INTERGENERATIONAL EQUITY AND THE DISCOUNT RATE FOR COST-BENEFIT ANALYSIS

JEAN-FRANÇOIS MERTENS & ANNA RUBINCHIK

Abstract. Current Office of Management and Budget (OMB) guidelines use the interest rate as a basis for the discount rate, and have nothing to say about an intergenerationally fair discount rate. A traditional approach leads to too high values for the latter, and in a wide range. We propose to apply Relative Utilitarianism to derive the discount rate, and find it should equal the growth rate of real per-capita consumption, independent of the interest rate.

1. Introduction

Many public policy decisions — whether about a housing project, about managing extraction of natural resources, about pension reform, etc. — typically involve trade-offs of economic costs and benefits, that are spread over time. Crucial, then, becomes the appropriate choice of the discount rate, i.e., how to translate benefits and costs into present (consumption) terms (either explicitly stated as a part of regulatory principles or implicitly embedded in a specific policy analysis). This choice should, naturally, be based on well-defined normative principles, i.e., for individuals making decisions on behalf of future generations, it has to be objective and justified. In this paper we use Relative Utilitarianism to derive an intergenerationally fair discount rate to evaluate benefits accruing from a public project in a general equilibrium model with overlapping generations.

Our purpose here is analytical: taking existing practices (say, summarized in the OMB Circulars) as given, we suggest a way to think about the underlying principles behind these practices, and to translate abstract "equity" requirements into concrete terms.

Circular A-4 of the U.S. Office of Management and Budget (2003) mandates that all executive agencies and establishments conduct a "regulatory analysis" for any new proposal, and more specifically (pp. 33–36), a cost-benefit analysis, at the rates of both 3% and 7%. Both rates are rationalized there as "the interest rate": the first one relative to private savings, the second one relative to capital formation and/or displacement, i.e., as the gross return on capital.

The OMB circular does refer explicitly to the requirement of equity vis-à-vis of future generations, and acknowledges it by requiring, for

Date: September 5, 2006.

projects that might have substantial long-term impact, a further analysis at a "lower but positive" discount rate (p. 36) — but more specific suggestions are hard to find.[1] This is the question we want to address.

The issue of discounting, and — more broadly — intergenerational justice, has been controversial in the literature since, probably, Sidgwick (1874).[2] Ramsey (1928) (p. 543) presents discounting future utility ('enjoyments') as a "practice which is ethically indefensible and arises merely from the weakness of the imagination."He suggests a way to overcome 'technical' difficulties of constructing a discount-free utilitarian social welfare criterion (based on the difference between actual and 'bliss' level of utility), later referred to as the "Ramsey criterion." Discounting utilities, or 'social impatience,' was axiomatised by Koopmans (1960). A growing literature in social choice and welfare economics is concerned with incorporating intergenerational justice principles in developing a social welfare criterion (among the most recent contributions, see Asheim, Mitra, and Tungodden (2006) who demonstrate existence of welfare functions satisfying some of Koopmans' (1960) postulates and principles of intergenerational equity, in particular, the axioms of 'sustainable development' by Chichilnisky (1996)); in addition a number of contributions are devoted to characterizing ethically acceptable (just) consumption allocations over time (Asheim (1991), Fleurbaey and Michel (2003) among others).

To adequately tackle questions of intergenerational equity we suggest to use Relative Utilitarianism, a welfare criterion introduced in Dhillon and Mertens (1999), that allows for a meaningful comparison of well-being across individuals born at different times and faced with different consumption choices and different economic environments.[3] The criterion explicitly requires equal treatment of individuals of different generations in its anonymity axiom. The importance of using an explicit criterion for this purpose was stressed in Drèze and Stern (1987), distinguishing this approach from that examining "potential

[1]Other applied sources share this view, e.g.: "Morally speaking, there is no difference between current and future risk. Theories which, for example, attempt to discount effects on human health in twenty years to the extent that they are equivalent to only one-tenth of present-day effects in cost-benefit considerations are not acceptable."Wildi, Appel, Buser, F.Dermange, Eckhardt, Hufschmied, and Keusen (2000)

[2]"How far we are to consider the interests of posterity when they seem to conflict with those of existing human beings? It seems, however, clear that the time at which a man exists cannot affect the value of his happiness from a universal point of view; and that the interests of posterity must concern a Utilitarian as much as those of his contemporaries, except in so far as the effect of his actions on posterity — and even the existence of human beings to be affected — must necessarily be more uncertain." (p. 414)

[3]Relative Utilitarianism is discussed in more detail in section 1.2. Of couse, its axiomatisation referred only to a finite set of individuals...

improvements" stemming from the project.[4] Formulating a social welfare function, the authors argue, provides greater transparency to the cost-benefit analysis, assures consistency of related choices and avoids a special preference for inaction.[5]

We focus on "small projects," viewed as "a disturbance to the economy, displacing it from some initial equilibrium to a new one"(Bell and Devarajan (1983), pp. 457–8). This linearisation is essential to cost-benefit analysis itself, both in order to be able to speak of costs and benefits, rather than welfare differences, and in order to be able to conduct a separate cost-benefit analysis for each project, rather than having to do an overall welfare optimization over all conceivable combinations of projects by all branches of the government. This means, projects are evaluated via 'shadow prices', and the discount rate is the shadow price for tomorrow's goods in terms of today's. Bell and Devarajan raised a concern that the shadow prices might not be well-defined if the corresponding policy is not fully specified. One way to avoid this is to translate the effect of a public project into its consumption equivalent for individuals. Viewing public projects this way, we have no reason to introduce public goods into the model, which makes the analysis more transparent. Moreover, this representation is closer to the practical guidance for conducting cost-benefit analysis suggesting that the impact of a public project be monetized (see Circular A-4). Thus, the relevant shadow price becomes the marginal social value created by an additional unit of consumption.

It is not uncommon to use prevailing prices to represent the shadow prices, and thus, the interest rate becomes the discount rate to be used. The welfare criterion supporting these shadow prices is very specific:

The status-quo is a given competitive equilibrium. Construct a social welfare function (SWF), W, as a weighted sum of individual utilities, $\sum \lambda_n u_n$, the weights being chosen such as to equalize the individual marginal utilities of consumption *at the given equilibrium*, so $\lambda_n \nabla u_n = \mu p$,[6] for the equilibrium price system p and some $\mu > 0$.[7]

---

[4]See Mishan (1976) for an in-depth discussion of "potential Pareto improvements" (traced back to Pigou (1932)) and their application to cost-benefit analysis. For a more recent overview of cost-benefit criteria see Coate (2000).

[5]"... a fundamental shortcoming of evaluation criteria based on Pareto improvements, whether actual or potential, is that, unless they are taken to imply that Pareto-improving changes are the only acceptable ones (a view which we regard as extremely unappealing and which attaches undue weight to the status quo), they provide no decision criterion for projects which cannot lead to Pareto improvements. It is difficult to overcome this problem without accepting the need to specify a social welfare function which embodies more definite judgements."(p.49)

[6]This condition is implied, for example, by one of Samuelson's (1954) optimality conditions, see his condition (3).

[7]Equivalently, assuming, e.g., concave utility functions, one can deduce from the First Welfare Theorem the existence of utility weights such that the given

Viewing projects as small perturbations of individual endowments, $\delta\omega_n$,[8] we are interested in the induced variation of social welfare, δW,

$$\begin{aligned} \delta W &= \sum \lambda_n \delta u_n = \sum \lambda_n \langle \nabla u_n, \delta c_n \rangle \\ &= \mu \langle p, \sum \delta c_n \rangle = \mu \langle p, \sum \delta \omega_n \rangle \end{aligned} \tag{1}$$

since $\langle p, \delta y \rangle = 0$, where y is the equilibrium production.[9]

Thus, with those specific weights, the prevailing prices are, indeed, the relevant shadow prices, reflecting the relative impact of the endowments on social welfare. In a dynamic interpretation, where goods become dated goods, the equilibrium price system includes, in particular, the interest rate, as the price of tomorrow's money in terms of today's.

One rationale for that approach is that cost-benefit analysis is to be carried out in a quite decentralized way by different government agencies, project by project. So the only way to ensure some coherence, and to ensure that each one stays within its area of competence, is to assume that the others do their job correctly — and in particular, that redistribution policy (most of which is determined by the legislature) is correct. Indeed, that implies directly that transfers are welfare-neutral at the margin, and hence the above weights. In this paper we are however interested in the implications of a specific view on intergenerational equity; hence we cannot assume that the prevailing interest rate is the correct discount rate. Thus we must depart from the above weights,[10], and use a SWF that explicitly embodies this concept of intergenerational equity.

In the next section we try to do this, in a toy-model, using the most traditional utilitarian methodology for policy analysis.

