The Sorting Effect of Price Competition*

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– *Preliminary draft* –

Abstract

We investigate under which conditions price competition leads to sorting of buyers and sellers. In a decentralized Walrasian market where sellers post prices and buyers choose whom to buy from positive assortative matching obtains only if there is a high enough degree of complementarity between buyer and seller types. The relevant condition is *root-supermodularity*; i.e., the square root of the match value function is supermodular. This condition is weaker than log-supermodularity, a condition required for positive assortative matching in markets with random search. Negative assortative matching obtains whenever the match value function is weakly submodular.

Keywords. Decentralized Walrasian markets. Price Competition. Two-Sided Matching. Root-Supermodularity.

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1 Introduction

We analyze decentralized Walrasian markets in which heterogeneous sellers post prices and buyers decide whom to buy from. Buyer valuations are determined by the match value function that depends both on the unobservable buyer type and the observable seller type. We derive general conditions under which decentralized price setting leads to assortative matching, where higher type buyers prefer to buy from strictly higher type sellers.

In decentralized Walrasian markets[1] sellers compete in prices, for example durables sold on platforms like eBay or Amazon. Unlike standard Walrasian markets where prices are processed through a centralized market institution, in decentralized Walrasian markets each contract, specifying the trading price for each seller, is traded in a separate market. Matching buyers and sellers across those separate decentralized markets injects frictions in this trading process, and the standard notion of market clearing is adjusted to accommodate them. These frictions stem from the lack of coordination of the buyers in their choice of seller, in conjunction with a limited capacity by the seller. This implies that there is a positive probability that the demand at a given good for sale is higher than the supply. These frictions are mediated through a competitive pricing process as the sellers' pricing strategies now affect their likelihood to trade through the impact on buyers' purchasing decisions who respond to changing prices. A seller thus faces a demand that is not perfectly elastic as he can attract more candidate buyers for his good by lowering the price.

The focus here is on a market with two-sided heterogeneity of buyers and sellers, where the valuation of the good consumed depends both on the buyer and the seller type. This is a realistic assumption in many markets. The value to the buyer of a house for example depends in general on the buyer's budget or family size and also on the characteristics of the house. Or the utility derived from a car will be determined by the car's quality and the driver's intended mileage. And the productivity of a worker depends on her qualifications and the type of occupation or job she is in. In general we investigate how the degree of complementarities in the valuation between buyer and seller types affects the equilibrium allocation of buyers and sellers.

Our main finding is that there is positive assortative matching if and only if the match values are root-supermodular[2]. There must be sufficient complementarity in the match values for positive sorting to obtain. This derives from the fact that the equilibrium allocation is governed by two forces, the degree of complementarity, and the frictions induced by the possibility of no trade. In a world without complementarities where match values are additive, it does not matter for efficiency who matches with whom if there are no frictions. Yet, with frictions, the equilibrium allocation is negatively assorted.

[1]See amongst others Peters (1997a) and Moen (1997).

[2]A function $f(x, y)$ is root-supermodular if its square root $\sqrt{f(x, y)}$ is supermodular.

High valuation buyers are willing to pay a high price in order to minimize the probability of no trade. That implies that the sellers they buy from and who set a high price have a high probability of not selling the good. The low type sellers are those who find it optimal to set high prices. The result is negative sorting with high type buyers matching with low type sellers. With some complementarities and no frictions, the equilibrium allocation is positively assorted: high buyer types match with high seller types. It is well known (Becker 1973) that supermodularity of the match value function leads to positive assortative matching. When there are frictions, sufficient supermodularity is needed for positive assortative matching in order to offset the tendency towards negative sorting induced by the frictions.

Our main result is consistent with existing results on assortative matching. Shimer and Smith (2000) show that in the context of random search, assortative matching requires log-supermodularity, which is stronger than root-supermodularity. In our decentralized Walrasian market, there are search frictions and hence the standard supermodularity requirement fails to deliver positive assortative matching. Because there is nonetheless some degree of competition between sellers, the requirement is less stringent. The degree of complementarity needed in decentralized Walrasian markets thus falls in between that needed for centralized markets (Becker) and markets with random search (Shimer and Smith).

Under weak submodularity – either the match value function exhibits zero complementarities or it has substitutability – we always get negative assortative matching, and weak submodularity is a necessary and sufficient conditions for negative assortative matching under all type distributions. The requirements for assortative matching, both positive and negative, are tight in the sense that for any match value function, there exist type distributions for which it will be strictly binding. In particular, the constraint will be binding whenever there exist markets where the ratio of buyers to sellers goes to zero, which is always the case when there are some buyers whose valuation is below the lowest price offered. There obviously always exist distributions for which the constraint is slack. This is typically the case when the lowest buyer valuations are high enough to ensure a strictly positive surplus and no buyer is rationed.

That also means that for some economies with given distributions, there may be positive assortative matching if the match value function is less than root-supermodular. More interesting, for some distributions we may have negative assortative matching even if the match values exhibit moderate degrees of supermodularity. To our knowledge, this is new in the literature.

Our set up is close to the one of McAfee (1993) and Peters (1997a) in which we analyze decentralized price setting in the Walrasian spirit rather than competition in auctions as they do. This seemingly minor difference of using competitive pricing rather than auctions leads to drastically different equilibrium outcomes. Competitive pricing leads to perfect sorting of buyers and sellers because sorting allows competing sellers to screen different valuation buyers *ex ante*. High valuation buyers are willing to pay more in order to avoid a long queue of potential competing buyers. We show that market frictions endogenously generate a single crossing property of expected payoffs in queue length and price. This property induces buyers to reveal their type by "voting with their feet". The equilibrium implication is that even with identical sellers, buyer heterogeneity leads competing sellers to offer different prices, each of which generate the same equilibrium profits. Sellers use prices to induce selection amongst buyers which leads to sorting. These sorting properties are also crucial for establishing existence and a notion of constrained efficiency.

Related Literature. There is an extensive literature modelling decentralized price setting. Our model falls into the broad class of Walrasian models of contract markets. Each of the contracts, i.e. a price and quality combination, are traded in a separate market. Pricing is competitive in the sense that sellers can affect the amount of trade by changing the offered price. Such Walrasian models of contract markets (Moen (1997)) assume that the expectations of traders about out of equilibrium market positions, i.e. in inactive markets, adjust consistent with rational expectations. These assumptions basically entail that all traders have common beliefs of what happens in the inactive markets, and they must believe that the number of traders in those markets is large.

Peters (1997b and 2000) and Burdett, Shi, Wright (2001) provide strategic foundations for those out-of-equilibrium beliefs by modeling deviations in the form of a (sub)game with a finite number of players. They show that equilibrium allocations in the Walrasian models coincide with the allocations of the exact equilibrium of the infinite version of the game with direct competition. These are the outcomes of a market with non-Walrasian direct competition, in which inactive markets consist of deviations by sellers who correctly anticipate the effects of those deviations. Those results are derived in the context of homogeneity of buyer valuations. The Walrasian models, including ours in this paper, use the matching function that is derived out of this exercise.

