

Conditional Moment Restrictions and Triangular Simultaneous Equations*

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Abstract

We examine whether a causal interpretation can be given to the function identified by the conditional moment restriction (CMR). It is shown that in general the CMR does not identify the average structural function (ASF) nor any other structural object. It is further shown that the CMR does not identify the average derivative either. We conclude that the CMR identifies a causal relation only if the model is structurally separable in observable covariates and unobservable random errors. We also provide a condition under which the nonseparable model is nonparametrically just identified from the population distribution of the observables, so that under this assumption the nonseparable triangular simultaneous equations model does not restrict the population distribution of the observables.

1 Introduction

Often we are interested in the relation between a (vector of) dependent variable(s) Y and a (vector of) independent variable(s) X . Because the list of independent variables is incomplete, the relation involves one or more unobservable “errors” that are treated as random variables. In the absence of prior knowledge of the nature of the relation, it is reasonable to adopt a nonparametric framework, and try to identify and estimate such a relationship. The literature has used two approaches within the nonparametric framework. In the first approach, the nonparametric relationship has an additively separable error that satisfies some conditional moment restriction. In the second approach, it has an error that is not separable, but satisfies

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some independence restriction. It is of interest to clarify the relationship between the two nonparametric approaches. See, e.g. Newey and Powell (2003), Hall and Horowitz (2005), and Carrasco, Florens, and Renault (2006).

Common sense suggests that there could be a relationship between the two approaches. In order to understand why, consider the simple case where X is exogenous. If we adopt the nonseparable error approach, we would write

$$Y = f(X, \varepsilon) \tag{1}$$

and assume that X and ε are independent of each other. As argued by Blundell and Powell (2004) and others, the main function/parameter of interest is the average structural function (ASF) $\phi(x)$ defined by

$$\phi(x) = \mathbb{E}[f(x, \varepsilon)] \tag{2}$$

where the expectation is over the marginal distribution of ε . The ASF is the average outcome if the value $X = x$ is assigned independently of the unobservable ε . This indicates that we can think of the ASF as the average causal relation between Y and X . In model (1) the ASF can be obtained using the following conditional moment restriction (CMR)

$$\mathbb{E}[Y - \phi(x)|X = x] = 0 \tag{3}$$

This follows¹ because by independence of X and ε

$$\mathbb{E}[Y|X = x] = \mathbb{E}[f(X, \varepsilon)|X = x] = \mathbb{E}[f(x, \varepsilon)] = \phi(x) \tag{4}$$

The main result of this paper is that such a relationship does *not* exist when X is endogenous. If we adopt the nonseparable error approach, the relation in which X is endogenous can be represented by a triangular system

$$Y = f(X, \varepsilon) \tag{5}$$

$$X = g(Z, V) \tag{6}$$

with ε and V correlated and Z an instrumental variable. The CMR in this model is

$$\mathbb{E}[Y - \psi(X) | Z = z]$$

and by analogy with the case with exogenous X we would expect that ψ and the ASF are related. However we show that the function $\psi(\cdot)$ identified by the CMR is different from the

¹See Hoderlein and Mammen (2007) for similar results in conditional quantile restrictions.

ASF.² Even stronger we argue that the CMR does *not* identify anything structural. To be specific, we show that the CMR approach is sensitive to the first stage in the sense that, if the correlation between the first stage error V and the structural error changes, then the function $\psi(\cdot)$ identified by the CMR changes as well. This sensitivity to the first stage implies that the CMR does not identify anything structural.

This paper is about the *interpretation* of the identified function. As such, we simply assume that the sufficient conditions for identification are satisfied for both approaches. The focus on interpretation means that we do not provide an in depth discussion of identification or estimation, since these topics are discussed in many existing papers.

2 Does The Conditional Moment Restriction Identify Anything Structural?

In this section, we argue that the conditional moment restriction does not identify the average structural function. Even stronger, we argue that the function that is identified by the conditional moment restriction is not structural.

