Unique Equilibria in Conflict Games with Incomplete Information (*Preliminary and Incomplete*)

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1 The Game with Privately Known Types

1.1 The basic model

We now introduce private information into two basic conflict games. We assume for simplicity that only the hostility parameters are private information, while all other parameters are commonly known. Player i's hostility parameter is his *type*. Each player knows his own type, but not the opponent's. The two parameters c and d are fixed and the same for both players. The payoff matrix is as follows. The row represents player i's choice and the column represents player j's choice. Only player i's payoffs are indicated.

$$\begin{array}{ccc} & H & D \\ H & h_i - c & h_i \\ D & -d & 0 \end{array} \tag{1}$$

We assume the hostility parameter h_i has a fixed publicly observed component k_i as well as a random privately observed component η_i. Thus, player i's type is $h_i = k_i + \eta_i$. The game of incomplete information is played as follows. First η_1 and η_2 are drawn from a symmetric joint distribution with support $[\underline{\eta}, \bar{\eta}] \times [\underline{\eta}, \bar{\eta}]$. Then player 1 is informed about η_1, but not about η_2. Similarly, player 2 is informed about η_2 but not about η_1. Finally, each player makes his choice simultaneously (H or D).

When the players make their choices, everything except η_1 and η_2 is commonly known. In particular, there is no uncertainty about the fixed parameters k_1 and k_2. The introduction of k_1 and k_2 is a convenient way to allow for *ex ante* asymmetries in the distribution of hostilities. If $k_1 = k_2$ then the two players are ex ante symmetric (i.e., before η_1 and η_2 are drawn there is nothing to distinguish them). If $k_1 \neq k_2$ then there is a publicly known ex ante asymmetry.

If η_1 and η_2 are correlated, then player i's knowledge of η_i can be used to update his beliefs about η_j. Formally, the cumulative distribution of η_j conditional on $\eta_i = y$ (where $i \neq j$) is denoted $F(\cdot|y)$. We assume $F(x|y)$ is continuously differentiable, with partial derivatives $F_1(x|y) \equiv \frac{\partial F(x|y)}{\partial x}$ and $F_2(x|y) \equiv \frac{\partial F(x|y)}{\partial y}$. Notice that $F_1(\cdot|y)$ is the density of η_j conditional on $\eta_i = y$. Since player j's hostility is $h_j = k_j + \eta_j$, uncertainty about η_j directly translates into uncertainty about player j's type. The types h_1 and h_2 are correlated if and only if η_1 and η_2 are correlated. If player i's type is $h_i = y$, then player i assigns probability $F(x - k_j|y - k_i)$ to the event that $h_j \leq x$. Indeed, $h_i = y$ if and only if $\eta_i = y - k_i$, and $h_j \leq x$ if and only if $\eta_j \leq x - k_j$. The least (resp. most) hostile type of player i has hostility parameter $\underline{h_i} = k_i + \underline{\eta}$ (resp. $\bar{h}_i = k_i + \bar{\eta}$).

We make the following assumption:

Assumption 1 (i) $F_1(x|y) > 0$ for all $x, y \in [\underline{\eta}, \bar{\eta}]$ and (ii) $F_2(x|y) \leq 0$ for all $x, y \in (\underline{\eta}, \bar{\eta})$.

Assumption 1 (i) says there is positive density. Assumption 1 (ii) says that $F(x|y)$ is not increasing in y. Therefore, as a player becomes more hostile, he becomes no less pessimistic about his opponent's hostility. This is true if η_1 and η_2 are affiliated (Milgrom, Theorem 5.4.3). Affiliation may occur if conflict concerns some resource such as oil. Of course, Assumption 1 (ii) also holds if η_1 and η_2 are independent. Independence is a natural assumption when the innate attitude of the two players towards hostility is uncertain. For future reference, we note that Assumption 1 implies that if $y > x$ then $F(y|x) - F(x|y) \geq F(y|y) - F(x|y) > 0$.

We classify types into four categories.

Definition 1 *Player i is a* dominant strategy hawk *if H is a dominant strategy ($h_i - c \geq -d$ and $h_i \geq 0$ with at least one strict inequality). Player i is a* dominant strategy dove *if D is a dominant strategy ($h_i - c \leq -d$ and*

$h_i \leq 0$ with at least one strict inequality). Player i is a coordination type *if H is a best response to H and D a best response to D ($c - d \leq h_i \leq 0$). Player i is an* opportunistic type *if D is a best response to H and H a best response to D (in this case, $0 \leq h_i \leq c - d$).*

Notice that coordination types exist only in games with strategic complements, and opportunistic types only in games with strategic substitutes.

1.2 Bayesian Nash Equilibrium

Suppose player i is of type h_i, and thinks player j will choose D with probability $\delta_j(h_i)$. His expected payoff from H is $h_i - (1 - \delta_j(h_i))c$, while his expected payoff from D is $-(1 - \delta_j(h_i))d$. Thus, if type h_i chooses H instead of D, his *net* gain is

$$h_i + (d - c)(1 - \delta_j(h_i)) \tag{2}$$

A *strategy* for player i is a function $\sigma_i : [\underline{h}_i, \bar{h}_i] \to \{H, D\}$ which specifies an action $\sigma_i(h_i) \in \{H, D\}$ for each type $h_i \in [\underline{h}_i, \bar{h}_i]$. In Bayesian Nash equilibrium (BNE), all types maximize their expected payoff. Therefore, $\sigma_i(h_i) = H$ if the expression in (2) is positive, and $\sigma_i(h_i) = D$ if it is negative. (If expression (2) is zero then type h_i is indifferent and can choose either H or D.) We say that player i uses a *cutoff strategy* if there is a *cutoff point* $x \in [\underline{h}_i, \bar{h}_i]$ such that $\sigma_i(h_i) = H$ for all $h_i > x$ and $\sigma_i(h_i) = D$ for all $h_i < x$.

If player j uses a cutoff strategy with cutoff point x, then $\delta_j(y) = F(x - k_j | y - k_i)$, so player i's net gain from choosing H instead of D when his type is $h_i = y$ is

$$\Psi^i(x, y) \equiv y + (d - c)\left(1 - F(x - k_j | y - k_i)\right). \tag{3}$$

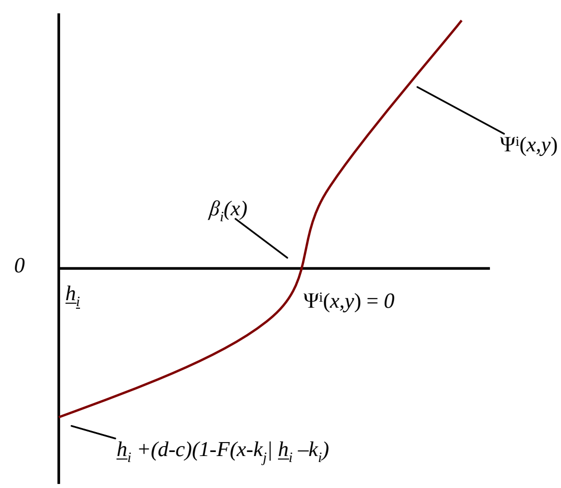
For a cutoff strategy to be a best response, player i should be more inclined to choose H the more hostile he is. That is, $\Psi^i(x, y)$ should be increasing in y :

$$\Psi_2^i(x, y) = 1 - (d - c)F_2(x - k_j | y - k_i) > 0 \tag{4}$$

Figure 1 illustrates this property:

In view of Assumption 1, (4) holds if $d > c$. It also holds if $d < c$ and the two types are not too strongly correlated. If c is much bigger than d and the two types are highly correlated, then (4) may be violated. The intuition is

Figure 1: $\Psi^i(x,y)$



that, if types are strongly correlated and the opponent uses a cutoff strategy, then a very hostile type thinks it is very likely that the opponent chooses H. If in addition c is much bigger than d, then the (H, H) outcome is very costly. In this situation, a very hostile type may be inclined to choose D, which would eliminate the possibility of cutoff equilibria.