These decentralized Walrasian market models have proven successful in analyzing labor and goods markets. Without attempting to be exhaustive, examples include Peters (1991), Acemoglu and Shimer (1999a and b), Shi (2001), Mortensen and Wright (2002), Galenianos and Kircher (2006), Kircher (2007), Delacroix and Shi (2006). All of these models mainly deal with homogeneous buyer types. Exemptions with two-sided heterogeneity include Mortensen and Wright (2002) who analyze a private valuation environment without complementarities under general specification of the matching frictions. Shi (2001) who anlyzes an economy with free entry and urn-ball matching. He uses free entry to derive conditions for assortative matching, yet their strength is hard to assess in terms of submodularity of a concave transform of the match value function.

Our focus of attention is on *heterogeneity* on both sides of the market. Buyer types are not observed, and sellers independently offer prices, which constitute complete contracts in the sense that seller valuations are not affected by unobservable buyer types. Peters (1997b) provides an example with diverse preferences on both sides of the market in which there are two types of buyers and two types of sellers. He constructs the equilibrium contracts under certain assumptions on the distribution of types on both sides of the market. He also conjectures that for other distributions equilibrium may not exist. He rightly points out that the price-queue length combinations offered must also be incentive compatible. Our result shows however that all offered price-queue length pairs will always be incentive compatible. The key insight here is that identical firms need not necessarily offer the same prices.

Price competition generates a prediction of equilibrium behavior under price competition that is fundamentally different from the behavior under competition in auctions. The private valuations version of our model is the same as that in McAfee (1993), Peters (1997a) and Peters and Severinov (1997). They show that the equilibrium in competing mechanisms is one where sellers compete in second price auctions and buyers randomize over sellers. There is no sorting *ex ante*, but rather screening by sellers *ex post*. The allocative outcome in competing auctions is therefore the opposite of perfect sorting. In equilibrium, high valuation buyers match both with high and low seller types. Under sorting, high valuation buyer types exclusively match with high seller types.

Shi (2002) and Shimer (2005)[3] consider a version of the ex post screening model that has perfectly observability of valuations, but that allows for complementarities. Because these are ex post screening models with observability (firms compete in price schedules) there is no perfect sorting in equilibrium. We return to the comparison of the *ex post* screening models with our *ex ante* sorting model in the discussion section. We analyze the efficiency properties and we show that despite the superior efficiency of posting auctions over price competition, prices are remarkably more efficient under competition than under monopoly. The efficiency loss of price posting relative to auctions is much smaller under competition than in a monopoly setting.

In our setting, both with private values and with complementarities, price posting leads to sorting in order to induce buyers to reveal their private valuations. This ex ante screening has desirable efficiency properties.

2 The Model

The economy consists of buyers and sellers. Each seller has one good for sale. Sellers are heterogeneous and indexed by a type y. Each seller draws her type from a distribution $S(y)$ with continuous density $s(y)$ on $\mathcal{Y} = [\underline{y}, \overline{y}] \subset \mathbb{R}_+^\infty$. Let $\mathcal{S}$ denotes the measure of all sellers. On the other side of the market

[3] Peters (2007) considers observable heterogeneity but lack of contractability on the part of the firms.

there is a unit mass of buyers. Each buyers draws his valuation x for the good i.i.d. from distribution $B(x)$ with continuous density $b(x)$ on $\mathcal{X}=[\underline{x},\overline{x}]\subset\mathbb{R}_+^\infty$. For trade to be viable, we assume $\overline{x}>\underline{y}$. S and B are C^2 and with strictly positive densities and because we are interested in assortative matching, we endow the function $f(x,y)$ with the properties $f_x>0, f_y>0$ on the ordered sets $\mathcal{X},\mathcal{Y}$. This is a normalization as we can redefine any function $\widetilde{f}$ such that $\widetilde{f}_x>0, \widetilde{f}_y>0$.[4]

Action sets. The market interaction has sellers simultaneously positing a price. Buyers observe the posted price and then decide which seller to buy from. Let $F_y(p)$ denote the CDF of price offers by sellers of type y. We can think of $F_y(p)$ as the symmetric distribution with which these sellers post prices, or as the outcome of pure strategies by the sellers. We assume measurability that allows the following CDF $F(p,y)=\int_{\underline{y}}^{y}F_y(p)\,s(y)dy$ of price-quality combinations in the market. We assume that buyers can perfectly observe the seller type y and the price p.

Given $F(y,p)$, each buyer decides on the price at which he attempts to trade. That is, each buyer type decides on a distribution $G_x(p,y)$ that describes his randomized choice of attempting to trade at price p if his realized type is x. Assume that $G_x(p,y)$ is measurable with respect to x for all p, and $G(p,y)=\int_{\mathcal{X}}G_x(p,y)\,b(x)dx$ denotes the distribution of sellers over prices. Buyer types are not observable by sellers.

Matching Technology. At a given price, the many buyers and sellers interact if the price is the joint support of $F(p)$ and $G(p)$. As is standard in this literature, we assume that buyers cannot coordinate their trading strategies, i.e., they use symmetric and anonymous strategies. This means in our environment, that buyers visit sellers of type y who post the same price with equal probability. We allow for a general formulation of constant returns to scale matching technology to allocate buyers to sellers. Because of constant returns, we can write the matching probabilities in terms of the ratio of buyers to sellers, denoted by λ. Let the matching probability of a seller be $m(\lambda)$ and the matching probability of a buyer is $q(\lambda)=\frac{m(\lambda)}{\lambda}$.[5] We consider constant returns to scale matching functions and make the standard assumptions on $m(\lambda)$ on its domain $[0,\infty)$: $0\le m(\lambda)\le\min\{\lambda,1\}, m'>0, m''<0$ and $q'<0$, and first and second derivatives of m and q are bounded.

The first assumption reflects that m is a probability and therefore confined to the unit interval. Moreover, one cannot match more sellers than buyers, i.e. $m(\lambda)\le\lambda$. The conditoin that $m'>0$ reflects the fact that it is easier for a seller to match when the ratio of buyers to sellers is high. The

[4]Below we consider an example where a firm's type is a cost of production c and the total value to the consumer of type x is $f(x,c)=x-c$. We can define $y=-c$ and write $\widetilde{f}(x,y)=x+y$, where $\widetilde{f}$ satisfies $\widetilde{f}_x>0, \widetilde{f}_y>0$.

