Multiplicity of Competitive Equilibria in Semi-Algebraic Exchange Economies*

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Abstract

In this paper, we examine multiplicity of equilibria in semi-algebraic exchange economies. We give examples of models that give rise to systems of polynomial equations that have few real solutions, independently of the degree of the polynomials or the number of unknown variables.

We introduce computational methods to find all competitive equilibria for semi-algebraic equilibrium models. We develop a test for uniqueness of equilibria over semi-algebraic and convex set of endowments and preference parameters. The computational methods allow us to bound the maximal number of competitive equilibria for all possible profiles of individual endowments, given semi-algebraic preferences.

We illustrate the method and some mathematical details in a model where all agents have constant elasticity of subsitution utility.

– VERY PRELIMINARY –

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1 Introduction

In this paper we show that Walrasian equilibria in semi-algebraic exchange economies can be characterized as subset of the finite set of solutions to a system of polynomial equations. We argue that from a practical point of view the assumption of semi-algebraic preferences imposes few restrictions on the economic fundamentals and explore mathematical results and algorithms that characterize all solutions to systems of polynomial equations. Under the assumption that all agents have CES utility we give examples that show that one can often show uniqueness of equilibria for a large set of profiles of individual endowments.

It is now well understood in general equilibrium analysis that sufficient assumptions for the global uniqueness of competitive equilibria are too restrictive to be applicable to models used in practice. However, it remains an open problem whether non-uniqueness of competitive equilibrium poses a serious challenge to applied equilibrium modeling or whether non-uniqueness is a problem that is unlikely to occur in so-called 'realistically calibrated' models. Given specifications for endowments, technology and preferences, the fact that the known sufficient conditions for uniqueness do not hold obviously does not imply that there must be several competitive equilibrium in the model economy. As a matter of fact, there seem to be few known examples of multiplicity for specifications of preferences, endowments and technologies commonly used in applied general equilibrium models.

However, given that algorithms which are used in practice to solve for equilibrium in applied models are never designed to search for all solutions of the model, there is no proof that there might not be several equilibria in these models after all. The fundamental problem is that for general preferences, one cannot prove that equilibria are unique for a given set of endowments. An obvious remedy for this problem is to consider semi-algebraic economies, i.e. to assume that preferences and technologies can be described by finitely many polynomial inequalities and equalities. In this case, the Tarski-Seidenberg theorem implies that it is decidable whether competitive equilibria are unique. In fact, it follows from the theorem that for any semi-algebraic class of economies, one can algorithmically determine whether there are economies in this class for which multiplicity of equilibria occurs. Unfortunately, the Tarski-Seidenberg procedure is known highly intractable and while it offers interesting theoretical results it is not applicable to even the smallest exchange economies.

On the other hand, if one can reduce the system of equations describing competitive equilibria to a polynomial system, all-solution algorithms can be used to approximate all roots to this system numerically. Recent advances in computational algebraic geometry have led to the development of relatively efficient algorithms for the computation of all zeros of a polynomial system of equations. In particular, if one can compute a Gröbner basis (see e.g. Cox et al. (1997) for a basic introduction) associated with a polynomial system the task of finding all roots to that system essentially reduces to finding all roots of a single polynomial equation in one unknown. With the development of fast computers and efficient algorithms Gröbner bases can now be computed even for fairly large systems of polynomial equations (see Faugère (1999)). To the best of our knowledge, there has so far not been an attempt to use these methods to make statements about the number of equilibria in general equilibrium models.

The first problem one faces when trying to apply all solution algorithms for polynomial systems to general equilibrium models is that these algorithms find all complex solutions to the system while in general only a subset of the real solutions describes competitive equilibrium (those who are associated with non-negative consumption and positive prices). Evidently, there are many different polynomial systems whose real solutions include the competitive equilibrium. The question is then whether one can find a 'minimal' system of equations that describes all equilibria of the economy but has not too many solutions which are not equilibria. Developing a general method for this is beyond the scope of this paper, but we give an example in Section 4 to show how to set up shuch minimal systems.

Having formulated any equilibrium system, it is easy to determine whether there are multiple equilibria for a given economy. However, in this paper we want to go a step further and show that within a given class of preferences, equilibrium is unique for 'most' realistic specifications of endowments, i.e. for some compact set of endowments. It turns out that in general the Gröbner base representation of the system does allow us to bound the number of zeros, but often does not guarantee that there is a unique equilibrium.

It is not clear how the idea that multiplicity of equilibria is rare in 'realistically calibrated' economies could possibly be formalized. The first observation is that one must impose joint restriction on preferences, endowments and technology in order to have any hope to guarantee uniqueness. For any profile of endowments, one can construct preferences such that the resulting economy has an arbitrary (odd) number of equilibria. Moreover, Gjerstad (1996) shows that in a pure exchange economy, for CES utility functions with elasticities of substitution above 2 (arguably realistically calibrated utility functions), multiplicity of equilibrium is a prevalent problem. The question then becomes whether for 'most' endowments and preference parameters, these economies have unique equilibria. Intuitively, in the case of Arrow-Debreu pure exchange economies one might think that since no-trade equilibria are always unique, one needs a large departure from Pareto-efficient endowments to obtain non-uniqueness. Balasko (1979) formalizes the idea that the set of endowments for which there are n equilibria, shrinks as n increases. Going beyond this result in the general case seems impossible. Instead, we use an idea due to Dakhlia (1999) and test whether there any critical economies in the specified convex set of parameters. In the absence of such, it suffices to prove uniqueness for a single economy in that set to infer that equilibrium must be unique in the entire set. Rouillier et al. (2000) and Aubry et al. (2002) develop algorithms which allows us to test for the presence of a critical point by investigating whether a positive dimensional system of equations has a solution in the set of interest.

The computations in this paper were all performed with the computer algebra system

SINGULAR, available free of charge at www.singular.uni-kl.de. In Section 4, we discuss a simple computational example in some detail. The rest of the paper is more conceptual and less geared towards particular applications.

The paper is organized as follows. In Section 2 we introduce semi-algbraic exchange economies and show that equilibria can be characterized as solutions to polynomial equations. In Section 3, we use results from real algebraic geometry to characterize all solutions to polynomial systems of equations. In Section 4, we examine uniqueness in Arrow Debreu economies with CES utility functions.

2 Semi-algebraic Arrow Debreu economies

There are H agents, $h \in \mathcal{H}$, trading L commodities. Agents have endowments $(e^h)_{h\in} \in \mathbb{R}_+^{HL}$ and preferences represented by utility functions

$$u^h : \mathbb{R}_+^L \to \mathbb{R}.$$

Commodity prices are denoted by $p \in \mathbb{R}_+^L$. Throughout we take commodity 1 as the numéraire and set $p_1 = 1$. A Walrasian equilibrium consists of a consumption allocation $(c^h)_{h\in\mathcal{H}}$ and prices p such that markets clear and each individual maximizes utility subject to the budget constraint, i.e.

$$\sum_{h\in\mathcal{H}} (c^h - e^h) = 0$$

and for each agent h

$$c^h \in \arg\max_{c\in\mathbb{R}_+^L} u^h(c) \text{ s.t. } p \cdot (c - e^h) = 0.$$

A function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is called semi-algebraic if its graph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = \phi(x)\}$ is a finite union and intersection of sets of the form

$$\{(x, y) \in \mathbb{R}^{m+n} : g(x, y) > 0\} \text{ or } \{(x, y) \in \mathbb{R}^{m+n} : f(x, y) = 0\}$$

for polynomials$f$ and g with real coefficients.