If condition (4) holds then player i's best response to player j's cutoff x is to use a cutoff point denoted $\beta_i(x)$. The best-response function $\beta_i(x)$ is defined as follows. (i) If $\Psi^i(x, \underline{h}_i) \geq 0$ then $\beta_i(x) = \underline{h}_i$ (so player i plays H with probability one). (ii) If $\Psi_i(x, \bar{h}_i) \leq 0$ then $\beta_i(x) = \bar{h}_i$ (so player i plays D with probability one). (iii) Otherwise, $\beta_i(x) \in (\underline{h}_i, \bar{h}_i)$ is the unique solution to the equation $\Psi^i(x, \beta_i(x)) = 0$ (all types above $\beta_i(x)$ play H, and all types below $\beta_i(x)$ play D). By the implicit function theorem, $\beta_i(x)$ is a continuous function as long as $\Psi^i(x, y)$ is increasing in y. In this case, the slope of the best response function is obtained by totally differentiating $\Psi^i(x, \beta_i(x)) = 0$,

$$\beta_i'(x) = -\frac{\Psi_1^i(x, \beta_i(x))}{\Psi_2^i(x, \beta_i(x))} = -\frac{(c-d)F_1(x - k_j|\beta_i(x) - k_i)}{1 - (d-c)F_2(x - k_j|\beta_i(x) - k_i)}. \qquad (5)$$

Notice that $\beta_i'(x) > 0$ if $d > c$ and $\beta_i'(x) < 0$ if $d < c$.

Proposition 2 *If $\Psi^i(x,y)$ is increasing in y for each $i \in \{1,2\}$, then a cutoff equilibrium exists.*

Proof. There are three possibilities:

(i) $\Psi^2(\beta_1(\underline{h}_2), \underline{h}_2) \geq 0$. In this case the cut-off points $(\beta_1(\underline{h}_2), \underline{h}_2)$ form a BNE;

(ii) $\Psi^2(\beta_1(\bar{h}_2), \bar{h}_2) \leq 0$. In this case the cut-off points $(\beta_1(\bar{h}_2), \bar{h}_2)$ form a BNE;

(iii) there is $x \in [\underline{h}_2, \bar{h}_2]$ such that $\Psi^2(\beta_1(x), x) = 0$. In this case the cut-off points $(\beta_1(x), x)$ form a BNE.

Since the function $\Psi^2(\beta_1(x), x)$ is continuous in x, one of the three cases must always occur. ■

1.3 Strategic Complements

The underlying complete information game displays strategic complementarity when $d > c$. In addition, it is a Stag Hunt game if $c > 0$. In this case, (4) holds, so a cutoff equilibrium exists by Proposition 2. We now derive a sufficient condition for this to be the *unique* BNE.

Theorem 3 *Suppose $d > c$, and $\bar{h}_i > 0$ and $\underline{h}_i < c - d$ so each player has dominant strategy hawks and doves. If for all $s, t \in (\underline{\eta}, \bar{\eta})$ it holds that*

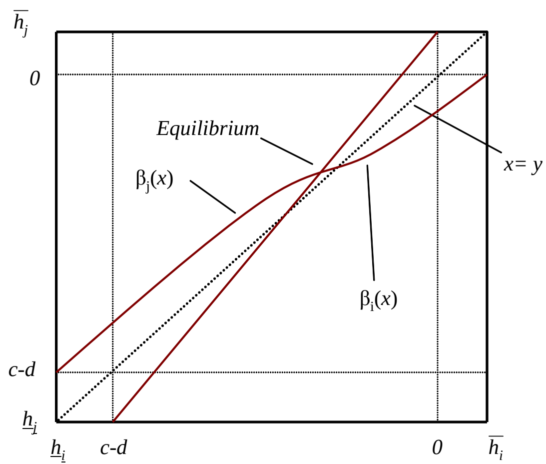
$$F_1(s|t) + F_2(s|t) < \frac{1}{d-c} \tag{6}$$

then there is a unique BNE. This BNE is a cutoff equilibrium.

Verifying this result requires two steps. The first step, which is rather technical and therefore relegated to the appendix, is to show that all BNE must be cut-off equilibria. The second step is to show that there cannot be more than one cut-off equilibrium. The second step is best explained diagrammatically. The fact that both players have dominant strategy hawks and doves means that for if player j uses cutoff x, $\beta_i(x)$ is interior. Then, we must have $\Psi^i(x, \beta_i(x)) = 0$ and so the expression (5) implies that the best response functions have a positive slope. A well-known condition for uniqueness is that the slope is *less than one.* Mathematically, this condition turns out to be $F_1(s|t) + F_2(s|t) < \frac{1}{d-c}$. To see this, use (5) to obtain

$$\beta_i'(x) = \frac{(d-c)F_1}{1-(d-c)F_2} = 1 - \frac{1-(d-c)(F_1+F_2)}{1-(d-c)F_2} = 1 - \frac{1-(d-c)(F_1+F_2)}{\Psi_2(x, \beta(x))}$$

Figure 2: Equilibrium for Strategic Complements



which is strictly less than one by (6). Therefore, $0 < \beta_i'(x) < 1$ for all $x \in [\underline{h}_i, \bar{h}_i]$, and the two best-response functions can cross only once. Therefore, there is only one cut-off equilibrium, which completes the proof:[1]

When there is a cut-off equilibrium, the reaction functions are upward sloping and the incomplete information game is a supermodular game. Hence, when there is a unique equilibrium, it can be obtained by iterated deletion of dominated strategies (see Athey, Milgrom and Roberts and Vives).

INSERT FIGURE 2 HERE.

If (6) holds, then the unique equilibrium is easy to characterize. First, if $\underline{h}_i \geq c - d$ for $i \in \{1, 2\}$ then there are no dominant-strategy doves, and it is certainly an equilibrium for each player to choose H regardless of type. Thus, this is the *unique* BNE. In a sense, this case represents an extreme case of the "Schelling dilemma". Conversely, if $\bar{h}_i \leq c - d$ for $i \in \{1, 2\}$ then

[1] It is possible to show, generalizing the results in Baliga and Sjostrom, that another sufficient condition for a unique equilibrium in cut off strategies is that ϕ is concave. To prove this, define the modified best response function $\hat{\beta}$ by using the function $\hat{\Psi}(x, y) \equiv y + (d - c)(1 - \phi(x))$ instead of the function Ψ. If ϕ is concave, then $\hat{\beta}$ intersects the 45 degree line exactly once. Moreover, β coincides with $\hat{\beta}$ on the 45 degree line, so β intersects the 45 degree line in a unique point.

there are no dominant-strategy hawks, so the unique equilibrium is for each player to choose D, regardless of type.