[5]Let the aggregate matching function of β buyers and σ sellers be $M(\beta,\sigma)$. Then the probability that a seller is matched is $\frac{M(\beta,\sigma)}{\sigma}$, and given consant returns we write $\frac{M(\beta,\sigma)}{\sigma}=M(\frac{\beta}{\sigma},1)=m(\lambda)$. The probability that a buyer is matched is $\frac{M(\beta,\sigma)}{\beta}=\frac{m(\lambda)}{\lambda}=q(\lambda)$.

opposite applies to the buyers. The concavity of m ensures the the second order condition of the firm's maximization is satisfied. Below, we will make use of another assumption which we will discuss further below:

Assumption A1. *$q(\lambda)^{-1}$ is convex at all $\lambda \in [0, \infty)$.*

One commonly used matching technology in the context of models of price posting is what is often referred to as the *urn-ball process* in which buyers use anonymous strategies and therefore randomize over the sellers with particular (y, p) combination and sellers randomly select present buyers with equal probability. Then $m(\lambda)$ is given by $m_1(\lambda) = 1 - e^{-\lambda}$ and is derived as follows. If the number of buyers and sellers that interact were finite, then the probability that a seller is approached by n buyers is binomially distributed. In the limit when there are many buyers and sellers with ratio $\lambda(y, p)$, the probability that a seller is approached by n sellers approaches a Poisson distribution with parameter $\lambda(y, p)$. Therefore, the probability that a seller has no buyer is $e^{-\lambda(y,p)}$, and with complementary probability $m(\lambda(y, p)) = 1 - e^{-\lambda(y,p)}$ she trades. Since there are $\lambda(y, p)$ buyers to each seller the probability for buyers to trade is $q(\lambda(y, p)) = (1 - e^{-\lambda(y,p)})/\lambda(y, p)$.[6] We show in the appendix that this function fulfills **A1**. Yet this is only one of many conceivable matching functions. If e.g. a fraction $1-\alpha$ of all intended trades falls through, then the matching function is $m_2(\lambda) = \alpha(1-e^{-\lambda})$. If a fraction $1-\beta$ of the buyers gets lost on their way to the sellers the matching function is $m_3(\lambda) = 1 - e^{-\beta\lambda}$. Or the microfoundations could be completely different, often referred to as matching on island economies. This is the matching technology typically assumed in made in models of monetary exchange. Assume agents who want to trade quality y at price p choose an island, on which they.randomly meet any other agent (both other buyers and sellers) with probability γ. If two buyers or two sellers meet, there is no trade. If a buyer and a seller meet, there is trade. This yields matching function $m_4(\lambda) = \gamma\lambda/(1-\lambda) = \gamma s/[s+b]$. All these functions fulfill our conditions, including **A1**.

Preferences. All agents maximize their expected utility. Both buyers and sellers have linear preferences over consumption. The value of a good consumed by buyer x and bought from seller y is $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$. Conditional on consuming and paying a price p, the ex-post utility of the buyer is $f(x, y) - p$ and that of the seller is p. In general the utility derived from consumption of the good depends both on the buyer and the seller type. Many goods like houses, durables,... depend both on the goods characteristics y and on the buyer's characteristics x (income, idiosyncratic preferences,...). We will consider the indices x and y to be ordered such they increase the utility of the buyer: $f_y > 0, f_y > 0$. We say the $f(x, y)$ is supermodular if for any $x' > x$ and $y' > y$ $f(x', y') + f(x, y) > f(x', y) + f(x, y')$

[6]For a careful but intuitive derivation of these matching probabilities as the limit of finite populations see Burdett, Shi and Wright (2001).

and $f(x,y)$ is submodular if the opposite inequality holds. We assume that f is twice continuously differentiable in both x and y. Then supermodularity is equivalent to $f_{xy} > 0$ and submodularity to $f_{xy} < 0$.

Expected utilities and profits are then given by

$$\begin{aligned} u_x(p,y) &= \frac{m(\lambda(p,y))}{\lambda(p,y)}(f(x,y)-p) && (1) \\ \pi_y(p) &= m(\lambda(p,y))p && (2) \end{aligned}$$

where $\lambda(p,y)$ is the queue length which depends on the price and the seller type. For the equilibrium condition and future notation, it will be convenient to denote the support of a distribution function with by a "hat", e.g., $\hat{F}_y$ is the support of F_y.

Assortative Matching. We define a matching $\mu : \mathcal{Y} \rightrightarrows \mathcal{X}$ as a correspondence that gives for every seller y the set of all buyer types x such that (y,p) is in the support of F_x for some p. Some sellers might not be able to trade, in which case $\mu(y) = \varnothing$. We are interested in the conditions on $f(x,y)$ that induce sorting, i.e. that generate a matching pattern μ that is single-valued and strictly monotonic on $(y, \bar{y}]$ and $\mu(y) = \varnothing$ on $[\underline{y}, y)$. In such a case we can treat μ as a function on $(y, \bar{y}]$. If the function μ is strictly increasing, then we say that it exhibits positive assortative matching, if μ is strictly decreasing, then we say that it exhibits negative assortative matching.

Equilibrium. We take the Walrasian approach to price setting. This approach rests on two assumptions on what agents believe about the market thickness in each market. First, all agents hold the same beliefs about the queue length in each market, and second, traders are supposed to hold well-defined beliefs for every possible market, including those that have no trading in equilibrium. The latter is equivalent to the complete markets assumption in centralized Walrasian markets.

Since buyers can condition their visit strategies on the seller type they want to meet, they have strategy $G_x(p,y)$, specifying the type of buyer and the price at which they want to interact. Let $G(p,y) = \int_{\mathcal{X}} G_x(p,y)dx$. For the equilibrium definition we then get

Definition. *An equilibrium is a set of distributions and queue lengths* $(F_y(p), G_x(p,y))$ such that there exists queue length function $\lambda(p,y) \in \bar{\mathbb{R}}$ *such that:*

1. (Profit Maximization) $\forall y \in \mathcal{Y} : \pi_y(p) \geq \pi_y(p'), \forall p \in \hat{F}_y, \forall p' \in [0, \overline{x}]$.

2. (Utility Maximization) $\forall x \in \mathcal{X} : u_x(p,y) \geq u_x(p',y'), \forall (p,y) \in \hat{G}_x, \forall (p',y') \in [0,\overline{x}] \times \mathcal{Y}$.

3. (Consistency of the Queue Length) For all $y \in \mathcal{Y}$ and all $p \in [0, \overline{x}]$

$$\mathcal{S} \int_{[\underline{y},y]\times[0,p]} \lambda(p,y)dF(y,p) = G(p,y). \tag{3}$$

Condition 1. states that the prices that sellers post yield a profit that is weakly higher than any other profit they could have achieved by setting a different price. Condition 2. states that buyers attempt to trade with type y at price p only if this yields expected profits at least weakly as high as if they had attempted to trade at any other seller-price combination. The equilibrium gives rise to some matching allocation which we will refer to as μ^*. Condition 3. is a consistency requirement that ensures that the queue length λ indeed reflects the ratio of buyers to sellers. It specifies that the "outflow" (how applications are dispersed over firms) equals inflow (where applicants up to p, y apply). We note that buyers can always attempt trade at price 0 and obtain weakly positive utilities, and sellers can always attempt to trade at price $\overline{x}$ and obtain weakly positive profits. We assume that buyers that with valuation below the minimum price in the support of F and sellers above the maximum price do exactly that. To reflect that some buyers try not to trade, we have to allow λ in the extended reals and specify $q(\infty) = \lim_{\lambda\to\infty} m(\lambda)$ and $m(\infty) = \lim_{\lambda\to\infty} m(\lambda)$, which are well defined as these functions are monotonic and bounded. We call buyers and sellers *active* if they choose prices with a strictly positive probability to trade.