1.1. A simple computation using the traditional methodology.

Let us start with a very simple model of an economy, in which individuals live for just one period, enjoying consumption $c_t > 0$ during their lifetime at t. Individual preferences over (lifetime) consumption are represented by a constant relative risk aversion utility function with coefficient, $\rho > 0$, so that $u(c) = c^{1-\rho}/(1-\rho)$; and suppose the economy is on a balanced growth path with per-capita consumption growing exponentially at rate $\gamma > 0$. Consider a policy that involves a variation in aggregate consumption of δC_t at each future date t and that is to be

---

equilibrium maximizes the corresponding weighted sum of utilities over all feasible allocations; this yields the same weights.

[8]Since we omit for simplicity public goods and externalities from our formal model, the $\delta\omega_n$ are assumed to include, in addition to the direct effect of the project, also the appropriate compensating variation (in real terms) for the different external effects.

[9]$\langle x, y \rangle$ denotes the inner product of two vectors x and y.

[10]As observed above, any such departure will make transfers non welfare-neutral, and hence will imply that aggregation (i.e., that everything depends only on $\sum \delta\omega_n$) is no longer possible. We will deal with that difficulty in sect. 4.2.

evaluated at time 0. The status-quo per-capita consumption at time t is $c_0 e^{\gamma t}$, where c_0 is the initial (time 0) per-capita consumption. Taking a traditional utilitarian criterion ($W = \sum_t e^{-\beta t} N_t u(c_t)$, where N_t is the number of agents at time t) as a guide for evaluating this policy, the net (social) benefit equals

$$\begin{aligned} & \sum_t e^{-\beta t} N_t \big[u\big(c_0 e^{\gamma t} + \tfrac{\delta C_t}{N_t} \big) - u(c_0 e^{\gamma t}) \big] \\ = \ & \sum_t e^{-\beta t} N_t u'\left(c_0 e^{\gamma t} \right) \cdot \tfrac{\delta C_t}{N_t} = \sum_t c_0^{-\rho} e^{-(\rho\gamma+\beta)t} \delta C_t \end{aligned}$$

This means that future consumption is discounted at the rate $\rho\gamma + \beta$ under this criterion. Even if we follow Sidgwick (1874) and Ramsey (1928) and set $\beta = 0$, to write explicitly that we want to treat future generations equally, the magnitude of the suggested discount rate, $\rho\gamma$, is far above any rates applicable in practice, and the estimated values have an extremely wide range, as the next subsection demonstrates.

1.2. **Orders of Magnitude for the Discount Rate.** To estimate γ one may use a measure of growth of real per-capita GDP. Based on the data from the Bureau of Economic Analysis, over the past 70 years the average in the U.S. is estimated to be around 2–2.5% per annum (with averages over various decades since 1950 ranging from 3% to 1.8%).

In the above model, individuals live for 1 period, so the *only* role of ρ is to determine the individuals' attitudes towards risk. And consistency with, e.g., Harsanyi's axiomatization(s) of such additive SWFs forces then to interpret u as the individual's von Neumann-Morgenstern utility function, and hence ρ as his coefficient of relative risk aversion. One of the most recent overviews compiling various (micro) estimates of the risk aversion coefficients is contained in Einav and Cohen (2005). Remarkable is both the range as well as the magnitude of the suggested values, ranging from single- to three-digit values. They measure relative risk aversion coefficients from individual-level data on car insurance and annual income, obtaining *two-digit* estimates. Clearly, cost-benefit analysis will then only allow for very short-sighted policies. This remains true even with more conservative estimates, like, say, derived by Drèze (1981) ($\rho \sim 12$–15), or like those which seem accepted as corresponding to "representative" (instead of individual) behaviour in financial markets — say 3, leading to $\rho\gamma \sim 6$–7%, way too high.

In sum, it is impossible to view the traditional methodology described above as a correct interpretation of "treating future generations equally" — which is exactly what the SWF tried to do, by using $\beta = 0$.

1.3. **Discount Rate under Relative Utilitarianism.** Since the traditional methodology failed so badly, producing unreasonably high discount factors within a wide range, let us now look at Relative Utilitarianism, introduced in Dhillon and Mertens (1999).

The axiomatisation consists basically of applying Arrow's axioms to preferences over lotteries, after "surgically removing" from them everything which is clearly objectionable — i.e., which anyone would expect a good social welfare functional to violate: the implications that variations in the intensity of preference of x over y don't matter.

After this removal, one can add anonymity (implying here also that individuals of different generations are treated equally) to obtain an axiomatization of a unique social welfare functional,[11] relative utilitarianism, that takes for each individual's preferences the unique von Neumann-Morgenstern representation having minimum 0 and maximum 1 over the feasible set, and sums those to obtain a representative of the corresponding social preferences.

It is stressed in that paper that this dependence on the feasible set implies that in actual use it should be applied with some universal feasible set, to quote "all alternatives that are feasible and just". In particular, in the present situation, the feasible set should consist not only of the "baseline" and the different proposals under consideration, but of all policies and policy-changes that might be considered by any agency of the government.

In (exogenous) growth models, the rate of growth is unaffected by any policy variable: policies affect only the height of the growth path, which, in the simple setup described in subsection 1.1, translates into multiplying per-capita consumption by some constant along the growth path. Therefore, the set of feasible policies at time t can be viewed as a range of induced per-capita consumption levels $(1-\eta)c_0e^{\gamma t}$ and $(1+\zeta)c_0e^{\gamma t}$ for some constants η and ζ. Applying relative utilitarianism to the simple model, we have to normalize individual utility $u(c_t)$ on the set of feasible policies:

$$v(c_0e^{\gamma t}+\delta c_t)=\frac{u(c_0e^{\gamma t}+\delta c_t)}{u\big((1+\zeta)c_0e^{\gamma t}\big)-u\big((1-\eta)c_0e^{\gamma t}\big)}$$

i.e., we divide by

$$\frac{e^{(1-\rho)\gamma t}}{c_0^{\alpha}}\left[\frac{-1}{(1+\zeta)^{(\rho-1)}}+\frac{1}{(1-\eta)^{(\rho-1)}}\right]\quad\sim e^{(1-\rho)\,\gamma\, t}$$

So the variation of our SWF becomes

$$\sum_t e^{(\rho-1)\,\gamma\, t}\delta C_t u'\left(c_0\, e^{\gamma t}\right)=\sum_t e^{(\rho-1)\,\gamma\, t}e^{-\rho\gamma t}\delta C_t=\sum_t e^{-\gamma\, t}\delta C_t$$

This implies that the previous discount rate of $\rho\gamma$ becomes now simply γ, $2-2\frac{1}{2}$ %, right in the ball-park of "positive and $< 3\%$".

One could argue that the example is not representative; in particular, since individuals live only one period they have no incentive to save, so there can be no capital accumulation and growth. In a real model where there is growth and savings, there is also an interest rate — and individuals would smooth the shock over their lifetime using the going

[11]The axiomatization assumes a finite number of agents.

interest rate: so one would expect the result to be driven back to the interest rate, to a large extent at least.

We will nevertheless show that the result (as well as that of sect. 1.1) does remain valid in the much more general framework of next section.

2. The model

We use a general-equilibrium overlapping generations model, cast in an exogenous growth framework. The main assumptions that we impose on the economy — homogeneity of utility functions with respect to (streams of) consumption, and constant returns to scale in production — are there to allow for a balanced growth path.[12]

2.1. **The Consumption Sector.**

2.1.1. *Population Dynamics.* Time is continuous, ranging from $-\infty$ to $+\infty$. There are several types of individuals. An individual of type τ lives up to age T_τ. The population dynamics are fully specified by non-decreasing right-continuous functions $P_{\tau,\tau'}$, defined on $[0, T_\tau]$ with $P_{\tau,\tau'}(s)$ being the number of children of type τ' an individual of type τ has at age s, and by saying that we are looking at a corresponding invariant distribution.[13] But as long as we dont introduce bequest motives or the like, it is only this distribution that matters. It is such that, at time t, the number of individuals of type τ in age-group $(s, s + ds)$ $(0 \leq s \leq T_\tau)$ is given by $N_\tau e^{\nu(t-s)}ds$. So, population grows at rate $\nu > 0$, keeping the proportion of each age group of each type constant.

2.1.2. *Preferences and Endowments.* At each instant of his life, s, an individual of type τ born at time x consumes non-negative quantities of n goods, $c^{\tau,x}(s) \in \mathbb{R}^n_+$ and allocates fractions of his time to h types of labour, $z^{\tau,x}(s) \in \mathbb{R}^h_+$.[14]

His preferences over integrable life-time consumption-streams in $\mathbb{R}^{n+h}$ are derived from a utility function U^τ (e.g., increasing in the goods, decreasing in labor, concave, differentiable). There is no bequest motive.[15]

For balanced growth to be at all possible, we assume U^τ to be homogeneous, say of degree $1 - \rho^\tau$, in the n streams of consumption-goods.

Endowments are 0 — except for the "endowment of time," which is unity at every instant (24h/day). This imposes an instantaneous

[12]See King, Plosser, and Rebelo (2002) and Arrow and Kurz (1970) for the relevant discussion.

[13]We keep everything deterministic here, just to avoid having to discuss irrelevant insurance markets for idiosyncratic risks.