Our main results are best understood in a simple example. Consider a triangular system of equations where

$$\begin{aligned} Y &= X^2\varepsilon \\ X &= Z + V \quad Z \perp V \end{aligned}$$

We assume that $\varepsilon \sim N(0, 1)$, and $V \sim N(0, 1)$. We also assume that the first stage error V is correlated with ε through

$$\varepsilon = \frac{1}{2}V + \frac{\sqrt{3}}{2}V^* \quad V^* \perp V, Z$$

and $V^* \sim N(0, 1)$. Finally, we will assume that Z has a normal distribution³.

The relationship between Y and X is obviously nonseparable. Because the ASF is defined

²Florens, Heckman, Meghir, and Vytlačil (2003) argued that the exclusion restriction along with the conditional moment restriction are not sufficient to identify the average treatment effect. Our result goes further, and establishes that the function identified by the conditional moment restriction is *different* from the average structural function.

³This ensures that the support condition in Imbens and Newey (2003) is satisfied, so the identification of the ASF does not present any conceptual challenge. It also ensures that the Newey and Powell (2003) completeness assumption is satisfied.

with respect to the marginal distribution of ε , we can see that the ASF $\phi(x)$ is

$$\phi(x) \equiv \mathbb{E} [x^2 \varepsilon] = x^2 \mathbb{E} [\varepsilon] = 0,$$

where the expectation is taken with respect to the marginal distribution of ε treating x as nonstochastic.

We now consider the conditional moment restriction

$$\mathbb{E}[Y - \psi(X) | Z = z] \tag{7}$$

Is there such a $\psi(X)$? If so, is it unique? The answer to both of these questions is affirmative because, under normality of Z , the Newey and Powell (2003) completeness assumption is satisfied. Because

$$\begin{aligned} \mathbb{E}[Y|Z] &= \mathbb{E} \left[(Z + V)^2 \left(\frac{1}{2}V + \frac{\sqrt{3}}{2}V^* \right) \middle| Z \right] \\ &= Z^2 \mathbb{E} \left[\frac{1}{2}V + \frac{\sqrt{3}}{2}V^* \middle| Z \right] + Z \mathbb{E} \left[V^2 + \sqrt{3}VV^* \middle| Z \right] \\ &\quad + \mathbb{E} \left[\frac{1}{2}V^3 + \frac{\sqrt{3}}{2}V^2V^* \middle| Z \right] = Z \end{aligned}$$

and

$$\mathbb{E}[X|Z] = \mathbb{E} [Z + V | Z] = Z$$

we have

$$\mathbb{E}[Y - X|Z] = 0$$

from which we find that $\psi(x) = x$. Therefore, we conclude that (7) does not identify the ASF $\phi(x) \equiv 0$.

The preceding analysis may be subject to the criticism that, even though the CMR may not identify the ASF, it may still identify some ‘structural’ object. In order to address such a concern, it is first necessary to define what the term ‘structural’ object means. We loosely define it as something solely based on the ‘second stage’. In other words, we argue that a structural object should not be affected by a change of the ‘first stage’ or its correlation with the ‘second stage’.

Now consider a slightly different relationship⁴ between the first stage error V and the second stage error ε

$$\varepsilon = \frac{1}{4}V + \frac{\sqrt{15}}{4}V^* \quad V^* \perp V, Z$$

⁴Note that the marginal distributions of ε and V are not changed.

We can see that

$$\begin{aligned}\mathbb{E}[Y|Z] &= \mathbb{E} \left[(Z + V)^2 \left(\frac{1}{4}V + \frac{\sqrt{15}}{4}V^* \right) \middle| Z \right] \\ &= Z^2 \mathbb{E} \left[\frac{1}{4}V + \frac{\sqrt{15}}{4}V^* \middle| Z \right] + Z \mathbb{E} \left[\frac{1}{2}V^2 + \frac{\sqrt{15}}{2}VV^* \middle| Z \right] \\ &\quad + \mathbb{E} \left[\frac{1}{4}V^3 + \frac{\sqrt{15}}{4}V^2V^* \middle| Z \right] = \frac{Z}{2}\end{aligned}$$