We call preferences semi-algebraically smooth if they can be represented by a utility function $u(.)$ that is C^2 on $\mathbb{R}_{++}^L$, strictly increasing and strictly concave and if $\partial_c u_h : \mathbb{R}_+^L \to \mathbb{R}^L$ is a semi-algebraic function. In a slight abuse of notation, we call an economy semi-algebraic if if each agent has semi-algebraically smooth preferences.

Throughout this paper, we will focus on interior equilibria. In order to guarantee that all equilibria are interior, one can make the additional assumption that indifference curves do not cut the axes, i.e. that for each h and all y, $cl(\{x : u(x) > y\}) \subset \mathbb{R}_{++}^L$. The example in Section 4 will satisfy this assumption, but it is not needed for the general analysis if one keeps in mind that we focus only on interior equilibria.

2.1 Semi-algebraic economies

How general is the assumption of semi-algebraic marginal utility? First note, that if a function is semi-algebraic, so are all its derivatives (the converse is not true, as the example $f(x) = \log(x)$ shows).

It follows from Blume and Zame (1993) that semi-algebraic preferences (i.e. the assumption that better sets are semi-algebraic sets) implies semi-algebraic utility.

From a practical point, it is easy to see that Cobb-Douglas and CES utility functions with rational elasticities of substitution, $\sigma \in \mathbb{Q}$, represent semi-algebraically smooth preferences.

From a theoretical point, by Afriat's theorem (Afriat (1967)), any finite number of observations that can be rationalized by arbitrary non-satiated preferences can be rationalized by a piece-wise linear, hence semi-algebraic function. While Afrait's construction does not yield a semi-algebraic, C^2, and strictly concave function, the construction in Chiappori and Rochet (1987) can be modified to our framework and we obtain the following lemma.

LEMMA 1 *Given N observations $(x^n, p^n) \in \mathbb{R}^{2l}_{++}$ with $p^i \neq p^j$ for all $i \neq j = 1, ..., N$, the following are equivalent.*

(1) There exists a strictly increasing, strictly concave and continuous utility function $u(x)$ such that
$$x^n = \arg\max_{x \in \mathbb{R}^l_+} u(x) \text{ s.t. } p^n \cdot x \leq p^n \cdot x^n.$$

(2) There exists a strictly increasing, strictly concave, semi-algebraic and C^2 utility function $v(x)$ such that
$$x^n = \arg\max_{x \in \mathbb{R}^l_+} v(x) \text{ s.t. } p^n \cdot x \leq p^n \cdot x^n.$$

To prove the lemma, observe that if statement (1) holds, the observations must satisfy the condition 'SSARP' from Chiappori and Rochet (1987). Given this one can follow their proof closely to show that there exists a C^2 semi-algebraic utility function that rationalizes the data. The only difference to their proof is that in the proof of their Lemma 2, one needs to use a polynomial 'cap'-function which is at least C^2. In particular, the argument in Chiappori and Rochet goes through if one replaces C^∞ everywhere with C^2 and uses the cap-function $\rho(x) = \max(0, 1 - \sum_l x_l^2)^3$. Since the integral of a polynomial function is polynomial, the resulting utility function is piece-wise polynomial, i.e. semi-algebraic.

2.1.1 The equilibrium set of semi-algebraic economies

The general assumption on semi-algebraic preferences imposes almost no restriction on the equilibrium set of exchange economies.

In the light of the theorems of Sonnenschein,Mantel and Debreu, Mas-Colell (1977) shows that for any compact (non-empty) set of positive prices $P \subset \Delta^{l-1}$ there exists an exchange economy with l households, $((u^h)_{h=1}^l, (e^h)_{h=1}^l)$, with u^h strictly increasing, strictly concave and continuous such that the equilibrium prices of this economy coincide precisely with P.

Given Lemma 1 above, this directly implies that for any finite set of prices $P \subset \Delta$, there exists an exchange economy $((u^h)_{h=1}^l, (e^h)_{h=1}^l)$, with u^h strictly increasing, strictly concave, semi-algebraic and C^2 such that the equilibrium prices of this economy coincide precisely with P. Therefore, the abstract assumption of semi-algebraic preferences imposes no restrictions of multiplicity of equilibria. Mas-Colell (1977) also shows that if the number of equilibria is odd, one can construct a regular economy and that there exist open sets of individual endowments for which the number of equilibria can be an arbitrary odd number.

Sufficient conditions for the uniqueness of Walrasian equilibrium are very restrictive. It is well known that equilibrium is unique if all agents have identical homothetic utility. More interestingly, W.E. is unique if for all agents and all $c \in \mathbb{R}^L_{++}$,

$$\frac{c'\partial^2 u^h(c)c}{c'\partial_c u^h(c)} < 1$$

or, if individual endowments are all collinear, if these expressions are smaller than 4. These are the bounds by Mijutshin and Polterovitch. Mas-Colell (1992) shows that the bounds on the expressions are tight.

2.2 Walrasian equilibria and polynomial systems of equations

It follows from the Tarski-Seidenberg theorem that for a given semi-algebraic economy it is decidable whether Walrasian equilibrium is unique(see e.g. Basu et al. (2003)). Unfortunately, the algorithmic complexity of quantifier-elimination based methods is too large for these methods to be of any use for economic applications.

Instead, we want to derive a system of polynomial equations that has finitely many solutions which include all Walrasian equilibria of the economy. An important complications comes in from the fact that available algorithms that find all solutions to systems of polynomial equations in fact find all complex solutions. We therefore need to ensure that the system of polynomial equations that characterizes equilibria has finitely many complex solutions and have to use some complex analysis.

An interior Walrasian equilibrium is characterized by the following equations.

$$\begin{aligned} \partial u^h(c) - \lambda p &= 0, h \in \mathcal{H} \\ \sum_l p_l(c_l - e_l^h) &= 0, h \in \mathcal{H} \\ \sum_h c_l^h - e_l^h &= 0, \quad l = 1, ..., L-1 \end{aligned}$$

The derivatives $\partial u^h(x)$ are semi-algebraic functions but of course in general not polynomial. Neyman (2003, Corollary 1) makes the following useful observation. The graph of any semi-algebraic function $\phi : V \to \mathbb{R}$, $V \subset \mathbb{R}^n$ can be written a the union of finitely many sets

$$G_i = \{f_i(x,y) = 0 \text{ and } g_i(x,y) > 0\}, \quad i = 1, ..., N,$$

with $f_i : V \times \mathbb{R} \to \mathbb{R}$ and $g_i : V \times \mathbb{R} \to \mathbb{R}^k$ polynomials. The polynomial function $h(x,y) = \Pi_{i=1}^N f_i(x,y)$ then is a non-zero polynomial that satisfies

$$h(x, \phi(x)) = 0 \text{ for all } x \in V.$$

Of course, for many (x,y) which satisfy $h(x,y) = 0$, we might have $y \neq \phi(x)$. However, it follows from the construction that we can assume without loss of generality that for any x, y with $f_i(x,y) = f_j(x,y) = 0$ for some i, j there is a $\tilde{x}$ arbitrarily close to x such that $f_i(\tilde{x}, \phi(\tilde{x})) = 0$ and $f_j(\tilde{x}, \phi(\tilde{y})) \neq 0$. Moreover, we can assume that for each i, $\frac{\partial f_i(x,\phi(x))}{\partial y} \neq 0$ whenever $f_i(x, \phi(x)) = 0$