If there are both dominant strategy hawks and dominant strategy doves, then certainly neither D nor H can be chosen by all types. Hence, the unique equilibrium must be interior: each player i chooses a cut-off point $h_i^* \in \left(\underline{h}_i, \bar{h}_i\right)$ which solves

$$h_i^* + (d-c)\left(1 - F(h_j^* - k_j | h_i^* - k_i)\right) = 0. \tag{7}$$

If in addition the players are symmetric, say $k_1 = k_2 = k$, then the best response curves must intersect at the 45 degree line, and the unique equilibrium is a symmetric cut-off equilibrium, $h_1^* = h_2^* = h^*$. The symmetric cut off point is the unique solution in $[\underline{h}_i, \bar{h}_i]$ to the equation

$$h^* + (d-c)\left(1 - F(h^* - k | h^* - k)\right) = 0$$

The comparative statics are as expected. For example, an increase in $d-c$ will lead to more aggressive behavior (a reduction of h^*).

The intuition behind Theorem 3 may be brought out by a standard "stability" argument. Suppose both players are at cutoff equilibrium (h_1^*, h_2^*) and suppose both simultaneously reduce their cut-off by ε (so a few more types use H). Then, consider type $h_i^* - \varepsilon$. If type $h_i^* - \varepsilon$ now prefers D, the initial equilibrium is stable, and this is what we want to verify. In fact there are two opposing effects. First, at the original cut-off h_i^*, type $h_i^* - \varepsilon$ strictly preferred D, so there is reason to believe he still prefers D. However, the opponent has now become more hostile. At the initial equilibrium, cutoff type h_i^* thought that the opponent would choose H with probability $1 - F(h_j^* - k_j | h_i^* - k_i)$. But after the perturbation, the new cutoff type $h_i^* - \varepsilon$ thinks that the opponent will choose H with probability $1 - F(h_j^* - k_j - \varepsilon | h_i^* - k_i - \varepsilon)$. If

$$F_1(h_j^* - k_j | h_i^* - k_i) + F_2(h_j^* - k_j | h_i^* - k_i) > \frac{1}{d-c}$$

then this second effect dominates, and type $h_i^* - \varepsilon$ will actually prefer H after the perturbation. In this case, the original equilibrium is unstable. Hence, stability requires (6).

An interesting special case occurs when types are independent, so $F_2(s|t) = 0$ for all $s, t \in [\underline{\eta}, \bar{\eta}]$. In this case, (6) requires that the density of the random variable η_i is sufficiently spread out, i.e., that there is "enough uncertainty" about types. For example, suppose $k_1 = k_2 = 0$ and η_1 and η_2 are independently drawn from a uniform distribution on $[\underline{\eta}, \bar{\eta}]$. In this case, $F(s|t) = \frac{s-\underline{\eta}}{\bar{\eta}-\underline{\eta}}$, so $F_1(s|t) + F_2(s|t) = \frac{1}{\bar{\eta}-\underline{\eta}} < \frac{1}{d-c}$ if and only if $\bar{\eta} - \underline{\eta} > d - c$. Thus, with a uniform distribution, there is a unique equilibrium if the support is not too small.

We now argue that adding affiliation will, in general, have ambiguous results. Consider, for simplicity, a symmetric case, where $k_1 = k_2 = 0$, and a symmetric cut-off equilibrium with cut-off point h^*. Affiliation impacts the stability of the equilibrium via the expression $F_1(h^*|h^*) + F_2(h^*|h^*)$. There are two contradictory effects. On the one hand, affiliation causes type h^* to think the opponent is likely to be similar to himself, so $F_1(h^*|h^*)$ is large. This effect makes uniqueness less likely. On the other hand, affiliation causes $F_2(h^*|h^*)$ to be negative. This effect makes uniqueness more likely. While the first effect is easy to understand, in terms of concentrating the density of types in a smaller area, the second effect is more subtle. Intuitively, in the above stability argument, affiliation causes type $h^* - \varepsilon$ to be less pessimistic about the opponent's hostility than type h^*, making him more likely to prefer D. That is, the best response curves are more likely to have the slopes that guarantee stability and uniqueness.

Example: the uniform independent case Suppose the idiosyncratic shocks η_1 and η_2 are independently drawn from a distribution which is uniform on $[\underline{\eta}, \bar{\eta}]$. Moreover, suppose $k_1 = 0$ and $k_2 = k \geq 0$. The flatness condition requires "sufficient uncertainty": $\bar{\eta} - \underline{\eta} > d - c$. Then, we are assured that a unique equilibrium exists. We assume $\underline{\eta} < c - d$. If this is not the case then neither player can be a dominant strategy dove and the unique equilibrium is for all types to play H. We also assume $\bar{\eta} + k > 0$, otherwise both players cannot be dominant strategy hawks and the unique equilibrium is for both to play D.

Let $\tilde{h}_1$ and $\tilde{h}_2$ be defined as solution to (7):

$$\begin{aligned}
\tilde{h}_1 &= \frac{\left[\bar{\eta}\left(\bar{\eta} - \underline{\eta} + (d-c)\right) + k\left(\bar{\eta} - \underline{\eta}\right)\right](c-d)}{\left(\bar{\eta} - \underline{\eta} - (d-c)\right)\left(\bar{\eta} - \underline{\eta} + (d-c)\right)} < 0 \text{ and} \\
\tilde{h}_2 &= \frac{\left[\bar{\eta}\left(\bar{\eta} - \underline{\eta} + (d-c)\right) - k\right](c-d)}{\left(\bar{\eta} - \underline{\eta} - (d-c)\right)\left(\bar{\eta} - \underline{\eta} + (d-c)\right)} > c - d.
\end{aligned}$$

Notice that as $k \geq 0$ and $c - d < 0$, $\tilde{h}_2 \geq \tilde{h}_1$.

If $\tilde{h}_i \in [c-d, 0]$ and $\tilde{h}_i \in [\underline{\eta} + k_i, \bar{\eta} + k_i]$, then the unique equilibrium is interior and has $h_i^* = \tilde{h}_i$. Therefore, for an interior equilibrium, we require

$$
\begin{aligned}
\frac{\left[\bar{\eta}\left(\bar{\eta}-\underline{\eta}-(c-d)\right)+k\left(\bar{\eta}-\underline{\eta}\right)\right]}{\left(\bar{\eta}-\underline{\eta}+(c-d)\right)\left(\bar{\eta}-\underline{\eta}-(c-d)\right)} &< 1 \text{ and} \\
\frac{\left[\bar{\eta}\left(\bar{\eta}-\underline{\eta}-(c-d)\right)-k\right]}{\left(\bar{\eta}-\underline{\eta}+(c-d)\right)\left(\bar{\eta}-\underline{\eta}-(c-d)\right)} &> 0.
\end{aligned}
$$

This implies

$$
\begin{aligned}
k &< \bar{\eta}\left(\bar{\eta}-\underline{\eta}-(c-d)\right) \text{ and} \\
k &< \frac{\left(c-d-\underline{\eta}\right)\left(\bar{\eta}-\underline{\eta}-(c-d)\right)}{\left(\bar{\eta}-\underline{\eta}\right)}.
\end{aligned}
$$

In other cases, there is a corner solution with either one or the other player always playing the same action.

1.4 Strategic substitutes

In this case $c > d$. In addition if $d > 0$, the game is chicken. Our main result for this case is the following.

