How Robust is the Folk Theorem with Imperfect Public Monitoring?*

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Abstract

In this paper, we prove that, under full rank, every perfect public equilibrium payoff (above some Nash equilibrium in the two-player case, and above the minmax payoff with more players) can be achieved with strategies that have bounded memory. This result is then used to prove that a version of the folk theorem remains valid under private but almost-public monitoring.

1. Introduction

The study of repeated games with imperfect public monitoring has significantly affected our understanding of intertemporal incentives. The concepts and techniques developed by Abreu et al. (1990) have proved especially and enormously helpful. Building on their work, Fudenberg et al. (1994) have shown that, when players are sufficiently patient, imperfect monitoring imposes virtually no restriction on the set of equilibrium payoffs, provided that some (generally perceived as mild) identifiability conditions are satisfied.

*Preliminary, but complete.

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Recent progress in the study of games with private monitoring has raised, however, some issues regarding the robustness of these concepts and techniques. When monitoring is private, players can no longer perfectly coordinate their continuation play. This inability is compounded over time, to the extent that the construction of equilibria from Abreu et al. may unravel under private monitoring no matter how close it is from being public.

Mailath and Morris (2002) show that *uniformly strict* perfect public equilibria are robust to sufficiently small perturbations in the monitoring structure provided that strategy profile has *finite memory*, i.e. it only depends on the last M observations, for some integer M. Unfortunately, the strategies used in establishing the folk theorem need not be uniformly strict, and fail to have finite memory. Indeed, a strategy profile as simple as grim-trigger does not have finite memory.

In a related contribution, Cole and Kocherlakota (2005) provide an example of a game with imperfect public monitoring in which the assumption of finite memory reduces the set of equilibrium payoff vectors. In their example, the set of equilibrium payoff vectors that can be achieved under finite memory, no matter how large the bound may be, is bounded away from the set of payoff vectors attainable with unbounded memory. Unfortunately, their analysis is restricted to strongly symmetric strategies, and their monitoring structure violates the identifiability conditions imposed by Fudenberg et al.

The purpose of this paper is to show that the restriction to finite memory Perfect Public Equilibria is to a large extent inconsequential. We establish a version of the folk theorem with imperfect public monitoring that only involves public strategies with finite memory. However, we strengthen the identifiability conditions of Fudenberg et al., and we further assume that a public randomization device is available. In the case of two players, our result is also weaker, in the sense that we show that every payoff vector that dominates a (static) Nash equilibrium payoff vector (as opposed to the minmax payoff vector) can be achieved in some equilibrium by sufficiently patient players. A final caveat involves the order of quantifiers. We prove that, for every sufficiently high discount factor, a given payoff can be achieved by using some strategy profile having finite memory. That is to say, the size of the memory depends on the discount factor, and tends to infinity as the discount factor tends to one.

We build on this result to prove that the folk theorem remains valid under almost-public private monitoring. This result is not a simple application of the results of Mailath and Morris, because our construction under imperfect public monitoring fails to be uniformly strict. Indeed, it is important for our construction that some players be indifferent across several actions after particular histories. Nevertheless, our proof is largely inspired by their insights and results. As in the case of imperfect public monitoring, we rely on a strong version of the identifiability conditions, on the existence of a public randomization device, and we only show that every payoff vector that dominates a *strict* Nash equilibrium payoff vector is achievable under private almost-public monitoring in the case of two players.

These results prove that the example of Cole and Kocherlakota either relies on the restriction to strongly symmetric strategies, or the failure of the identifiability conditions (or a combination of both). More importantly, our construction relies on the key ideas from Abreu et al. and Fudenberg et al., so that our result does not only establish the robustness of (a version of) their folk theorem, but also shows that their methods, in particular the strategy profiles they use, require only minor adjustments to be robust as well.

The present folk theorem with imperfect private monitoring differs from those from the existing literature. Mailath and Morris (2002) assume that monitoring is not only almost public but also almost perfect. More importantly, they incorrectly claim that the profile described in the proof of their Theorem 6.1 has bounded recall.[1] Hörner and Olszewski (2005) also assume almost perfect monitoring; moreover, they use substantially different strategies than those from Abreu et al. and Fudenberg et al. On the other hand, almost public signals are typically very correlated (even conditionally on action profiles), while Hörner and Olszewski (2005) allow for any pattern of correlation conditional on action profiles. Matsushima (2004) and Yamamoto (2006)

[1]See Section 13.6 in Mailath and Samuelson (2006) for a detailed discussion of the flaw in their proof.

do not assume almost perfect monitoring either, but they obtain the folk theorem only for the Prisoner's Dilemma and games with very similar structure; also, they assume conditionally independent signals (or signals that differ from public monitoring by conditionally independent components). Finally, other constructions of equilibria under private not almost perfect monitoring either do not yield the folk theorem (Ely et al. (2005)), or allow players to communicate (Compte (1998), Kandori and Matsushima (1998), Aoyagi (2002), Fudenberg and Levine (2004), Obara (2006)), or allow players to obtain, at a cost, perfect signals (Miyagawa et al. (2005)).

2. The Model

In the present paper, we study both public and private (but almost-public) monitoring. We first introduce the notation for public monitoring, used throughout Section 3, and then slightly modify the notation to accommodate private monitoring, studied in Section 4.

2.1. Public Monitoring

We follow most of the notation from Fudenberg et al. (1994). In the stage game, player $i = 1, ..., n$ chooses action a_i from a finite set A_i. We denote by m_i the number of elements of A_i, and we call a vector $a \in A = \prod_{i=1}^{n} A_i$ a profile of actions. Profile a induces a probability distribution over the possible public outcomes $y \in Y$, where Y is a finite set with m elements. Player i's realized payoff $r_i(a_i, y)$ depends on the action a_i and the public outcome y. We denote by $\pi(y \mid a)$ the probability of y given a. Player i's expected payoff from action profile a is

$$g_i(a) = \sum_{y \in Y} \pi(y \mid a) r_i(a_i, y).$$

A mixed action α_i for each player i is a randomization over A_i, i.e. an element of $\triangle A_i$. We denote by $\alpha_i(a_i)$ the probability that α_i assigns to a_i. We define $r_i(\alpha_i, y)$, $\pi(y \mid \alpha)$, $g_i(\alpha)$ in the standard manner. We often denote the profile in which player i plays α_i and all other players play a profile α_{-i} by (α_i, α_{-i}).

Throughout the paper, we assume that different action profiles induce linearly independent probability distributions over the set of possible public outcomes. That is, we assume that the set $\{\pi(y \mid a)\colon a \in A\}$ consists of linearly independent vectors; we call this assumption *full-rank condition.*

Note that the full-rank condition implies that every (including mixed) action profile has both *individual and pairwise full rank* in the sense of Fudenberg et al. (1994). The full-rank condition also implies that there are at least as many outcomes y as action profiles a, i.e. $m \geq \prod_{i=1}^{n} m_i$. Thus, the full-rank condition is stronger than the conditions typically imposed in the studies of repeated games with imperfect public monitoring. However, given a number of signals $m \geq \prod_{i=1}^{n} m_i$, it is satisfied by all monitoring structures $\{\pi(y \mid a) : a \in A,\, y \in Y\}$ but a non-generic set of them.

We do not know if the folk theorems presented in this paper hold true if the full-rank condition gets relaxed to a combination of individual and pairwise full rank conditions assumed by Fudenberg et al. (1994).

For each i, the *minmax payoff* $\underline{v}_i$ of player i (in mixed strategies) is defined as

$$\underline{v}_i := \min_{\alpha_{-i} \in \triangle A_{-i}} \max_{a_i \in A_i} g_i\left(a_i, \alpha_{-i}\right).$$

Pick $\underline{\alpha}_{-i} \in \triangle A_{-i}$ so that

$$\underline{v}_i = \max_{a_i \in A_i} g_i\left(a_i, \underline{\alpha}_{-i}\right).$$

The payoff $\underline{v}_i$ is the smallest payoff that the other players can keep player i below in the static game, and the profile $\underline{\alpha}_{-i}$ is an action profile that keeps player i below this payoff.

Let:

$$U := \{(v_1, \ldots, v_n) \mid \exists a \in A,\, \forall i,\; g_i(a) = v_i\},$$
$$V := \text{Convex Hull of } U,$$

and

$$\underline{V} := \text{ Interior of } \left\{(v_1, \ldots, v_n) \in V \mid \forall i,\; v_i > \underline{v}_i\right\}.$$

The set V consists of the feasible payoffs, and $\underline{V}$ is the set of payoffs in the interior of V that strictly Pareto-dominate the minmax point $\underline{v} := (\underline{v}_1, \ldots, \underline{v}_n)$. We assume throughout that $\underline{V}$ is non-empty.

Given a stage-game (pure or mixed) equilibrium payoff vector $v^* := (v_1^*, \ldots, v_n^*)$, we also study the set

$$V^* := \text{ Interior of } \left\{ (v_1, \ldots, v_n) \in V \mid \forall i,\ v_i > v_i^* \right\},$$

and we then assume that V^* is also non-empty.

We assume that players have access to a public randomization device. We will henceforth suppose that in each period, all players observe a public signal $x \in [0, 1]$; the public signals are *i.i.d.* draws from the uniform distribution. We do not know whether the folk theorem presented in this paper is valid without a public randomization device.

We now turn to the repeated game. At the beginning of each period $t = 1, 2, ...,$ players observe a public signal x^t. Then the stage game is played, resulting in a public outcome y^t. The public history at the beginning of period t is $h^t = (x^1, y^1, ..., x^{t-1}, y^{t-1}, x^t)$; player i's private history is $h_i^t = (a_i^1, ..., a_i^{t-1})$. A strategy σ_i for player i is a sequence of functions $(\sigma_i^t)_{t=1}^{\infty}$ where σ_i^t maps each pair (h^t, h_i^t) to a probability distribution over A_i.

Players share a common discount factor $\delta < 1$. All repeated game payoffs are discounted and normalized by a factor $1 - \delta$. Thus, if $(g_i^t)_{t=1}^{\infty}$ is player i's sequence of stage-game payoffs, the repeated game payoff is

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} g_i^t.$$

A strategy σ_i is *public* if in every period t it depends only on the public history h^t and not on player i's private history h_i^t. We focus on equilibria in which all players' strategies are public, called perfect public equilibria (PPE). Given a discount factor δ, we denote by $E(\delta)$ the set of repeated game payoff vectors that correspond to PPE's when the discount factor is δ.

A strategy σ_i has *finite memory of length* T if it depends only on public (randomization device) outcomes and signals in the last T periods, i.e. on $(x^{t-T}, y^{t-T}, ..., x^{t-1}, y^{t-1}, x^t)$ and $(a_i^{t-T}, ..., a_i^{t-1})$. We denote by $E(\delta)$ the set of repeated game payoff vectors that can by achieved by a PPE, and by $E^T(\delta)$ the set of repeated game payoff vectors achievable by PPE's in which all players' strategies have finite memory of length T.