Finally, we relate our equilibrium concept to the Market Utility Assumption that is common for directed and competitive search models to capture a notion of subgame perfection (see e.g. Shimer (2005), Moen (1997), Montgomery (1991)). Define $U(x) = \max_{p,y} u_x(p, y)$ as the Market Utility, i.e. the utility that buyer type x obtains in equilibrium. The Market Utility Assumption states that a seller of type y that offers some price p that is not offered by any other seller with type y, i.e. $(p, y) \notin \hat{F}_y$, will have a queue length $\lambda(y, p)$ that is low enough such that some buyers want to apply but high enough such that no buyer buyer type finds it strictly positive.[7] It is easy to see that this is the maximal queue length consistent with equilibrium condition 2. Since any lower queue length makes it more attractive for firms to deviate, it can be shown that any equilibrium in our environment is also an equilibrium under the Market Utility Assumption.

3 The Main Results

The problem that a firm resolves is to set prices in order to maximize profits. In the ensuing subgame where buyers make their visiting decisions, the firm's pricing strategy will determine the queue length λ and hence the matching probability $m(\lambda)$, and also the type of buyers x that will visit the seller.

Fix any equilibrium. In this equilibrium $U(x)$ is the utility that workers of type x obtain. We want to show a condition that establishes that the equilibrium can only entail positive assortative matching.

[7]Formally, if $(p, y) \notin \hat{F}_y$ then $\lambda(p, y) = \max_{x\in\mathcal{X}} \lambda_x(p, y)$ where $\lambda_x(p, y) \in \bar{\mathbb{R}}_+$ such that $\frac{1-e^{-\lambda_x(p,y)}}{\lambda_x(p,y)}[f(x, y) - p] = U(x)$ if $\lambda_x(p, y) > 0$. This entails that the buyer who is most eager to trade with combination (p, y) determines the queue length.

Consider a firm that wants to optimize its profits by setting the appropriate price p. Let's assume it could specify the type x of buyer it wants to trade with. Then it achieves at least

$$\max_{x,p} \pi = m(\lambda)p$$
$$\text{s.t.} \qquad \frac{m(\lambda)}{\lambda}[f(x,y) - p] = U(x).$$

Let π_y^* be the equilibrium profit that type y sellers obtain, and let C_y be the set of buyer types which are part of a solution to this maximization problem. Let $\tilde{x}$ be the maximum of C_y. Since buyer types are private $U(x)$ is continous, and equilibrium condition 2 requires $U'(x)$ to be well defined for all types that are trading with positive probability. Continuity implies that $\max C_y$ is well defined. Consider a buyer with positive equilibrium profits $\pi_y^* > 0$. We will show that any buyer type $\hat{y} > y$ will prefer to trade with weakly higher types. That is: $\min C_{\hat{y}} \geq \tilde{x}$. To show this, observe that the maximization problem for a given x above yields after substitution for p

$$\max_{x,\lambda \geq 0} \pi = m(\lambda)f(x,y) - \lambda U(x).$$

This gives the FOC

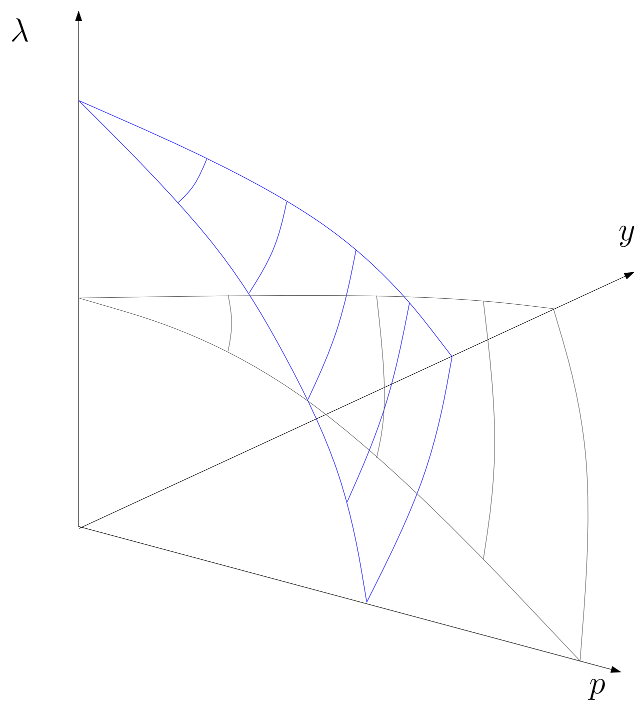
$$m'(\lambda)f(x,y) = U(x) \tag{4}$$
$$q(\lambda)f_x(x,y) = U'(x). \tag{5}$$

Profits conditional on trading with the optimal x are given by

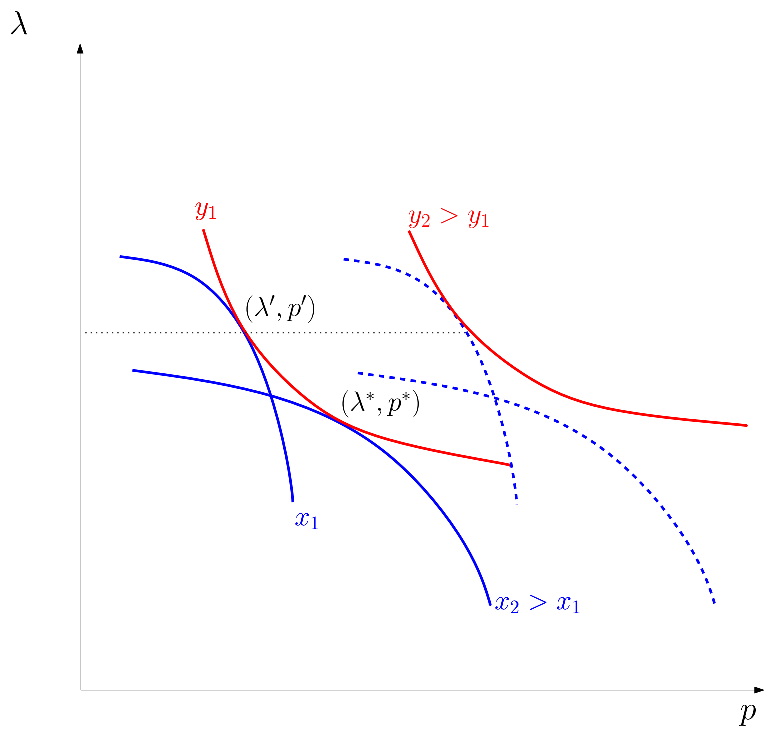
$$\pi = \left[m(\lambda^{x,y}) - \lambda^{x,y} m'(\lambda^{x,y})\right] f(x,y), \tag{6}$$

where $\lambda^{x,y}$ is defined by $U(x)$ via equation (4).

The main argument informally. Before stating and proving the main result, we informally summarize why supermodularity *per se* is not enough to induce positive assortative matching. Buyers care about the probability of matching reflected by λ, the price p, and the type of the seller y. Their preferences (equation (1)) are over a three-dimensional space as illustrated in Figure 1.

