*Proof of Step 4. Statement.* There exists a $T$ such that $G_i^n(T) = 1 - z_i$.

*proof.* For some time $s$ suppose that $(\sigma_i(s), \sigma_j(s))$ has been played in all periods $\Delta k < s$ and consider the strategy of $i$ that continues to play $\sigma_i(s)$ for all periods $n$ such that $\Delta n \in [s, s+t]$, given that $(\sigma_i(s), \sigma_j(s))$ has been played in all prior periods. For this strategy to be considered, it must do better for player $i$ than revealing rationality which guarantees her $-\rho K(\Delta)$ by loosing at most $M(\Delta)$. For this strategy to do better than revealing rationality for $i$, the probability with which player $j$ plays $\sigma_j(s)$ in all periods $\{n : \Delta n \in [s, s+t]\}$, given that $(\sigma_i(s), \sigma_j(s))$ has been played in all prior periods, $\mathbb{P}\{a_{n,j} = \sigma_j(s) \; \forall \Delta n \in [s, s+t] | a_{k<n} = (\sigma_i(s), \sigma_j(s))\}$, must satisfy the following:

$$-\rho K(\Delta) - M(\Delta) \leq \mathbb{P}\{a_{n,j} = \sigma_j(s) \; \forall \Delta n \in [s, s+t] | h_n\} (\bar{g} e^{-rt} - l_i(1 - e^{-rt}) + M(\Delta)) \\ +(1 - \mathbb{P}\{a_{n,j} = \sigma_j(s) \; \forall \Delta n \in [s, s+t] | h_n\}) \bar{g}$$

where play is according to $(\sigma_i(s), \sigma_j(s))$ in all histories $h_n$ under consideration. Consequently,

$$\mathbb{P}\{a_{n,j} = \sigma_j(s) \; \forall \Delta n \in [s, s+t] | h_n\} \leq \frac{\bar{g} + \rho K(\Delta) + M(\Delta)}{(\bar{g} + l_i)(1 - e^{-rt})}$$

Observe that for $t$ large, $\frac{\bar{g}+\rho K(\Delta)+M(\Delta)}{(\bar{g}+l_i)(1-e^{-rt})-M(\Delta)} < 1$. This implies that for $i$ to be willing to play $\sigma_i(s)$ for all $\Delta n \in [0, tk]$

$$\begin{aligned} z_j &\leq \mathbb{P}\{\sigma_j(s) \; \forall tk\} = \prod_{s=0}^{k} \mathbb{P}\{a_{n,j} = \sigma_j(s) \; \forall \Delta n \in [s, s+t] | h_n\} \\ &\leq \left( \frac{\bar{g} + \rho K(\Delta) + M(\Delta)}{(\bar{g} + l_i)(1 - e^{-rt})} \right)^k \end{aligned}$$

However for $t$ and $k$ sufficiently large this is not possible. □

*Proof of Step 6. Statement.* The distribution functions $\hat{G}_1(t)$ and $\hat{G}_2(t)$ do not have any common points of discontinuity.

*proof.* Assume that $\hat{G}_1$ and $\hat{G}_2$ have a common point $t$ where they are both discontinuous. Let $J_1 = \hat{G}_1(t) - \lim_{s \nearrow t} \hat{G}_1(s)$ and let $J_2 = \hat{G}_2(t) - \lim_{s \nearrow t} \hat{G}_2(s)$. We can pick $\zeta$, arbitrarily close to $t$, such that both $\hat{G}_1$ and $\hat{G}_2$ are continuous at $t + \zeta$ and $t - \zeta$. This implies that for each $\epsilon > 0$, there is a $N$ such that the game is played at least once in each interval of length $2\zeta$ and $G_i^n[t - \zeta, t + \zeta] = G_i^n(t + \zeta) - G_i^n(t - \zeta) \geq J_i - \epsilon > 0$, for all $n \geq N$. In words, the probability that 1 plays an action different than $a_i(s)$ in the interval $[t-\zeta, t+\zeta]$ is greater than $J_1 - \epsilon$. Also, pick $N$ such that the value to any player after she has played an action other than the commitment action is less than $\epsilon$, the payoff to any player who has not played an action different than the commitment action against an opponent known to be rational is greater than $\bar{g} - \epsilon$, and $M(\Delta^n) < \epsilon$ . Pick the first period $k$ such that $\Delta^n k \in [t-\zeta, t+\zeta]$ and $\mathbb{P}\{a_{k1} \neq \sigma_1(s)\} > 0$ or $\mathbb{P}\{a_{k2} \neq \sigma_2(s)\} > 0$. Since $G_i^n[t-\zeta, t+\zeta] > 0$ such a period exists for all $n > N$.