2.2. Private (Almost-Public) Monitoring

Studying games with private monitoring, we shall henceforth make the assumption (as in Mailath and Morris (2002)) that the public monitoring structure has full support.

Assumption. $\pi(y|a) > 0$ for all $y \in Y$ and $a \in A$.

For each i, the *minmax payoff* $\underline{v}_i^P$ in pure strategies of player i is defined as

$$\underline{v}_i^P := \min_{\alpha_{-i} \in A_{-i}} \max_{a_i \in A_i} g_i(a_i, \alpha_{-i}).$$

We further define the set $\underline{V}^P$ by

$$\underline{V}^P := \text{ Interior of } \left\{(v_1, \ldots, v_n) \in V \mid \forall i,\ v_i > \underline{v}_i^P\right\}.$$

In Section 4, we assume that v^* corresponds to the equilibrium payoff vector of a *strict* Nash equilibrium.

A private monitoring structure is a collection of distributions $\{\rho(\cdot|a) : a \in A\}$ over Y^n. The interpretation is that each player receives a signal $y_i \in Y$, and each signal profile $(y_1, \ldots, y_n)$ obtains with some probability $\rho(y_1, \ldots, y_n|a)$. Player i's realized payoff $r_i(a_i, y_i)$ depends on the action a_i and the private signal y_i. Player i's expected payoff from action profile a is therefore

$$g_i(a) = \sum_{(y_1,\ldots,y_n) \in Y^n} \rho(y_1, \ldots, y_n|a) r_i(a_i, y_i).$$

The private monitoring structure ρ is ϵ-close to the public monitoring structure π if $|\rho(y,\ldots,y|a)-\pi(y|a)|<\epsilon$ for all y and a. Let $\rho_i(y_{-i}|a,y_i)$ denote the implied conditional probability of $y_{-i}:=(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n)$.

We maintain the assumption that players have access to a public randomization device in every period.

3. The Finite-Memory Folk Theorem under Public Monitoring

3.1. Result

We establish the following theorem.

Theorem 3.1.

(i) ($n=2$) For any payoff $v\in V^*$, there exists $\bar{\delta}<1$, for all $\delta\in\left(\bar{\delta},1\right)$, there exists $T<\infty$ such that $v\in E^T\left(\delta\right)$.

(ii) ($n>2$) For any payoff $v\in \underline{V}$, there exists $\bar{\delta}<1$, for all $\delta\in\left(\bar{\delta},1\right)$, there exists $T<\infty$ such that $v\in E^T\left(\delta\right)$.

Observe that the statement is stronger in the case of more than two players. Also, observe the order of quantifiers. In particular, we do not know whether we can take T independently of δ. In the proof presented below, the length of memory T is such that

$$\lim_{\delta\to 1} T=\infty.$$

3.2. Sketch of the Proof

The proofs of the two cases are very similar. In both cases, we modify the proof of the folk theorem from Fudenberg et al. (1994), hereafter FLM. In particular, we take a set W that contains v and show that $W\subset E^T\left(\delta\right)$. Our proof starts from the observation that matters would be relatively simple if players could publicly communicate, and communication was (i) cheap (i.e., not directly payoff-relevant), (ii) perfect (i.e., players observe precisely the opponents' message) and (iii) rich (i.e., players' message space is as rich as the set of possible continuation payoff vectors). In this case, players could regularly communicate each others' continuation payoff, and they could then ignore the history that led to those continuation payoffs.

However, even when communication is cheap, rich and perfect, one cannot ignore the issue of truth-telling incentives. This is particularly problematic with two players: since no player i has an incentive to report truthfully his own continuation payoff in the case it is the lowest possible in the set W that is considered, each player must have a message that would force his opponent's continuation payoff to be as low as possible in the set W. In turn, this implies that the set W that is considered includes a payoff vector $\underline{w}$ that is simultaneously the lowest possible for both players, and therefore it must have a 'kink', precluding the smoothness of W assumed in the construction of equilibria from FLM. Observe however that a 'kink' of that kind is not a problem if it only occurs at the "bottom left" of the set W, provided that it Pareto-dominates a Nash-equilibrium payoff vector of the stage game. If the continuation payoff vector is equal or close to $\underline{w}$, then players play the stage-game equilibrium profile, followed by a continuation payoff vector from the set W that is independent of the public outcome observed in the current period. This, roughly speaking, explains why we only have a Nash-threat folk theorem in the case of two players.

We will proceed in steps. We first show that for some particular sets W, if communication is cheap, perfect and rich, then there exist protocols in which players truthfully reveal the continuation payoff vector in equilibrium; moreover, we construct a protocol where players take turns in sending messages, as it is easier to extend such protocols to the case of imperfect communication. Of course, there is no cheap, perfect and rich communication in the repeated game. Players can use actions as signals (the meaning of which may depend on the outcome of the public randomization), but they satisfy neither of the three properties. Therefore, even if players wish to truthfully reveal continuation payoff vectors, they can only reveal a finite number of those, they do so imperfectly, and possibly at the cost of flow payoffs. [As it turns out, the major problem is that communication is imperfect, and actions used to send signals have direct payoff consequences. This is where the strengthening of full-rank is convenient.] We will show that this does not create any essential problem, but this will require modifying the construction from FLM and performing some computations.

The strategy profile that we specify alternates phases of play signaling future continuation payoffs, with phases of play generating those payoffs:

Regular Phase (of length $M(\delta)$): in the periods of this phase, play proceeds as in FLM, given a continuation vector;

Communication Phase (of length $N(\delta)$): in the periods of this phase, players use their actions to signal a continuation payoff vector.

As, the public histories from a communication phase contain information regarding the continuation payoff vector for the following regular phase, the strategies need only to have memory of length $2N(\delta) + M(\delta)$. One can easily check that then we obtain the folk theorem in finite-memory strategies by the argument from FLM, assuming that the communication phase is short enough and the regular phase is long enough; more precisely, it suffices to obtain the following condition:

$$\lim_{\delta \to 1} \delta^{N(\delta)} = 1 \text{ and } \lim_{\delta \to 1} \delta^{M(\delta)} = 0. \tag{3.1}$$

Indeed, disregarding the flow payoffs and incentives during communication phases, one can slightly modify the construction from FLM to prove an analogue of their folk theorem for repeated games in which players' discount factor changes over time: it is δ for $M(\delta) - 1$ periods, and then it drops, for one period, to $\delta^{N(\delta)}$. Alternatively, let $\delta' := \delta^{N(\delta)}$; taking any set W that is self-generated both for discount factors δ and δ', we can mimic the proof of Theorem 1 in Abreu et al. (1990) to show that this implies that any payoff in W is an equilibrium payoff of our repeated game with the discount factor changing over time.

Condition (3.1) also guarantees that the flow payoffs during a communication phase do not affect the equilibrium payoff vector in the limit as $\delta \to 1$.

To deal with the problem that communication is not cheap, we prescribe strategies such that the play in a communication phase affects the continuation payoff vector differently in two complementary events, determined by public randomization. In one event, players will proceed as in FLM, given the continuation payoff vector that just has been communicated; in the other event, the continuation payoff vector will be contingent on the public outcomes observed in the communication phase, so that it makes every player indifferent (with respect to flow payoffs of the communication phase) across all actions (in every period of the communication phase), given any action profile of his opponents. The probability of this former event will be of the order of $\delta^{N(\delta)}$, and the probability of this latter event will be only of the order of $1 - \delta^{N(\delta)}$. This latter event can be therefore disregarded (in the analysis of the repeated game that begins in the communication phase) as it does not affect the equilibrium payoff vector in the limit as $\delta \to 1$, and its presence can be treated as an additional discount factor (that also tends to 1 with δ) in the period that follows the communication phase.

It is not a serious problem that communication is not rich, because it suffices to communicate only a finite number of continuation payoff vectors; indeed, given any discount factor δ, any payoff vector from the boundary of W can be represented as the weighted average of a vector of flow payoffs in the current period and a continuation payoff from the interior of W. The set of all those continuation payoff vectors from the interior of W is a compact subset of W. Therefore there exists a (convex) polyhedron $P(\delta) \subseteq W$ that strictly contains all those vectors. Thus, one can communicate any such continuation payoff vector from the interior of W by randomizing (using the public randomization device from the first period of the communication phase) over the vertices of $P(\delta)$ with appropriate probabilities.

However, when the discount factor δ increases, the number of vertices of $P(\delta)$ typically increases as well, and so does the length of the communication phase. So, calculations must be performed to ensure that (3.1) is satisfied.

Finally, to deal with the problem that communication is not perfect, notice that any desired vertex of $P(\delta)$ can be communicated with high probability by repeating for a number of times the message (playing for a number of times the action profile) that corresponds to this vertex. Suppose that players communicate in a similar manner vertices of another polyhedron $Q(\delta)$ that contains $P(\delta)$ in its interior. When the number of repetitions of the message that corresponds to any desired vertex of $Q(\delta)$ increases, the expected value of the distribution over the vertices of $Q(\delta)$ induced by this sequence of messages converges to the desired vertex. Ultimately, the polyhedron spanned by those expected values contains $P(\delta)$, and so the continuation payoff vectors for all payoff vectors from the boundary of W.

However, the length of the communication phases raises as well with the number of repetitions of any message. Thus, we again need some calculations to make sure that the condition (3.1) will be satisfied.

4. Details of the Proof

4.1. The definition of the set W

Given any payoff vector v in V^* ($n = 2$) or $\underline{V}$ ($n > 2$), we define a set W as follows.

- ($n = 2$) The set W contains v and is contained in V^*. It is assumed closed and convex, and having a non-empty interior. Let

$$\underline{w}_i := \min\{w_i : w \in W\} \quad (i = 1, 2).$$

That is, $\underline{w}_i$ is the i-th coordinate of any vector in W that minimizes the value of this coordinate over the set W. Given any vector $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$ in W, let also: $\tilde{w}_2^1 = \tilde{w}_2$ and

$$\tilde{w}_1^1 := \min\{w_1 : w = (w_1, w_2) \in W \text{ and } w_2 = \tilde{w}_2\}.$$

That is, $\tilde{w}^1 = (\tilde{w}_1^1, \tilde{w}_2^1)$ is the horizontal projection of $\tilde{w}$ onto the boundary of W that minimizes the first coordinate over W. Finally, given $\tilde{w}$ in W, and thus $\tilde{w}^1$, let $\tilde{w}_1^{12} = \tilde{w}_1^1$ and

$$\tilde{w}_2^{12} := \min\left\{w_2 : w = (w_1, w_2) \in W \text{ and } w_1 = \tilde{w}_1^1\right\}.$$

That is, $\tilde{w}^{12} = (\tilde{w}_1^{12}, \tilde{w}_2^{12})$ is the vertical projection of $\tilde{w}^1$ onto the boundary of W that minimizes the second coordinate over W. See the right panel of Figure 1.