Theorem 4 *Suppose $d < c$, and $\bar{h}_i > c-d$ and $\underline{h}_i < 0$ so each player has dominant strategy hawks and doves. If for all $x, s, t \in (\underline{\eta}, \bar{\eta})$ it holds that*

$$
F_1(s|t) - F_2(s|t) < \frac{1}{c-d} \quad \text{and} \quad F_1(s|x) - F_2(x|t) < \frac{1}{c-d} \tag{8}
$$

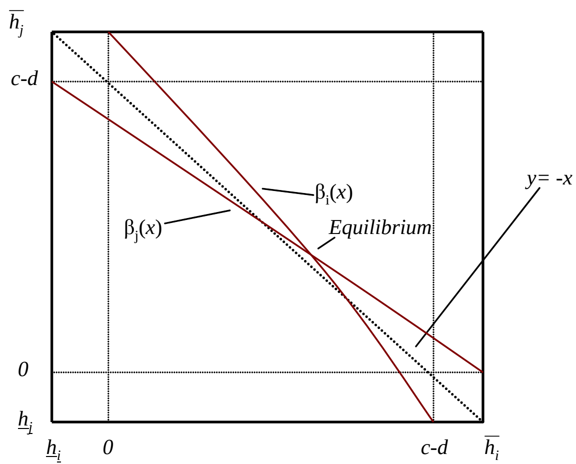
then there is a unique BNE. This BNE is a cutoff equilibrium.

Verifying this result again requires two steps. The first step, which shows that all BNE must be cut-off equilibria, is relegated to the appendix. The second step is to show that there exists one and only one cut-off equilibrium. This is again best explained diagrammatically. Notice that the assumption $F_1(s|x) - F_2(x|t) < \frac{1}{c-d}$ implies

$$
(c-d)F_2(x|t) > (c-d)F_1(s|x) - 1 \geq -1.
$$

Therefore, $\Psi_2^i(x, y) > 0$ so a cutoff equilibrium exists by Proposition 2. The fact that both players have dominant strategy hawks and doves means

Figure 3: Equilibrium for Strategic Substitutes



that for if player j uses cutoff x, $\beta_i(x)$ is interior. Then, we must have $\Psi^i(x, \beta_i(x)) = 0$ and so we can use (5) to study the best-response function β. Since $\Psi_1^i(x, y) > 0$, the best response functions are downward-sloping: $\beta_i'(x) < 0$. From (5), we can conclude

$$1 + \beta_i'(x) = 1 + \frac{(d-c)F_1}{1-(d-c)F_2} = \frac{1+(c-d)(F_2-F_1)}{1-(d-c)F_2}$$

which is strictly positive as $F_1(s|t) - F_2(s|t) < \frac{1}{c-d}$ by hypothesis. This implies $-1 < \beta_i'(x) < 0$ for all $x \in [\underline{h}_j, \bar{h}_j]$. This implies the two best-response functions cannot cross more than once, so there can be only one cutoff equilibrium:

The incomplete information game with strategic substitutes is a submodular game. In the two player case, we can invert one player's strategy set to make the game supermodular. Hence, when there is a unique equilibrium, we can again invoke the result that a supermodular game with a unique equilibrium can be solved by iterated deletion of (interim) dominated strategies.

The sufficient condition for uniqueness with strategic substitutes contains one inequality that is symmetric with strategic complements (upto two negative signs). But it also contains a second condition that is used to prove the non-existence of non-cut-off equilibria. The issue of non-cutoff equilibria does not arise in a complete information setting where there is a full equivalence between two player supermodular and submodular games.

As in the case of strategic complements, it is easy to characterize the unique equilibrium. If the unique equilibrium is interior, then player i chooses the cut-off point $h_i^* \in \left(\underline{h}_i, \bar{h}_i\right)$ which solves

$$h_i^* + (d-c)\left(1 - F(h_j^* - k_j | h_i^* - k_i)\right) = 0. \tag{9}$$

In other cases, there are corner solutions.

The uniqueness conditions for the two classes of games, (8) and (6), can be compared. If types are independent, then $F(s|t)$ is independent of its second argument, so (6) and (8) both reduce to the condition

$$F_1(s|t) < \left|\frac{1}{d-c}\right|$$

for all $s, t \in (\underline{\eta}, \bar{\eta})$. If in addition η_1 and η_2 are independently drawn from a uniform distribution with support $[\underline{\eta}, \bar{\eta}]$, then the sufficient condition for uniqueness of equilibrium is

$$\bar{\eta} - \underline{\eta} > |d - c|\,.$$

However, if types are affiliated, so that $F_2(s|t) < 0$, then the uniqueness conditions for strategic substitutes are more stringent than those for strategic complements, because$F_2(s|t)$ enters with a *negative* sign in (8).

While stag hunt captured the idea of Schelling's "reciprocal fear of surprise attack," chicken, a game with strategic substitutes captures a different logic of "escalating fear of conflict". Coordination types in chicken want to *mis*-coordinate with the opponent's action, particularly if he plays H. Coordination types with low a low hostility level h are near indifferent between H and D if they are certain that the opponent plays D. But if there is positive probability that the opponent is a dominant strategy type, the "almost dominant strategy doves" strictly prefer to back off and play D. This in turn emboldens coordination types who are almost dominant strategy hawks to play H and the cycle continues. This escalation is more powerful if there is negative correlation between types and dovish coordination types with low h put high probability on hawkish coordination types with high h and vice-versa. But it is more natural to assume independence, if there is no fundamental connection between the two players, or positive correlation, if they are both fighting over a common resource. In the latter case, the uniqueness condition for chicken is less likely to hold.

Example: the uniform independent case Suppose the idiosyncratic shocks ε_1 and ε_2 are independently drawn from a distribution which is uniform on $[\underline{\varepsilon}, \bar{\varepsilon}]$. The flatness condition requires "sufficient uncertainty": $\bar{\varepsilon} - \underline{\varepsilon} > c - d$. Then, we are assured that a unique equilibrium exists. Notice that the flatness condition implies that player 1 either must have dominant strategy hawks or dominant strategy doves with positive probability. We assume $c - d > \underline{\varepsilon}$, otherwise we have a trivial equilibrium where both players play H whatever their type. We also assume $\bar{\varepsilon} > 0$, otherwise there is a trivial equilibrium where player 1 always plays D and player 2 plays h iff $h_2 \geq 0$.

Let

$$\begin{aligned}
\tilde{h}_1 &= \frac{\left[\bar{\varepsilon}\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right) + k\left(\bar{\varepsilon} - \underline{\varepsilon}\right)\right](c-d)}{\left(\bar{\varepsilon} - \underline{\varepsilon} + (c-d)\right)\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right)} > 0 \text{ and} \\
\tilde{h}_2 &= \frac{\left[\bar{\varepsilon}\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right) - k\right](c-d)}{\left(\bar{\varepsilon} - \underline{\varepsilon} + (c-d)\right)\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right)} < c - d
\end{aligned}$$

where these equations solve (9). Notice that as $k \geq 0$, $\tilde{h}_1 \geq \tilde{h}_2$.