but we continue to have $\mathbb{E}[X|Z] = Z$. Therefore, we have

$$\mathbb{E} \left[Y - \frac{1}{2}X \middle| Z \right] = 0,$$

and the CMR (7) now identifies the function $\psi(x) = \frac{1}{2}x$. The CMR continues to identify a function different from the ASF. Moreover, the function changes as a result of the change in the correlation between the first stage and second stage errors. We may therefore conclude that the CMR does not identify a ‘structural’ function. This should not come as a surprise because in a sense the ASF is the only meaningful CMR-based structural object. After all, if we consider the degenerate first stage where $X = Z$, so that there is no endogeneity problem, then the CMR does identify the ASF.

It should be stressed that our discussion here does not rely on a failure of the completeness assumption. The result above suggests interpretational problems with the CMR, and it has nothing to do with the identifiability of the function $\psi(\cdot)$ using the CMR.

We now provide an explanation of the example by providing an interpretation of the function $\psi(X)$ identified by the CMR. We do so under the assumption that the joint population distribution of Y, X, Z is such that there exists a triangular simultaneous equations model $Y = f(X, \varepsilon)$, $X = g(Z, V)$ with the ‘instrument’ Z jointly independent of (ε, V) . We provide a justification of this assumption in the next section.

Our explanation is very simple. We note that the CMR is inherently defined by the joint density of (ε, V) whereas the ASF is defined by the marginal density of ε . This fundamental tension in their definitions naturally leads to the discrepancy that we observed in the previous section. More specifically, we have

$$\mathbb{E}[Y|Z = z] = \mathbb{E}[\psi(X)|Z = z] = \mathbb{E}[f(g(z, V), \varepsilon)] = \int \int f(g(z, v), \varepsilon) p(\varepsilon, v) dv d\varepsilon \quad (8)$$

The conditional expectation of the ASF in (2) given $Z = z$ is

$$\mathbb{E}[\phi(X)|Z = z] = \mathbb{E}[\mathbb{E}[f(g(Z, V), \varepsilon)]] = \int \int f(g(z, v), \varepsilon) p_V(v) p_\varepsilon(\varepsilon) dv d\varepsilon \quad (9)$$

We conclude that $\mathbb{E}[\psi(X)|Z = z] \neq \mathbb{E}[\phi(X)|Z = z]$, because the former expectation is over the joint distribution of (the dependent) ε, V while the latter is over the marginal distributions of ε and V . In both conditional expectations setting $Z = z$ does not fix X so that we at most identify some average causal object. The problem with the CMR (7) is that if Z is fixed, there is still variation in X due to variation in V that is correlated with variation in ε . The ASF as a proper causal response function is for the case that X varies independently of V as in (9).

3 Does the Conditional Moment Restriction At Least Identify the Average Derivative?

Having concluded that the CMR does not identify anything structural, we may want to ask if the CMR at least identifies an interesting reduced form parameter. This is too general a question, so we restrict our attention to a reduced form parameter that was considered by, e.g., Imbens and Newey (2003). Given a triangular system $Y = f(X, \varepsilon)$, $X = g(Z, V)$, the average derivative is defined to be

$$\mathbb{E} \left[\frac{\partial f(X, \varepsilon)}{\partial x} \right] = \int \int \frac{\partial f(x, \varepsilon)}{\partial x} p(x, \varepsilon) dx d\varepsilon$$

which corresponds to the effect of increasing X by a unit, holding fixed the relationship between X and ε . Obviously it is defined with respect to the joint distribution of (X, ε) , and it is not ‘structural’ according to our loose definition in the previous section. The question is whether the $\psi(\cdot)$ identified by the CMR $\mathbb{E}[Y - \psi(X) | Z = z]$ is such that $\mathbb{E}[\psi'(X)] = \mathbb{E}[\partial f(X, \varepsilon) / \partial x]$, in which case we can argue that the CMR does identify something interesting. We show that this is not the case by a counter-example.