[14]Sometimes we will use the notation $c^\tau(s,t)$ and $z^\tau(s,t)$ to stand for consumption and labor of an individual of type τ who is of age s at time t, so that $x = t - s$.

[15]We index consumption streams by age, in $[0, T_\tau]$, so all individuals of the same type have the same consumption set, $C[0, T_\tau]$, and utility function, independently of their birth-date.

constraint on the individual feasible set requiring the sum of fractions of time devoted to all possible occupations to be always less than unity.

In what follows, a policy will be associated with a perturbation of endowments of consumption goods, $(\delta\omega)_i$ for $i = 1 \dots n$ — cf footnote 8.

2.2. **Production.**

2.2.1. *Instantaneous production.* Instantaneous production is described by a closed convex cone $Y \subset \mathbb{R}^{h+m+n+m}$, $t \in \mathbb{R}$, describing feasible production plans transforming $h+m$ inputs (h types of (effective) labour, $L(t) \in \mathbb{R}^h_+$, and m types of capital) into n consumption goods and m investment goods. Assume no free lunch, $Y \cap \mathbb{R}^n_+ = \{0\}$.

Individuals supply labour (time) to the firms, and their productivity changes with time and age. The amount of effective labour of type i received at time t by a production firm from an individual of type τ and of age s is $e^{\gamma t}\varepsilon_i^\tau(s)z_i^\tau(s,t)$, where $\varepsilon_i^\tau(s)$ is this individual's life-cycle 'productivity' (in occupation i),[16] and where γ is (labour-enhancing) technological progress. Recall $z_i^\tau(s,t)$ is the amount of labour (time) supplied by an individual (of type τ) born at time $t-s$.

Thus, (exogenous) growth in this model is driven by a steady increase in labour productivity.

2.2.2. *Capital accumulation.* There are m capital goods K^i $(i = 1 \dots m)$, each with its corresponding investment good I^i, depreciation rate δ_i, and capital-accumulation equation $\frac{dK^i(t)}{dt} = I^i(t) - \delta^i K^i(t)$,[17] together with the "initial condition" that $\limsup_{t\to-\infty} e^{-(\gamma+\nu)t}K^i(t) < \infty$.

The rest of this subsection is devoted to verifying that the production set is well-defined (proofs in appendix).

The capital accumulation condition and the initial condition have the following implication:

Lemma 1. *$K^i(t) = e^{-\delta^i t}\int_{-\infty}^t e^{\delta^i s}I^i(s)ds$ for all t, where the integral is a Lebesgue integral.*

To ensure bounded production possibilities, capital cannot reproduce itself ("rabbit economy").[18] Lemma 1 ensures that $K^i(\cdot)$ is uniquely

[16] For example, in the textbook OLG models going back to Diamond (1965) ε would be 1 during the first half of life and 0 after.

[17] Assumed to hold a.e., and implying that K_t^i is assumed locally a Perron primitive and I_t^i locally Perron-integrable.

[18] For instance, assume a single good, a single type of labour, a CES production function $(AK^\alpha+BL^\alpha)^{1/\alpha}$, and a policy where all agents work full-time and consume nothing (e.g., in order to get an upper bound on capital and investment). Assume also $A^{1/\alpha} \geq R$ with $R = \gamma+\nu+\delta$. Note that $L_t = L_0\exp(\gamma+\nu)t$, so for $D = BL_0^\alpha$, $K'(t) = (AK^\alpha(t) + De^{(\gamma+\nu)t})^{1/\alpha} - \delta K(t)$; or with $x(t) = K(t)e^{-(\gamma+\nu)t}$, $x'(t) = (Ax^\alpha(t)+D)^{1/\alpha} - Rx(t) \geq D^{1/\alpha} > 0$. Since $x(t) \geq 0$, there is no solution, i.e., the upper bound of $K(t)$ is infinity. And even if $B = 0$, the solutions are $x(t) = Ce^{(A^{1/\alpha}-R)t}$, with $C \geq 0$ arbitratily large, so $K(t)$ is unbounded in this case too.

determined by $I^i(\cdot)$. However it might not be sufficient to guarantee that any investment policy (e.g., I is a function of current K instead of time) has a well-determined outcome, without either using the full strength of the "initial condition" (Lemma 2 below), or slightly reinforcing the assumption that capital cannot reproduce itself (Lemma 3).

Lemma 2. *Assume Y is such that no investment good can be produced without some form of labour input. Assume $R \equiv \gamma + \nu + \delta > 0$. Then the set of all feasible functions $K^i(t)$ and $I^i(t)$ is bounded above by $\bar{K}e^{(\gamma+\nu)t}$ for some $\bar{K}$.*

In fact, at least with a slightly stronger condition on Y, the formula of lemma 1 suffices, without the initial condition:

Lemma 3. *Assume $\exists \varepsilon > 0, A, B\colon (-L, -K, C, I) \in Y \implies \|I\| \leq A\|L\| + B\|K\|^{1-\varepsilon}\|L\|^{\varepsilon}$. Then the conclusions of lemma 2 hold, assuming just lemma 1, without the need for the "initial condition".*

3. Equilibria (Solution Concepts)

In addition to the classical Arrow-Debreu equilibrium concept, there are other possible solution concepts for this economy, to which our theorem is applicable too.

3.1. **Time Invariance.** The economy we have described possesses a convenient time-invariance property that will prove to be useful later. We consider the effect on the economy (i.e., the description of the population, the feasible consumption and production plans, and individual preferences thereon) of shifting the origin of time by h.

Definition 4. *The transformation T_h of the economy ('time-shift by h')*

(1) *shifts all consumption, production and endowment vectors (both goods and labour) forward in time by h.*
(2) *multiplies all non-labour individual quantities (endowments of goods, allocations of goods) in the economy by $\exp(\gamma h)$,*
(3) *multiplies the aggregate quantities of population and labour*[19] *in the economy by $\exp(\nu h)$.*

Now we claim that the economies we consider are time-invariant in the sense of this transformation:

Lemma 5. *Each T_h is an automorphism of the model:*[20]

- *it maps feasible production plans in a 1-to-1 way onto feasible production plans.*

[19] And hence aggregate quantities of all non-labour goods (consumption, capital, investment) are multiplied by $\exp((\gamma + \nu)h)$.

[20] Hence, if endowments are invariant under T_h, i.e., if the endowment of goods of an agent of type τ born at time t is of the form $\omega_\tau \exp(\gamma t)$ (in particular, 0), then T_h is even an automorphism of the economy.

- *it maps the preferences of each consumer between different consumption bundles (consumption=goods+labour) to the preferences of his image, born time h later. And his initial endowment is mapped as well to the initial endowment of his image.*

Remark 6. *T_h induces a map from allocations in the initial economy to the allocations in the image economy.*

Definition 7. *A (set-valued) solution concept is time-invariant if time-shifts T_h map solutions to solutions of the image economy.*

3.2. **Examples of Time-Invariant Solution Concepts.** Next, let us consider several examples of time-invariant solution concepts. First we discuss Arrow-Debreu equilibrium, and then briefly mention a couple of other examples: an adaptation of Diamond's (1965) equilibrium to this framework, and a 'social planner' solution, allocating goods to maximize a 'time invariant' objective, e.g., the specific Relative Utilitarian criterion that we use to evaluate welfare perturbations.

3.2.1. *The Arrow-Debreu Equilibrium.* To describe the Arrow-Debreu equilibrium for the our economy we have to define profits of a firm. It is convenient to think of two types of firms: a single firm that handles the instantaneous production and has Y as technology,[21] and one firm per capital good that handles the corresponding investment and has the capital accumulation equation as technology.

Given $p_c(t) \in \mathbb{R}^n$, $p_I(t) \in \mathbb{R}^m$, the equilibrium prices for consumption and investment goods, and $p_k \in \mathbb{R}^m$, $p_l \in \mathbb{R}^h$, the equilibrium rental rates for capital and labour, the *production firm*[22] chooses the amount of inputs to rent from the investment firms (aggregate capital, $K(t) \in \mathbb{R}^m_+$) and consumers (aggregate efficient labour, $L(t) \in \mathbb{R}^m_+$) as well as outputs of final (aggregate consumption, $C(t) \in \mathbb{R}^n_+$ and aggregate investment, $I(t) \in \mathbb{R}^m_+$) goods to maximize its profits, Π_C,

$$\langle p_c(t), C(t)\rangle + \langle p_I(t), I(t)\rangle - \langle p_k(t), K(t)\rangle - \langle p_l(t), L(t)\rangle,$$
$$(-L(t), -K(t), C(t), I(t)) \in Y$$

which are zero.

The *investment firms* can choose a time-path of investment and rent out their accumulated capital (uniquely determined by Lemma (1)) to the production firm.

[21]A choice of production plan at time t involves no implications for profits of the firm at other dates, so the profit maximization problem of this firm is static.

[22]One could introduce many production firms that have access to the technology described by Y without changing the results. Indeed, the composition of the industry is irrelevant as long as the total production set is preserved.