The set W is chosen so that:

(i) for any vector $\tilde{w} \in W$,

$$\tilde{w}_2^{12} = \underline{w}_2,$$

and

(ii) the boundary of W is smooth except at the point $\underline{w} := (\underline{w}_1, \underline{w}_2)$.

Observe that (i) implies that $\underline{w} \in W$. To be concrete, we assume that the boundary of W consists of three quarters of a circle and a quarter of a square, as it is depicted in the right panel of Figure 1.

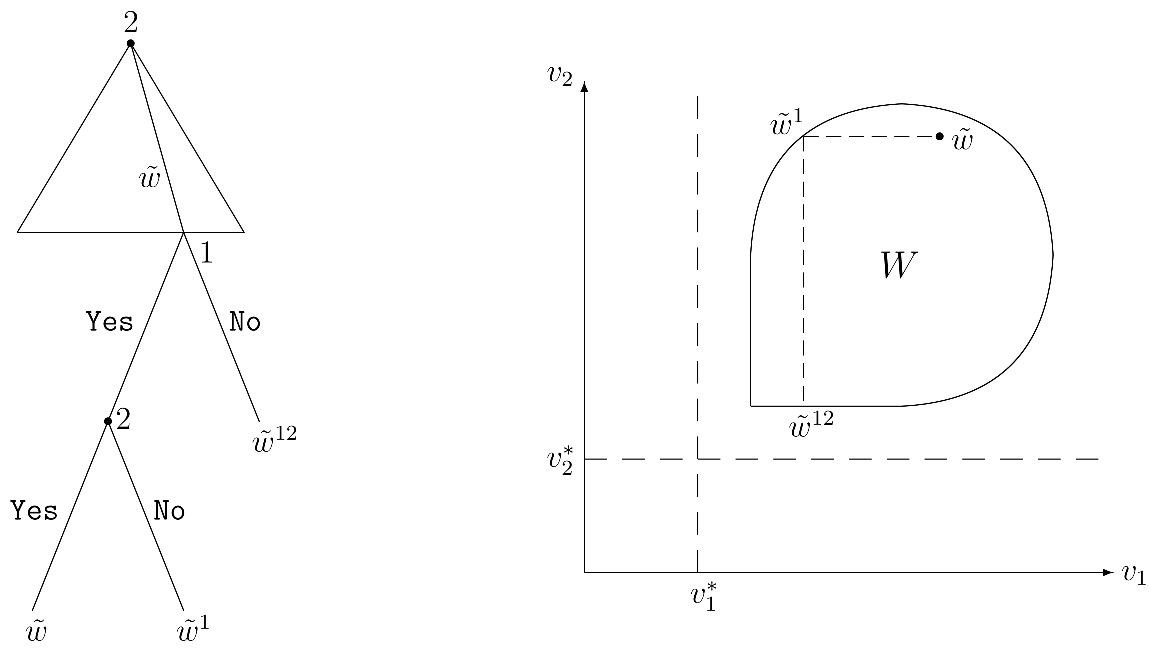

Figure 1

- ($n > 2$) The set W is any closed ball such that $v \in W$ and $W \subseteq \underline{V}$. In what follows, we let $\underline{w}^i$ denote the vector that minimizes the i-th coordinate over W :

$$\underline{w}^i := \min \{w_i : w \in W\} \quad (i = 1, 2, 3).$$

Also, given any vector $\tilde{w}$ in W, we let $\tilde{w}^2$ denote the vector in W that minimizes the second coordinate over all vectors w in W such that $w_1 = \tilde{w}_1$. Formally, $\tilde{w}_1^2 = \tilde{w}_1$,

$$\tilde{w}_2^2 := \min \{w_2 : w = (w_1, w_2, ..., w_n) \in W \text{ and } w_1 = \tilde{w}_1\},$$

and $\tilde{w}^2 \in W$ is the unique vector whose the first two coordinates are equal to $\tilde{w}_1^2$ and $\tilde{w}_2^2$, respectively. Observe that

$$\forall w \in W, \ \underline{w}_2^3 \geq \tilde{w}_2^2. \tag{4.1}$$

That is, identifying coordinate i with player i's payoff, player 2 always weakly prefers the payoff vector $\underline{w}^3$ to the vector $\tilde{w}^2$. See the right panel of Figure 2.

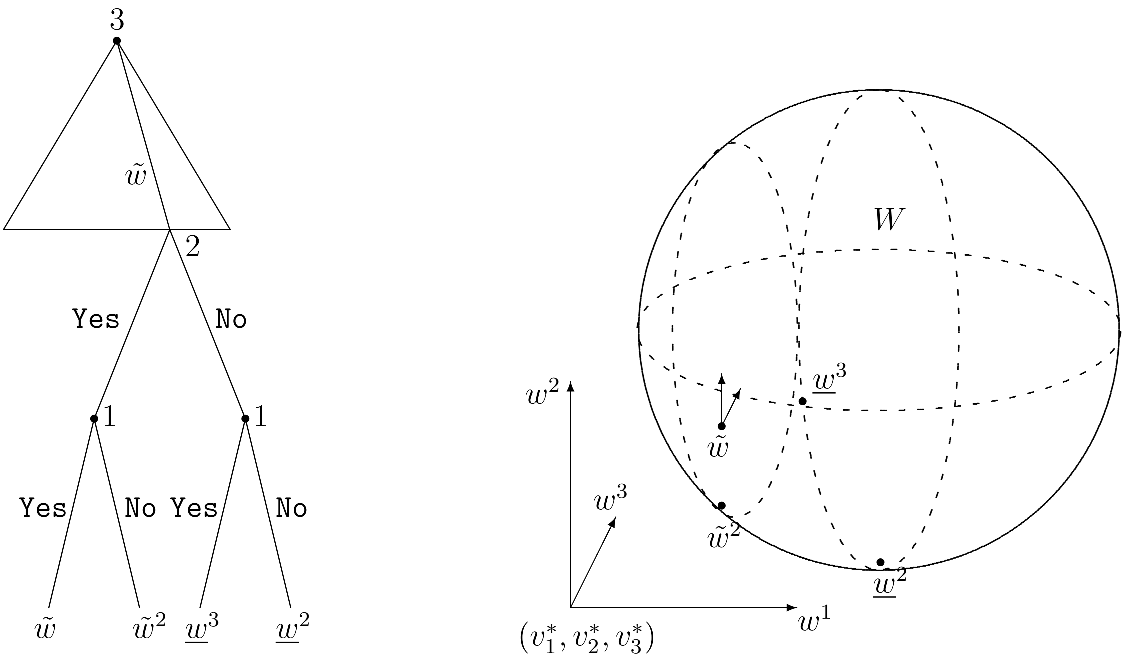

Figure 2

4.2. Truthful Communication of Continuation Payoffs

4.2.1. Cheap, rich and perfect *sequential* communication

The problem we address here is the following: Can we design an extensive-form, perfect information game, such that, for any $w \in W$, there exists a (subgame-perfect) equilibrium of this game in which the equilibrium payoff is equal to w? The difficulty, of course, is that the extensive form itself does not depend on the specific w. The equilibrium strategies, on the other hand, are allowed to depend on w.

The answer is affirmative, for sets W as defined above. In fact, the properties of W are essentially motivated by this problem.

Notice first that it is straightforward to give a simultaneous-move game with the required properties. It suffices if players report a payoff vector, and the actual payoff vector coincides with the reported one if all reports are identical. In the case of two players, the payoff vector minimizes the payoff of both players over the set W (i.e. it is equal to $\underline{w}$) if the two reports differ; in the case of three or more players, the payoff vector minimizes the payoff of the player whose report differs from otherwise identical reports (it is unimportant what payoff vector is prescribed in other cases).

We shall now slightly modify this simultaneous-move game to obtain a game in which only one player moves at a time. Consider both cases in turn.

- ($n = 2$) The extensive form is described in the left panel of Figure 1. Player 2 moves first, choosing any action $\tilde{w} \in W$. Player 1 moves second, and chooses `Yes` or `No`. If player 1 chose `Yes`, player 2 moves a last time, and chooses `Yes` or `No`. Payoff vectors are as follows. If the history is ($\tilde{w}$, `Yes`, `Yes`), the payoff vector is $\tilde{w}$. If it is ($\tilde{w}$, `Yes`, `No`), the payoff vector is $\tilde{w}^1$. If the history is ($\tilde{w}$, `No`), the payoff vector is $\tilde{w}^{12}$.

Given w, the equilibrium strategies $\sigma := (\sigma_1, \sigma_2)$ are as follows:

$$\sigma_2(\varnothing) = w,$$

$$\sigma_2(\tilde{w}, \texttt{Yes}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w}, \\ \texttt{No} \text{ otherwise,} \end{cases}$$

$$\sigma_1(\tilde{w}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w}, \\ \texttt{No} \text{ otherwise.} \end{cases}$$

To check that this is an equilibrium, observe that player 2 is indifferent between both outcomes if the history reaches ($\tilde{w}$, `Yes`) by definition of $\tilde{w}^1$. Therefore, his continuation strategy is optimal. Given this, after history $\tilde{w} = w$, player 1 chooses between getting $\tilde{w}_1$ (if he chooses `Yes`) or $\tilde{w}_1^{12}$ (if he chooses `No`), so that choosing `Yes` is optimal; after history $\tilde{w} \neq w$, he chooses between $\tilde{w}_1^1$ (if he chooses `Yes`) or $\tilde{w}_1^{12}$ (if he chooses `No`), and choosing `No` is optimal as well. Given this, player 2 faces an initial choice between getting w_2 by choosing $\tilde{w} = w$ or getting $\tilde{w}_2^{12} = \underline{w}_2$ by choosing another action. Since $w_2 \geq \underline{w}_2$, choosing $\tilde{w} = w$ is optimal.

- ($n > 2$) The extensive form is described in the left panel of Figure 2. While there may be more than three players, only three of them take actions. Player 3 moves first, choosing any action $\tilde{w} \in W$. Player 2 moves second, and chooses `Yes` or `No`. Player 1 then moves a last time, and chooses either `Yes` or `No`. Payoff vectors are as follows. If the history is ($\tilde{w}$, `Yes`, `Yes`), the payoff vector is $\tilde{w}$. If it is ($\tilde{w}$, `Yes`, `No`), the payoff vector is $\tilde{w}^2$. If the history is ($\tilde{w}$, `No,Yes`), the payoff vector is $\underline{w}^3$. If the history is ($\tilde{w}$, `No`, `No`), the payoff vector is $\underline{w}^2$.