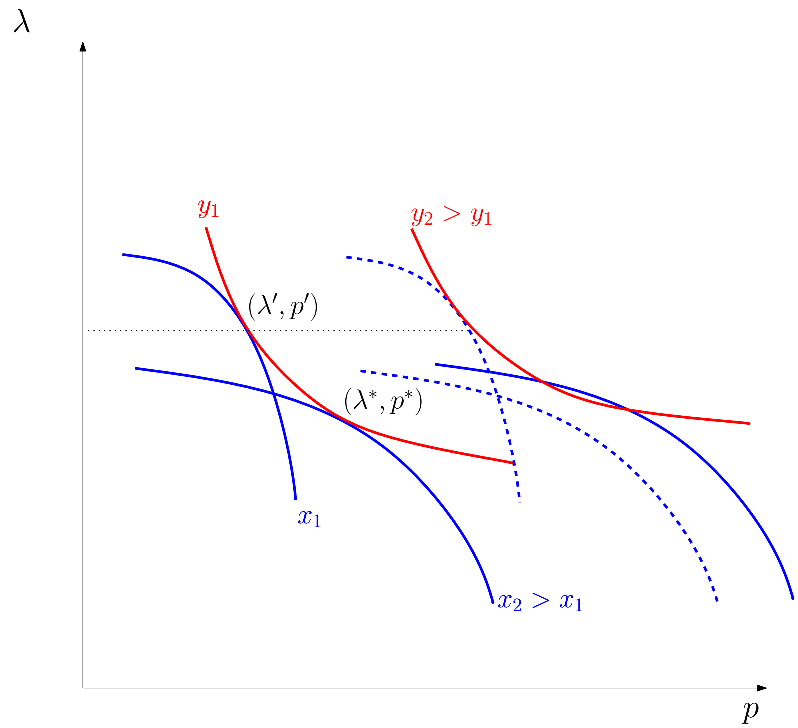


For clarity of exposition, in Figure 2 we project those preferences into a two-dimensional space (p, λ) while indexing each of the indifference curves (in blue) by y. The solid concave curves on the left refer to the indifference curves for buyers trading with a seller type y_1. They exhibit single crossing because high valuation buyers value fast trade more, and are therefore willing to pay a higher price to avoid no-sale. The dotted curves on the right represent the same level of utility for the buyers when they match with a higher buyer type $y_2 > y_1$. All indifference curves of a given type x when matching with different y are parallel shifts of the solid curves: since $f(x, y) - p$ stays constant along some level λ, a higher y is associated with an equal increase in p for all points on a given indifference curve. That shift is proportional to the change in the valuation from an increase in y: $f_y(x, y)$. Moreover, in the absence of complementarities, e.g. when $f(x, y) = x + y$, the change f_y is the same for different x, since $f_{xy} = 0$. As a result, in that case the curves for different buyer types shift by the same amount, as shown in the figure.



One can think of an individual seller's problem as taking the utility of the buyers as given and choosing the point on their iso-profit curve that yields the highest profit. The red convex curves represent iso-profit curves for given sellers. We have drawn the case where seller type y_1 is indifferent between trading with either buyer. To the right is another iso-profit curve, corresponding to a higher type sellers. Since a higher seller type induces higher utility for buyers, he attracts more buyers and therefore obtains higher profits. Because at higher prices and same λ a seller is less willing to reduce the price, higher iso-profit curves are flatter at each λ. The red curve on the right is therefore not a parallel shift. This implies that the highest utility for a type y_2 seller can be achieved when trading with the low seller type x_1. There is negative assortative matching as high types match with low types and vice versa. Recall that the dashed indifference curves in the figure are those when there are no complementarities, e.g. when $f(x, y) = x + y$. Observe that in a frictionless environment, the allocation is indeterminate in the absence of complementarities. The market frictions therefore generate a force that leads to negative sorting.

When there is supermodularity in $f(x, y)$, the change f_y is larger for larger x since $f_{xy} > 0$. Therefore, because the shift in the indifference curves is proportional to f_y, buyer type x_2's indifference curve will shift more than that of x_1. In Figure 2, the solid blue line is x_2's indifference curve in an economy with complementarities. Only if $f(x, y)$ is sufficiently supermodular will the indifference curve for the high buyer type shift enough to intersect with the iso-profit curve that is illustrated for y_2. Then the high type sellers are better off by trading with the high buyer types, which leads to positve assortative matching. The condition that induces such a large enough shift is root-supermodularity.



3.1 Positive Assortative Matching under Root-Supermodularity

We now state and prove our main result. First, we define the notion of root-supermodularity which is used in the main theorem.

Definition 1 *A function $f(x, y)$ is root-supermodular if $\sqrt{f(x, y)}$ is supermodular.*

Root-supermodularity requires that its cross-partial derivative of $\sqrt{f(x, y)}$ is positive, i.e. that $\frac{\partial^2}{\partial x \partial y}\sqrt{f(x, y)} = \frac{1}{2}f^{-\frac{1}{2}}f_{xy} - \frac{1}{4}f^{-\frac{3}{2}}f_x f_y > 0$ or

$$f_{xy}(x, y) - \frac{1}{2}\frac{f_x(x, y)f_y(x, y)}{f(x, y)} > 0.$$

Theorem 2 *Assume **A1**. The function $f(x, y)$ is root-supermodular if and only if there is Positive Assortative Matching of μ^* for all permissible distributions $B(x), S(y)$.*

To prove the main theorem, we first establish two Lemmas. We then derive two Propositions, one establishing the necessary condition, the other the sufficient condition. This also highlights the role played by Assumption **A1**. As will become apparent, this assumption is not needed for establishing necessity. The first Lemma formalizes the graphical illustration above.

Lemma 3 *Along the equilibrium allocation* μ^*, $\frac{\partial^2 \pi}{\partial y \partial x} > 0$ *if and only if* $f_{xy}(x,y) > a(\lambda)\frac{f_y(x,y) f_x(x,y)}{f(x,y)}$, *where*

$$a(\lambda) = -\frac{m'(\lambda)\left[m(\lambda) - \lambda m'(\lambda)\right]}{\lambda m''(\lambda) m(\lambda)}.$$

Likewise, the statement holds when all strict inequalities are reversed, and when the strict inequalities are replaced by equalities.