Without loss of generality assume that $\mathbb{P}\{a_{k1} \neq \sigma_1(s)\} > 0$. The payoff that player 1 receives from deviating from $\sigma_1(s)$ in period $k$ must be at least as well as playing $\sigma_1(s)$ throughout the interval $[t-\zeta, t+\zeta]$. Let $U_i$ denote the payoff that player i receives in equilibrium conditional on both players not playing $\sigma_i(s)$ in period $k$. Consequently,

$$\mathbb{P}\{a_{k2} \neq \sigma_2(s)\}U_1 + (1 - \mathbb{P}\{a_{k2} \neq \sigma_2(s)\})\epsilon \geq e^{-r4\zeta}(\bar{g} - \epsilon)G_2^n[t-\zeta, t+\zeta] - l(1 - e^{-r4\zeta}) + \epsilon$$

Redefine $\epsilon' = \epsilon + \frac{\epsilon}{e^{-r4\zeta}(J_i - \epsilon)} + \frac{l(1-e^{-r4\zeta}) - \epsilon}{e^{-r4\zeta}(J_i - \epsilon)}$ and rewrite the above equation as follows:

$$\mathbb{P}\{a_{k2} \neq \sigma_2(s)\}U_1 \geq e^{-r4\zeta}(\bar{g} - \epsilon')G_2^n[t-\zeta, t+\zeta].$$

For $\epsilon$ and $\zeta$ sufficiently small, the right hand side of the equation is approximately $J_2\bar{g}$ and the left hand side is $\mathbb{P}\{a_{k2} \neq \sigma_2(s)\}U_1$. Consequently, for this inequality to hold, $\mathbb{P}\{a_{k2} \neq \sigma_2(s)\} > 0$. Also, by definition, $\mathbb{P}\{a_{k2} \neq \sigma_2(s)\} \leq G_2^n[t-\zeta, t+\zeta]$, and consequently, $U_1 \geq e^{-r(4\zeta)}(\bar{g} - \epsilon')$.

$\mathbb{P}\{a_{k2} \neq \sigma_2(s)\} > 0$ implies, by a symmetric argument as in the case of player 1, that

$$\mathbb{P}\{a_{k1} \neq \sigma_1(s)\}U_2 \geq e^{-r4\zeta}(\bar{g} - \epsilon')G_1^n[t-\zeta, t+\zeta]$$

Consequently, $U_2 \geq e^{-r4\zeta}(\bar{g}-\epsilon')$. However, $U_1 \geq e^{-r4\zeta}(\bar{g}-\epsilon')$, implies that $U_2 \leq \rho(\bar{g}-U_1) = \rho(\bar{g}(1-e^{-r4\zeta})+\epsilon' e^{-r4\zeta})$. So,

$$\rho(\bar{g}(1-e^{-r4\zeta})+\epsilon' e^{-r4\zeta}) \geq e^{-r4\zeta}(\bar{g}-\epsilon')$$

Taking the limit first with respect to $\epsilon$ and then with respect to $\zeta$ gives $0 \geq \bar{g}$, which is a contradiction. □

*Proof of Step 7. Statement.* If $(G_1^n(t), G_2^n(t)) \to (\hat{G}_1(t), \hat{G}_2(t))$, then

$$\lim U_i(\sigma^n)(1-z_i) = \int\int \bar{U}_i(t,k)d\hat{G}_j(k)d\hat{G}_i(t) = \int\int \underline{U}_i(t,k)d\hat{G}_j(k)d\hat{G}_i(t).$$

*proof.* If $G_1^n$ converges to $\hat{G}_1$ and $G_2^n$ converges to $\hat{G}_2$, then the product measure $G_1^n \times G_2^n$ converges to $\hat{G}_1 \times \hat{G}_2$, see Billingsley (1995), Page 386, Exercise 29.2. Observe that the functions $\bar{U}_1(t,k)$ and $\underline{U}_1(t,k)$ are continuous at all points except on the set $\{t=k\}$. By the previous lemma, $\int_{\mathbb{R}^2} 1_{\{t=k\}} d(\hat{G}_1 \times \hat{G}_2) = 0$. Consequently, the $\hat{G}_1 \times \hat{G}_2$ measure of the points of discontinuity of $\bar{U}_1(t,k)$ and $\underline{U}_1(t,k)$ is zero. Billingsley (1995), Theorem 29.2, shows that if the set of discontinuities of a measurable function $h$, $D_h$, has $\mu$ measure zero, i.e., $\mu(D_h)=0$ and $\mu_n \to \mu$, then $\int h d\mu_n \to \int h d\mu$. So,

$$\lim_n \int_{t_1}\left(\int_{t_2} \bar{U}_1(t_1,t_2)dG_2^n(t_2)\right)dG_1^n(t_1) = \lim_n \int_{\mathbb{R}^2} \bar{U}_1 d(G_1^n \times G_2^n) = \int_{\mathbb{R}^2} \bar{U}_1 d(\hat{G}_1 \times \hat{G}_2)$$

and similarly for $\underline{U}_1$. Also, since $\bar{U}_1$ and $\underline{U}_1$ differ only on a set of zero measure, $\int_{\mathbb{R}^2} \bar{U}_1 d(\hat{G}_1 \times \hat{G}_2) = \int_{\mathbb{R}^2} \underline{U}_1 d(\hat{G}_1 \times \hat{G}_2)$. □