An investment firm i owns capital $K^i(t) = e^{-\delta^i t}\int_{-\infty}^{t} e^{\delta^i s} I^i(s)ds$ of type i and chooses an investment policy $I^i(\cdot)$ to maximize its profits

$$\Pi_I^i(I(\cdot)) \equiv \int_{-\infty}^{+\infty} I^i(t)[-p_I^i(t) + \int_0^{+\infty} e^{-\delta^i s} p_k^i(t+s)ds]dt$$

which should be zero.[23] This condition implies

$$p_I^i(t) = \int_0^{\infty} e^{-\delta^i s} p_k^i(t+s)ds \tag{2}$$

Finally, we have to define the life-time budget constraint of an individual of type τ born at time x:

$$\int_0^{T_\tau} \langle p_z(s+x), z^{\tau,x}(s)\rangle - \langle p_c(s+x), c^{\tau,x}(s)\rangle \, ds = 0 \tag{3}$$

$$z^{\tau,x}(s), c^{\tau,x}(s) \geq 0, \quad \langle 1, z^{\tau,x}(s)\rangle \leq 1 \tag{4}$$

Clearly, the price for efficient unit of labour, $p_l(t)$, is proportional to the price of labour time, $p_z(t)$, at each instant t: $p_z(t) = e^{\gamma t}\langle \varepsilon^\tau(s), p_l(t)\rangle$.

After the above definitions, the definition of an Arrow-Debreu equilibrium is standard.

3.2.2. *Diamond Equilibrium.* One could also reproduce an equilibrium concept suggested by Diamond (1965) for this model. There are no investment firms, consumers hold the capital stock of different types and rent it out to the production firm. As in Arrow-Debreu equilibrium, consumers can also lend to each other (IOU's) — e.g., if they must borrow when young. So, the value of the total net savings of the consumers at each point in time equals the total value of the accumulated capital.

3.2.3. *Selection.* Observe that the above equilibrium concepts are typically multi-valued, so to get from them a single-valued time-invariant solution concept, as our main result below (Theorem 9) requires, one has to make a selection in a neighbourhood of the given balanced growth equilibrium — cf. footnote 27 below for this.

3.2.4. *Maximizing Welfare.* Maximizing welfare — where the utilities can be discounted, but must be normalised as in Relative Utilitarianism, cf. section 4.1 below[24] — is also a time-invariant solution concept. Since however time varies from $-\infty$ to $+\infty$, it is not immediately obvious that a maximum exists (whether the discount factor used equals 0 or not), but arguably reasonable social welfare functions (discount

[23]We don't investigate here the important question as to under what conditions (on I, p_I, p_k) the order of integration can be changed. The definition is written this way to lead as easily as possible to a definition of equilibrium.

[24]Except of course if all individuals' relative risk aversion coefficients (i.e., degrees of homogeneity) are the same, then this is irrelevant. But else the relative weight on the types with higher risk aversion would, without the normalisation, effectively go down to 0 as the economy grows.

rates) should ensure this existence. Now, the concept being time-invariant, one might expect that, if a maximum exists, it is achieved at some balanced growth path. It should thus suffice to maximise the welfare of any fixed generation over all feasible balanced paths[25] — and then to show that, for reasonable discount rates, when the utility levels on that path are subtracted from each individual's utility function, the social welfare is indeed maximised. But this criterion to which we were led — to maximise for any fixed generation over all balanced paths — is completely independent of the discount rate we started with! This clearly suggests there might very well be no 'reasonable' discount rates (for utilities) beyond the obvious candidate, ν, the rate of growth of the population (this being the only one to weigh equally the past and the future). Observe that this leads to a discount rate of $\nu + \gamma$ on real consumption, i.e., the interest rate, in the framework of golden rule rule equilibria…Clearly the above heuristics need confirmation by a full proof, but in that case, they might conceivably form the basis for an argument of inadequacy of our criterion of intergenerational fairness in the framework of economic models with growing population…

To return to our subject, one could then maximise, for perturbed endowments too, this same ν-discounted sum of normalized utilities (where the normalisation includes the subtraction of the utility level on the status-quo path), giving thus another example of time-invariant solution concept (and typically single-valued this time).

3.2.5. *Unanticipated shocks.* All the above deal with fully anticipated shocks. One can, for the same concepts, consider the polar case, where all contracts have already been signed for the unperturbed economy, so the effective initial endowment that gets perturbed is the final allocation for the unperturbed economy with that solution concept (and it is from that perturbed endowment that individuals re-trade). This being a balanced path, by the assumption of the theorem, it suffices then to observe that the theorem remains applicable as is to economies with such initial endowments, the time-invariance property being preserved.

4. The Relative Utilitarian Welfare Function

4.1. **The Set of Alternatives.** To formulate the social welfare function we need the feasible set, and the simplistic formulation used in section 1.3 is no longer adequate in view of the multiple goods, and several types of consumers. Ideally it should be defined in the space of policies, but since one of our aims is to prove that our result is completely independent of it, we will define it as the corresponding set in the space of (final — i.e., after all equilibrium readjustments) allocations.

The set of available allocations should be *time-invariant*, i.e., it should be mapped to itself by any time-shift T_h.

[25]Leading thus to a natural generalisation of the 'golden rule' paths.

So, the time invariance is here to capture the previous idea that policies affect only the height of the growth path — while leaving the geometry of the feasible set completely arbitrary in all other respects.

Further, an obvious implication of the justice requirement on the feasible set is that each individual's utility is bounded below.

4.2. **The distribution of costs and benefits.** We associate with any policy-change a corresponding perturbation of individual endowments of consumption goods over time. We want to evaluate the corresponding variation of social welfare, after individuals trade to a new equilibrium.

Let $\omega_y^\tau(t)$ be a perturbation of consumption (vector) of individual of type τ who was born at time y. It is clear that by just taking on a given day consumption away from the old and giving it to the young one could achieve artificial welfare increases: indeed, since their utilities at birth are weighted equally in the social welfare function, their own time-impatience will have for effect that the benefits of the transfer to the young is much greater than the disutility to the old.

Thus our variation of welfare will in general depend on the whole perturbation of endowments, not only on the aggregate.

One faces this problem as soon as one uses individual weights in the social welfare function for which the given equilibrium path is not optimal.

One may want to approach this problem (problem — in as much one wants to adhere to this idea of intergenerationally fair social welfare function —) in at least two different ways.

The first would be to argue that individual preferences must be respected — by the model, by the "state" —; that if somebody goes to the casino and loses all his money (or robs a bank and gets to jail), it would break all incentives for the state to bail him out afterwards — and similarly if he exhibits such time-preferences as to spend all his money in his youth. This is roughly the point of view of the present model, and the reason for insisting that the social welfare function be formulated in the terms of the individuals' *expected utilities at birth*.

In this vein, one would want to reformulate individual utilities in the model to encompass both a "bequest motive" (e.g. in the form of a utility depending recursively on that of one's children too), and some form of altruism vis-à-vis of one's parents: both effects tend to lengthen individuals time horizon, i.e., to decrease their impatience, hence probably to reduce claims of inadequacy of this approach. And de facto, it seems that in traditional societies those 2 aspects prevented any form of gross injustice.

If the above approach is not sufficiently adequate, – or anyway, since it is not a solution in principle –, one might want to take a more paternalistic approach, and argue that, in the same way the state has to protect future generations against short-sightedness of the current generation, it also has to protect each individual against the consequences in his old age of his own short-sightedness when young. The various policy instruments used to achieve this (in the extreme, some forms of forced savings, etc.) should then be incorporated into the model, to get rid of the problem — i.e., to reduce in effect to the same model, but where individual time-preferences have been corrected to fit with the social welfare function.

It is clear that such things require much more work, and thought, and lead us astray from our subject — the discount rate for cost-benefit analysis. Hence, to be able to pursue our analysis, in a way unaffected by this problem, we will assume that somehow this problem is being taken care of by current policy, and that the aggregate perturbation $\Omega(t)$ gets distributed in a fixed (i.e., time- and commodity-independent) way across age groups and types. So the variation in welfare will be a function just of the aggregate Ω.

Let thus $\vartheta^{\tau}(s)$ be some integrable function, the distribution of endowments, with $\vartheta^{\tau}(s) = 0$ for $s < 0$ and $s > T_{\tau}$, and with $\sum_{\tau} \int_{-\infty}^{+\infty} \vartheta^{\tau}(s) ds = 1$. Then, a perturbation of consumption (vector) of individual of type τ who was born at time y is related to the aggregate perturbation $\Omega(t)$ in the following way,

$$\omega_y^{\tau}(t) = \vartheta^{\tau}(t-y) \frac{\Omega\left(t\right)}{N^{\tau} e^{\nu y}}$$

Recall, the population (within each type of individuals) grow at a constant rate ν, so total population of people of type τ who were born at time y is $N^{\tau} e^{\nu y}$ with N^{τ} being population of type τ born at time 0. Thus, we assume that the endowment is shared equally within each age-type category of individuals.