Suppose that the first stage is such that the conditional density of X given $Z = z$ is exponential with mean equal to z .⁵ We can then write the first stage as

$$X = ZV$$

where V is independent of Z , and has a unit exponential distribution. Now, we let

$$Y = X^2\varepsilon$$

where

$$\varepsilon = V + V^*$$

⁵By Theorem 2.2 in Newey and Powell (2003), the completeness condition is satisfied if Z has a support that contains an open interval.

and V^* is unit exponential independent of V .

We first show that the ASF is different from the function identified by the CMR. Because V is unit exponential, we have $\mathbb{E}[V] = 1$, $\mathbb{E}[V^2] = 2$, and $\mathbb{E}[V^3] = 6$. Likewise, we have the same result for V^* . It follows that the ASF is equal to $2X^2$. Now, note that $X^2\varepsilon = (ZV)^2(V + V^*) = V^3Z^2 + V^2Z^2V^*$, and therefore, $\mathbb{E}[Y|Z] = 8Z^2$. We also note that $\mathbb{E}[X^2|Z] = 2Z^2$, and hence, we conclude that

$$\mathbb{E}[Y - 4X^2|Z] = 0$$

which identifies $\psi(X) = 4X^2$, which is different from the ASF.

Now, we calculate the average derivative by noting that

$$\begin{aligned} \mathbb{E}\left[\frac{\partial f(X, \varepsilon)}{\partial x}\right] &= \mathbb{E}[2X\varepsilon] = 2\mathbb{E}[(ZV)(V + V^*)] \\ &= 2\mathbb{E}[ZV^2 + ZVV^*] = 2(\mathbb{E}[Z]\mathbb{E}[V^2] + \mathbb{E}[Z]\mathbb{E}[V]\mathbb{E}[V^*]) \\ &= 2\mathbb{E}[Z](2 + 1 \cdot 1) = 6\mathbb{E}[Z] \end{aligned}$$

On the other hand, we have

$$\mathbb{E}[\psi'(X)] = \mathbb{E}[8X] = 8\mathbb{E}[ZV] = 8\mathbb{E}[Z]\mathbb{E}[V] = 8\mathbb{E}[Z]$$

from which we obtain that

$$\mathbb{E}\left[\frac{\partial f(X, \varepsilon)}{\partial x}\right] \neq \mathbb{E}[\psi'(X)]$$

In other words, the CMR does not identify the average derivative in general.

We do note that there is an interesting exception, in which the CMR does identify the average derivative. Suppose that the first stage is separable in Z and V and can be written $X = g(Z) + V$ with g differentiable and $V \perp Z$. The CMR identifies

$$\begin{aligned} 0 &= \mathbb{E}[Y - \psi(X)|Z = z] = \mathbb{E}[f(g(Z) + V, \varepsilon) - \psi(g(Z) + V)|Z = z] \\ &= \mathbb{E}[f(g(z) + V, \varepsilon) - \psi(g(z) + V)] \end{aligned}$$

or

$$0 = \int \int (f(g(z) + v, \varepsilon) - \psi(g(z) + v))p(v, \varepsilon) dv d\varepsilon$$

Now differentiate both sides with respect to z , and we obtain

$$0 = \int \int \left(\frac{\partial f(g(z) + v, \varepsilon)}{\partial x} - \psi'(g(z) + v) \right) g'(z)p(v, \varepsilon) dv d\varepsilon$$

Dividing by $g'(z)$ we obtain

$$0 = \mathbb{E}\left[\left(\frac{\partial f(X, \varepsilon)}{\partial x} - \psi'(X)\right)\middle|Z = z\right]$$

Now, using the law of iterated expectations, we obtain

$$0 = \mathbb{E} \left[\frac{\partial f(X, \varepsilon)}{\partial x} - \psi'(X) \right]$$

or

$$\mathbb{E} \left[\frac{\partial f(X, \varepsilon)}{\partial x} \right] = \mathbb{E} [\psi'(X)]$$

Note that if the first stage relation between X and Z is expressed as a projection of X on Z we apparently obtain a relation that is separable. However, only if the first stage is structurally separable, in the sense defined below, can we identify the average derivative by the CMR.