Given w, the equilibrium strategies $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ are as follows:

$$\sigma_3(\varnothing) = w,$$
$$\sigma_2(\tilde{w}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w}, \\ \texttt{No} \text{ otherwise,} \end{cases}$$
$$\sigma_1(\tilde{w}, \texttt{Yes}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w}, \\ \texttt{No} \text{ otherwise,} \end{cases}$$
$$\sigma_1(\tilde{w}, \texttt{No}) = \begin{cases} \texttt{No} \text{ if } w = \tilde{w}, \\ \texttt{Yes} \text{ otherwise.} \end{cases}$$

Again, it is straightforward to verify that this is an equilibrium. Player 1 is indifferent between his actions after both histories he may face, by definition of $\tilde{w}^2$, and since $\underline{w}_1^3 = \underline{w}_1^2$ (because W is a ball). Given this, after history $\tilde{w} = w$, player 2 chooses between getting $\tilde{w}_2$ (if he chooses `Yes`) or $\underline{w}_2^2$ (if he chooses `No`), so that choosing `Yes` is optimal; after history $\tilde{w} \neq w$, he chooses between $\tilde{w}_2^2$ (if he chooses `Yes`) or $\underline{w}_2^3$ (if he chooses `No`), and choosing `No` is optimal as well given (4.1). Given this, player 3 faces an initial choice between getting w_3 by choosing $\tilde{w} = w$ or getting $\underline{w}_3^3$ by choosing another action. Choosing $\tilde{w} = w$ is optimal.

It is not difficult to see that similar constructions exist for some sets W which do not have the properties specified earlier. However, the only feature that has been assumed which turns out to be restrictive for our purposes (namely, that $\underline{w} \in W$ in case $n = 2$) cannot be dispensed with: if $\underline{w} \notin W$, there exists no game (extensive-forms or simultaneous-move) solving the problem described in this subsection. Indeed, since there must be an equilibrium in which player 1 receives his minimum over the set W, player 2 must have a strategy s_2 such that player 1's payoff equals (at best) to this lowest payoff independently of his own strategy; similarly, player 1 must have a strategy s_1 such that player 2's payoff equals to his lowest payoff independently of his own strategy. The pair of strategies (s_1, s_2) must yield a payoff vector from the set

W, and so it must be the vector that minimizes simultaneously the payoffs of both players.

4.2.2. Sequential communication with simultaneous actions

One problem with the repeated game is that all players take actions in every period, not just whichever player is most convenient for the purpose of our construction. To address this issue, we must ensure that there exists a mapping from public signals to messages such that a player can (at least probabilistically) get his intended message across, through his choice of actions, independently of his opponents' actions.

Binary messages, say a or b, are good enough. We address next the following problem: Given some mapping $f : Y \to [0, 1]$, consider the one-shot game where each player j chooses an action $\alpha_j \in \triangle A_j$, but his payoff depends only on the message, which is equal to a with probability $f(y)$, where y is the public signal that results from the action profile that is chosen; in the repeated game, the message is equal to a if the public randomization device takes (in the following period) a value $x \in [0, f(y)]$. Fix one player, say player i. Can we find f and action profiles α^* and β^* such that

(i) for every $j \neq i$, actions of player j do not affect the probability of message a, given that his opponents take the action profile α^*_{-j} or β^*_{-j};

(ii) player i maximizes the probability of message a by taking action α^*_i, given that his opponents take the action profile α^*_{-i}, and player i maximizes the probability of message b by taking action β^*_i, given that his opponents take the action profile β^*_{-i};

(iii) the probability of message a is (strictly) higher under α^* than under β^*?

We argue here that the answer is positive, under the identifiability assumption (full-rank condition) that we have imposed. We take first an arbitrary pure action profile α^*. Then we define β^* as an arbitrary pure action profile that differs from α^* only by the action of player i. Finally, we will define the values $\{f(y) : y \in Y\}$ as a solution of a system of

$$m_i + 2\sum_{j\neq i}(m_j - 1)$$

linear equations. In this system of equations, there is one equation that corresponds to the action profile α^*, one equation that corresponds to β^*, one equation for every pure action profile that differs from α^* (and so from β^*) only by the action of player i, and one equation for every pure action profile that differs from α^* or β^* only by the action of one of the players $j \neq i$. Every pure action profile determines a probability distribution over public signals $y \in Y$. We take the probability assigned to the public signal y as the coefficient of the unknown $f(y)$, and the right-hand constant is equal to one of two numbers l or h, where $l < h$. It is h for α^* and any pure action profile that differs from α^* only by the action of one of the players $j \neq i$, and it is l for the remaining action profiles.

This system of equations has a solution because the vectors of coefficients of particular equations are linearly independent (by the full-rank condition). If it happens that the values $\{f(y) : y \in Y\}$ do not belong to $[0, 1]$, we replace them with $\{f'(y) : y \in Y\}$ where

$$f'(y) = cf(y) + d;$$

we pick some positive c and d to make sure that the values $\{f'(y) : y \in Y\}$ belong to $[0, 1]$. This changes the right-hand constants to $l' = cl + d$ and $h' = ch + d$, but the property that $l' < h'$ is preserved.

By construction, if player i takes action α_i^* the probability of message $\mathtt{a}$ is equal to h given the other players take the action profile α_{-i}^*; if he takes any other action, it is equal to l. On the other hand, no other player $j \neq i$ can unilaterally affect the probability of message $\mathtt{a}$; it is h independently of his action. Similarly, if player i takes action β_i^* (or any action other than α_i^*) the probability of message $\mathtt{a}$ is equal to l given that the other players take the action profile $\beta_{-i}^* = \alpha_{-i}^*$ (it is h if he takes action α_i^*), and no other player $j \neq i$ can unilaterally affect the probability of message $\mathtt{a}$. This yields properties (i)-(ii); the probability of message $\mathtt{a}$ is equal to h and l contingent on α^* and β^*, respectively; thus, (iii) follows from the assumption that $l < h$.

Notice finally that the method of interpreting signals as messages described in this section could be relatively simple due to the communication protocol that allowed only one player to speak at a time. If several players were allowed to speak at a time, we would have to design a method of interpreting signals as message profiles such that players can choose their own messages but cannot affect the messages of other players.

4.2.3. The communication phase

We now design the communication phase in the repeated game. Flow payoffs are still ignored, so we assume that the payoff vector is equal to the continuation payoff that will be communicated.

Suppose players want to communicate the vector w (a vertex of the polyhedron $P(\delta)$). In the communication phase, players send binary messages by the means of a number of periods in which they play α^* or β^*. This should be interpreted as follows: In every single period, the public outcome translates into a message (`a` or `b`) in the way we have just described. We fix a threshold number of messages `a` and we interpret the message of the communication phase as `a` if the number of single periods in which `a` has been received exceeds this threshold. The threshold number and the number of periods of the communication phase are chosen such that if players play α^* (or respectively, β^*) in all periods, then the message of the communication phase will be `a` (respectively `b`) with high probability. (We will specify later what we mean by high probability.)

Case $n = 2$ Begin with the case $n = 2$. Ignore first that every message consists of a sequence of binary messages. However, recall that information transmission is imperfect, messages received are equal to message prescribed by the equilibrium strategy only with probability close to, but not equal to one, so that players face lotteries. The extensive form is defined as follows. See also Figure 3.

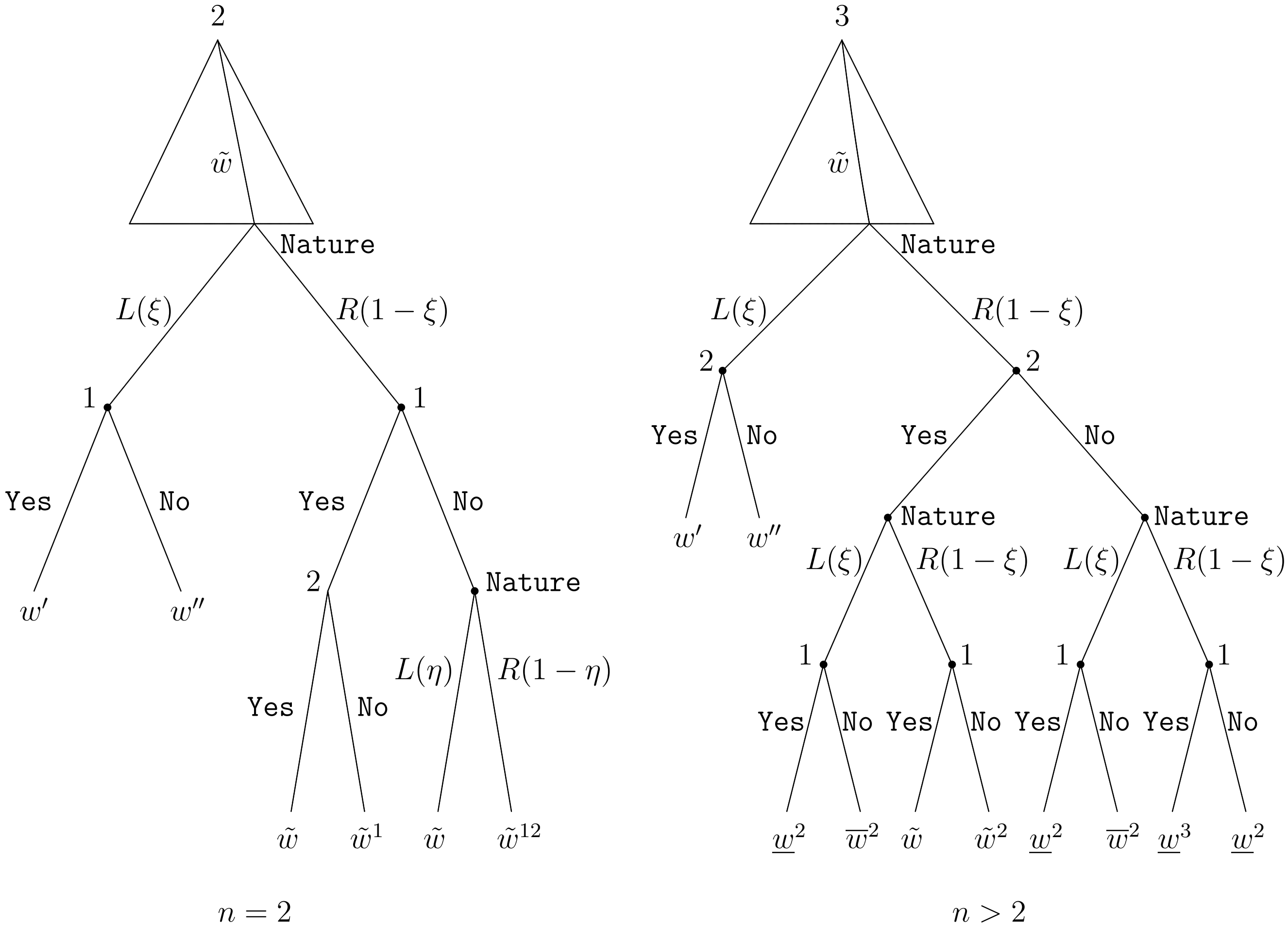

Figure 3

- Player 2 moves first, choosing any $\tilde{w}$ (a vertex of the polyhedron $P(\delta)$).
- Nature (the public randomization device) moves second, choosing `Left` and `Right` with probability ξ and $1-\xi$, respectively (to be defined).
- Player 1 moves third, choosing `Yes` or `No`.
- If Nature has chosen `Right` and Player 1 has chosen `No`, Nature chooses `Left` and `Right` with probability $\eta(\tilde{w})$ and $1-\eta(\tilde{w})$, respectively (to be defined).