5. The Main Statement

To demonstrate the result, we need to evaluate the effect of a small aggregate consumption perturbation Ω at a balanced growth path on the social welfare function. The perturbation should affect equilibrium allocations, which, in turn, alter individual well-being and the social welfare. Thus, the objective is to compute the differential of the map from consumption perturbations to welfare and to prove that whenever it exists it is of the form $\int \langle q, \Omega(t) \rangle e^{-\gamma t} dt$ for some $q \in \mathbb{R}^n$ — i.e., that the discount rate used equals γ.

Technically, to make the main statement as strong as possible, we need to use the weakest notion of differential, i.e., that of Gateaux.[26]

[26] A function f from a subset D of a topological vector space X to $\mathbb{R}$ is Gateaux-differentiable at zero, if $\forall x \in X$ the set $\{t \in \mathbb{R} \mid tx \in D\}$ is a neighbourhood of zero in $\mathbb{R}$, say V_x, and if $t \mapsto f\left(tx\right)$ is differentiable at $t = 0$, say with derivative d_x, and if $x \mapsto d_x$ is a continuous linear functional on X.

We also need to specify the space of perturbations and its topology; we will use the space K (defined below), because that way the statement implies the same statement for about any other space of perturbations, since K embeds continuously as a dense subspace in about any other space.

We follow Schwartz (1957-59) and Gelfand and Shilov (1959) in defining K and the space K^* of continuous linear functionals on K (i.e., generalized functions):

Definition 8. *K is the space of infinitely differentiable functions with compact support, and a sequence of functions $\varphi_n \in K$ converges to zero if $\exists h \in \mathbb{R}$: $|x| \geq h \Longrightarrow \varphi_n(x) = 0$ for all n, and φ_n and all its successive derivatives converge uniformly to zero.*

K^* is the space of linear functionals ψ on K s.t. $\psi(\varphi_n) \to 0$ whenever $\varphi_n \to 0$ in K.

The economic meaning of $\Omega \in K^n$ is that the endowments are perturbed only over a bounded interval of time. Note that the status-quo (zero endowment) point also belongs to this space, so we can view the social welfare function W as being defined on K^n.

Next step is to define precisely the map from endowments to social welfare, given a solution concept that satisfies time-invariance. Let us consider a single-valued time-invariant solution concept ψ, which maps consumption endowments, $\Omega \in K^n$ to final allocations.[27] Assume that its domain, D, contains zero, which corresponds to the economy we described, in which individuals are born with no consumption endowments. As the solution concept is time-invariant, $\psi(0)$ describes a balanced growth path. Define the social welfare function W of relative utilitarianism by subtracting from each individual's normalised utility function its value at $\psi(0)$. (Thus, a constant is subtracted from each of the individual utilities to assure that welfare is well-defined on the growth path.) Denote by $\aleph'$ the subset of the space $\aleph$ of allocations

[27]Given a perturbation $\Omega(t)$ of consumption endowments, it is true that several equilibria might emerge. If dealing with a solution concept that does not guarantee local uniqueness, we choose e.g. out of those the one closest to the initial stable growth path in terms of the L_p norm $\sum_i \left\|\ln p_i(t) - \ln p_i^{\Omega}(t)\right\|_p$, where $p(t)$ is the price vector at time t prevailing at the initial equilibrium and $p^{\Omega}(t)$ is the price vector of a perturbed economy: though the price system does not necessarily fully specify an equilibrium, it does specify the individual utility levels, which is all we need. The effect of the logarithms is to make the distance independent of price normalization, hence to induce a distance between equilibria (or: between price-rays): for any multiple of p_i the minimum, over all multiples of p_i^{Ω} (clearly there is at most one such multiple where the value is finite, when $p < \infty$), will be achieved at the corresponding multiple, and the value of the minimum is independent of this multiple, and remains the same when permuting the roles of p_i and p_i^{Ω}. Finally, because of the L_p norms (i.e., Lebesgue measure being shift-invariant), the selection will be time-invariant. Obviously there will remain to show that there is some equilibrium at a finite distance, and that locally the minimum is achieved at a unique point.

where W is well defined (i.e., the integral converges). This set, for example, might include allocations that are not 'too different' from those on the balanced growth path $\psi(0)$, say, those that deviate from it over a bounded interval of time. Let us focus on the subset of consumption endowments, D', for which W is well defined: $D' = \psi^{-1}(\aleph')$, and note that $0 \in D'$, i.e., the status-quo belongs to this set, as by construction the welfare function is zero as evaluated at the initial balanced growth path, $\psi(0)$. Finally, we define the map from endowments to social welfare corresponding to the chosen solution concept ψ: W_ψ is the composite map $W \circ \psi$ from D' to $\mathbb{R}$.

Now the main result can be stated in the following succinct form:[28]

Theorem 9. *Consider a point-valued time-invariant solution concept ψ. If W_ψ is Gateaux-differentiable at 0, then its differential equals $\int \langle q, \Omega(t) \rangle e^{-\gamma t} dt$ for some $q \in \mathbb{R}^n$.*

This implies that the discount rate is the growth rate γ of per capita output. A rationale behind the proof is that the cost of consumption (in terms of inputs used in its production) becomes cheaper with time, due to the enhancement of labour productivity. Individual productivity grows at a rate γ, so this is exactly the rate of decrease in the real price of per-capita consumption.

6. Discussion of the Main Result

6.1. **A policy re-interpretation.** Real-life policies rarely involve direct consumption transfers (changes in endowments). The model can be re-interpreted to incorporate more realistic policies as follows. Assume a set[29] of basic policies. Let a policy be a specification of such a basic policy as a function of time. Assume that shifting a policy forward in time by h transforms its effect on the economy (through the solution concept) by T_h[30]. Then the result still holds, in the sense that, at a given status-quo stationary policy π^*, the welfare effect of a small policy variation $\delta\pi_t \in K$ is given by $\int e^{\nu t} \langle q, \delta\pi_t \rangle dt$.[31]

[28] A similar result could be shown in the traditional set-up, provided (the multi-dimensional analog of) risk-aversion, ρ_τ, is independent of the type τ — giving then a discount factor of $\rho\gamma$, and hence showing the robustness of our conclusions from the mini-model in the introduction.

[29] More precisely, a manifold, to make differentiability meaningful. . .

[30] In particular, any constant policy leads to some balanced growth path. So basic policies might be for example linear taxes — or non-linear (sales or income) tax-schedules indexed by average income.

[31] q is typically a 'small' vector (dimension that of the set of basic policies), so its computation is *much* easier than to follow all the equilibrium dynamics of the model for an arbitrary policy. E.g., it might be computable by changing one policy variable at a time to some close-by stationary value, and computing the resulting balanced growth path.

6.2. **On non-vacuity.** The Theorem relies on differentiability of the map from endowments to welfare. Indeterminacy is known to plague some classes of OLG models;[32] hence one would need to show that this problem is avoided in our case — in particular, making a policy change meaningful, in the sense that it generates predictable (determinate) changes in the economy. To demonstrate the non-vacuity of the statement, one has to show that (1) solution is non-empty valued, i.e., there is a balanced growth path which belongs to the set of solutions; (2) the map from consumption endowments to allocations (under the solution concept) is differentiable; (3) the map from allocations to welfare is differentiable. Verifying each of the requirements (even in a model with fully specified preferences and technology) might not be a trivial — however tractable — problem, and lies beyond the scope of this paper, but it will be dealt with in the subsequent research.

6.3. **What makes the Traditional Approach fare so badly?** If one accepts Harsanyi's theory, that the social (von Neumann-Morgenstern) utility should be a positive linear combination of individual vNM utilities, by respect of unanimity, then the only arbitrary choice we have made here is to assign to all individuals the same weight, since they have the same utilities. This is standard in every application. And Harsanyi himself too argues that when two individuals have the same preferences, it is reasonable to assign them the same utilities. But that argument is for preferences and utilities over the (common) set of alteratives, while here — and in most applications — it is about preferences and utilities over an individual set of personal consequences. Even the sets are not directly comparable, since in economics goods are indexed by date and location — and there is even no economic reason to use the same physical units at different times or places.

This weakens substantially the argument for treating identically individuals having the same preferences — and that is what went wrong. It is also why relative utilitarianism fares much better, by deriving those different weights from a normalisation over *the common set of alternatives.*

Alternatively, if one were to reject Harsanyi's theory, and be ready to live with social preferences which are completely irrational over lotteries, a welfarist approach is still useable, ignoring lotteries, and using some axiom system that relates individual *utilities*[33] to social utilities. So the cardinal representation of each individual's preferences are left to the choice of the user, to represent what he thinks are or should be the preferences of society. In this spirit, one could arrive at the "correct" result in the following way:

[32] See Geanakoplos and Brown (1985), Geanakoplos and Polemarchakis (1991)

[33] A fiction in the "observing mathematician"'s (Rawls (1971)) mind, furthermore without any uniqueness property.

- first apply a monotone transformation to each utility function, in such a way that they become all homogeneous of degree $1-\rho$ in the consumption goods (it is crucial — for aggregation — to have the same ρ for all individuals, cf. footnote 28 below). ρ must be chosen here such as to induce the correct social preferences for equality within a generation.
- next discount future utilities with the (negative, typically) discount rate $(1-\rho)\gamma$.