Proof. We repeat the same exercise as above and calculate the cross-partial derivative of profits of the equilibrium profits. First we obtain

$$\frac{\partial \pi}{\partial y} = \left[-\lambda^{x,y} m''(\lambda^{x,y})\right] \frac{\partial \lambda^{x,y}}{\partial y} f(x,y) + \left[m(\lambda^{x,y}) - \lambda^{x,y} m'(\lambda^{x,y})\right] f_y(x,y).$$

Since only y changes but x stays constant, we use $U(x) = m'(\lambda^{x,y}) f(x,y)$ to determine

$$\frac{\partial \lambda^{x,y}}{\partial y} = -\frac{m'(\lambda^{x,y}) f_y(x,y)}{m''(\lambda^{x,y}) f(x,y)}$$

and get

$$\frac{\partial \pi}{\partial y} = m(\lambda^{x,y}) f_y(x,y).$$

Then derivation with respect to x gives

$$\frac{\partial^2 \pi}{\partial y \partial x} = m'(\lambda^{x,y}) \frac{\partial \lambda^{x,y}}{\partial x} f_y(x,y) + m(\lambda^{x,y}) f_{xy}(x,y).$$

At the x that is the optimal choice for y, we have $d\pi/dx = 0$. Using this we get along the equilibrium allocation

$$\frac{\partial \lambda^{x,y}}{\partial x} = -\frac{\left[m(\lambda^{x,y}) - \lambda^{x,y} m'(\lambda^{x,y})\right] f_x(x,y)}{\left[-\lambda^{x,y} m''(\lambda^{x,y})\right] f(x,y)}.$$

Therefore

$$\frac{\partial^2 \pi}{\partial y \partial x} = \frac{m'(\lambda^{x,y})\left[m(\lambda^{x,y}) - \lambda^{x,y} m'(\lambda^{x,y})\right]}{\lambda^{x,y} m''(\lambda^{x,y})} \frac{f_x(x,y) f_y(x,y)}{f(x,y)} + m(\lambda) f_{xy}(x,y)$$

which is positive provided

$$f_{xy}(x,y) > -\frac{m'(\lambda^{x,y})\left[m(\lambda^{x,y}) - \lambda^{x,y} m'(\lambda^{x,y})\right]}{\lambda^{x,y} m''(\lambda^{x,y}) m(\lambda^{x,y})} \frac{f_y(x,y) f_x(x,y)}{f(x,y)}.$$

Denote $a(\lambda)$ by

$$a(\lambda) = -\frac{m'(\lambda)\left[m(\lambda) - \lambda m'(\lambda)\right]}{\lambda m''(\lambda) m(\lambda)}.$$

Then since $q(\lambda) = \frac{m(\lambda)}{\lambda}$, it follows that $q'(\lambda) = \frac{1}{\lambda^2}\left[\lambda m'(\lambda) - m\right]$ and we can write $a(\lambda)$ as

$$a(\lambda) = \frac{m'(\lambda) q'(\lambda)}{m''(\lambda) q(\lambda)}. \tag{7}$$

which establishes the Lemma. ■

Lemma 3 formalizes the graphical argument above. Only if $f(x, y)$ is sufficiently supermodular will high type sellers be better off trading with high type buyers: the profits of higher seller types y are increasing in buyer types x, i.e. $\frac{\partial^2 \pi}{\partial y \partial x} > 0$. Lemma 3 highlights that this is the case if and only if

$$f_{xy} > a(\lambda) \frac{f_x f_y}{f} \tag{8}$$

If this inequality is fulfilled, equilibrium behavior is such that high buyer types gain more by moving towards higher seller types, and therefore positive assortative matching results.

The properties of the function $a(\lambda)$ are key to understanding condition (8). Because distributions B and S can always be suitably chosen, this condition must hold for all possible $\lambda \in \mathbb{R}_+$. Even at zero, $a(\lambda)$ is well defined, since $q(0) > 0$ and $m(0) < 0$. It turns out that at $\lambda = 0$, the function has a property that holds for any matching function in the class that we consider, namely $a(0) = 1/2$. This is the second Lemma:

Lemma 4 *Under any permissible matching function* $a(0) = 1/2$.

Proof. This result follows from the properties of the class of matching functions we consider. Constatn returns to matching implies $\lambda q(\lambda) = m(\lambda)$, which readily yields after some rearranging that $q'(\lambda) = \frac{m'(\lambda) - q(\lambda)}{\lambda}$ and $q''(\lambda) = \frac{m''(\lambda) - 2q'(\lambda)}{\lambda}$. Our class of matching functions also assumes boundedness of first and second derivatives of q. This necessarily implies $m'(0) = q(0)$ and $m''(0) = 2q'(0)$, since both q' and q'' are divided by zero for $\lambda = 0$. Boundedness thus obtains only if the numerator is zero as well: $m'(0) - q(0) = 0$ and $m''(0) - 2q'(0) = 0$. Since

$$a(\lambda) = \frac{m'(\lambda) q'(\lambda)}{m''(\lambda) q(\lambda)}$$

it follows that

$$a(0) = \frac{m'(0) \frac{m''(0)}{2}}{m''(0) q(0)} = \frac{1}{2}.$$

■

We are now in a position to prove the two Propositions.

Proposition 5 *(Necessary) If there is Positive Assortative Matching of* μ^* *for all permissible distributions* $B(x), S(y)$ *then* $f(x, y)$ *is root-supermodular.*

Proof. For the purposes of our proof we will endow all types $x \leq \tilde{x}$ with the hypothetical utility $\tilde{U}(x)$ which defines queue length $\tilde{\lambda}_{xy}$ according to the first order condition $m'(\tilde{\lambda}_{xy}) f(x, y) = \tilde{U}(x)$ such that according to this hypothetical situation all $x \leq \tilde{x}$ give the optimal utility to type y, i.e.

$\pi_y^* = \left[m(\tilde{\lambda}_{xy}) - \tilde{\lambda}_{xy} m'(\tilde{\lambda}_{xy})\right] f(x,y)$ for all types for which this is possible. Very low types for whom this is not possible obtain $\tilde{U}(x)$. Clearly the hypothetical utilities are lower than equilibrium ones, i.e. $\tilde{U}(x) \geq U(x)$, because these types are now more attractive to y then they were before, and $U(x) = \tilde{U}(x)$ for $x \in C_y$. In particular, in any neighborhood of $\hat{x}$ there are buyer types for which actual and hypothetical utilities coincide. Since utilities are lower, every buyer type weakly prefers the hypothetical situation to the actual equilibrium utility. If we can show that under the hypothetical situation buyer types above y prefer $\hat{x}$, we know that under the equilibrium utilities there choice has to be weakly higher than $\hat{x}$.

Lemma 4 above establishes that $a(0) = 1/2$. Establish that if PAM, then there always exist distributions such that $\lambda \longrightarrow 0$ and therefore $a(\lambda)$ must be at least $1/2$. ■

Before deriving tje sufficient condition, we first show the following property of $a(\lambda)$. We know from Lemma 4 that $a(0) = 1/2$, but that does not rule out that $a(\lambda)$ cannot be larger than $1/2$ for larger values of λ, in which case root-supermodularity would not be sufficient. It turns out that when $1/q(\lambda)$ is convex, $a(\lambda)$ is never larger than $1/2$.

Lemma 6 *$a(\lambda) \leq 1/2$ if and only if $\frac{1}{q(\lambda)}$ is convex in λ (**A1**).*

Proof. Since $\frac{1}{q} = \frac{\lambda}{m}$, we have $(\lambda/m)' = \frac{1}{m^2}[m - m'\lambda]$. Therefore

$$\begin{aligned}(\lambda/m)'' &= \frac{1}{m^4}[m'm^2 - m'm^2 - \lambda m'' m^2 - 2mm'(m - m'\lambda)] \\ &= \frac{1}{m^3}[-\lambda m'' m - 2m'[m - m'\lambda]] \\ &= \frac{\lambda}{m^3}[-m''q + 2m'q'].\end{aligned}$$

This is positive if and only if $-m''q + 2m'q' \geq 0$ or $m'q' \geq \frac{1}{2} m''q$ or $\frac{m'q'}{m''q} \leq \frac{1}{2}$. The left hand side of the inequality is $a(\lambda)$. ■

Lemma 6 provides a necessary and sufficient condition for $a(\lambda)$ to be less than $1/2$, therefore establishing a condition under which root-supermodularity is the hightest possible degree of supermodularity possibly need to ensure positive assortative matching. Proposition 7 thus establishes the sufficient condition.