- If Nature has chosen `Right` and Player 1 has chosen `Yes`, Player 2 chooses `Yes` or `No`.

  Payoff vectors: The payoff vector is defined as follows. If the history is ($\tilde{w}$, `Left`, `Yes`), the payoff vector is w' to be defined; if it is ($\tilde{w}$, `Left`, `No`), the payoff vector is w'' to be defined; if it is ($\tilde{w}$, `Right`, `Yes`, `Yes`), the payoff vector is $\tilde{w}$; if it is ($\tilde{w}$, `Right`, `Yes`, `No`), the payoff vector is $\tilde{w}^1$; if the history is ($\tilde{w}$, `Right`, `No`, `Left`), the payoff vector is $\tilde{w}$; finally, if the history is ($\tilde{w}$, `Right`, `No`, `Right`), it is $\tilde{w}^{12}$.

  The vectors w' and w'' are vertices of $P(\delta)$ such that player 1 is indifferent between both, but player 2 strictly prefers w' to w'' (thus, any strategy of player 1 in the event that Nature picks `Left` will be optimal).

  Notice two differences between the communication protocol studied in this section compared to that studied in Section 3.2.1. First, the protocol has been extended by two moves of Nature; this turns out necessary to give players incentives if players can get their messages across only probabilistically. Second, players communicate only vertices of the polyhedron $P(\delta)$. In particular, the payoff vectors $\tilde{w}^1$ and $\tilde{w}^{12}$ are defined with respect to the polyhedron $P(\delta)$, instead of the set W, and it is assumed that for every vertex $\tilde{w}$, those vectors as well as the vectors w' and w'' are themselves vertices of $P(\delta)$. This of course requires some regularity assumptions about the polyhedron $P(\delta)$, but it is easy to see that such assumptions can be made with no loss of generality.

  Strategies: Given w, the strategy profile $\sigma := (\sigma_1, \sigma_2)$ defined as follows is a sequential equilibrium.

$$\sigma_2(\varnothing) = w,$$

$$\sigma_2(\tilde{w}, \texttt{Right}, \texttt{Yes}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w}, \\ \texttt{No} \text{ otherwise,} \end{cases}$$

$$\sigma_1(\tilde{w}, \texttt{Right}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w}, \\ \texttt{No} \text{ otherwise,} \end{cases}$$

$$\sigma_1(\tilde{w}, \texttt{Left}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w} \text{ and } w \text{ minimizes player 2's payoff over } P(\delta), \\ \texttt{No} \text{ otherwise.} \end{cases}$$

Observe that, given these strategies, if the history is $(\tilde{w}, \texttt{Right})$, with $w \neq \tilde{w}$, player 1 faces a lottery between getting $\tilde{w}$ with low probability and $\tilde{w}^1$ with high probability (by choosing `Yes`), and a lottery between $\tilde{w}$ and $\tilde{w}^{12}$ (by choosing `No`), with probability η and $1-\eta$, respectively. We can thus pick η so as to guarantee that player 1 is indifferent between both choices in this event. Note that η is accordingly low (it is equal to the probability of receiving message `Yes` when players intend to communicate message `No`).

In case w minimizes player 2's payoff over $P(\delta)$, player 2's incentive for revealing truthfully $\tilde{w} = w$ comes from the event that Nature picks `Left` in the second round, since in that case truth-telling (resp. lying) gives a lottery between w' and w'' with high (resp. low) probability on w'. Since in the event that Nature picks `Right` in the second round, player 2 gets $\tilde{w}_2^{12} = \underline{w}_2^2$ with very high probability, we can ensure truthful revelation by picking $\xi > 0$, but low nevertheless (again, this probability can be set proportional to the probability of receiving the opposite message to one that players intend to communicate).

The remaining equilibrium conditions are immediate to verify, given the definitions of $\tilde{w}^1$and $\tilde{w}^{12}$. The argument is analogous to that from Section 3.2.1. It is, however, important to emphasize that, for that argument to work, the probability of receiving the opposite message to one that players intend to communicate (through the sequence of action profiles α^* or β^*) must be low compared to the differences in coordinates of $P(\delta)$ whenever they are different.

Notice that the action profiles α^* and β^* have been constructed so that the actions of players other than one who is supposed to move do not affect messages. Thus, only the player who is supposed to move has to be given incentives. Further, recall that any message $\tilde{w}$ consists of a sequence of binary messages. It may therefore happen that player 2 who is supposed to communicate $\tilde{w} = w$ learns for sure that he failed to send the prescribed binary message in one of the rounds. Then player 2's strategy prescribes, in the remaining rounds, binary messages that maximize player 2's payoff (among payoff vectors that still can be communicated).

Case $n > 2$ Consider now the case $n > 2$. As before, ignore that any message $\tilde{w}$ requires several periods of binary messages. This problem can be dealt with in a manner similar to that for the two-player case. Recall again that messages are only equal to the prescription of the equilibrium strategy with probability close to, but not equal to one, so that players face lotteries. The extensive form is defined as follows. See also Figure 3.

- Player 3 moves first, choosing any $\tilde{w}$ (a vertex of the polyhedron $P(\delta)$).
- Nature (the public randomization device) moves second, choosing `Left` and `Right` with probability ξ and $1-\xi$, respectively (to be defined).
- Player 2 moves third, choosing `Yes` or `No`.
- If Nature has chosen `Right` in the second stage, Nature then moves again, choosing `Left` and `Right` with probability ξ and $1-\xi$, respectively.
- If Nature has chosen `Right` in the second stage, player 1 moves last, choosing `Yes` or `No`.

Payoff vectors: The payoff vector is defined as follows. If the history is ($\tilde{w}$, `Left`, `Yes`), the payoff vector is w' to be defined; if it is ($\tilde{w}$, `Left`, `No`), the payoff vector is w'' to be defined; if it is ($\tilde{w}$, `Right`, `Yes`, `Right`, `Yes`), the payoff vector is $\tilde{w}$; if it is ($\tilde{w}$, `Right`, `Yes`, `Right`, `No`), the payoff vector is $\tilde{w}^2$; if the history is ($\tilde{w}$, `Right`, $\mathtt{m}_2$, `Left`, `Yes`), the payoff vector is $\underline{w}^2$ for $\mathtt{m}_2 = \mathtt{Yes,No}$; if the history is ($\tilde{w}$, `Right`, $\mathtt{m}_2$, `Left`, `No`), it is $\bar{w}^2$ (to be defined), ; if the history is ($\tilde{w}$, `Right`, `No`, `Right`, `Yes`), it is $\underline{w}^3$; finally, if the history is ($\tilde{w}$, `Right`, `No`, `Right`, `No`), it is $\underline{w}^2$. Modifying the polyhedron $P(\delta)$ if necessary, we can assume that all payoff vectors are vertices of this polyhedron.

The vertices w', w'' are such that both player 1 and 2 are indifferent between them and player 3 strictly prefers w' to w''. Therefore, any strategy of player 2 when Nature chooses `Left` in the second stage will be optimal.

The vertex $\bar{w}^2$ is defined as the element of W that gives the highest payoff to player 2 within the ball W; therefore, player 1 is indifferent between $\bar{w}^2$ and $\underline{w}^2$. Since player 1 is also indifferent between $\underline{w}^3$ and $\underline{w}^2$ as well as between $\tilde{w}$ and $\tilde{w}^2$, this ensures that any strategy of player 1 will be optimal for every possible history.

Similarly to the case of two players, vectors $\tilde{w}^2$ are defined with respect to the polyhedron $P(\delta)$, instead of the set W, and it is assumed that all terminal payoff vectors are vertices of $P(\delta)$.

- *Strategies*: Given w, the strategy profile $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ defined as follows is a sequential equilibrium.

$$\sigma_3(\varnothing) = w,$$

$$\sigma_2(\tilde{w}, \texttt{Left}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w} \text{ and } w \text{ minimizes 3's payoff over } P(\delta), \\ \texttt{No} \text{ otherwise,} \end{cases}$$

$$\sigma_2(\tilde{w}, \texttt{Right}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w}, \\ \texttt{No} \text{ otherwise,} \end{cases}$$

$$\sigma_1(\tilde{w}, \texttt{Right}, \texttt{Yes}, \texttt{Right}) = \begin{cases} \texttt{Yes} \text{ if } w = \tilde{w} \\ \texttt{No} \text{ otherwise,} \end{cases}$$

$$\sigma_1(\tilde{w}, \texttt{Right}, \texttt{Yes}, \texttt{Left}) = \begin{cases} \texttt{No} \text{ if } w = \tilde{w} = \underline{w}^2 \\ \texttt{Yes} \text{ otherwise,} \end{cases}$$

$$\sigma_1(\tilde{w}, \texttt{Right}, \texttt{No}, \texttt{Right}) = \begin{cases} \texttt{No} \text{ if } w = \tilde{w} \\ \texttt{Yes} \text{ otherwise,} \end{cases}$$

$$\sigma_1(\tilde{w}, \texttt{Right}, \texttt{No}, \texttt{Left}) = \begin{cases} \texttt{No} \text{ if } w \neq \tilde{w} \text{ and } \tilde{w} = \hat{w}^1 \text{ or } \hat{w}^2 \\ \texttt{Yes} \text{ otherwise,} \end{cases}$$

where $\hat{w}^1$, $\hat{w}^2$ are to be defined.

To verify incentives of player 2 when Nature has chosen Right, we consider the cases $w = \tilde{w}$ and $w \neq \tilde{w}$ separately.