If $\tilde{h}_i \in [0, c-d]$ and $\tilde{h}_i \in [\underline{\varepsilon} + k_i, \bar{\varepsilon} + k_i]$, then the unique equilibrium is interior and has $h_i^* = \tilde{h}_i$. Therefore, for an interior equilibrium, we require

$$\begin{aligned}
\frac{\left[\bar{\varepsilon}\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right) + k\left(\bar{\varepsilon} - \underline{\varepsilon}\right)\right]}{\left(\bar{\varepsilon} - \underline{\varepsilon} + (c-d)\right)\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right)} &< 1 \text{ and} \\
\frac{\left[\bar{\varepsilon}\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right) - k\right]}{\left(\bar{\varepsilon} - \underline{\varepsilon} + (c-d)\right)\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right)} &> 0.
\end{aligned}$$

This implies

$$\begin{aligned}
k &< \bar{\varepsilon}\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right) \text{ and} \\
k &< \frac{\left(c - d - \underline{\varepsilon}\right)\left(\bar{\varepsilon} - \underline{\varepsilon} - (c-d)\right)}{\left(\bar{\varepsilon} - \underline{\varepsilon}\right)}.
\end{aligned}$$

In other cases, there is a corner solution with either one or the other player always playing the same action.

1.5 Global games

We follow a key example in Carlsson and van Damme [1] and assume the players' types are generated from an underlying parameter θ as follows. First, θ is drawn from a uniform distribution on $\Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$. Then, η_1 and η_2 are independently drawn from a uniform distribution on $[\theta-\varepsilon, \theta+\varepsilon]$. Player i knows his own draw η_i but not the opponent's draw η_j. Neither player can observe θ. Conditional on θ, η_1 and η_2 are independent random variables. If player i draws $\eta_i \in [\underline{\theta}+\varepsilon, \bar{\theta}-\varepsilon]$, then his posterior beliefs about θ are given by a uniform distribution on $[\eta_i-\varepsilon, \eta_i+\varepsilon]$. Therefore, player i's beliefs about η_j are given by a symmetric, triangular distribution around η_i with support $[\eta_i - 2\varepsilon, \eta_i + 2\varepsilon]$. Notice that this triangular distribution is symmetric about η_i. If $s, t \in [\underline{\theta}+\varepsilon, \bar{\theta}-\varepsilon]$ then

$$F(t|s) = \begin{cases} 1 & \text{if} \quad t \geq s + 2\varepsilon \\ 1 - \frac{1}{2}\left(1 - \frac{t-s}{2\varepsilon}\right)^2 & \text{if} \quad s \leq t \leq s + 2\varepsilon \\ \frac{1}{2}\left(1 - \frac{s-t}{2\varepsilon}\right)^2 & \text{if} \quad s - 2\varepsilon \leq t \leq s \\ 0 & \text{if} \quad t \leq s - 2\varepsilon \end{cases}$$

which implies

$$F(t|s) + F(s|t) = 1. \tag{10}$$

We also compute

$$F_1(t|s) = \begin{cases} 0 & \text{if} \quad t > s + 2\varepsilon \\ \frac{1}{4\varepsilon^2}(s - t + 2\varepsilon) & \text{if} \quad s \leq t \leq s + 2\varepsilon \\ \frac{1}{4\varepsilon^2}(t - s + 2\varepsilon) & \text{if} \quad s - 2\varepsilon \leq t \leq s \\ 0 & \text{if} \quad t \leq s - 2\varepsilon \end{cases}$$

so

$$F_2(t|s) = F_2(s|t) = -F_1(t|s) = -F_1(s|t).$$

Assume $d > c$ so we are in the case of strategic complements. Also assume

$$\begin{aligned} c - d &> \underline{\theta} + k + \varepsilon \text{ and} \\ 0 &< \bar{\theta} - \varepsilon. \end{aligned}$$

This guarantees both players have dominant strategy hawks and doves and that we can use the triangular distribution above to determine the beliefs of coordination types. Notice the structure of F implies

$$0 = F_1(s|t) + F_2(s|t) < \frac{1}{d-c}$$

and the sufficient condition for uniqueness (6) is satisfied. That is, there is always a unique equilibrium for all ε.

Now, assume $c > d$ and we are in the case of strategic substitutes. For $|s - t| < 2\varepsilon$ we have

$$F_1(s|t) - F_2(s|t) = 2F_1(s|t) = \frac{1}{2\varepsilon^2}\left(|s - t| + 2\varepsilon\right)$$

which reaches a maximum $2/\varepsilon$ when $|s - t| = 2\varepsilon$. Also,

$$F_1(s|x) - F_2(x|t) = F_1(s|x) + F_1(t|x)$$

which also reaches a maximum $2/\varepsilon$. Therefore, our uniqueness condition (8) is $\varepsilon > \frac{c-d}{2}$. This is the implication of "large idiosyncratic uncertainty" which guarantees uniqueness. Also, we know that the game can be solved by iterated deletion of (interim) dominated strategies.

Proposition 5 *If $\varepsilon > \frac{c-d}{2}$ then there is a unique equilibrium which can be obtained by iterated deletion of dominated strategies.*

When uniqueness obtains, it is a stark consequence of the dynamic triggered by "deterrence by fear." The chance of conflict with dominant strategy hawks triggers dovish behavior among the coordination types. When $2\varepsilon > c - d$, type 0 of player i puts positive probability on player j being a dominant strategy hawk. This implies that type 0 and types just higher play D. Similarly, type $c - d$ of player i puts positive probability on player j being a dominant strategy dove. This implies type $c - d$ and types just below play H. This process of iterated deletion of dominated strategies identifies a unique equilibrium.

However, the global games literature shows that a unique equilibrium exists if the idiosyncratic uncertainty is sufficiently small. We will verify this for our game.

Assume $k_1 = 0$ and $k_2 = k > 0$. Suppose

$$k + \underline{\theta} + \varepsilon < 0$$

$$\bar{\theta} - \varepsilon > c - d$$

This implies that both dominant strategy hawks and doves exist for each player and that we can use the triangular distribution to determine the beliefs of opportunistic types.

Conditional on θ, player 1's type $h_1 = \eta_1$ is uniformly distributed on $[\theta - \varepsilon, \theta + \varepsilon]$, while player 2's type $h_2 = k + \eta_2$ is uniformly distributed on $[k+\theta-\varepsilon, k+\theta+\varepsilon]$. A strategy $\sigma_i : [\underline{h}, \bar{h}] \to \{H, D\}$ for player i is a measurable function which specifies a choice $\sigma_i(h_i) \in \{H, D\}$ for each type h_i. If player i's type h_i thinks player j will choose D with probability $\delta_j(h_i)$, then his net gain from choosing H instead of D is

$$h_i + (d - c)(1 - \delta_j(h_i)). \tag{11}$$

Remark 6 *If $\eta_1 < -k - 2\varepsilon$, then player 1 knows that $h_2 = k + \eta_2 < 0$, so player 2 must be a dominant strategy dove. If $\eta_1 > c - d - k + 2\varepsilon$, then player 1 knows that $h_2 = k + \eta_2 > c - d$, so player 2 must be a dominant strategy hawk. If $\eta_2 < -2\varepsilon$, then player 2 knows that $h_1 = \eta_1 < 0$, so player 1 must be a dominant strategy dove. If $\eta_2 > c - d + 2\varepsilon$, then player 2 knows that $h_1 = \eta_1 > c - d$, so player 1 must be a dominant strategy hawk.*

Consider the process of eliminating (interim) dominated strategies. In the first "round" of elimination, D is eliminated for dominant strategy hawks ($h_i \geq c - d$) and H for dominant strategy doves. Now consider the second round.