Denote by $m^h(c,y)$ the L-vector of polynomials constructed as above that satisfy $m_l^h(c, \partial_{c_l} u^h(c)) = 0$ for each $l = 1, ..., L$. Since these are polynomials, we can write them as function from complex space, $m^h : \mathbb{C}^{L+1} \to \mathbb{C}^L$. Define the 'demand system' to be

$$D^h(c, \lambda, p) = \begin{pmatrix} m_1^h(c_1, \ldots, c_L, \lambda) \\ m_l^h(c_1, \ldots, c_L, \lambda p_l), \quad l = 2, \ldots, L \\ \sum_{l=2}^L p_l(c_l - e_l^h) + c_1 - e_1^h \end{pmatrix}. \tag{1}$$

It is then clear that Walrasian equilibria are solutions to the following polynomial system of equations

$$\begin{aligned} D^h(c^h, \lambda^h, p) &= 0, \quad h = 1, \ldots, H && (2) \\ \sum_{h=1}^H (c_l^h - e_l^h) &= 0 \quad l = 1, ..., L-1 && (3) \end{aligned}$$

Note that throughout the paper, we work with first order conditions. Of course, Tarski-Seidenberg implies that aggregate excess demand is also semi-algebraic. While it is often difficult to compute demand analytically it can always be written implicitly as a solution of a triangular polynomial system. However, it turns out that using aggregate excess demand function, although it reduces the number of unknowns and equations considerably, usually does not lead to efficiency gains in computing all Walrsian equilibria.

In order to guarantee that the system (2)-(3) only has finitely many complex solutions, we need to add the requirement that $\partial_{c,\lambda} D^h(c^h, \lambda^h, p)$ has full rank $L+1$. This gives the following additional equation.

$$1 - t^h \cdot \det\left(\partial_{c,\lambda} D^h(c^h, \lambda^h, p)\right) = 0, \quad h = 1, \ldots, H \tag{4}$$

We have the following theorem that characterizes the solutions to these equations.

THEOREM 1 *There is an open set of full measure of* $(e^1, ..., e^H) \in \mathbb{R}_+^{HL}$ *such that the system of equations (2)-(4) has at most finitely many complex solutions and such that all Walrasian equilibria are solutions to the equations.*

To prove the theorem we need two lemmas.

LEMMA 2 *Consider the function* $f : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$. *Suppose that* $f(z, e) = 0$ *with* $e \in \mathbb{R}^m$ *implies that* $\partial_{z,e} f(z, e)$ *has full rank. Then for generic* $\bar{e} \in \mathbb{R}^m$, 0 *is a regular value of* $f_{\bar{e}}(x)$.

LEMMA 3 *Suppose* $\phi : \mathbb{R}^L \to \mathbb{R}$ *is a differentiable semi-algebraic function and* $f_i : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ *are non-zero polynomials such that whenever* $f_1(x, \phi(x)) = f_2(x, \phi(x)) = 0$, *there is a* $\bar{x}$ *arbitrarily close to* x *such that* $f_1(\bar{x}, \phi(\bar{x})) = 0$ *and* $f_2(\bar{x}, \phi(\bar{x})) \neq 0$.

For an open and full measure set of $e^1, ..., e^H$ *and for each* $h = 1, ..., H$ *there is no Walrasian equilibrium for which* $f_1(x^h, \phi(x^h)) = f_2(x^h, \phi(x^h)) = 0$.

Proof of the theorem

We first prove that there are finitely many complex solutions. Using Lemma (2), it suffices to prove that the derivative of the system of equations under consideration with respect to $x^1, \lambda^1, ..., x^H, \lambda^H, p, e^1$ has full rank $H(L+1) + H + L - 1$. Equation (4) ensures that the derivative of each $D^h(x^h, \lambda^h, p)$ with respect to $x_1, ..., x_L$ and λ has rank $L + 1$. The derivatives of Equation (3) with respect to the t^h, $h = 1, ..., H$ give rank H. Following Debreu (1972) and considering directional derivatives for those e_l^1 for which $p_l \neq 0$ as well as direct derivatives for the other e_l^1, the derivatives with respect to e^1 give additional rank $L - 1$.

To prove that each Walrasian equilibrium solves equations (2)-(4), recall that we can write each each $m_l^h(c, y)$, $h \in \mathcal{H}$, $l = 1, ..., L$, as a product of finitely many polynomials $m_l^h(c, y) = \Pi_{j=1}^k f_{j,l}^h(c, y)$ with $\{(c, y) : f_{j,l}^h(c, y) = 0\} \neq \{(c, y) : f_{j',l}^h = 0\}$ for all j, j'. Moreover for each c and each l, there is a j_l such that $f_{j_l,l}^h(c, \partial_{c_l} u^h(c)) = 0$. If for each l, $f_{j',l}^h(c, \partial_{c_l} u^h(c)) \neq 0$ for all $j' \neq j_l$, we obtain, by the implicit function theorem that the matrix

$$\begin{pmatrix} -\frac{1}{\partial_y m_1^h(c,\lambda^h)} \partial_c m_1^h(c, \lambda^h) \\ \vdots \\ -\frac{1}{\partial_y m_L^h(c,\lambda^h p_L)} \partial_c m_L^h(c, \lambda^h p_L) \end{pmatrix}$$

is negative definite. Therefore, $D_{c,\lambda} D^h(c, \lambda)$ must have full rank $L + 1$.

By lemma 3, the set of individual endowments (e^h) for which there are h, j, j' and y with $f_j^h(c^h, y) = f_{j'}^h(c^h, y) = 0$ for some Walrasian equilibrium consumption c^h is a closed set with Lebesgue measure zero. $\square$.

In addition to solving the equilibrium equations, competitive equilibrium will be characterized by a system of polynomial inequalities $g_i(x) \geq 0$, $i = 1, ..., M$. Given individual endowments (e^h), we are thus interested in the set of competitive equilibria,

$$\mathcal{E} = \{((x^h), p) \in \mathbb{R}^{H(L+1)+L-1} \text{ that solve } (2) - (4) : g((x^h, \lambda^h), p) \geq 0\} \tag{5}$$

Obviously we can obtain an upper bound on the number of equilibria by bounding the number of complex solutions of Equations $(2) - (4)$.

2.3 Maximal number of solutions to polynomial systems

The following theorem provides a well known upper bound for the number of locally isolated solutions that is easy to compute.

THEOREM 2 (BÉZOUT) *Suppose $f : \mathbb{C}^n \to \mathbb{C}^n$ are n polynomials of degrees $d_1, ..., d_n$. The number of locally isolated solutions in $\mathbb{C}^n$ is bounded by $d_1 \dots d_n$.*

Unfortunately, even for very simple economies, the Bézout bound can be quite large. For example, consider an economy with two agents and two commodities and Cobb-Douglas utility functions. Walrasian equilibrium is unique, but the equilibrium equations read as

$$\begin{aligned}
1 - \lambda^1 c_1^1 &= 0 \\
1 - \lambda^1 p c_2^1 &= 0 \\
c_1^1 - e_1^1 + p(c_2^1 - e_2^1) &= 0 \\
1 - \lambda^2 c_1^2 &= 0 \\
1 - \lambda^2 p c_1^2 &= 0 \\
c_1^1 + c_1^2 - e_1^1 - e_1^2 &= 0 \\
c_2^1 + c_2^2 - e_2^1 - e_2^2 &= 0
\end{aligned}$$

The Bezout bound on the number of solutions is 72! This seems to indicate that even for simple economies, it is hopeless to try to solve for all complex solutions and then identify those who correspond to a Walrasian equilibrium.