4 Related Discussion

In this section, we will make a small digression, and examine some other objects considered in the literature.

Local IV In an exactly identified linear simultaneous equations model, the IV estimator can be written as a ratio of two least squares estimators. Perhaps motivated by such an indirect least squares interpretation, the local IV⁶ of the form

$$\lambda(z) = \frac{\partial \mathbb{E}[Y|Z=z]/\partial z}{\partial \mathbb{E}[X|Z=z]/\partial z}$$

has been suggested as a way of summarizing some causal relation between Y and X . Our analysis suggests that $\lambda(z)$ is not a structural object either. In the first example above, we have $\mathbb{E}[Y|Z] = Z$ and $\mathbb{E}[X|Z] = Z$ so that $\lambda(z) = 1$. On the other hand, in the second example with slightly different correlation between the first stage and second stage errors, we have $\mathbb{E}[Y|Z] = \frac{1}{2}Z$ and $\mathbb{E}[X|Z] = Z$ so that $\lambda(z) = \frac{1}{2}$. It follows that the local IV is affected by the first stage, and therefore cannot be a structural object.

With the example considered in Section 3, we can examine whether the local IV identifies the average derivative. Because $\mathbb{E}[Y|Z] = 8Z^2$ and $\mathbb{E}[X|Z] = Z$, we can see that $\lambda(z) = 16z$. It follows that $\mathbb{E}[\lambda(Z)] = 16\mathbb{E}[Z] \neq 6\mathbb{E}[Z] = \mathbb{E}[\partial f(X, \varepsilon)/\partial x]$, so the local IV does not identify the average derivative either.

Conditional Quantile Restriction The conditional quantile restriction (CQR) solves $\Pr[Y \leq \psi(X)|Z=z] = \tau$ for some $\tau \in (0, 1)$, whereas the quantile structural function (QSF) ϕ is such

⁶See Florens, Heckman, Meghir, and Vytlacil (2003), e.g., for related discussion on local IV.

that $\Pr[f(x, \varepsilon) \leq \phi(x)] = \tau$. Note that the latter is with respect to the marginal distribution of ε . Chernozhukov, Imbens, and Newey (2007) showed that, if the ε is a *scalar* random variable, then the CQR identifies the QSF, i.e., $\psi = \phi$. It is difficult to give a plausible interpretation to the QSF with two or more dimensional errors, which may be a justification to focus on cases with scalar errors, but the question remains whether the CQR identifies anything structural in the case that ε has two or more components⁷. Our preliminary analysis shows that it does not.

For this purpose, we now consider a model

$$Y = X\varepsilon_1 + \varepsilon_2 \quad \varepsilon_1, \varepsilon_2 \stackrel{iid}{\sim} N(0, 1)$$

so that the median structural function $\phi(x) \equiv 0$. We consider the first stage

$$X = Z + V \quad Z \perp V, \quad \varepsilon_2 = tV + \sqrt{1-t^2}V^* \quad V \perp V^*$$

and $V \sim N(0, 1)$, $V^* \sim N(0, 1)$ for some t such that $|t| \leq 1$. We assume that ε_1, V and V^* are independent of each other. The $\psi(\cdot)$ identified by the conditional median restriction solves $\Pr[Y - \psi(X) \leq 0 | Z = z] = \frac{1}{2}$. It can be shown that this can be rewritten as

$$\frac{1}{2} = \mathbb{E} \left[\Phi \left(\frac{-tV + \psi(z + V)}{\sqrt{(z + V)^2 + 1 - t^2}} \right) \right] \quad \forall z \quad (10)$$

where $\Phi(\cdot)$ denotes the CDF of $N(0, 1)$.