But this procedure sounds a bit arbitrary (and looks suspiciously like a very ad-hoc social welfare functional): it may be better to use a social welfare functional from the outset.

Finally, a welfarist might want to defend that discount rates as high as $\rho\gamma$ do correctly represent "treating future generations equally". While one can understand intellectually how that logic might lead to this, we just note that most people would agree with the OMB's position that "treating future generations equally" requires a discount factor lower than the interest rate.

6.4. **The Value of a Human Life.** We show now that the issue is not only that $\rho\gamma$ is not of the correct order of magnitude as compare to γ, but even that the former formula is conceptually wrong, and the latter exactly correct.

The value of life, according to any criteria [e.g., each of the four in Mishan's (1971) introduction, or even judicial criteria in assessing damages], is proportional to his life-time income, or to average life-time income at his time: anyway, proportional to $e^{\gamma t}$ in an exogenous growth model. Further, one should note that if our full-fledged model further down were extended such as to allow for variable life-spans — so, individual "consumption-sets" are of the type $\cup_T C[0,T]$ —, then this conclusion would also formally follow from the model, given the homogeneity assumption on individual utilities (which is forced upon us to get a balanced growth model).

Hence, if we want 1 human life one generation down the road to count as much as 1 now, we must discount further consumption *exactly* by $e^{-\gamma t}$.

6.5. **Reinterpreting the Traditional Utilitarian Approach.** In their fundamental work Arrow and Kurz (1970) offer a criterion, or a social welfare function, that has been widely used to evaluate public investments in the literature since then. Denote by N_t population at time t, let c_t be per-capita consumption and β be a (subjective) discount rate, then the criterion (in its simplest formulation)[34] is

$$\tilde{W}\left(\left(c_t\right)_t\right) \equiv \int_0^{\infty} e^{-\beta t} N_t u\left(c_t\right) dt, \tag{5}$$

where u is a concave and increasing function of per-capita consumption. To put it in their own words,

> The flow of felicity to society is the sum over individuals at a given time; the total utility from a policy is taken to be the sum over all time of the felicities of each time, discounted back to the present at a constant rate.

Criterion (5) can be presented as a true social welfare function, i.e., a function of individual (lifetime) utilities. Indeed, assume, for example, all individuals live for a fixed period of time (unity), and that an individual born at time t has a life-time utility of the form

$$U_t\left(c.\right) = \int_t^{t+1} e^{-\alpha(s-t)} u\left(c_s\right) ds,$$

where $c.$ is the time-path of consumption (as a function of age), and α is the individual time preference. Assume also that population grows exponentially at a rate ν. Then, aggregating over all individuals (integrate over t from $-\infty$ to $+\infty$) when discounting their life-time utilities at a rate β, one gets the following criterion:[35]

$$\begin{aligned}
W &\equiv \int_{-\infty}^{\infty} N_t e^{-\beta t} U_t\left(c.\right) dt && (6)\\
&= \int_{-\infty}^{\infty} N_t e^{-\beta t} \int_t^{t+1} e^{-\alpha(s-t)} u\left(c_s\right) ds dt && (7)\\
&= N_0 \int_{-\infty}^{\infty} e^{-\alpha s} u\left(c_s\right) \int_{s-1}^{s} e^{(\alpha+\nu-\beta)t} dt && (8)\\
&= M \int_{-\infty}^{\infty} e^{-\beta t} N_t u\left(c_t\right) dt, \text{ where } M \equiv \int_0^1 e^{(\alpha+\nu-\beta)x} dx && (9)
\end{aligned}$$

Observe that, under the above assumptions, the two criteria (W and $\tilde{W}$) rank the policies that affect per-capita streams of consumption only after time zero in the same fashion. The advantage of using criterion W is its generality: it encompasses the Arrow and Kurz (1970)

[34] More generally, the utility, u, can depend directly on government investment, k_g in a given period.

[35] The discount rate $\beta > 0$ is often introduced only for the 'technical' reason of making sure the social welfare function returns a finite number for strictly positive consumption profiles. Ramsey (1928) avoided this difficulty by suggesting to use a bounded function $u(c)$ and, then to minimize the difference between u and the 'bliss', B, or the highest attainable utility, using that as the criterion, cf the discussion in the introduction. This is indeed what we do, when $\rho > 1$.

criterion and also allows for life-time utilities that are not necessarily time-separable. Finally, re-interpreting the criterion in this way allows to separate the individual time preference, α, from the social discount rate, β, and it allows as well to separate completely attitudes towards risk from the time preferences (and, in particular, from the inter-temporal substitution).

Interestingly, this social welfare function is identical to the "traditional utilitarian" criterion mentioned in the introduction and in footnote 28.

7. Conclusions

We show that if one is to accept relative utilitarian criterion, which incorporates equal treatment of different generations, the discount rate for projects that have long-term impact is the per-capita growth rate. The conclusion is true under any 'time-invariant' solution concept, and, more importantly, the discount rate is independent of a particular equilibrium (and the associated prices) that the economy is in. The per-capita growth rate represents the 'true' shadow cost of (consumption-generating) resources today in terms of those in the future, so the prescribed discount rate is based solely on the fundamentals of the economy.

Interestingly, Ramsey (1928) conjectured that population growth and "future inventions and improvements in organisation" might have an effect on the trade-off between current and future consumption. This model might be viewed as a support for his conjecture.

Let us stress that the result is independent of individual impatience, the model does not require time-separable preferences. Crucial assumptions are those needed for stable growth to be feasible, i.e., (1) homogeneity of individual utility functions (over consumption life-time streams, for fixed life-time streams of labour activities), and (2) constant returns to scale in production.

The next step is to prove that the main statement is non-vacuous, as it would be, e.g., in case of indeterminacy. Based on our work in progress we can conjecture that the differentiability is not a very restrictive assumption in terms of underlying parameters of the model (γ, ρ, ν) at least for the case of Diamond-like economies with inelastic labor supply and time-separable constant relative risk aversion instantaneous utility function.

Appendix A

Proof of Lemma 1. Continuity yields that K_t^i is bounded on any interval $(-\infty, t_0)$. The production technology implies a similar upper bound for I_t^i. In particular I_t^i is locally-integrable and thus K_t^i locally absolutely continuous. Letting $M_t = e^{\delta^i t} K_t^i$, the differential equation

equation becomes $M_t' = e^{\delta^i t} I_t^i$, hence, by the local absolute continuity, $M_t = M_0 + \int_0^t e^{\delta^i s} I_s^i ds$. Therefore $K_t^i \geq 0$ yields $\int_{-T}^0 e^{\delta^i s} I_s^i ds \leq M_0 \, \forall T$, and $K_t^i \leq \bar{K} e^{(\gamma+\nu)t}$ for $t \leq 0$ yields $\int_{-T}^0 e^{\delta^i s} I_s^i ds \geq M_0 - \bar{K} e^{-(\gamma+\nu+\delta^i)T}$ for $T \geq 0$. In particular, the (locally integrable, as just seen) function $h(t) = e^{\delta^i t} I_t^i$ is such that $\int_{-T}^0 h(s) ds$ converges to M_0 when $T \to \infty$. Since our upper bound for I_t^i implies $h(t) \leq \bar{h} e^{(\gamma+\nu+\delta^i)t}$ for $t \leq 0$, we conclude that $h(t)$ is (absolutely) integrable on $(-\infty, t)$ for all t, and thus $K_t^i = e^{-\delta^i t} \int_{-\infty}^t e^{\delta^i s} I_s^i ds$ for all t, where the integral is a Lebesgue integral. □

Proof of Lemma 2. Clearly, we can set consumption to zero and assume that all agents work full-time. Fix a vector $L_0 \in \mathbb{R}^h$ such that any feasible vector of labour inputs $L(t) \leq L_0 e^{(\gamma+\nu)t}$. Enlarge the instantaneous production cone Y by allowing all investment goods to be perfect substitutes for each other and the same for the capital goods. Let $F \colon \mathbb{R}_+^2 \to \mathbb{R}_+ \colon (K, l) \mapsto \sup\{\sum_i I^i \mid \exists K^i \geq 0, \sum_i K^i \leq K, \big(-lL_0, -(K^i)_i, 0, (I^i)_i\big) \in Y\}$. The supremum is achieved, else with bounded inputs unbounded outputs would be feasible and, as Y is convex and closed, the same would be true for zero inputs, thus contradicting the assumption $Y \cap \mathbb{R}_+^n = \{0\}$. In particular, $F(K, l)$ is finite. Clearly, F is positively homogeneous of degree one, concave and continuous. Further, by the assumptions of the lemma, $F(K, 0) = 0$.

Let us, finally, improve the possibilities of capital accumulation by lowering each δ^i to $\delta = \min_i \delta^i$.