However, the Bezout bound on the number of solutions is generically only obtained for so-called 'dense' systems of equations, i.e. for polynomials for which all monomial terms appear with non-zero coefficients (see Sturmfels (2002)).

The root count developed by Bernshtein, Kushnirenko and Khovanskii counts the number of isolated zeros of a 'sparse' polynomial system (see Sturmfels (2002)). In the following we refer to this as 'BKK-bound'. The theory underlying this bound turns out not to be relevant for this paper. Although the resulting sharper upper bounds a much more difficult to obtain than just multiplying degrees, there are freely downloadable software packages for computing these bounds in arbitrary high dimensions (the computations in this paper were performed with the package by Tangan Gao, T. Y. Li, and Xing Li). For the Cobb-Douglas economy, this actually gives a bound of at most one complex solutions, which obviously implies that equilibrium is always unique. Unfortunately, it is easy to see that in economic problems even the BKK bound is often not very good.

For Pareto-optimal endowments, equilibria are always unique. The BKK bound is uniform over profile of endowments. We show in examples below, that even uniformly across profiles of endowments, the bound is often not sharp.

We therefore now turn to algorithms which compute all (complex) solutions to polynomial systems.

3 Gröbner bases for the computation of all solutions

Given a polynomial system of equations $f : \mathbb{C}^M \to \mathbb{C}^M$ there are now a variety of algorithm to approximate numerically all complex and real zeros of f. Sturmfels' (2002) monograph provides an excellent overview. The two most important approaches are homotopy continuation methods and solution methods based on Gröbner bases. Both approaches are too inefficient to be applicable to large economic models, but they can be used for models with 4-5 households and 9-10 commodities. To find all equilibria for a given economy, homotopy methods seem slightly more efficient, while Gröbner bases allow for statements about entire classes of economies. In this paper, we therefore focus on Gröbner bases.

A Gröbner basis is a set of multivariate polynomials which has desirable algorithmic properties - in particular, given a Gröbner basis it is often possible to solve polynomial equations by solving a univariate polynomial. Every set of polynomials can be transformed into a Gröbner basis. Loosely speaking, this process generalizes Gaussian elimination for solving linear equations.

The following theorem provides the basis for our algorithms.

THEOREM 3 *Given utility functions $(u^h)_{h\in\mathcal{H}}$ that represent semi-algebraically smooth preferences, there exist equations*

$$\mathcal{G} = \{x_1 - q_1(x_n; (e^h)), x_2 - q_2(x_n; (e^h)), \ldots, x_{n-1} - q_{n-1}(x_n; (e^h)), r(x_n; (e^h))\}$$

where for each $(e^h) \in \mathbb{R}_+^{HL}$, r is a polynomial of some fixed degree d (independent of (e^h)) and the q_i are polynomials of at most degree $d-1$, such that for generic $(e^h) \in \mathbb{R}_+^{HL}$, the solutions of equations (2)-(4) are identical to the common solutions of the equations in $\mathcal{G}$.

The theorem is proved in the appendix, using methods from real algebraic geometry. It turns out that $\mathcal{G}$ forms a so-called 'Gröbner' basis under the lexicographic monomial ordering. Buchberger's algorithm is guaranteed to produce this basis in a finite number of steps. Computer algebra systems such as SINGULAR have implementations of this algorithm. It is noteworthy to stress that the q_i and r are rational functions in e^h and that the calculations produce an exact Gröbner basis if all coefficients in Equations (2) - (4) are rational numbers.

We call the function $r(.)$ the 'univariate representation' of the class of Arrow-Debreu economies with utilities (u^h).

The 'rational univariate representation' of Rouiller (1999) often has much smaller coefficients and is therefore numerically better behaved. However, for the purposes of this paper, we use the lexicographic Groebner basis to examine a system of polynomial equations. More sophisticated methods are subject to further research.

3.1 Algorithms

3.1.1 Finding all competitive equilibria

It follows directly from Theorem 3 that in order to find all equilibria for a given generic semi-algebraic economy, it suffices to compute the lexicographic Groebner basis and to find all real solutions to a univariate polynomial equation. Sturm's algorithm provides an exact method to determine the number of solutions to a univariate polynomial in the interval $[0, \infty)$. Therefore, one can determine the exact number of solutions of the univariate polynomial. Using simple bracketing, one can then approximate all solutions numerically, up to arbitrary precision! Given the solutions to the univariate representation, the other solutions can then be computed with arbitrary precision by evaluating polynomials up to arbitrary precision. Therefore, equilibria in this model are Turing computable (in contrast, see Richter and Wong (199x) who show that without restrictions on preferences Walrasian equilibria are generally not Turing computable).

3.1.2 A test for uniqueness

Dakhlia (1999) makes the following observation for Arrow-Debreu exchange economies. Given smooth utility functions $(u^h)_{h\in\mathcal{H}}$, if for a convex set of individual endowments $E \subset \mathbb{R}_+^{HL}$, there exists an $(\bar{e}^h)_{h\in\mathcal{H}}$ such that the economy $(u^h, e^h)_{h\in\mathcal{H}}$ has a unique equilibrium and if there are no critical economies in E, then Walrasian equilibrium must be unique for all profiles of individual endowments $(e^h) \in E$. This follows directly from the implicit function theorem and is in itself not hugely helpful, since it is generally not feasible (although for possible for semi-algebraic economies) to determine if such a critical economy in E exists.

However, for a single polynomial $r(x, (e^h))$ it often is feasible to determine if $r(x, (e^h)) = 0$ and $r'(x, (e^h)) := \frac{\partial r(x,(e^h))}{\partial x} = 0$ has a solution for positive x and for $(e^h) \in \Omega$, with

$$\Omega = \{e^h \in \mathbb{R}_+^{HL} : g_i(e^h) \leq 0, i = 1, ..., J\}, \tag{6}$$

where g_i are polynomials for all i. The trouble is that in order for this test to be of any use, one needs to ensure that real positive solutions to the univariate representation correspond to competitive equilibria or at least one needs to be able to easily identify those that do not correspond to Walrasian equilibria.

Rouillier et al. (2000) and Aubry et al. (2002) develop algorithms which find one point in each connected component of the variety by minimizing the distance function between the variety and a point. Their main result is used in the Appendix to prove the following theorem.

THEOREM 4 *Suppose that Walrasian equilibria are characterized by the positive real solutions of the univariate representation $r(x,(e^h))$ that satisfy $h(x) \leq 0$ for polynomials $h : \mathbb{R} \to \mathbb{R}^M$ and suppose that there cannot be solutions with $h(x) = 0$. Suppose that there exists a $(e^h) \in \Omega$, as defined in Equation 6 for which there is a unique Walrasian equilibrium. Then there cannot be an open set of endowments in Ω with multiple equilibria unless the following system of equations has a solution for some $(e^h) \in \Omega$.*

$$\begin{aligned}(e^h) - (\bar{e})^h - \lambda D_{(e^h)} r(x,(e^h)) - \mu D_{(e^h)} r'(x,(e^h)) - \nu D_{(e^h)} k(e^h) &= 0 \\ r(x,(e^h)) &= 0 \\ r'(x,(e^h)) &= 0 \\ 1 - tk(e^h) &= 0\end{aligned}$$

where $k(e^h) = \Phi_i g_i(e^h)$.