- $w = \tilde{w}$: By choosing Yes, player 2 faces a compound lottery; with probability $1 - \xi$, the compound lottery yields a lottery on $\{\tilde{w}, \tilde{w}^2\}$ with high probability on $\tilde{w}$; with probability ξ, the compound lottery yields a lottery on $\{\bar{w}^2, \underline{w}^2\}$; by choosing No, player 2 also faces a compound lottery; with probability $1 - \xi$, it yields a lottery on $\{\underline{w}^3, \underline{w}^2\}$ with high probability on $\underline{w}^2$; with probability ξ, it yields a lottery on $\{\bar{w}^2, \underline{w}^2\}$. Therefore, it is clear that player 2 prefers Yes, unless $w = \underline{w}^2$. In that case, however, choosing Yes is optimal because it guarantees that the probability of $\bar{w}^2$ is high, while it is low otherwise.
- $w \neq \tilde{w}$: By choosing No, player 2 faces a compound lottery; with probability $1 - \xi$, it yields a lottery on $\{\underline{w}^3, \underline{w}^2\}$ with high probability on $\underline{w}^3$; with probability ξ,

it yields a lottery on $\{\bar{w}^2, \underline{w}^2\}$; by choosing `Yes`, player 2 also faces a compound lottery; with probability $1-\xi$, it yields a lottery on $\{\tilde{w}, \tilde{w}^2\}$ with high probability on $\tilde{w}^2$; with probability ξ, it yields a lottery on $\{\bar{w}^2, \underline{w}^2\}$. Therefore, player 2 prefers `No`, unless $\tilde{w}$ is one of the two extreme points $\hat{w}^1$, $\hat{w}^2$ for which both $\tilde{w}_2^2 = \underline{w}_2^3$. In that case, however, choosing `No` is optimal because it guarantees that the probability of $\bar{w}^2$ is high, while it is low otherwise.

Finally, the incentives of player 3 are immediate to verify.

4.3. The Effect of Flow Payoffs on Incentives

Finally, repeated game payoffs depend also on flow payoffs in the communication phase. Those flow payoffs may affect players' incentives to communicate the continuation payoff vector truthfully. However, this problem is easy to deal with.

Suppose that in the first period that follows the communication phase, the players play with some probability $\phi > 0$ not strategies that yield the continuation payoff vector that they just have communicated, but strategies that yield another payoff vector $v \in W$. It is decided by the public signal x observed in the first period that follows the communication phase which of the two continuation payoff vectors will be played. The vector v is contingent on the public outcomes observed in the communication phase, so that it makes every player indifferent (in every period of the communication phase) across all actions given any action profile of the opponents. The full-rank condition and the full-dimensionality condition guarantee the existence of vector v, provided that the differences in flow payoffs in the communication phase are small enough compared to the continuation payoff vector v, even if it is received only with the probability ϕ. However, it is indeed the case for every $\phi > 0$ if δ is large enough. Further, if ϕ is small and communication phases are sufficiently infrequent, the repeated game payoff vector converges to that for $\phi = 0$.

More precisely, we can assume that ϕ is of order proportional to the flow payoffs during the communication phase, i.e. it is proportional to

$$1 - \delta^{N(\delta)}.$$

4.4. The Length of Communication Phase

We have prescribed strategies that give players incentives to communicate the continuation payoff vector truthfully, assuming that the communication phase is sufficiently short (i.e. $N(\delta)$ satisfies the first part of (3.1)[2]). We have to check now that players indeed need only this short communication phase for the prescribed protocol. It suffices to show that

$$\lim_{m\to\infty} (\delta_m)^{N(\delta_{m+1})} = 1 \tag{4.2}$$

where

$$\delta_m := 1 - \frac{1}{2^m}.$$

Indeed, we shall construct a specific sequence of polyhedra $P(\delta_m)$, $m = 1, 2, ...$, and we shall define $P(\delta)$ as $P(\delta_{m+1})$ for every δ such that $\delta_m \le \delta < \delta_{m+1}$. Then

$$(\delta_m)^{N(\delta_{m+1})} \le (\delta)^{N(\delta)} \le 1,$$

and the first part of (3.1) follows from (4.2).

Observe first that the polyhedron $P(\delta_m)$ can be chosen so that its number of vertices is proportional to k^m, i.e. it is equal to $c \cdot k^m$ where c and k are constants, independent of m. Indeed, denote the boundary of the set W by bdW; consider the standard Euclidean distance in the set W, and define the distance of a vector $w \in W$ from a compact set $V \subset W$ by

$$dist(w, V) = \min_{v\in V} dist(w, v).$$

It follows directly from FLM (or the proof of Theorem 1 in APS) that any given payoff vector from the boundary of W is represented as the weighted average of the current period payoff vector taken with weight $1 - \delta_m$ and a linear (convex) combination of continuation payoff vectors taken with weight δ_m. Those continuation

[2]The second part of (3.1) is straightforward, as we can specify the regular phase to be arbitrarily long.

payoff vectors converge (as $\delta_m \to 1$) to the given payoff vector from the boundary approximately along straight lines, and so their distance from the boundary of W is proportional to $1-\delta_m$. Recall that we modify in the present paper the construction of FLM to repeated games in which players discount payoffs by δ_m for $M(\delta_m)-1$ periods, and then their discount factor drops, for one period, to $\delta_m^{N(\delta_m)}$. One can easily see, however, that this makes the distance of continuation payoff vectors (in the period follows the communication phase) from the boundary of W even of a larger order that $1-\delta_m$.

By continuity and compactness arguments, the set of continuation payoff vectors for the entire boundary of W is contained in the set

$$W'_m = \{w \in W : dist(w, bdW) \geq d \cdot (1-\delta_m)\},$$

where d is a constant, independent of m. We shall now construct a polyhedron $P(\delta_m)$ such that

$$W'_m \subset P(\delta_m) \subset W.$$

Consider the case $n=2$. In this case, we assume that the boundary of W (and so the boundary of W'_m) consists of three quarters of a circle and a quarter of a square, as it is depicted in the right panel of Figure 1. We shall first construct the polyhedron $P(\delta_m)$ with the required properties under the assumption that the boundary of W is a circle, and then we shall argue that the construction generalizes to the shape in which one of the quarters of this circle is replaced with a quarter of a square.

Define $P(\delta_m)$ so that it has 2^m vertices that belong to the boundary of the ball W, and the edges of the boundary of $P(\delta_m)$ have equal length, as it is shown in Figure 4(a). Denote by x_m the distance between the middle of an edge and the boundary of W. Notice that

$$x_m = r\left(1-\cos\frac{\pi}{2^m}\right),$$

where r denotes the radius of W, and so

$$\frac{x_m}{2^{-m}} = r\frac{1-\cos\frac{\pi}{2^m}}{\frac{1}{2^m}};$$

this expression converges to 0 (as $m \to \infty$), because

$$\frac{1 - \cos \pi x}{x} \to_{x \to 0} 0.$$

Thus, x_m converges to 0 faster than 2^{-m}, which yields that $W'_m \subset P(\delta_m)$ (at least when m is large enough).

If the boundary of W consists of three quarters of a circle and a quarter of a square, then the polyhedron $P(\delta_m)$ may have even fewer vertices, as we need only three vertices, independently of m, for the entire square quarter. In the case of $n > 2$, we can take as $P(\delta_m)$ a polyhedron whose boundary consists of regular simplices (i.e. simplices with equal edges) of the same size, such that all vertices of $P(\delta_m)$ belong to the boundary of W. Actually, the number of vertices $P(\delta_m)$ need not be exponential; there exist polyhedra $P(\delta_m)$ with the required properties, and a polynomial number of vertices. See Böröczky (2001), which also contains the proof for the general case of $n \geq 2$.[3]

Our construction from Section 3.2.3 requires that for every vertex $\tilde{w}$, the payoff vectors $\tilde{w}^1$ and $\tilde{w}^{12}$ (or $\tilde{w}^2$) as well as the vectors w' and w'' (and $\underline{w}^2$, $\bar{w}^2$, $\underline{w}^3$) are themselves vertices of $P(\delta_m)$. It can, however, be seen that the construction of $P(\delta_m)$ given above can easily achieve this goal.[4]

Thus, players need only a number of binary message that is proportional to m (say $c \cdot m$, for a constant $c > 0$) to communicate a desired vertex of $P(\delta_m)$. Indeed, the set of all vertices of $P(\delta_m)$ can be divided into halves, and the first binary message reveals which half contains the vertex to be communicated; then each half is divided into halves again, and the second binary message reveals which half within that half contains the vertex, and so forth. If players were able to communicate binary messages accurately, this would already conclude the proof of (4.2), as

$$\left(1 - \frac{1}{2^m}\right)^{m+1} \to_m 1.$$

[3] We are indebted to Rakesh Vohra for this reference.

[4] Of course, there cannot then be only three vertices corresponding to the square quarter, but this number of vertices must be equal to that corresponding to every circle quarter.

However, players communicate only by means of sequences of action profiles α^* and β^*; in every single period, the public outcome translates into a single-period message (`a` or `b`). When α^* is played the probability of message `a` is higher compared to when β^* is played; the message of the communication phase is interpreted as `a` if the number of single periods in which `a` has been received exceeds a threshold number.

This sort of communication imposes the following problem: Since players do not communicate accurately, when they take the sequence of action profiles that corresponds to a vertex of $P(\delta_m)$, they end up only with a probability distribution over the messages that correspond to various vertices of $P(\delta_m)$, and the expected value of this probability distribution typically does not coincide with the vertex to be communicated.

To address this problem, we first slightly modify the polyhedron $P(\delta_m)$ so that the distance of any vertex of $P(\delta_m)$ to the boundary of W can be bounded by a number proportional to $1 - \delta_m$. In the case of $n = 2$, replace each vertex of $P(\delta_m)$ with a vertex that lies on the line joining the original vertex with the origin of W whose distance from the boundary of W is equal to a half of the distance between W and W'_m; moreover, for every edge of the boundary of $P(\delta_m)$, add a vertex that lies on the line joining the origin of W and the middle of this edge whose distance from the boundary of W is also equal to a half of the distance between W and W'_m (see Figure 4(b)). The modified polyhedron $P(\delta_m)$ contains W'_m, because the point from its edge that is closest to bdW'_m coincides with the intersection point of this edge and an edge of the original $P(\delta_m)$. Moreover the modified $P(\delta_m)$ has only twice as many vertices as the original $P(\delta_m)$, so its number of vertices is proportional to k^m for a constant k.

The argument is analogous in the case of $n > 2$; the distance of each vertex of the modified $P(\delta_m)$ from the boundary of W is equal to a half of the distance between W and W'_m, and each regular simplex that the boundary of the original $P(\delta_m)$ is composed of is replaced with 2^{n-1} regular simplices in the modified $P(\delta_m)$. Again, see Böröczky (2001) for a rigorous proof that a modified polyhedron $P(\delta_m)$ with the required properties exists in the general case of $n \geq 2$. It can also be assumed without loss of generality that all terminal payoff vectors of the communication game described in Section 3.2.3 are vertices of $P(\delta_m)$.