Then the capital accumulation equation becomes $K'(t) = F(K(t), e^{(\gamma+\nu)t}) - \delta K(t)$. Let $x(t) \equiv K(t) e^{-(\gamma+\nu)t}$, and $f(x) \equiv F(x, 1)$ — then $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and concave. Then the differential equation becomes $x'(t) = f(x(t)) - Rx(t)$. As $\lim_{x \to \infty} \frac{f(x)}{x} = 0$ (because $F(1, 0) = 0$ and continuity), there is $\bar{x} \geq 0$ such that $f(x) - Rx \leq -1$ iff $x \geq \bar{x}$.

Let now $y(t) = e^{-(\gamma+\nu)t} \sum_i K_t^i$ along some feasible path in the original economy: a fortiori $y(t) \geq \bar{x}$ implies $y'(t) \leq -1$. Since, by the initial condition, $\exists \bar{y} \colon y(t) \leq \bar{y}$ for $t < 0$, it follows that for all t, $y(t) \leq \bar{x}$. Hence our bound on each K_t^i, which themselves imply (via Y) a similar upper bound for the I_t^i. □

Proof of Lemma 3. All norms on $\mathbb{R}^n$ being equivalent, we can assume the ℓ_1 norm in the statement. The right hand member of the inequality is then concave, and we can proceed as in the proof of lemma 2: now $f(x) = A + Bx^{1-\varepsilon}$, and, since, as seen above, if $x(t) > \bar{x}$ then $x(s)$ must have been decreasing (and hence $x(s) > \bar{x}$) for all $s \leq t$, we can assume $A = 0$, by majorising f on $[\bar{x}, +\infty]$ by another such function (and if necessary increasing $\bar{x}$ to an appropriate value for that new function). Thus we have to show that over all feasible paths (k_t, i_t) ($= e^{-(\gamma+\nu)t)}(K_t, I_t)$) the k_t are uniformly bounded. And feasibility means $i_t \leq B k_t^{1-\varepsilon}$ and $e^{Rt} k_t = \int_{-\infty}^t e^{Rs} i_s ds$ (and $k_t \geq 0$, $e^{Rs} i_s$ integrable on

$[-\infty, t]$). I.e., letting $y_t = \int_{-\infty}^{t} e^{Rs} i_s ds$, we have $y_{-\infty} = 0$, $k_t = e^{-Rt} y_t$, $i_t = e^{-Rt} y'_t$, so our inequality becomes $y'_t \le B e^{\varepsilon Rt} y_t^{1-\varepsilon}$, i.e., $\frac{dy_t^{\varepsilon}}{de^{\varepsilon Rt}} \le \frac{B}{R}$. Since $y_{-\infty} = 0$, this yields $y_t^{\varepsilon} \le \frac{B}{R} e^{\varepsilon Rt}$, i.e., $k_t \le \left(\frac{B}{R}\right)^{1/\varepsilon}$. □

Proof of Lemma 5. The second part is obvious: for the endowments, it holds by definition of the transformation, and for the preferences, it follows because all agents of the same type have the same utility function over their consumption set, which is homogeneous in the goods: so multiplying the "goods-component" by a constant just multiplies to whole utility function by a constant, and hence doesn't change preferences.

For the first part, note that the capital-accumulation equations are not affected, since they are linear and homogeneous in the aggregate goods. Remains to check for the "instantaneous production cone" Y that it too is preserved by the transformation. Assume thus for some t a vector $(-L, y)$ in Y_t — i.e., $(-\exp(\gamma t)L, y)$ in Y — before the transformation — where the coordinates of $y = (-K, C, I)$ are all aggregate consumption and investment outputs and capital inputs, and those of L are the aggregate labour input. Then, after the transformation, this vector becomes $[-\exp(\nu h)L, \exp\left((\gamma+\nu)h\right) y]$, and we have to show that this belongs to Y_{t+h} — i.e., that $[-\exp(\gamma(t+h))\exp(\nu h)L, \exp((\gamma+\nu)h)y]$ belongs to Y. Since this vector equals $\exp((\gamma+\nu)h)\,[-\exp(\gamma t)L, y]$, this follows straight from the fact that Y is a cone. □

Proof of Theorem 9. By definition of a Gateaux differential,

$$DW\left(\Omega^0\right) = \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon W\left(\Omega^0\right)}{\varepsilon},$$
$$\delta_\varepsilon W\left(\Omega^0\right) = W\left(\Omega^0 + \varepsilon\Omega\right) - W\left(\Omega^0\right)$$

By assumption,

$$DW(0) = \langle \Omega, \mu \rangle$$

where $\mu \in (K^*)^n$, i.e., the differential at $\Omega^0 = 0$ is linear in Ω. It is sufficient for what follows to describe $\delta_\varepsilon W\left(\Omega^0\right)$, i.e., the change in the social welfare function caused by the perturbation of endowments, which amounts to subtracting a constant from each agent's utility, the utility on the baseline, thus the criterion of interest is the difference δW.

To construct δW let us first normalise life-time utilities. Recall the set of available allocations is time-invariant. We have to compute w_t^τ, the difference between the sup and the inf over this set of the utility of an agent of type τ born at time t. By time-invariance, the set of consumption and labor allocations of this agent equals that for an agent of the same type born at time 0, except for rescaling the consumption component by $e^{\gamma t}$. Therefore, by the homogeneity of U^τ of degree $1-\rho^\tau$ with respect to consumption, $w_t^\tau = e^{(1-\rho^\tau)\gamma t} w_0^\tau$. Let $w^\tau \equiv (w_0^\tau)^{-1}$. Then we get for normalised utility $U_t^{*\tau}$ (that enters the social welfare function)

$$U_t^{*\tau} = e^{(\rho^\tau - 1)\gamma t} w^\tau U^\tau$$

We, therefore, can write the social welfare function in the following form

$$\delta W(\cdot) \equiv \int_{-\infty}^{\infty} \sum_\tau N_t^\tau \left(\delta U_t^{*\tau}\right) dt \tag{10}$$

Let us define $V_t^\tau : \Omega_t \mapsto \mathbb{R}$ to be the utility level of individual of type τ born at time t, under an equilibrium with the perturbed endowments.

$$\begin{aligned} W\left(\Omega^0\right) &= \sum_\tau w^\tau W^\tau\left(\Omega^0\right) \\ W^\tau\left(\Omega^0\right) &\equiv \int_{-\infty}^{\infty} N_t^\tau e^{(\rho^\tau - 1)\gamma t} V_t^\tau\left(\Omega_t^0\right) dt \end{aligned}$$

Consider now the perturbation $\tilde{\Omega}_t$, where

$$\tilde{\Omega}_{t+h} = e^{(\gamma+\nu)h}\Omega_t \tag{11}$$

By Lemma 5 the corresponding "response" of the system is obtained from the response to Ω_t by delaying everything by h, multiplying all aggregate quantities of goods by $e^{(\gamma+\nu)h}$, and all per-capita quantities by $e^{\gamma h}$, and correspondingly for prices.

Hence, for utilities, by their homogeneity property in goods $1, \ldots, n$,

$$V_{t+h}^\tau\left(\tilde{\Omega}\right) = e^{(1-\rho^\tau)\gamma h} V_t^\tau(\Omega)$$

and, in particular, when $\tilde{\Omega} = \Omega = 0$,

$$V_{t+h}^\tau\left(\tilde{\Omega}\right) - V_{t+h}^\tau(0) = e^{(1-\rho^\tau)\gamma h}\left(V_t^\tau(\Omega) - V_t^\tau(0)\right)$$

Therefore,

$$\begin{aligned} &W^\tau\left(\tilde{\Omega}\right) - W^\tau(0) = \\ &\quad = \int_{-\infty}^{+\infty} N_0^\tau e^{\nu(t+h)} e^{(\rho^\tau-1)\gamma(t+h)} \left[V_{t+h}^\tau\left(\tilde{\Omega}\right) - V_{t+h}^\tau(0)\right] d(t+h) \\ &\quad = \int_{-\infty}^{+\infty} N_0^\tau e^{\nu(t+h)+(1-\rho^\tau)\gamma h} e^{(\rho^\tau-1)\gamma(t+h)} \left[V_t^\tau(\Omega) - V_t^\tau(0)\right] dt \\ &= e^{\nu h} \int_{-\infty}^{+\infty} N_t^\tau e^{(\rho^\tau-1)\gamma t} \left[V_t^\tau(\Omega) - V_t^\tau(0)\right] dt = e^{\nu h}\left[W^\tau(\Omega) - W^\tau(0)\right] \end{aligned}$$

(i.e., the factor $(1-\rho^\tau)\gamma$ drops out). As a consequence, the total change in welfare is

$$\begin{aligned} W\left(\tilde{\Omega}\right) - W(0) &= \sum_\tau w^\tau \left(W^\tau\left(\tilde{\Omega}\right) - W^\tau(0)\right) \\ &= e^{\nu h}\left[W(\Omega) - W(0)\right] \end{aligned}$$

Therefore, applying the definition of the derivative, we get

$$\langle\tilde{\Omega},\mu\rangle = \lim_{\varepsilon\to 0}\frac{W\left(\varepsilon\tilde{\Omega}\right)-W(0)}{\varepsilon} = \tag{12}$$
$$= e^{\nu h}\lim_{\varepsilon\to 0}\frac{W(\varepsilon\Omega)-W(0)}{\varepsilon} = e^{\nu h}\langle\Omega,\mu\rangle$$