Suppose $\varepsilon < k/2$. This is the case of highly correlated types. When $c - d - k + 2\varepsilon < \eta_1 < c - d$, player 1 knows that player 2 is a dominant strategy hawk (by remark 6). Hence, H can be eliminated for player 1. Indeed, even if η_1 is slightly below $c - d - k + 2\varepsilon$, H can be eliminated, because player 2 is highly likely to be a dominant strategy hawk. Let η_1' be the largest η_1 such that H cannot be eliminated for player 1's type η_1 in round 2. Notice that $\eta_1' < c - d - k + 2\varepsilon$. Now if $h_2 = \eta_2 + k$ is slightly below $c - d$, then player 2 knows that player 1 has a positive probability of having a type between η_1' and $c - d$. Such types of player 1 had H removed in round 2 of the elimination of interim dominated strategies. Therefore, in round 3, D must be eliminated for types of player 2 slightly below $c - d$. Let η_2' be the largest η_2 such that D cannot be eliminated for player 2 in round 3. In round 4, player 1's types slightly below η_1' will be able to remove D, etc.

We claim that this process must eventually eliminate H for all $h_1 \in (0, c - d)$, and D for all $h_2 \in (0, c - d)$. If this were not true, then the process cannot proceed below some $h_1^* > 0$ and $h_2^* > 0$. Now, $h_2^* > h_1^* + k - 2\varepsilon$, otherwise type h_1^* knows that $h_2 \geq h_2^*$, and all such types have eliminated D, but then H must be eliminated for types slightly below h_1^*.

Consider player 2's type h_2^*, with private component $\eta_2^* = h_2^* - k$. He knows that $h_1 = \eta_1 \leq h_2^* - k + 2\varepsilon < c - d - k + 2\varepsilon \leq c - d$. Now H has been eliminated for all $h_1 \in (h_1^*, c-d)$, and according to type h_2^*, the probability that player 1's type lies in this interval is $1 - F(h_1^*|h_2^* - k)$. Therefore, if D cannot be eliminated for type h_2^*, it must be the case that type h_2^* weakly prefers D when the opponent uses H with probability at most $F(h_1^*|h_2^* - k)$. This implies

$$h_2^* + (d-c)F(h_1^*|h_2^* - k) \leq 0$$

By a similar argument, if H cannot be eliminated for type h_1^*, then type h_1^* must prefer H when the opponent uses H with probability at least $1 - F(h_2^* - k|h_1^*)$. That is,

$$h_1^* + (d-c)\left(1 - F(h_2^* - k|h_1^*)\right) \geq 0$$

Subtracting the first inequality from the second yields

$$h_1^* - h_2^* \geq (c-d)\left(1 - F(h_2^* - k|h_1^*) - F(h_1^*|h_2^* - k)\right) = 0$$

where the last equality uses 10. However, this contradicts $h_2^* > h_1^* + k - 2\varepsilon$. We summarize the argument as follows.

Proposition 7 *If $\varepsilon < k/2$ then there is a unique BNE, which can be obtained by the iterated elimination of dominated strategies. In this BNE, player 1 plays H iff $h_1 \geq c - d$ and player 2 plays H iff $h_2 \geq 0$.*

If ε is small then the types are highly correlated. In this case, Proposition 7 shows that the process of iterated elimination of interim dominated strategies selects a unique outcome, where all opportunistic types of player 1 "fold" and play D, and all opportunistic types of player 2 play H. The process begins with the aggressive opportunistic types of player 1 backing off as they put high probability on player 2 being a dominant strategy hawk and the process continues. This is the reverse of the logic underlying Proposition 5. Finally, since (D, H) is the risk-dominant outcome for the complete information game with $\varepsilon = 0$ and $k > 0$, this result agrees with the conclusion of Carlsson and van Damme [1].

The previous two propositions show that a unique equilibrium exists for sufficiently small and sufficiently large ε. For intermediate ε, multiple equilibria can exist, as we now show.

Suppose $\varepsilon > k/2$. Here types are not highly correlated, and the process of iterated elimination of dominated strategies cannot achieve anything after the first "round". Consider type $h_1 = c - d$. In the second "round", he cannot rule out the possibility that player 2 will choose D when $h_2 < c - d$. Moreover, the event that $h_2 = \eta_2 + k < c - d$ has positive probability when $\eta_1 = h_2 = c - d$ and $k < 2\varepsilon$. Therefore, we cannot eliminate H for type $h_1 = c - d$. Player 1's type $h_1 = 0$ cannot eliminate H, because H has not been eliminated for $h_2 > 0$. Since neither of the "boundary" opportunistic types can eliminate H, no opportunistic type at all can eliminate H. Clearly, they cannot eliminate D either. Thus, no opportunistic type of player 1 can eliminate any action in round 2. A similar argument applies to player 2.

Let $h^* = \frac{(c-d)(2\varepsilon-k)^2}{8\varepsilon^2}$. Notice that

$$h^*+k-2\varepsilon = \frac{(c-d)\left(2\varepsilon-k\right)^2}{8\varepsilon^2} - (2\varepsilon-k) = \left(\frac{(c-d)\left(2\varepsilon-k\right)}{8\varepsilon^2} - 1\right)(2\varepsilon-k) > 0$$

as long as

$$\frac{(c-d)\left(2\varepsilon-k\right)}{8\varepsilon^2} > 1$$

and

$$h^* + k + 2\varepsilon = \frac{(c-d)\left(2\varepsilon-k\right)^2}{8\varepsilon^2} + k + 2\varepsilon < c - d$$

as long as

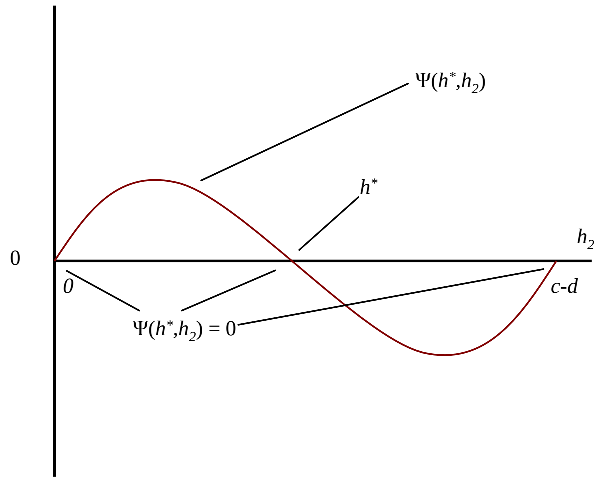
$$\frac{(c-d)\left(2\varepsilon-k\right)}{8\varepsilon^2} < \frac{c-d-(k+2\varepsilon)}{2\varepsilon-k}$$

Players' strategies are as follows: player 1 plays D iff $h_1 \leq h^*$; player 2 plays D iff $h_2 \leq 0$ or $h_2 \in [h^*, c-d]$.

Consider player 1 first. For player 1 of type h^*, the probability that player 2 plays H is $F(h^*|h^*-k) = \frac{(2\varepsilon-k)^2}{8\varepsilon^2}$ and he is indifferent between H and D. Higher types are more aggressive and assess a lower probability that player 2 plays H. These types strictly prefer to play H and, by a symmetric argument, lower types prefer to play D.