Now, suppose that players communicate the vertices of another, larger and homothetic polyhedron $Q(\delta_m)$.[5] If the communication is sufficiently accurate, the sequence of action profiles that corresponds to a vertex of $Q(\delta_m)$ induces a probability distribution over the messages that correspond to various vertices of $Q(\delta_m)$ with the expected value that is close to the desired vertex; if the desired vertex is communicated successfully at least with probability $1-\varepsilon$, then the distance of this expected value to the desired vertex is proportional to ε (of course, this expected value belongs to the (convex) polyhedron $Q(\delta_m)$). Denote by $Q'(\delta_m)$ the polyhedron spanned by the expected values of the probability distributions induced by sequences of action profiles that correspond to all vertices of $Q(\delta_m)$. By taking ε small enough, but still proportional to $1-\delta_m$, we can make sure that the polyhedron $Q'(\delta_m)$ contains $P(\delta_m)$. Thus, the vertices of $P(\delta_m)$ can be communicated in expectation in the communication phase in which players communicate the vertices of $Q(\delta_m)$, and the vertex to be communicated is determined by the public randomization device from the first period of the communication phase.

Recall from Section 3.2.3 that we require ε to be small also for another reason: The probability of receiving the opposite message to one that players intend to communicate (through the sequence of action profiles α^* or β^*) must be low compared to the differences in coordinates of $Q(\delta)$ whenever they are different. However, $Q(\delta_m)$ can be easily constructed such that those differences are of order

$$x'_m = r'\left(1 - \cos\frac{\pi}{2^{m+1}}\right),$$

where r' stands for the "radius" of the polyhedron $Q(\delta_m)$, or higher (in Figure 4(c), we depict the case in which the difference is the lowest possible if the extreme points

[5]The assumption that $Q(\delta_m)$ and $P(\delta_m)$ are homothetic guarantees all payoff vectors used the communication phase described in Section 3.2.3 are vertices of this polyhedron $Q(\delta_m)$, as so they were for the polyhedron $P(\delta_m)$.

of the circle are vertices of $Q(\delta_m)$). Since

$$\frac{x'_m}{r'(\frac{\pi}{2^{m+1}})^2} \to_m \frac{1}{2},$$

it suffices to take an ε proportional to 2^{-2m-2}.

The sufficient accuracy of communication can be achieved only if the action profiles α^* and β^* (corresponding to each binary message) are played repeatedly for a sufficiently large number of periods. We shall now estimate that number of periods.

By Hoeffding's inequality, players need a number of periods (in which they play α^* or β^*) that is proportional to $-\log\varepsilon$ to make sure that the binary message is interpreted as $\mathtt{a}$ or $\mathtt{b}$ at least with probability $1-\varepsilon$. Therefore it suffices to have a number of periods that is proportional to

$$-m\log\left(\frac{\varepsilon}{m}\right)$$

to communicate successfully any vertex of $Q(\delta_m)$ at least with probability $1-\varepsilon$; indeed, if every binary message (out of $c\cdot m$) is successful at least with probability

$$1-\frac{\varepsilon}{cm},$$

then the desired vertex is communicated successfully at least with probability

$$\left(1-\frac{\varepsilon}{cm}\right)^{cm} \geq 1-\varepsilon.$$

Since we take an ε proportional to $(2^{-m-1})^2$, one needs only a number of periods that is proportional to

$$-m\log\left(\frac{1}{m2^{2m+2}}\right) = m\log\left(m2^{2m+2}\right)$$

to communicate any desired vertex of $P(\delta_m)$ in expectation, and so (4.2) follows from the fact that

$$\left(1-\frac{1}{2^m}\right)^{(m+1)\log\left[(m+1)2^{2m+4}\right]} \to_m 1.$$

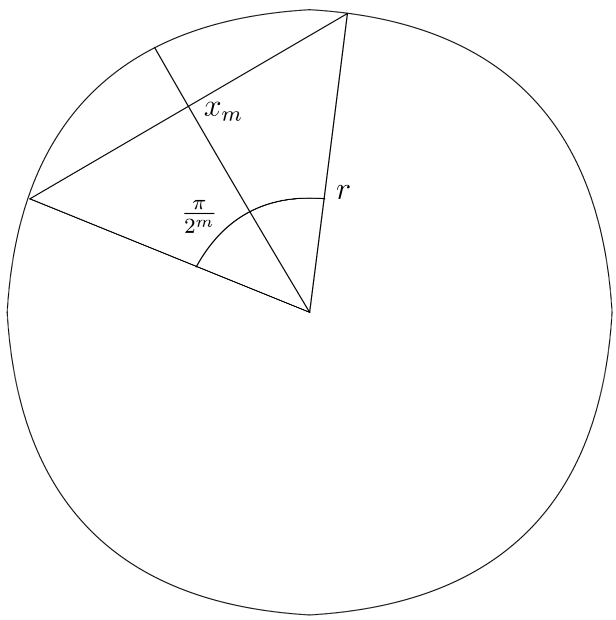

Figure 4(a): The polyhedron $P(\delta_m)$

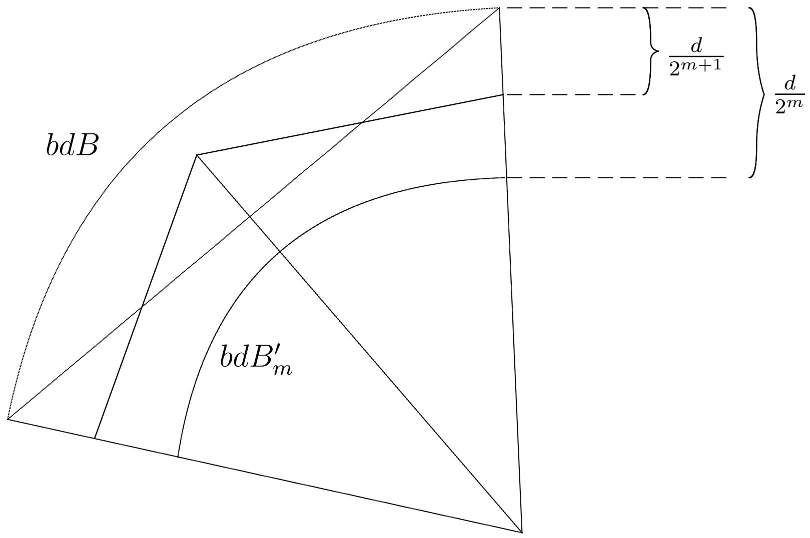

Figure 4(b): The modified polyhedron $P(\delta)$. The boundary of the original $P(\delta)$ is depicted by the straight line, and the boundary of the modified $P(\delta)$ is depicted by the kinked line.

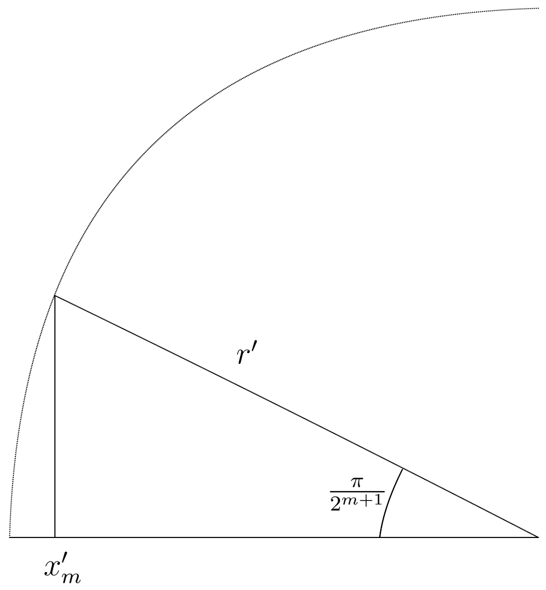

Figure 4(c)

5. The Folk Theorem under Private (Almost-Public) Monitoring

5.1. Result

In this section, we use the folk theorem under imperfect public memory and bounded memory to prove that a folk theorem remains valid under almost-public monitoring. The proof follows the line of reasoning of Mailath and Morris (2002), although their result is not directly applicable, as the equilibrium with bounded memory used in the previous section is not uniformly strict. Indeed, our construction critically relies on the indifference of some players during the communication phase.

We may now state the main result of this section.

Theorem 5.1. *Fix a public monitoring structure π satisfying full rank and full support.*

(i) $(n = 2)$ *Suppose that* v^* *corresponds to the equilibrium payoff vector of a strict Nash equilibrium. For any payoff* $v \in V^*$, *there exists* $\bar{\delta} < 1$, *for all* $\delta \in (\bar{\delta}, 1)$, *there exists* $\epsilon > 0$ *such that* $v \in E(\delta)$ *for all private monitoring structures that are* ϵ*-close to* π.

(ii) $(n > 2)$ *For any payoff* $v \in \underline{V}^P$, *there exists* $\bar{\delta} < 1$, *for all* $\delta \in (\bar{\delta}, 1)$, *there exists* $\epsilon > 0$ *such that* $v \in E(\delta)$ *for all private monitoring structures that are* ϵ*-close to* π.

5.2. A Modification of the Public Monitoring Strategies

Before proving the result, we modify the construction in the case of public monitoring. First, we take the action profiles α^* and β^* to be pure; notice that we have actually constructed them as pure action profiles in Section 3.2.2. Additionally, we wish to modify the construction from Section 3.3 ensuring that flow payoffs do not affect incentives during the communication phase. Recall that, with some small probability $\phi > 0$, right after the communication phase (say in period t), the realization of the public randomization device, x_t, is such that continuation payoff vector is not equal to one determined in the communication phase. Rather, as before, continuation play in this event is determined so as to guarantee that all players are indifferent over all actions during all periods of the communication phase that has just finished. Now, we will require the value x_t also to determine a period of the communication phase and a player. The interpretation is that continuation play in the event that period τ and player i have been determined is supposed to guarantee that player i is indifferent across all his actions in period τ.

Observe that there exists at most one player, say player j, whose action in period τ depends on the continuation payoff vector to be communicated (namely, the player who is sending a binary message in period τ). Pick some player k as follows (in the interpretation player k will control that player i is indifferent across all his actions in period τ of the communication phase). If $i \neq j$, let $k = j$, and if $i = j$, let k be some player different than i. It is an important property that player k always knows all equilibrium actions that were taken in period τ, except possibly the action of player

i (when $j = i$).

Fix some pure action profiles $a, a' \in A$, with $a_{-k} = a'_{-k}$, $a_k \neq a'_k$; these will be action profiles to be taken in period t, by means of which player k controls that player i is indifferent across all his actions in period τ. By full-rank, we can find continuation payoff vectors $\gamma(y) \in \mathbb{R}^n$, for all $y \in Y$, such that:

(i) for all players $j \neq k$, a_j strictly maximizes

$$(1-\delta)\, u_j(\tilde{a}_j, a_{-j}) + \delta \sum_y \pi(y|\tilde{a}_j, a_{-j})\, \gamma_j(y),$$

and

$$(1-\delta)\, u_j\left(\tilde{a}_j, a'_{-j}\right) + \delta \sum_y \pi\left(y|\tilde{a}_j, a'_{-j}\right) \gamma_j(y);$$

(ii) player k's payoff

$$(1-\delta)\, u_k(\tilde{a}_k, a_{-k}) + \delta \sum_y \pi(y|\tilde{a}_k, a_{-k})\, \gamma_k(y)$$

is maximized both by a_k and a'_k;

(iii) player i's payoff is under a exceeds that under a':

$$(1-\delta)\, u_i(a) + \delta \sum_y \pi(y|a)\, \gamma_i(y) - (1-\delta)\, u_i(a') + \delta \sum_y \pi(y|a')\, \gamma_i(y) > K,$$

for some constant $K > 0$ to be specified.