Now, we consider $t = 0$. This is the case with X independent of $\varepsilon_1, \varepsilon_2$, and we can easily see that $\psi(\cdot)$ should be identically equal to zero if it has to satisfy (10). We now ask whether (10) is satisfied with $\psi(\cdot) = 0$ for other values of t . With $t = \frac{1}{2}$ and $z = 1$, we found (by Monte Carlo with 100,000 runs) that the expectation on the right is equal to 0.5381.

We note that the discrepancy between the CQR and QSF can be potentially used as a basis of a specification test, because there should not be any discrepancy if the second stage error is indeed a scalar.

5 Creation of Nonparametric Triangular Simultaneous Equations Model

In Section 2, we made an assumption that the joint population distribution of Y, X, Z is such that there exists a triangular simultaneous equations model $Y = f(X, \varepsilon)$, $X = g(Z, V)$ with

⁷See Hoderlein and Mammen (2007) for related discussion when X is exogenous.

the ‘instrument’ Z independent of both (ε, V) . We now provide a justification by showing that, under some reasonable assumption, it is always possible to represent the population distribution of Y, X, Z by such a triangular system.

We first begin by the construction of the first stage. The most general model for the relation between a dependent variable X and a vector of independent variables Z has $g(Z, V)$ monotone in V for all values of Z . If we assume that V and Z are independent and we normalize the distribution of V as uniform on $[0, 1]$, then as shown by Matzkin (2003)

$$g(z, V) = G^{-1}(V|z) \quad (11)$$

with $G(x|z)$ the conditional CDF of X given $Z = z$. The model (11) can be ‘constructed’ from the joint distribution of X, Z by defining the random variable V as

$$V = G(X|Z) \quad (12)$$

Because

$$\Pr(V \leq e|Z = z) = \Pr(G(X|Z) \leq e|Z = z) = \Pr(X \leq G^{-1}(e|z)|Z = z) = e \quad (13)$$

the error V has a uniform distribution that is independent of Z . Upon inversion we obtain (11). This construction implies that there is a one-one correspondence between the model (11) with V independent of Z and the observed joint distribution of X, Z . In other words, the model with V independent of Z is nonparametrically just identified.

Now, let $H(y|z, v) = \Pr(Y \leq y|Z = z, V = v)$ be the conditional CDF of Y given (Z, V) , and let $U = H(Y|Z, V)$. Note that U and V have a uniform distribution that is independent of Z . Hence if we define $h(z, v, u) = H^{-1}(u|z, v)$, we have

$$Y = h(Z, V, U) \quad U \perp Z, V \quad (14)$$

In the same way

$$X = g(Z, V) \quad Z \perp V \quad (15)$$

Note that by construction g, h are increasing in their last argument and that Z, U, V are mutually independent.

To construct a triangular simultaneous equations model we would like to invert (15) with respect to Z and express Z as a function of X, V . The simplest case is that $g(z, v)$ is strictly monotonic, without loss of generality strictly increasing, in z for (almost all) v . This is equivalent to assuming that the joint distribution of X, Z is such that if $z > z'$ then $G(x|z) < G(x|z')$

for all x on the union of the supports of these distributions. If $Z = z$ is the assigned level of Z , then $X(z) = g(z, V)$ is the resulting level of X and the assumption is equivalent to the assumption that this level is strictly increasing in z for all members of the population. If $g(z, v)$ is strictly increasing in z for (almost all) v , then

$$Z = g^{-1}(X, V) \tag{16}$$

where g^{-1} is the inverse with respect to the first argument. Note that in (16) X and V are not independent. Substitution in equation (14) gives

$$Y = h(g^{-1}(X, V), V, U) = f(X, \varepsilon) \tag{17}$$

with ε the vector (U, V) . Hence we have constructed a triangular system (15) and (17) with errors ε, V that are independent of Z . Because monotonicity is equivalent to $G(x|z)$ being decreasing in z for all x we can check whether this assumption holds.