We must also show player 2's strategy is at a best-response. Assume $k < 2\varepsilon\left(1 - \frac{2\varepsilon}{c-d}\right)$. For player 2, if he is a dominant strategy type, the specified strategy is clearly optimal. For $h_2 \in [h^* + k - 2\varepsilon, h^*]$, $\Pr\{h_1 < h^*|h_2\} =$

Figure 4: $\Psi^2(h^*,h_2)$



$1 - \frac{1}{8\varepsilon^2}\left((h_2 - k + 2\varepsilon) - h^*\right)^2 = \delta_1(h_2)$. Substituting this into (11), the net gain from playing H rather than D becomes

$$h_2 + \frac{(d-c)}{8\varepsilon^2}\left(h_2 - h^* - k + 2\varepsilon\right)^2. \tag{12}$$

This is quadratic in h_2 and equals zero when $h_2 = h^*$. It reaches a maximum at

$$\hat{h} = h^* + k - 2\varepsilon + \frac{4\varepsilon^2}{c-d}$$

which is interior to the interval $[h^* + k - 2\varepsilon, h^*]$ as long as $k < 2\varepsilon\left(1 - \frac{2\varepsilon}{c-d}\right)$. In fact, (12) is clearly strictly positive for $h_2 \in [h^* + k - 2\varepsilon, h^*)$. For $h_2 \in [0, h^* + k - 2\varepsilon]$, player 2 knows his opponent plays D and then it is optimal to play H as (11) is equal to $h_2 \geq 0$. There is a similar argument for $h_2 \in (h^*, c-d]$ and so the entire $\Psi^2(h^*, h_2)$ picture is:

There is another equilibrium with the roles of players 1 and 2 reversed.[2]

[2] Let $h^* = (c-d)\left(1 - \frac{(2\varepsilon - k)^2}{8\varepsilon^2}\right)$. Notice that

$$h^* - k - 2\varepsilon = (c-d)\left(1 - \frac{(2\varepsilon - k)^2}{8\varepsilon^2}\right) - 2\varepsilon - k > 0$$

There is also an equilibrium where player 2 plays H when $h_2 \geq 0$ and player 1 plays H when $h_1 \geq c-d$.

Now suppose $2\varepsilon > c-d$. Then, the above argument fails as $\hat{h} > h^*$. Indeed, Proposition 5 shows that a unique equilibrium exists in this case. Large ε approximates independence and is the polar opposite of the "global games" conclusion.

To summarize, deterrence by fear leads to a unique equilibrium in two cases. The chance of conflict with dominant strategy hawks triggers dovish

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as long as

$$(c-d)\left(1-\frac{(2\varepsilon-k)^2}{8\varepsilon^2}\right) > k+2\varepsilon$$

and

$$h^* - k + 2\varepsilon = (c-d)\left(1-\frac{(2\varepsilon-k)^2}{8\varepsilon^2}\right) + 2\varepsilon - k < c-d$$

as long as

$$(c-d)\left(1-\frac{(2\varepsilon-k)^2}{8\varepsilon^2}\right) < c-d-(2\varepsilon-k)$$

Players' strategies are as follows: player 2 plays D iff $h_2 \leq h^*$; player 1 plays D iff $h_1 \leq 0$ or $h_1 \in [h^*, c-d]$.

Consider player 2 first. For player 2 of type h^*, the probability that player 1 plays H is $F(h^*|h^*-k) = 1 - \frac{(2\varepsilon-k)^2}{8\varepsilon^2}$ and he is indifferent between H and D. Higher types are more aggressive and assess a lower probability that player 1 plays H. These types strictly prefer to play H and, by a symmetric argument, lower types prefer to play D.

We must also show player 1's strategy is at a best-response. Assume $k < 2\varepsilon\left(1-\frac{2\varepsilon}{c-d}\right)$. For player 1, if he is a dominant strategy type, the specified strategy is clearly optimal. For $h_1 \in [h^*, h^*+2\varepsilon-k]$, $\Pr\{h_2 < h^*|h_1\} = \frac{1}{8\varepsilon^2}\left(h^*-k-(h_1-2\varepsilon)\right)^2 = \delta_2(h_1)$. Substituting this into (11), the net gain from playing H rather than D becomes

$$h_1 + (d-c)\left(1-\frac{1}{8\varepsilon^2}\left(h^*-k-(h_1-2\varepsilon)\right)^2\right). \tag{13}$$

This is quadratic in h_1 and equals zero when $h_1 = h^*$. It reaches a minimum at

$$\hat{h} = h^* - k + 2\varepsilon - \frac{4\varepsilon^2}{c-d}$$

which is interior to the interval $[h^*, h^*+2\varepsilon-k]$ as long as $k < 2\varepsilon\left(1-\frac{2\varepsilon}{c-d}\right)$. In fact, (13) is clearly strictly negative for $h_1 \in (h^*, h^*+2\varepsilon-k]$. For $h_1 \in (h^*+2\varepsilon-k, c-d]$, player 1 knows his opponent plays H and then it is optimal to play D as (11) is negative. There is a similar argument for $h_1 \in [0, h^*)$.

behavior among the opportunistic types. For this to occur, either types must be independent and the prior diffuse or it must be common knowledge that one player is inherently more aggressive than the other. This in turn persuades opportunistic types to play H, as they put positive probability of their opponent playing D, and the process continues.

2 Joint normal distribution

TBD

3 Appendix

Proofs of non-existence of non-cutoff equilibria. Since the proofs for chicken and stag-hunt are parallel, it is convenient to combine them. If some player uses a constant strategy, then each player must be using a cut-off strategy. Thus, we may assume neither player uses a constant strategy. Let

$$x_i = \inf \{h_i : \sigma_i(h_i) = H\}$$

and

$$y_i = \sup \{h_i : \sigma_i(h_i) = D\}$$

By definition, $x_i \leq y_i$.

Since player i's type x_i weakly prefers H, (2) implies

$$x_i + (1 - \delta_j(x_i))(d - c) \geq 0 \tag{14}$$

By definition, $\sigma_j(h_j) = D$ for all $h_j < x_j$ and $\sigma_j(h_j) = H$ for all $h_j > y_j$, so

$$G_i(x_j|x_i) \leq \delta_j(x_i) \leq G_i(y_j|x_i). \tag{15}$$

We will show that one player must be using a cutoff strategy. Hence, as (4) holds, so must the other.

Case 1: $c < d$. In this case, (14) and the first inequality of (15) imply

$$\begin{aligned} x_i + (1 - G_i(x_j|x_i))(d - c) &\geq 0 \text{ or} \\ x_i + (1 - F(x_j - k_j|x_i - k_i))(d - c) &\geq 0 \end{aligned} \tag{16}$$

By the same reasoning,

$$\begin{aligned} y_i + (1 - G_i(y_j|y_i))(d-c) &\leq 0 \text{ or} \\ y_i + (1 - F(y_j - k_j|y_i - k_i))(d-c) &\leq 0 \end{aligned} \tag{17}$$

Combining the (16) and (17) expressions for players i and j respectively yields

$$F(y_j - k_j|y_i - k_i) - F(x_j - k_j|x_i - k_i) \geq \frac{1}{d-c}(y_i - x_i) \text{ and} \tag{18}$$

$$F(y_i - k_i|y_j - k_j) - F(x_i - k_i|x_j - k_j) \geq \frac{1}{d-c}(y_j - x_j) \tag{19}$$

Combining (16) for player i and (17) for player j, we obtain

$$F(y_i - k_i|y_j - k_j) - F(x_j - k_j|x_i - k_i) \geq \frac{1}{d-c}(y_j - x_i). \tag{20}$$

Combining (16) for player j and (17) for player i, we obtain

$$F(y_j - k_j|y_i - k_i) - F(x_i - k_i|x_j - k_j) \geq \frac{1}{d-c}(y_i - x_j). \tag{21}$$

Assume $x_i - k_i \leq x_j - k_j$ w.l.o.g.
If $y_i - k_i \geq y_j - k_j$, (18) implies

$$F(y_i - k_i|y_i - k_i) - F(x_i - k_i|x_i - k_i) \geq \frac{1}{d-c}(y_i - x_i). \tag{22}$$

By the mean value theorem, there is a $z \in [\underline{\eta}, \bar{\eta}]$ such that $(F_1(z|z) + F_2(z|z))(y_i - x_i) = F(y_i - k_i|y_i - k_i) - F(x_i - k_i|x_i - k_i)$. The hypothesis of Theorem 3 implies $F_1(z|z) + F_2(z|z) < \frac{1}{d-c}$, so (22) can only hold if $y_i = x_i$ and player i uses a cutoff strategy.