Proposition 7 *(Sufficient) Assume **A1**. If the function $f(x,y)$ is root-supermodular then there is Positive Assortative Matching of μ^* for all permissible distributions $B(x), S(y)$.*

Proof. Assumption **A1** is equivalent to $a(\lambda) \leq 1/2$ and root-supermodularity ensures that $f_{xy} > \frac{1}{2}\frac{f_x f_y}{f}$. Therefore it follows that $f_{xy} > a(\lambda)\frac{f_x f_y}{f}$. Since $\frac{\partial^2 \pi}{\partial y \partial x}$ and (8) are equivalent, it follows that there is Positive Assortative Matching of μ^* for all permissible distributions $B(x), S(y)$. ■

It remains to be evaluated how important a restriction assumption **A1** is. All the examples of the matching functions we mentioned above satsify assumption **A1**. In fact, so far we have not been able to come up with a counterexample of a matching function that is in our class and that violates assumption **A1**. It has been possible to come up with matching functions that violate **A1**, but those examples violate the conditions we imposed on the class of matching functions. For example, $m(\lambda) = \lambda^b/(1+\lambda^b)$ for $b < 1$ violates assumption **A1**, but $q(\lambda) = \lambda^{b-1}/(1+\lambda^b)$ is no longer bounded as $\lim_{\lambda \to 0} q(\lambda) = \infty$. This violates the restriction that $q \in [0,1]$, which needs to be satisfied since q is interpreted as a probability.

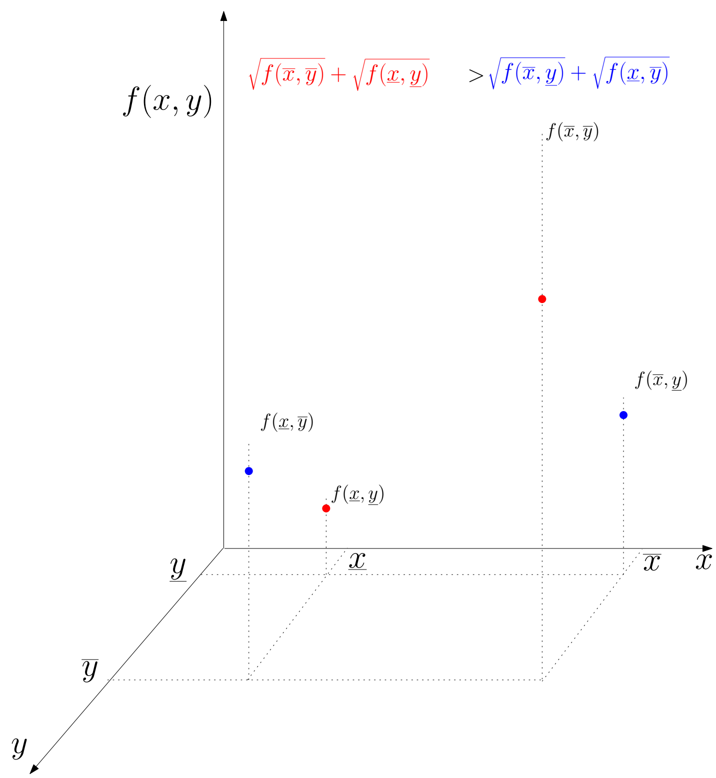
We note here that assumption **A1** has several equivalent interpretations. In the following let $\mathcal{E}_g$ be the elasticity of function g with respect to its argument.

Lemma 8 *Assumption **A1** holds if and only if*

1. *The elasticity of the the derivative of the buyers matching function is larger than for the sellers matching function, i.e.* $\mathcal{E}_{q'} \geq \mathcal{E}_{m'}$ *at all* $\lambda \in [0, \infty)$.
2. $|\mathcal{E}_{m'}| \geq 2[1 - \mathcal{E}_m]$ *at all* $\lambda \in [0, \infty)$*; or equivalently* $\mathcal{E}_{q'} \geq 2\mathcal{E}_q$ *at all* $\lambda \in [0, \infty)$.
3. $x(M)$ *is concave; where* $x(M)$ *is the buyers matching probability if sellers match with probability* M*. That is, for given* M *there is* λ_M *such that* $m(\lambda_M) = M$*, and* x *is defined as* $x(M) = q(\lambda_M)$.

Proof. See appendix. ■

Finally, it is useful at this point to compare the condition of root-supermodularity with log-supermodularity, the condition derived in Shimer and Smith (2000) for the case of random search. A function $f(x,y)$ is log-supermodular if $\log(f(x,y))$ is supermodular and therefore $f_{xy}(x,y) - \frac{f_x(x,y)f_y(x,y)}{f(x,y)} > 0$. Inspection immediately reveals that root-supermodularity is weaker, as the cross-partial derivative must be only at least half as big. Therefore, a function that is log-supermodular is necessarily also root-supermodular, but not vice versa. For an alternative point of view, observe that supermodularity requires that for $x' > x$ and $y' > y$ (and $f_x, f_y > 0$) $f(x',y') + f(x,y) > f(x',y) + f(x,y')$. Any concave transformation of $f(x,y)$ then implies a stronger restriction. The sum of the extreme values of f on the left-hand side is reduced more than the intermeidate values of f on the right-hand side of the inequality. The higher the degree of concavity of the transformation, the more stringent the new condition is. Both the logarithm and the square root are concave transformations, but the logarithm is more concave in the sense that we can we can apply $g(y)$ to $\sqrt{f}$ such that $\log f = g(y) \circ \sqrt{f}$. Solving for $g(y)$, we get $g(y) = 2\log y$, which is a strictly concave. Therefore $\log f$ is "more concave" than $\sqrt{f}$. This is illustrated in the next figure.



The next Corollary then follows immediately from Proposition 7.

Corollary 9 *If $f(x, y)$ is log-supermodular for all (x, y) then the matching pattern exhibits positive assortative matching.*

3.2 Negative Assortative Matching under Weak Submodularity

While "sufficient" supermodularity is needed for positive assortative matching, we now show that negative assortative matching obtains under weak submodularity, i.e. $f_{xy}(x, y) \leq 0$. In the informal graphical argument at the outset, we argue that even under amodularity (e.g. $f(x, y) = x + y$) there is negative assortative matching. While in the frictionless world, the equilibrium allocation under amodularity is indeterminate, the market frictions impose a force towards negative assortative matching. In this sense, there is an asymmetry between positive and negative sorting as there is no need for "sufficient" submodularity. There is no such asymmetry for example under random search (Shimer and Smith (2000)), as negative assortative matching obtains provided $f(x, y)$ is log-submodular.