While the theorem obviously only provides a sufficient condition, it turns out that the method can be applied to a variety of examples. We illustrate this point in Section XXY below.

3.2 Bounding the Number of Real Zeros

While we explained above that bounds on the number of solutions to polynomial equations are usually bounds on the number of complex solutions, it turns out that the use of Gröbner basis sometimes allows for the derivation of bounds on the number of positive real solutions. The reason for this is that the number of real solutions to $f_1 = \ldots = f_n = 0$ is equal to the number of real roots of $r(x) = 0$, where $r(x)$ is the representing polynomial from Theorem 3 and that there exist simple bounds on the number of real solutions for univariate polynomials.

Given any univariate polynomial, $\sum_{i=0}^{d} a_i x^i$, with $a_i \in \mathbb{R}$ for all i, the number of its complex zeros is obviously bounded by its degree d. However, there a better bounds available for the number of real zeros. Define the number of sign changes of r to be the number of elements of $\{a_i \neq 0, i = 0, \ldots, d-1 : \text{sign}(a_i) = -\text{sign}(a_{i+1})\}$. The classical Descartes's Rule of Signs, see Sturmfels (2002), states that the number of real positive zeros of r does not exceed the number of sign changes. This bound is remarkable because it bounds the number of real zeros. It is possible that a polynomial system is of very high degree and has many solutions but the Descartes bound on the number of zeros of the representing polynomial proves that the system has a single real positive solution.

4 Application to CES utility

From now on we make the restrictive (but in policy work very common) assumption that utility exhibits constant elasticities of substitution and is of the form

$$u_h(c) = \sum_{l=1}^{L} \frac{1}{1-\sigma_h} \alpha_{hl}^{-\sigma_h} c_l^{1-\sigma_h}, \tag{7}$$

with rational $\sigma_h \neq 1$ and $\alpha_{hl} > 0$.

With $\sigma_h = \frac{N^h}{M^h}$, the partial derivatives can be written as

$$m_l^h(c, \frac{\partial u_h(c)}{\partial c_l}) \equiv 0,$$

where $m_l^h(c,y) = y^{M^h} \alpha_{hl}^{N^h} c_l^{N^h} - 1$. Note that this is also the correct representation for the Cobb-Douglas case with $\sigma_h = 1$. Note that for M^h even, the equation $m_l^h(c,y) = 0$ also has a solution with $y < 0$ which does not describe the correct marginal utility. Therefore, one would expect additional solutions to Equation (2)-(4) which do not correspond to Walrasian equilibria.

In the CES-framework the Equations (2)-(3) can be written as follows

$$\begin{aligned}
\alpha_{h1}^{N^h} (c_1^h)^{N^h} (\lambda^h)^{M^h} - 1 &= 0, \quad h = 1, ..., H \\
\alpha_{hl}^{N^h} (c_l^h)^{N^h} (\lambda^h)^{M^h} p_l^{M^h} - 1 &= 0, \quad h = 1, ..., H \\
c_1^h - e_1^h + \sum_{l=1}^{L} p_l (c_l^h - e_l^h) &= 0, \quad h = 1, ..., H \\
\sum_{h=1}^{H} c_l^h - e_l^h &= 0, \quad l = 1, ..., L-1
\end{aligned}$$

Note that since $\frac{\partial m^h(c,y)}{\partial y} \neq 0$ for all $c > 0$ and all y for which $m^h(c,y) = 0$, Equation (4) is satisfied automatically.

Without loss of generality we can write $\sigma_h = \frac{N}{M^h}$ for some N constant across h. Defining $q_l = p_l^{1/N}$, and eliminating the λ^h, we obtain a similar system of equations, which has the same real positive solutions but often fewer complex solutions.

$$\alpha_{h1}^{M^h} c_1^h - \alpha_{hl}^{M^h} c_l^h q_l^{M^h} - 1 = 0, \quad h \in \mathcal{H}, l = 2, ..., L \tag{8}$$

$$c_1^1 - e_1^1 + \sum_{l=2}^{L} q_l^N (c_l^h - e_l^h) = 0, \quad h = 1, ..., H \tag{9}$$

$$\sum_{h=1}^{H} c_l^h - e_l^h = 0, \quad l = 1, ..., L-1 \tag{10}$$

The following theorem is indicative in how to order the variables

THEOREM 5 *All positive and real $(c^h), q$ that solve (8)-(10) satisfy $c^h \gg 0$ whenever $q \gg 0$. Moreover, if N and M^h are odd for all $h \in \mathcal{H}$, all real solutions satisfy $q \gg 0$.*

Proof. Suppose $(c^h), q$ solve (8)-(10), $q \gg 0$ but $c_l^h < 0$ for some h, l. Then Equation (8) implies that $c^h \ll 0$, but then (9) cannot hold.

Now assume N , M^h odd and $q_l < 0$ for at least one l. Define $\bar{\mathcal{H}} = \{h : c_1^h > 0\}$. Market clearing implies that this set and its complement are non-empty. By (8), whenever $q_l < 0$, for all $h \in \bar{\mathcal{H}}$, $c_l^h < 0$ and for all $h \notin \bar{\mathcal{H}}$, $c_l^h > 0$. Again using market clearing, we obtain $\sum_{h \in \bar{\mathcal{H}}} c_l^h > \sum_{h \in \mathcal{H}} e^h$ whenever $q_l > 0$. Adding the budget constraint (9) for all $h \in \bar{\mathcal{H}}$ then yields a contradiction, since $\sum_{\bar{\mathcal{H}}} p_l^h (c_l^h - e_l^h)$ is positive for all l $\square$.

4.1 Two classes of economies with few equilibria

To examine whether there are interesting classes of economies with 'few' Walrasian equilibria we first present two examples. In the examples we give upper bounds on the number of Walrasian equilibria across all possible profiles

4.2 Example 1

First suppose that $H = L = 2$, and both agents have CES utility functions. As above, let $\gamma_h = N/M^h$ and define $\xi^h = (\alpha_2^h/\alpha_1^h)^{M^h}$ for $h = 1, 2$. Define $K_1 = N + |M_2 - M_1|$, $K_2 = N$, $K_3 = N - \min[M_1, M_2]$, $K_4 = \max[M_1, M_2]$ and $K_5 = |M_2 - M_1|$ and assume that

$$K_1 > K_2 > K_3 > K_4 > K_5. \tag{11}$$

Let $y = p_2^{1/N}$. The univariate representing polynomial is then given by

$$r(y) = -e_2^2 \xi_2 y^{K_1} - e_2^1 \xi_1 y^{K_2} + (e_1^2 + e_1^1) y^{K_3} - \xi_1 \xi_2 (e_2^1 + e_2^2) y^{K_4} + e_1^1 \xi_2 y^{K_5} + e_1^2 \xi_1.$$

By Descartes' bound, the number of positive real solutions is uniformly bounded by three! Evidently for large Nm but also for large M^h this is substantially below the BKK bound which goes to infinity as some $\gamma_h \to \infty$. The result is intuitivly appealing: one would not expect the number of Walrasian equilibria to go up, if some γ^h changes from being 2 to being 180/179. In the univariate representation, this is indeed not the case.