6 Structural Separability

We argue that ‘additive separability’ can be obtained by mathematical manipulation with vacuous economic content, and some care is needed to interpret an additively separable model. Consider the example discussed in Section 2. If we “define” our error to be $e \equiv X^2\varepsilon - X$, we can easily obtain an additive separable model

$$Y = X + e$$

where e satisfies the conditional moment restriction $\mathbb{E}[e|Z = z] = 0$. Despite the superficial familiarity, the function $\psi(x) = x$ thus identified does not have any structural interpretation.

In the previous section, we concluded that we can construct a nonseparable model $Y = h(g^{-1}(X, V), V, U) = f(X, \varepsilon)$ under monotonicity. If the model thus obtained is additively separable $h(g^{-1}(X, V), U) = m(X) + k(U, V)$, we call the model *structurally separable*. Therefore under monotonicity the nonseparable triangular simultaneous equations model is the natural alternative to the separable model

$$Y = m(X) + \varepsilon \tag{18}$$

$$X = g(Z, V) \tag{19}$$

Under monotonicity the separable model is obtained if $h(g^{-1}(X, V), U) = m(X) + k(U, V)$ and we define $\varepsilon \equiv k(U, V)$. Note that in this case Z is independent of ε, V by construction. If

the model has an additive error, we call the model *structurally separable*. In the structurally separable model the ASF is $m(\cdot)$ and under the completeness assumption the ASF is the unique ψ that satisfies $\mathbb{E}[Y|Z] = \mathbb{E}[\psi(X)|Z]$. However, if the population model is not structurally separable, then, as we have seen in Section 2, the CMR corresponding to mean independence does not recover the ASF.

Imbens and Newey (2003) propose an approach to the identification and estimation of the nonseparable triangular simultaneous equation model in (5) and (6) that also applies if (5) is not structurally separable. Their method recovers the function f in the case that the random error ε is a scalar random variable. The construction in Section 6 gives a model in which ε is a vector and in that case we do not recover f , but the ASF. The key insight is that because Z and V are independent, the distribution of X given $V = v$ is the same as that of $g(v, Z)$. Because ε, V are independent of Z , we have that

$$X \perp \varepsilon | V = v \tag{20}$$

The variable V is called the control variate. The control variate approach seems to use the representation of the first-stage relation in (6). However, this equation does not impose any restriction on the joint distribution of X, Z .

The results in Blundell and Powell (2004) and Imbens and Newey (2003) imply that the control variate V allows us to obtain the ASF by a conditional moment restriction, even if the model is not structurally separable. To see this we observe that

$$\mathbb{E}[Y|X = x, V = v] = \mathbb{E}[f(X, \varepsilon)|X = x, V = v] = \mathbb{E}[f(x, \varepsilon)|V = v]$$

To obtain the ASF ϕ we integrate this expression with respect to the marginal distribution of V that is the uniform distribution on $[0, 1]$. Hence ϕ satisfies the average conditional moment restriction (ACMR)

$$\int_0^1 \mathbb{E}[Y - \phi(X)|X = x, V = v]dv \tag{21}$$

It is important to note that this works even if ε is a vector.

If the model is structurally separable, we can use the ACMR to recover m . The control variate method adds the control variate V to the set of regressors in the nonparametric regression of Y on X . The nonparametric regression is then averaged over the control variate.

If the model is not structurally separable, then the CMR based on mean independence does not identify the ASF. However if the monotonicity assumption holds, then the ACMR still recovers the ASF. Hence the ACMR is robust against misspecification of the model, because it will recover the ASF if the population relation between Y and X is not separable.

7 Conclusion

Our main conclusion is that in the nonparametric triangular simultaneous equations model, the usual CMR based on mean independence of the error from the instrument identifies a causal relation between the dependent variable and an endogenous covariate, only if the model is structurally separable in the observable covariates and the unobservable random error. If the CMR is used in a population where the relation is nonseparable, the average structural function that gives the average response given an exogenously assigned level of $X = x$, is not recovered, nor is any other structural object.

We also provided a condition under which the nonseparable model is nonparametrically just identified from the population distribution, so that if that condition holds, the nonseparable model does not impose any restriction on the population distribution and hence is the natural alternative to the separable model.

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