Define $T_h : t \longmapsto \xi(t+h)$. By (11)

$$\tilde{\Omega} = e^{(\gamma+\nu)h}T_h\Omega$$

Combining with (12), we get

$$e^{(\gamma+\nu)h}\langle T_h(\Omega),\mu\rangle = \langle\tilde{\Omega},\mu\rangle = e^{\nu h}\langle\Omega,\mu\rangle$$

and, due to arbitrariness of h and Ω, the following condition holds for all $h\in\mathbb{R}$ and all perturbations $\Omega\in K^n$,

$$\langle\Omega - e^{\gamma h}T_h(\Omega),\mu\rangle = 0$$

for $\mu\in(K^*)^n$. Dividing by h and taking limit as $h\to 0$, we get

$$\langle\gamma\Omega-(\Omega)',\mu\rangle = 0$$

Rearranging and using the definition of a derivative of a generalized function,

$$\langle\mu',f\rangle = -\langle\mu,f'\rangle,\ f\in K,\ \mu\in K^*$$

results in

$$\langle\gamma\mu+\mu',\Omega\rangle = 0,\ \forall\Omega\in K^n$$

so we have to solve a differential equation $\gamma\mu+\mu'=0$, which, by Lemma (10) has only the solutions of the form $\mu = q\otimes e^{-\gamma t}$ for some $q\in\mathbb{R}^n$, therefore,

$$DW = e^{-\gamma t}\langle q,\Omega\rangle = \int_{-\infty}^{+\infty} e^{-\gamma t}\langle q,\Omega_t\rangle dt,\ \forall\Omega\in K^n.$$

□

Lemma 10. *Consider a homogeneous differential equation of the form*

$$y' = \lambda y, \tag{13}$$

for a given constant λ. Then every solution of that system in the class K^* of generalized functions is of the form

$$y = Ce^{\lambda t},\ C\in\mathbb{R}$$

i.e., is a "classical solution".

Proof. From (13) we have that for any $\varphi\in K$, $\langle y',\varphi\rangle = \lambda\langle y,\varphi\rangle$; by definition of the derivative of a generalized function this implies $\langle y,-\varphi'\rangle = \lambda\langle y,\varphi\rangle$, and so $\langle y,\lambda\varphi+\varphi'\rangle = 0$. Let $K_\lambda = \{\psi\in K \mid \int_{-\infty}^{\infty} e^{\lambda t}\psi(t)\,dt = 0\}$. Observe that $\forall\psi\in K_\lambda\exists\varphi\in K\colon \psi = \lambda\varphi+\varphi'$: take $\varphi(t) = \int_{-\infty}^{t} e^{\lambda s}\psi(s)ds$ (the converse is true as well, but we won't use it). So $y=0$ on K_λ.

Note that any $\varphi \in K$ can be represented in the form $\varphi = \psi + c\varphi_0$, where $\psi \in K_\lambda$, c is a constant and $\varphi_0 \in K \setminus K_\lambda$ is fixed: choose $c = \frac{\int_{-\infty}^{\infty} e^{\lambda t}\varphi(t)dt}{\int_{-\infty}^{\infty} e^{\lambda t}\varphi_0(t)dt}$, then $\psi = \varphi - c\varphi_0 \in K_\lambda$.

Thus $\langle y, \varphi\rangle = c\langle y, \varphi_0\rangle$, so, letting the constant $C = \frac{\langle y,\varphi_0\rangle}{\int_{-\infty}^{\infty} e^{\lambda t}\varphi_0(t)dt}$, we get $\langle y, \varphi\rangle = C \int_{-\infty}^{\infty} e^{\lambda t}\varphi(t)\, dt$, $\forall\varphi \in K$, i.e., $y = Ce^{\lambda t}$. □

References

Arrow, K. J., and M. Kurz (1970): *Public Investment, the Rate of Return, and Optimal Fiscal Policy*. The Johns Hopkins Press, Baltimore and London.

Asheim, G. B. (1991): "Unjust intergenerational allocations.," *Journal of Economic Theory*, 54, 350–371.

Asheim, G. B., T. Mitra, and B. Tungodden (2006): "Sustainable recursive social welfare functions," mimeo.

Bell, C., and S. Devarajan (1983): "Shadow prices for project evaluation under alternative macroeconomic specifications," *Quarterly Journal of Economics*, 98, 454–477.

Bergstrom, T. C. (2006): "Benefit-Cost in a Benevolent Society," *The American Economic Review*, 96(1), 339–351.

Bliss, R. R., and N. Panigirtzoglou (2004): "Option-Implied Risk Aversion Estimates," *Journal of Finance*, LIX(1).

Chichilnisky, G. (1996): "An axiomatic approach to sustainable development," *Social Choice and Welfare*, 13, 231–257.

Coate, S. (2000): "An Efficiency Approach to the Evaluation of Policy Changes," *Economic Journal*, 110(463), 437–455.

Dhillon, A., and J.-F. Mertens (1999): "Relative Utilitarianism," *Econometrica*, 67(3), 471–498.

Diamond, P. (1965): "National Debt in a Neoclassical Growth Model," *American Economic Review*, LV(5), 1126–1150, Part 1.

Drèze, J., and N. Stern (1987): "The Theory of Cost-Benefit Analysis," in *Handbook of Public Economics*, ed. by A. Auerbach, and M. Feldstein, vol. II, chap. 14. Elsevier Science Publishers.

Drèze, J. H. (1981): "Inferring Risk Tolerance from Deductibles in Insurance Contracts," *The Geneva Papers on Risk and Insurance*, pp. 48–52.

Einav, L., and A. Cohen (2005): "Estimating Risk Preferences from Deductible Choice," NBER WP 11461.

Erosa, A., and M. Gervais (2001): "Optimal Taxation in Infinitely-Lived Agent and Overlapping Generations Models: A Review," *Federal Reserve Bank of Richmond Economic Quarterly*, 87(2), 23–44.

Fleurbaey, M., and P. Michel (2003): "Intertemporal equity and the extension of the Ramsey criterion," *Journal of Mathematical Economics*, 39, 777–802.

Geanakoplos, J. D., and D. J. Brown (1985): "Comparative Statics and Local Indeterminacy in OLG Economies: An Application of the Multiplicative Ergodic Theorem," Cowles Discussion Paper 773.

Geanakoplos, J. D., and H. M. Polemarchakis (1991): "Overlapping Generations," in *Handbook of Mathematical Economics*, ed. by W. Hildenbrand, and H. Sonnenschein, vol. 4, chap. 35, pp. 1899–1960. Elsevier.

Gelfand, I., and G. Shilov (1959): *Obobshennuye funkzii i deistviya nad nimi*. Fizmatgiz, Moscow, 2 edn.

Hicks, J. R. (1939): "Foundations of welfare economics," *The Economic Journal*, 49(196), 696–712.

——— (1942): "Consumers' Surplus and Index-Numbers," *The Review of Economic Studies*, 9(2), 126–137.

Kaldor, N. (1989): "Welfare propositions and interpersonal comparisons of utility," *Economic Journal*, 49, 549–552.

King, R. G., C. I. Plosser, and S. T. Rebelo (2002): "Production, Growth and Business Cycles: Technical Appendix," *Computational Economics*, 20(1-2), 87–116.

Koopmans, T. (1960): "Stationary ordinal utility and impatience," *Econometrica*, 28, 287–309.

Kotlikoff, L. J. (1998): "The A-K Model — Its Past, Present, and Future," NBER WP 6684.

——— (2002): "Generational Policy," in *Handbook of Public Economics*, ed. by A. J. Auerbach, and M. Feldstein, vol. 4, chap. 27, pp. 1873–1932. Elsevier.

Mishan, E. J. (1971): "Evaluation of Life an Limb: A Theoretical Approach," *The Journal of Political Economy*, 79(4), 687–705.

——— (1976): *Cost-benefit analysis*. Praeger Publishers.

Pigou, A. C. (1932): *The Economics of Welfare*. Macmillan, 4th edn.

Ramsey, F. (1928): "A Mathematical Theory of Saving," *The Economic Journal*, 38(152), 543–559.

Rawls, J. (1971): *A Theory of Justice*. Harvard University Press.

Samuelson, P. A. (1954): "The Pure Theory of Public Expenditure," *The Review of Economics and Statistics*, 36(4), 387–389.

Schwartz, L. (1957-59): *Théorie des distributions*, vol. I and II. Hermann, Paris.

Sidgwick, H. (1874): *The Methods of Ethics*. Macmillan and Company, London.

U.S. Office of Management and Budget (2003): "OMB Circular A-4: Regulatory Analysis," .

Wildi, W., D. Appel, M. Buser, F.Dermange, A. Eckhardt, P. Hufschmied, and H.-R. Keusen (2000): "Disposal Concepts for Radioactive Waste," written on behalf of the Federal Department for the Environment, Transport, Energy and Communication

of Switzerland.