If the above conditions (11) on N and M_1, M_2 do not hold, the results are very similar. A notable special case results if one agent has log-utility, e.g. if $M_1 = N$. In this case the representing polynomial simplifies as follows

$$r(x_3) = -e_2^2 \xi_2 y^{K_1} - e_2^1 \xi_1 y^{K_2} + (e_1^2 + e_1^1) y^{K_3} + e_1^2 \xi_1,$$

and Descartes' bound implies that equilibria are unique for all endowments and all ξ_1, ξ_2. This is independent of γ_2, the elasticity of substitution of the second agents.

For arbitrary parameters the bound of three equilibria is tight, as the following simple case illustrates

Suppose $\gamma_1 = \gamma_2 = 3$, $\xi_1 = 4, \xi_2 = 1/4$ and $e_2^1 = e_1^2 = 1$. If $e_1^1 = e_2^2 = f > 44$ the economy has three equilibria, the univariate representation is given by

$$r(y) = (f + 16)y^3 - (4f + 4)y^2 + (4f + 4)y - f - 16$$

whose 3 positive real solutions for $f > 44$ correspond to 3 Walrasian equilibria.

The fact that in the example, there are always at most 3 equilibria, independently of preference parameters or endowments can only be explained by the fact that we looked at a very special class of preferences, CES utility is both homothetic and separable! Moreover, as the next example shows, it is crucial for the result that there are two goods and two agents.

4.2.1 Example 2

Now suppose H and L is arbitrary, but assume that $N = \gamma_h = \gamma_1 \in \mathbb{Z}_{++}$ for all $h = 2, ..., H$, i.e. all agents have identical integer- valued and identical elasiticicities of substitution. As the above example shows this is not a guarantee for uniqueness.

Using standard software for the computation of the BKK bound, it can be easliy verified that the BKK bound on the number of complex solutions of this system is given by γ_1^{H-1}. Interestingly, this bound is independent of the number of commodities, L, but increases exponentially in the number of agents.

In this example, it cannot be easily shown that the number of competitive equilibria always lies below the number of complex solutions to the equations. In the univariate representation, Descartes' bound does not have any bite since the number of sign-changes cannot be bounded. Furthermore, even for moderate H and L, it is not computationally feasible to use SINGULAR to compute a univariate representation as a function prices and allocations **and** profiles of endowments. For specific given endowments and ξ^h, computations can be performed on models with up to 5 commodities and 5 agents. In all examples considered, the number of Walrasian equilibria was always not larger than 3, in most cases, Walrasian equilibria were unique.

4.3 A test for uniqueness

In order to illustrate our test for uniqueness, we return to Example 1 and assume furthermore that $\xi_1 = 1.25$, $\xi_2 = 0.75$. For $\gamma_1 = \gamma_2 = 3$, the univariate representation becomes

$$r(y; e_2^1, e_1^2, e_2^2) = (-20e_2^2 - 12e_1^2)y^3 + (16e_1^2 + 16) * y^2 + (-15e_2^2 - 15e_2^1)y + 12e_1^2 + 20$$

and we want to examine the real solutions to $r(y, e) = 0$ and $\partial r / \partial y = 0$. Clearly, given Theorem 5, if this system has no real solution, equilibria are unique for all endowment profiles. Moreover, if there is a real solution, multiplicty of equilibria is likely in a neighnorhood of these solutiosn.

As explained above, for this we consider the first order conditions for minimizing the distance between the solution set and a point. The solution set to this is zero-dimensional, but unfortunately, there are many negative solutions. We need to rule out that they are semi-algebraically connected to some point in the positive orthant. For this, we consider the following system of equations

$$
\begin{aligned}
e_2^1 - 10 - \lambda(-12y^3 - 15y) - \mu(-36y^2 - 15) &= 0 \\
e_1^2 - 10 - \lambda(16y^2 + 12) - \mu 32 * y - \kappa t e_2^2 &= 0 \\
e_2^2 - 10 - \lambda(-20y^3 - 15y) - \mu(-60y^2 - 15) - \kappa t e_1^2 &= 0 \\
y - 1 - \lambda(-60e_2^2 - 36e_2^1)y^2 + 2(16e_1^2 + 16)y + (-15e_2^2 - 15e_2^1)) - & \\
\mu(-120 * e_2^2 - 72e_2^1)y + 32e_1^2 + 32 &= 0 \\
(-20e_2^2 - 12e_1^2)y^3 + (16e_1^2 + 16)y^2 + (-15e_2^2 - 15e_2^1)y + 12e_1^2 + 20 &= 0 \\
(-60e_2^2 - 36e_1^2)y^2 + (32e_1^2 + 32)y + (-15e_2^2 - 15e_2^1) &= 0 \\
-1 + te_1^2 e_2^2 &= 0 \\
(1 - t) - \kappa e_2^2 e_1^2 &= 0
\end{aligned}
$$

It turns out that this system does not have a real solution with positive y, e_2^2 and e_1^2. This implies that there cannot be a profile of endowments for which multiplicity arises. The solutions with negative e^2 cannot be connected to solutions with positive e^2 because we impose $-1 + te_1^2 e_2^2 = 0$ for some t. Furthermore, negative prices cannot be connected to positive prices.

We have thus proven that equilibrium is globally unique for the economy with 'small' taste-shocks

Appendix A: Basic Algebraic Geometry

For the description of a polynomial f in the n variables $x_1, x_2, \ldots, x_n$ we first need to define monomials. A monomial in $x_1, x_2, \ldots, x_n$ is a product $x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ where all exponents α_i, $i = 1, 2, \ldots, n$, are nonnegative integers. It will be convenient to write a monomial as $x^\alpha \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^N$, the set of nonnegative integer vectors of dimension n. A polynomial is a linear combination of finitely many monomials with coefficients in a field $\mathbb{K}$. We can write a polynomial f as

$$f(x) = \sum_{\alpha \in S} a_\alpha x^\alpha, \quad a_\alpha \in \mathbb{K}, \quad S \subset \mathbb{Z}_+^N \text{ finite.}$$

We denote the collection of all polynomials in the variables $x_1, x_2, \ldots, x_n$ with coefficients in the field $\mathbb{K}$ by $\mathbb{K}[x_1, \ldots, x_n]$, or, when the dimension is clear from the context, by $\mathbb{K}[x]$. The set $\mathbb{K}[x]$ satisfies the properties of a commutative ring and is called a polynomial ring.

In this paper we do not need to allow for arbitrary fields of coefficients but instead we can focus on three commonly used fields. These are the field of rational numbers $\mathbb{Q}$, the field of real numbers $\mathbb{R}$, and the field of complex numbers $\mathbb{C}$.

Throughout this paper we order monomials according to the lexicographic ordering, that is,

$$x^\alpha > x^\beta \iff \alpha > \beta \iff \text{The left-most non-zero entry of } \alpha - \beta \text{ is positive.}$$

For this particular monomial order we can define for any polynomial $f \in \mathbb{K}[x]$ the multidegree of $f = \sum_\alpha a_\alpha x^\alpha$, $\mathrm{md}(f) = \max\{\alpha \in \mathbb{Z}_+^n : a_\alpha \neq 0\}$. That is, the multidegree of f is the largest vector of exponents among the monomials in f according to the monomial (here lexicographic) ordering. The monomial with the multidegree as its vector of exponents give rise to the leading term of f, $\mathrm{LT}(f) = a_{\mathrm{md}(f)} x^{\mathrm{md}(f)}$.