*Proof of Step 8. Statement.* The distribution functions $(\hat{G}_1(t), \hat{G}_2(t))$ solve the War of Attrition and consequently $(\hat{G}_1(t), \hat{G}_2(t)) = (F_1(t), F_2(t))$

*proof.* In the continuous time war of attrition, if player 1 is behaving according to $\hat{G}_1$, then for each $\epsilon$, there is a $N$ such that for all $n > N$, $G_2^n$ is an $\epsilon$ best response to $\hat{G}_1$ and consequently, since $\epsilon$ is arbitrary $\hat{G}_2$ is a best response to $\hat{G}_1$. Also, the symmetric argument is true for player 2 showing that $\hat{G}_1$ is a best response to $\hat{G}_2$. Proving that $\hat{G}_1$ and $\hat{G}_2$ form an equilibrium for the continuous time war of attrition. Since the war of attrition has a unique equilibrium $\hat{G}_1 = F_1$ and $\hat{G}_2 = F_2$. This argument is identical to Abreu and Gul (2000), proof of Proposition 4, on page 114 where a more detailed proof may be found. □

## References

Abreu, D., P. Dutta, and L. Smith (1994): "The Folk Theorem for Repeated Games: A Neu Condition," *Econometrica*, 62(4), 939–948.

Abreu, D., and F. Gul (2000): "Bargaining and reputation," *Econometrica*, 68(3), 85–117.

Abreu, D., and D. Pearce (2007): "Bargaining, reputation, and equilibrium selection in repeated games with contracts," *Econometrica*, 75(3), 653–710.

Battigalli, P., and J. Watson (1997): "On "Reputation" Refinements with Heterogeneous Beliefs," *Econometrica*, 65(2), 369–374.

Billingsley, P. (1995): *Probability and Measure*. John Wiley and Sons, New York, NY.

Celantani, M., D. Fudenberg, D. Levine, and W. Pesendorfer (1996): "Maintaining a Reputation Against a Long-Lived Opponent," *Econometrica*, 64, 691–704.

Chan, J. (2000): "On the non-existence of reputation effects in two-person infinitely repeated games," Working paper.

Cripps, M., E. Dekel, and W. Pesendorfer (2005): "Reputation with equal discounting in repeated games with strictly conflicting interests," *Journal of Economic Theory*, 121, 259–272.

Cripps, M., and J. Thomas (1997): "Reputation and perfection in repeated common interest games," *Games and Economic Behavior*, 18, 141–158.

Evans, R., and J. Thomas (1997): "Reputation and Experimentation in Repeated Games With Two Long-Run Players," *Econometrica*, 65(5), 1153–1173.

Fudenberg, D., and D. Levine (1989): "Reputation and equilibrium selection in games with a patient player," *Econometrica*, 57, 759–778.

——— (1992): "Maintaining a reputation when strategies are imperfectly observed," *Review of Economic Studies*, 59, 561–579.

Fudenberg, D., and E. Maskin (1986): "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica*, 54(3), 533–554.

Kreps, D., and R. Wilson (1982): "Reputation and Imperfect Information," *Journal of Economic Theory*, 27, 253–279.

Mailath, G., and L. Samuelson (2006): *Repeated Games and Reputations: Long-run Relationships*. Oxford University Press, Oxford, UK.

Milgrom, P., and J. Roberts (1982): "Predation, Reputation, and Entry Deterrence," *Journal of Economic Theory*, 27, 280–312.

Myerson, R. (1991): *Game Theory: Analysis of Conflict*. Harvard University Press, Cambridge, MA.

Rubinstein, A., and A. Wolinsky (1995): "Remarks on infinitely repeated extensive-form games," *Games and Economic Behavior*, 9, 110–115.

Schmidt, K. (1993a): "Commitment through incomplete information in a simple repeated bargaining game," *Journal of Economic Theory*, 60, 114–139.

——— (1993b): "Reputation and equilibrium characterization in repeated games with conflicting interests," *Econometrica*, 61, 325–351.

Sorin, S. (1995): "A note on repeated extensive games," *Games and Economic Behavior*, 9, 116–123.

Wen, Q. (2002): "A Folk Theorem for Repeated Sequential Games," *Review of Economic Studies*, 69, 493–512.

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