Next, suppose $y_j - k_j \geq y_i - k_i$. Then, (20) implies

$$F(y_i - k_i|y_i - k_i) - F(x_i - k_i|x_i - k_i) \geq \frac{1}{d-c}(y_j - x_i). \tag{23}$$

If $y_j \geq y_i$, (23) implies

$$F(y_i - k_i|y_i - k_i) - F(x_i - k_i|x_i - k_i) \geq \frac{1}{d-c}(y_i - x_i).$$

But using the mean value theorem and the hypothesis of Theorem 3, we can again show this implies $y_i = x_i$ so player i uses a cutoff strategy.

If $y_j < y_i$, notice that our assumption $y_j - k_j \geq y_i - k_i$ implies $k_j < k_i$. There are two cases. First, suppose $x_j \geq x_i$. Then, substitutions in (21) yield

$$F(y_i - k_j|y_i - k_i) - F(x_i - k_j|x_i - k_i) \geq \frac{1}{d-c}(y_i - x_i).$$

By the mean value theorem, there is a $z \in [\underline{\eta}, \bar{\eta}]$ such that

$$(F_1(z + k_i - k_j|z) + F_2(z + k_i - k_j|z))(y_i - x_i) = F(y_i - k_j|y_i - k_i) - F(x_i - k_j|x_i - k_i).$$

The hypothesis of Theorem 3 implies $F_1(z+k_i-k_j|z)+F_2(z+k_i-k_j|z) < \frac{1}{d-c}$ and this inequality only holds if $x_i = y_i$ and so player i uses a cut-off strategy. Second, suppose $x_j < x_i$. Then, after substitutions (19) becomes

$$F(y_i - k_j|y_i - k_i) - F(x_i - k_j|x_i - k_i) \geq \frac{1}{d-c}(y_i - x_i)$$

Again, using the mean value theorem and the hypothesis of Theorem 3, this inequality only holds if $y_i = x_i$ and player i uses a cut-off strategy. This completes the proof of Theorem 3.

Case 2: $c > d$. In this case, (14) and the second inequality of (15) imply

$$\begin{aligned} x_i + (1 - G_i(y_j|x_i))(d - c) &\geq 0 \text{ or} \\ x_i + (1 - F(y_j - k_j|x_i - k_i))(d - c) &\geq 0 \end{aligned} \tag{24}$$

By the same reasoning,

$$\begin{aligned} y_i + (1 - G_i(x_j|y_i))(d - c) &\leq 0 \text{ or} \\ y_i + (1 - F(x_j - k_j|y_i - k_i))(d - c) &\leq 0 \end{aligned} \tag{25}$$

Combining (24) and (25) for player i yields

$$F(y_j - k_j|x_i - k_i) - F(x_j - k_j|y_i - k_i) \geq \frac{1}{c-d}(y_i - x_i). \tag{26}$$

Combining (24) for player i and (25) for player j yields

$$F(y_j - k_j|x_i - k_i) - F(x_i - k_i|y_j - k_j) \geq \frac{1}{c-d}(y_j - x_i). \tag{27}$$

Combining (24) for player j and (25) for player i yields

$$F(y_i - k_i|x_j - k_j) - F(x_j - k_j|y_i - k_i) \geq \frac{1}{c-d}(y_i - x_j). \tag{28}$$

Assume $x_i - k_i \leq x_j - k_j$ w.l.o.g.

If $y_i - k_i \geq y_j - k_j$, then (26) implies

$$F(y_i - k_i|x_i - k_i) - F(x_i - k_i|y_i - k_i) \geq \frac{1}{c-d}(y_i - x_i) \tag{29}$$

But by the mean value theorem,

$$\begin{aligned} & F(y_i - k_i|x_i - k_i) - F(x_i - k_i|x_i - k_i) + F(x_i - k_i|x_i - k_i) - F(x_i - k_i|y_i - k_i) \\ = \; & \left(F_1(t|x_i - k_i) - F_2(x_i - k_i|s)\right)(y_i - x_i) \end{aligned}$$

for some $s, t \in [\underline{\eta}, \bar{\eta}]$. By the hypothesis of Theorem 4 then (29) implies $x_i = y_i$ so player i uses a cut-off strategy.

If $y_i - k_i < y_j - k_j$, then (28) implies

$$F(y_j - k_j|x_j - k_j) - F(x_j - k_j|y_j - k_j) \geq \frac{1}{c-d}(y_i - x_j). \tag{30}$$

If $y_i \geq y_j$, then via the mean value theorem and (30), the hypothesis of Theorem 4 implies $y_j = x_j$ and player j uses a cut-off strategy. If $y_j > y_i$, there are two cases. First, if $k_j \geq k_i$, from (27) we obtain

$$F(y_j - k_i|x_i - k_i) - F(x_i - k_i|y_j - k_i) \geq \frac{1}{c-d}(y_j - x_i)$$

and a now familiar argument implies that $y_j = x_i$. But this contradicts $y_j > y_i \geq x_i$. Second, if $k_j < k_i$, from (28), we obtain

$$F(y_i - k_i|x_j - k_i) - F(x_j - k_i|y_i - k_i) \geq \frac{1}{c-d}(y_i - x_j)$$

and now a familiar argument implies $y_i = x_j$. Substituting this into (28), we obtain

$$F(y_i - k_i|y_i - k_j) - F(y_i - k_j|y_i - k_i) \geq 0.$$

But, as $k_j < k_i$ and $F_2(x|y) \leq 0$, the left-hand-side is negative, a contradiction. This completes the proof of Theorem 4. QED

References

[1] Hans Carlsson and Eric van Damme (1993): "Global Games and Equilibrium Selection," *Econometrica*, 989-1018.

[2] Robert Jervis (1978): "Cooperation Under the Security Dilemma," *World Politics*, Vol. 30, No. 2., pp. 167-214.

[3] Stephen Morris and Hyun Shin (2003): "Global Games: Theory and Applications," in *Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Society)*, edited by M. Dewatripont, L. Hansen and S. Turnovsky. Cambridge: Cambridge University Press, 56-114.

[4] Stephen Morris and Hyun Shin (2003): "Heterogeneity and Uniqueness in Interaction Games," in *The Economy as an Evolving Complex System III*, edited by L. Blume and S. Durlauf. Oxford: Oxford University Press, 207-242.

[5] Thomas Schelling (1963): *Strategy of Conflict*, New York: Oxford University Press.