A subset I of the polynomial ring $\mathbb{K}[x]$ is called an ideal if it is closed under sums, $f + g \in I$ for all $f, g \in I$, and it satisfies the property that $h \cdot f \in I$ for all $f \in I$ and $h \in \mathbb{K}[x]$. For given polynomials $f_1, \ldots, f_k$, the set

$$I = \{\sum_{i=1}^{k} h_i f_i : h_i \in \mathbb{K}[x]\} = \langle f_1, \ldots, f_k \rangle,$$

is an ideal. It is called the ideal generated by $f_1, \ldots, f_k$. This ideal $\langle f_1, \ldots, f_k \rangle$ is the set of all linear combinations of the polynomials $f_1, \ldots, f_k$, where the "coefficients" in each linear combination are themselves polynomials in the polynomial ring $\mathbb{K}[x]$. The Hilbert Basis Theorem states that for any ideal $I \subset \mathbb{K}[x]$ there exist finitely many polynomials that generate I.

We denote by $LT(I)$ the set of leading terms of elements of I, that is, $LT(I) = \{cx^\alpha : \exists f \in I \text{ with } LT(f) = cx^\alpha\}$ and by $\langle LT(I) \rangle$ the ideal generated by all the elements of $LT(I)$.

For an ideal I the radical of I is defined as $\sqrt{I} = \{f \in \mathbb{K}[x] : \exists m \geq 1 \text{ such that } f^m \in I\}$. The radical $\sqrt{I}$ is itself an ideal and contains I, $I \subset \sqrt{I}$. We call an ideal I radical if $I = \sqrt{I}$.

Gröbner Basis

Observe that if $I = \langle f_1, \ldots, f_k \rangle$, it is true $\langle LT(f_1), \ldots, LT(f_k) \rangle \subset \langle LT(I) \rangle$ but the converse often does not hold. The question is if there are some $g_1, \ldots, g_k$ which generate I and for which in fact $\langle LT(g_1), \ldots, LT(g_k) \rangle = \langle LT(I) \rangle$. One can show that these polynomials exist and they are called a Gröbner basis for I.

Definition 1 *A finite subset $g_1, \ldots, g_s$ of an ideal I is called a Gröbner basis of I if*

$$\langle LT(g_1), \ldots, LT(g_k) \rangle = \langle LT(I) \rangle$$

While the definition does not require that $g_1, \ldots, g_k$ forms a basis for I this can be shown fairly easily.

A Gröbner basis, G, is called 'reduced' if for all distinct $p, q \in G$ no monomial appearing in p is a multiple of $LT(q)$. Each ideal in $\mathbb{K}[x_1, \ldots, x_n]$ has a unique reduced Gröbner basis in which the coefficient of the leading term of every polynomial is one.

(Lexicographic) Gröbner bases are interesting because they reduce the problem of finding all solutions of a polynomial system of equations to finding all zeros of a single univariate polynomial.

Buchberger's Algorithm

There are now a variety of methods to compute Gröbner basis. The original algorithm by Buchberger implies a constructive existence proof for Gröbner basis and allows us to derive some important properties. Therefore we briefly outline the algorithm in this section.

Given any k polynomials $f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n]$, every $f \in \mathbb{K}[x_1, \ldots, x_n]$ can be written as

$$f = a_1 f_1 + \ldots + a_k f_k + r, \quad a_i, r \in \mathbb{K}[x_1, \ldots, x_n],$$

where for each i, $a_i f_i = 0$ or $LT(f) \geq LT(a_i f_i)$ and where either $r = 0$ or r is a linear combination of monomials, none of which is divisible by $LT(f_i)$ for any $i = 1, \ldots, k$. The polynomial r is called the remainder of f on division by $(f_1, \ldots, f_k)$. A simple generalization of the one-dimensional algorithm for polynomial division constructs the above terms. See e.g. Cox et al. (1997) for a detailed description.

To outline Buchberger's algorithm, we need to define an S-polynomial. For this, let $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ with $\mathrm{md}(f) = \alpha$ and $\mathrm{md}(g) = \beta$. Define γ by $\gamma_i = \max\{\alpha_i, \beta_i\}$, $i = 1, \ldots n$ and define

$$S(f, g) = \frac{x^\gamma}{LT(f)} f - \frac{x^\gamma}{LT(g)} g$$

It is relatively easy to prove that the following algorithm always produces a Gröbner basis in finitely many steps (see e.g. Cox et al (1997)). Let $F = f_1, \ldots, f_k$ be a basis for the ideal I. We construct a set G which is a Gröbner basis.

1. Set $G := F$
2. $G' := G$
3. For each pair $p, q \in G'$, $p \neq q$, let S denote the remainder of $S(f, g)$ on division by G'. If $S \neq 0$ then $G := G \cup \{S\}$
4. If $G \neq G'$ goto step 2

Note that while this algorithm is well defined independently of the field $\mathbb{K}$, it can be performed exactly over $\mathbb{Q}$. Furthermore, if the coefficients in the polynomials $f_1, \ldots, f_k$ are parameters, the algorithm can be applied to obtain a a set of polynomials $g_1, \ldots, g_m$ whose coefficients themselves are polynomial functions of the parameters. If the coefficients of $f_1, \ldots, f_k$ are real parameters, the coefficients of $g_1, \ldots, g_m$ will be polynomial functions in these parameters. The result of Buchberger's algorithm forms a Gröbner basis for $\langle f_1, \ldots, f_k \rangle$ for all values of the parameters, except for a set that is a finite union of sets defined by polynomial equations. The division set is generic in that for specific values of the parameters (satisfying some polynomial equation) it implies division by zero and is therefore not valid. However, it is clear that if we take the parameters to lie in $\mathbb{R}^k$, the polynomials resulting from Buchberger's algorithm for a Gröbner basis for a Zariski-open subset of $\mathbb{R}^k$. Unless some of the polynomial functions are identical equal to zero (and the subset of valid parameters is the empty set), the set of parameters for which the resulting functions do not form a Gröbner basis has k-dimensional Lebesgue measure zero. This does not change if one considers a reduced Gröbner basis. In this case, one simply eliminates some of the generating polynomials.

The following lemma (see e.g. Becker et al. (1994)) is key to the proof of Theorem 3.

LEMMA 4 (SHAPE LEMMA) *Let I be a zero-dimensional radical ideal in $\mathbb{Q}[x_1, \dots, x_n]$ with all d complex roots of I having distinct x_n coordinates. Then the reduced Groebner basis of I in the lexicographic term order has the shape*

$$\mathcal{G} = \{x_1 - q_1(x_n), x_2 - q_2(x_n), \dots, x_{n-1} - q_{n-1}(x_n), r(x_n)\}$$

where r is a polynomial of degree d and the q_i are polynomials of degree $d-1$.

In parts of the analysis, we need to test whether a positive dimensional system of equations has real solutions. The basic idea to do this is to consider the first order conditions of minimizing the distance between the variety defined by the equations and some point in the reals. For an ideal I we denote by $V(I)$ the affine variety of I, the set of points where all the elements of I vanish. If $I = \langle f_1, \dots, f_k \rangle$ then we can simply write $V(I) = \{y \in \mathbb{K}^n : f_1(y) = \dots = f_k(y) = 0\}$. Aubry et al. (2002) prove the following result which we use for the test.