That is, the action a_j is players $j \neq k$'s best-reply to $a_{-j} = a'_{-j}$ in the game in which public-outcome dependent continuation payoffs are given by γ; player k is indifferent between both actions a_k and a'_k, given $a_{-k} = a'_{-k}$, and player i strictly prefers action profile a to a'.

The constant K is assumed not to be too small for a given δ, but any positive K is sufficient for our purposes and sufficiently large discount factors. More precisely, we require that there exists $f_k : A_k \times Y \to (0,1)$, mapping the action played by player k in period τ and the signal player k received in period τ into the probability of him taking action a_k in period t with the following property: Player i is indifferent in period τ across all his actions, given that in the event that x_t picks player i and period τ in period t, player k chooses action a_k (rather than a'_k) with the probability given by f_k (and given that, in this event, continuation payoff vectors are given from period $t+1$ on by γ). Notice that, by full-rank, f_k can be chosen so that the actions (in period τ) of players other than i and k do not affect the expected value of f_k.

Thus, in the event that x_t has picked player i and period τ, player k randomizes in period t between actions a_k and a'_k so as to ensure that player i is indifferent across all his actions in period τ. Meanwhile, players $j \neq k$ play their strict best-reply $a_j = a'_j$. Continuation payoffs are then chosen according to γ, which can, without loss of generality, always be picked inside the polyhedron $P(\delta)$. Play then proceeds as before, given the public signal y observed in period t, players use the outcome x_{t+1} of the public randomization device to coordinate on some vertex of $P(\delta)$ giving $\gamma(y)$ as an expected value (where the expectation is taken over x_{t+1}), and we may as well assume that each vertex is assigned strictly positive probability.

5.3. Proof

We now return to private monitoring. The proof begins with the observation that the strategies in regular phases can be chosen uniformly strict, in the sense that there exists an $\nu > 0$ such that each player prefers his equilibrium action by at least ν to any other action. Indeed, Theorem 6.4 of FLM (and the remark that follows that theorem) establish that for any smooth set $W \subseteq V^*$, or $W \subseteq V^P$, there exists a minimal discount factor above which all elements of W correspond to strict perfect public equilibria. (A PPE is strict if each player strictly prefers his equilibrium action to any other action). Because in our construction, continuation payoffs can be drawn from a finite set of vertices (of the polyhedron $P(\delta)$), it follows that all elements of W correspond to uniformly strict perfect public equilibria, i.e. there exists $\nu > 0$ such that for any history of the regular phase for which the continuation value is not given by $\underline{w}$ (in the case of $n = 2$), each player prefers his equilibrium action by at least ν to any other action.[6] The assumption that v^* corresponds to the equilibrium

[6]McLean, Obara and Postlewaite (2005) provides an alternative proof that the equilibrium strategies used in the construction of FLM can be chosen to be uniformly strict.

payoff vector of a strict Nash equilibrium implies that some $\nu > 0$ also applies to the histories for which the continuation value is given by $\underline{w}$.

The following lemma is borrowed from Mailath and Morris (2002):

Lemma 1. *(Mailath and Morris) Given η, there exists $\varepsilon > 0$ such that if ρ is ε-close to π, then for all $a \in A$ and $y \in Y$,*

$$\rho_i(y, \ldots, y|a, y) > 1 - \eta.$$

This lemma states that each player assigns probability arbitrarily close to one to all other players having observed the same signal as he has, independently of the action profile that has been played and the signal he has received, provided that private monitoring is sufficiently close to public monitoring. [Note that this result relies on the full support assumption.] For each integer T, this lemma immediately carries over to finite sequences of action profiles and signals of length no more than T; it implies that, if all players use strategies of finite memory at most T, each player assigns probability arbitrarily close to one to all other players following the continuation strategy corresponding to the same sequence of signals he has himself observed (see Theorem 4.3 in Mailath and Morris).

This implies that we can pick $\varepsilon > 0$ so that player i's continuation value to any of his strategies, given a private history h_i^t, is within $\nu/3$ of the value obtained by following the equilibrium strategy under public monitoring; we identify here the private history h_i^t under imperfect private monitoring with the corresponding history under public monitoring. (This is Lemma 3 of Mailath and Morris (2002)). This implies that, for this or lower $\varepsilon > 0$, by the one-shot deviation principle, players' actions in the regular phase under public monitoring remain optimal under imperfect private monitoring, since incentives were uniformly strict under public monitoring by the constant of ν.

It remains to prove that we can preserve players' incentives to play their public monitoring strategies during the communication phase under private monitoring, as well as in the period t right after the communication phase if the realization of x_t is such that continuation play will be making players indifferent over all actions during all periods of the communication phase that has just finished.

Consider first the communication phase. We claim that player k can still pick his action, a_k or a'_k, according to some function $\tilde{f}_k : A_k \times Y \to (0,1)$ (presumably different from f_k under public monitoring), so that to ensure that player i is indifferent between all his actions in period τ, and so that the action of players other than i and k do not affect the expected value of $\tilde{f}_k$. Indeed, player k knows precisely the actions of all players $j \neq i$ in period τ, and the event that the outcome x_t selects player i and period τ is commonly known in period t, so that the expected continuation payoff vector in this event is independent of private histories up to period τ, given the prescribed strategies.

Consider now a realization of x_t in period t such that player k makes player i indifferent over all actions in period τ. First, observe that in period t all players' incentives were strict, except for player k. So, the same action a_j remains optimal for all players $j \neq k$ under private monitoring (even if vertices in the next period are chosen according to distributions that are only close to those used under public monitoring). Recall that, under public monitoring, the signal y observed in period t determines a vertex v of the polyhedron according to a distribution $q(v|y)$ on which all players coordinate by means of the public randomization x_{t+1}.

Let $v_1, \ldots, v_L$ denote the set of vertices. Observe that, under public monitoring, each public history determines a vertex, and the corresponding vertex determines the players' continuation play. Therefore, we can identify each player's strategy with a finite automaton, as in Mailath and Morris, and the private states can be identified with the vertices. Let $v(x_{t+1}|y_i)$ denote the function mapping the outcome of the randomization device in period $t+1$ given i's signal in period t into the vertex determining his continuation play (the same function applies for all players). For definiteness, assume that, if $y \neq y'$,

$$\Pr\{x_{t+1}|v(x_{t+1}|y) = v, v(x_{t+1}|y') = v'\} =$$

$$= \Pr\{x_{t+1}|v(x_{t+1}|y) = v\} \Pr\{x_{t+1}|v(x_{t+1}|y') = v'\},$$

that is, conditional on two different signals, players choose their vertices independently. To see that this is possible, we can view the randomization device as consisting of m independent uniform draws from $[0,1]$ (recall that m is the number of public outcomes), with players using one distinct draw for each distinct public outcome. (Of course, the randomization device has been assumed so far to be a unique draw x from $[0,1]$, but to see that this is equivalent, we can consider the dyadic expansion of x and split this expansion in m infinite expansions, from which we obtain m independent and uniform draws). While this assumption is inessential, it simplifies the argument. Let $q(v|y) := \Pr\{x_{t+1}|v(x_{t+1}|y) = v\}$.

Under imperfect private monitoring, players do not necessarily observe the same signal. Therefore each player j may select different vertices (or states) $v(x_{t+1}|y_j)$ based on the same realization of x_{t+1}, since the signal y_j is private.

Our objective is to show that, if monitoring is sufficiently close to public, we can find distributions $\{q(v|y) : v, y\}$ such that, if players use these distributions in period $t+1$ (and in all corresponding periods for other communication phases), player k is indifferent between the two actions a_k or a'_k (in the event that he must make player i indifferent), and prefers those actions to all others. We shall prove that such distributions exist, using the implicit function theorem, given that ρ is in a neighborhood of π.

Observe that player k's payoff from, say, action profile a is:

$$(1-\delta) g_k(a) + \delta \sum_{(y_1,\ldots,y_n)} \rho(y_1,\ldots,y_n|a) \mathbb{E}_{x_{t+1}} \{V_k(v(x_{t+1}|y_1),\ldots,v(x_{t+1}|y_n))\}, \tag{5.1}$$

where $V_k(v(x_{t+1}|y_1),\ldots,v(x_{t+1}|y_n))$ is player k's expected continuation payoff from the equilibrium strategy, given that each player j is in private state $v(x_{t+1}|y_j)$. By making player k's payoffs from all actions equal to those with public monitoring, we obtain the number of equations equal to m_k, the number of actions of player k. The variables of choice are $\{q(v|y) : v = v_1,\ldots,v_{L-1}, y = y_1,\ldots,y_m\}$; they will be a function of the monitoring structure $\{\rho(y_1,\ldots,y_n|a) : a \in A, (y_1,\ldots,y_n) \in Y^n\}$. (Observe that we omit v_L, since $q(v_L|y)$ is pinned down by the other values). Consider the Jacobi matrix whose rows correspond to the equations and whose columns correspond to the variables of choice, evaluated at the public monitoring structure $\{\pi(y|a) : y, a\}$. The entry corresponding to the equation for action $\widetilde{a}_k$ and column for $v = v_l$ and $y = y_{m'}$ is equal to:

$$\pi\left(y_{m'}\right|\left(\widetilde{a}_k, a_{-k}\right))\left(V_{k,l} - V_{k,L}\right) + \lambda,$$

where $V_{k,l}$ is player k's continuation payoff under public monitoring if vertex l is selected, and λ is the remainder involving only terms premultiplied by $\delta^{M(\delta)}$. This expression is easy to obtain due to the assumption that conditional on two different signals players choose their vertices independently.

By construction, $V_{k,l} \neq V_{k,L}$ for some l. Since $\delta^{M(\delta)} \to 0$, also $\lambda \to 0$. Thus, since π satisfies individual full rank, the rank of our Jacobi matrix is equal to m_k. It is easy to see that the expression (5.1) is continuously differentiable with respect to each $q(v|y)$ (it is actually analytic). That is, the assumptions of the implicit function theorem are satisfied. This guarantees the existence of distributions $\{q(v|y) : v, y\}$ such that the payoff to player k under public and almost-public monitoring are equal (for each of his actions). Finally, since the values of $q(v|y)$ under public monitoring are in $(0, 1)$, it follows that there exists a neighborhood of the public monitoring structure in which the values of $q(v|y)$ under private monitoring are in $(0, 1)$ as well.

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