LEMMA 5 *Let $V \subset \mathbb{C}^n$ be a variety of dimension d with $I(V) = \langle f_1, ..., f_s \rangle$, $f_i \in \mathbb{Q}[x]$ for all $i = 1, ..., s$. Given a point $a \in \mathbb{Q}^n$, $a \notin V$, let*

$$\mathcal{C}(V, a) = \{x \in V : \text{rank} \begin{pmatrix} \partial_x f(x) \\ a - x \end{pmatrix} \leq n - d\}.$$

The set $\mathcal{C}(V, a)$ meets every semi-algebraically connected component of $V \cap \mathbb{R}^n$, moreover, for generic $a \in \mathbb{Q}^n$, the dimension of $\mathcal{C}(V, a)$ is smaller than d.

Appendix B: Proofs

In this appendix, we give detailed proofs of the results not proven in the main body of the paper.

Proof of Lemma 2

Given a polynomial function $g : \mathbb{C}^n \to \mathbb{C}$ one can define partial derivatives with respect to complex numbers in the usual way. Write

$$g = c_0(z_{-j}) + c_1(z_{-j})z_j + \dots + c_d(z_{-j})z_j^d,$$

where the c_i are polynomials in the variables $z_{-j} = (z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$. Then,

$$\frac{\partial g}{\partial z_j} := c_1(z_{-j}) + \ldots + dc_d(z_{-j})z_j^{d-1}.$$

Given a system of polynomial equations $f : \mathbb{C}^n \to \mathbb{C}^n$, the Jacobian $\partial_x f(x)$ is defined as usual as the matrix of partial derivatives. A solution $\bar{x} \in \mathbb{C}^n$, $f(\bar{x}) = 0$, is called locally unique if $det(\partial_x f(\bar{x})) \neq 0$. We say zero is a regular value of f if all solutions are locally unique.

Instead of working with complex derivatives, one can alternatively consider the real expansion of f defined as the map $\hat{f} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ which maps real and imaginary parts to real an imaginary parts.

The Cauchy-Riemann equations (see any textbook on complex analysis) imply that if $g(z_1, ..., z_n)$ is a complex polynomial with $z_j = x_j + iy_j$ and $g = g^r + ig^i$ then

$$\frac{\partial g}{\partial z_j} = \frac{\partial g^r}{\partial x_j} + i\frac{\partial g^i}{\partial x_j}$$

and

$$\frac{\partial g^r}{\partial x_j} = \frac{\partial g^i}{\partial y_j} \text{ and } \frac{\partial g^r}{\partial y_j} = -\frac{\partial g^i}{\partial x_j}.$$

Therefore the Jacobian of a polynomial system has full rank if and only if the Jacobian of the real expansion has full rank. We will work alternately with the original system and the expansion. In order to prove the parametric transversality theorem for functions from complex space with real parameters, consider the function $f : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$. We are interested in (z, e) for which $e \in \mathbb{R}^m$ and $f(z, e) = 0$. Defining $z = x + iy$ and $e = u + iv$, we can define a real function $\hat{f} : \mathbb{R}^{2n} \times \mathbb{R}^{2m} \to \mathbb{R}^{2n+m}$ as follows.

$$\hat{f}(x, y, u, v) = \begin{cases} \hat{f}^r(x, y, u, v) \\ \hat{f}^i(x, y, u, v) \\ v \end{cases}$$

Clearly $f(z, e) = 0, e \in \mathbb{R}^m$ if and only if $\hat{f}(x, y, u, v) = 0$. The parametric transversality theorem states that if $\partial_{x,y,u,v}\hat{f}(x, y, u, v)$ has full rank $2n + m$ whenever $\hat{f}(x, y, u, v) = 0$ then for a set of full Lebesgue measure of $\bar{u} \in \mathbb{R}^m$, 0 is a regular value of $\hat{f}_{\bar{u}}$. This implies in particlar that for generic $\bar{e} \in \mathbb{R}^m$, 0 is a regular value of f since if 0 is a regular value of $\hat{f}_{\bar{u}}$, it must also be true that

$$\partial_{x,y}\begin{pmatrix} \hat{f}^r(x, y, \bar{u}, 0) \\ \hat{f}^i(x, y, \bar{u}, 0) \end{pmatrix} \text{ has full rank whenever} \hat{f}(x, y, \bar{u}, 0) = 0.$$

Proof of Lemma 3

Consider the system of equations in $(c^h), p, e^h$

$$\begin{aligned} \partial_c u^h(c^h) - \lambda^h p &= 0, \quad h \in \mathcal{H} && (12)\\ p \cdot (c^h - e^h) &= 0, \quad h = 2, ..., H && (13)\\ \sum_{h=1}^{H} c^h - e^h &= 0 && (14) \end{aligned}$$

It suffices to show that that

$$\Phi = \{(e^h) \in \mathbb{R}_+^{HL} : \exists (c^h), p \text{ s.t. } (c^h), p \text{ solve } (12)\text{–}(14) \text{ and } f_1(c^1, \phi(c^1)) = f_2(c^1, \phi(c^1)) = 0\}$$

is a closed set of zero Lebesgue measure. By assumption $\partial_y f_1 \neq 0$ and there exists a direction δ_c such that the n'th directional derivative of $f_2(c^1, phi(c^1)) - f_1(c^1, \phi(c^1))$ is non-zero. We can partition Φ into finitely many sets of the form

$$\{(e^h) \in \mathbb{R}_+^{HL} : \exists (c^h), p \text{ s.t. } (c^h), p \text{ solve } (12)\text{–}(14) \text{ and } \partial^{(n)} f_2 - f_1 \neq 0, \partial_{\delta_c}^{(i)} f_2 - f_1 = 0, i < n\},$$

where ∂^n consists of all n'th partial derivatives. By the parametric transverality theorem each of these sets has measure zero. $\square$

Proof of Theorem 3

Given Lemma 4, it suffices to show that the equilibrium equations generate a zero dimensional radical ideal with all d complex roots having distinct x_n coordinates for generic (e^h). Given that Buchberger's algorithm gives the correct Groebner basis for generic (e^h) the theorem then follows from that. But in the proof of Theorem 1 we already showed that generically in e^h, 0 is a regular value for the equilibrium system. The preimage theorem then implies that generically in e^h all complex solutions have distinct x_i coordinates for all $i = 1, ..., n$. the ideal generated by the equations must then be radical since To see that the ideal generated by the equations must then be radical, let $I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[x]$, following Becker et al. (1994, Proof of Proposition 5), denote its zeros by $(a_{1j}, ..., a_{nj})$, $j = 1, ..., J$. Since all zeros are locally unique, I must be the intersection of D ideals of the form $\langle x_1 - a_{1i}, x_2 - a_{2i}, x_n - a_{ni} \rangle$, $i = 1, ..., D$ – if any of the primary components had as a basis function $(x_m - a_{mi})^d$ for some m, i and some $d = 2, 3, ...$, all derivatives with respect to x_m at the i'th zero would be zero which violates local uniqueness. But this intersection must be radical.

Proof of Theorem 4

By Lemma (5), the solution of the system meets every semi-algebraically connected component of the set of critical economies for which $g(e^h) \neq 0$. The result follows.

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