We now establish the Theorem on negative assortative matching.

Theorem 10 *Assume **A1**. The function $f(x, y)$ is weakly submodular if and only if there is Negative Assortative Matching of μ^* for all permissible distributions $B(x), S(y)$.*

We establish the result by showing the necessary and sufficient condition separately.

Proposition 11 *(Necessary) If the function $f(x, y)$ is weakly submodular then there is Negative Assortative Matching of μ^* for all permissible distributions $B(x), S(y)$.*

Proof. Using (8): $f_{xy}(x, y) < a(\lambda)\frac{f_y(x,y)f_x(x,y)}{f(x,y)}$ where the right-hand side is always positive. Weak submodularity of $f(x, y)$ requires $f_{xy} \leq 0$, implying this is always satisfied and from Lemma 3, it follows that $\frac{\partial^2 \pi}{\partial y \partial x} < 0$ and therefore negative assortative matching. ■

Proposition 12 *(Sufficient) Assume **A1**. The function $f(x, y)$ is weakly submodular if there is Negative Assortative Matching of μ^* for all permissible distributions $B(x), S(y)$.*

Proof. We will show that $\lim_{\lambda\to\infty} a(\lambda) = 0$, which means that $f_{xy}(x, y) > 0$ violates $f_{xy}(x, y) < a(\lambda)\frac{f_y(x,y)f_x(x,y)}{f(x,y)}$ if the queue length at that pair x, y is very high. By shifting the distribution such that this pair is at this pair the buyer has nearly no option to trade this will be the case.

To show that $\lim_{\lambda\to\infty} a(\lambda) = 0$, observe that $m'(\lambda) = \lambda q'(\lambda) + q(\lambda)$ and $m''(\lambda) = \lambda q''(\lambda) + 2q(\lambda)$ allows us to write $a(\lambda) = [\lambda q'(\lambda)/q(\lambda) + 1] / [\lambda q''(\lambda)/q'(\lambda) + 2]$. Under constant returns to scale the elasticities of the matching function add to unity, so that $-\lambda q'(\lambda)/q(\lambda) + \lambda m'(\lambda)/m(\lambda) = 1$. Since a doubling of the queue length does hardly change the matching probability for sellers at long queues (because it is nearly unity), we have $\lim_{\lambda\to\infty} \lambda m'(\lambda)/m(\lambda) = 0$ which implies $\lim_{\lambda\to\infty} \lambda q'(\lambda)/q(\lambda) = -1$. Assumption $A1$ can be rearranged to yield $-\lambda q''(\lambda)/q'(\lambda) \geq -\frac{2}{q(\lambda)}\frac{\lambda q'(\lambda)}{q(\lambda)}$, which by $\lim_{\lambda\to\infty} q(\lambda) = 0$ means that $-\lambda q''(\lambda)/q'(\lambda)$ is unbounded as $\lambda \to \infty$. ■

Interestingly, it is possible for specific distributions to have negative assortative matching for moderate degrees of supermodularity. To see this, refer back to the initial informal graphical argument. Let there be a very small degree of supermodulairty, say $f(x, y) = x + y + \varepsilon xy$, with ε small. Then the solid indifference curve on the right will only be slightly more to the right of the dashed indifference curve. Provided the distributions are such that $a(\lambda)$ is nowhere zero, the iso-profit curve for y_2 will be flatter everywhere and the equilibrium allocation will exhibit negative assortative matching.

Proposition 13 *There exist distributions $B(x), S(y)$ and supermodular functions $f(x, y)$ such that there is Negative Assortative Matching of μ^*.*

Proof. It is sufficient to find some λ such that $a(\lambda)$ is bounded away from zero and some $f(x, y)$ such that $0 < f_{xy}(x, y) < a(\lambda)\frac{f_y(x,y)f_x(x,y)}{f(x,y)}$. There always exist distributions such that $a(\lambda)$ is bounded away from zero (in particular with not too large λ). ■

Likewise, it is possible to obtain positive sorting even if the match value function is not root-supermodular.

Proposition 14 *Assume* ***A1****. There exist distributions* $B(x), S(y)$ *and functions* $f(x, y)$ *that are not root-supermudular such that there is Positive Assortative Matching of* μ^**.*

Proof. It is sufficient to find some λ that is bounded away from zero such that $a(\lambda)$ is strictly smaller than $\frac{1}{2}$ and some $f(x, y)$ such that $a(\lambda)\frac{f_y(x,y)f_x(x,y)}{f(x,y)} < f_{xy}(x, y) < \frac{1}{2}\frac{f_y(x,y)f_x(x,y)}{f(x,y)}$. There always exist distributions B, S such that λ is always large enough. ■

3.3 Existence

We now establish existence of equilibrium.

Proposition 15 *Assume* ***A1****. If the function* $f(x, y)$ *is root-supermodular, then there exists an equilibrium matching* μ^* *for all permissible distributions* $B(x), S(y)$*.*

Proof. An equilibrium is characterized by a queue length function $\tilde{\lambda}(y)$ for each seller $y \in \mathcal{Y}$ and an assignment function $\mu(y) \in \mathcal{X} \cup \emptyset$ that specifies the type of buyer to which seller y is matched, where $\mu(y) = \emptyset$ means that the seller is not matched. If $x \in \mathcal{X}$ but $x \notin \mu(\mathcal{Y})$ then the buyer is not matched to any seller. We are looking for an assignment function with $\mu'(y) > 0$ for all $y > \widehat{y}$, where $\widehat{y}$ is the lowest seller type that is matched. Let $\widehat{x}$ be the lowest buyer type that is matched. Since sellers maximize the problem

$$\max_{x,\lambda} m(\lambda)f(x, y) - \lambda U(x)$$

and the problem for almost all types $y > \widehat{y}$ has to be given by the first order conditions

$$\begin{aligned} m'(\lambda)f(x, y) &= U(x) \\ \frac{m(\lambda)}{\lambda}f_x(x, y) &= U'(x). \end{aligned}$$

We immediately have the following necessary conditions for an equilibrium

$$m'(\tilde{\lambda}(y))f(\mu(y), y) = U(\mu(y)) \tag{9}$$

$$q(\tilde{\lambda}(y))f_x(\mu(y), y) = U'(\mu(y)) \tag{10}$$

Moreover, for the boundary types it has to hold that

$$U(\mu^{-1}(\widehat{y})) = m'(\tilde{\lambda}(\widehat{y}))f(\mu^{-1}(\widehat{y}), \mu(\widehat{y})) \geq 0, \text{ with equality if } \mu^{-1}(\widehat{y}) > \underline{x}, \tag{11}$$

and

$$\pi(\widehat{y}) = [m(\tilde{\lambda}(\widehat{y})) - \tilde{\lambda}(\widehat{y})m'(\tilde{\lambda}(\widehat{y}))]f(\mu(\widehat{y}), \widehat{y}) \geq 0, \text{ with equality if } \widehat{y} > \underline{y}, \tag{12}$$

where the notation μ^{-1} is the inverse of $\mu(y)$. Additionally, the queue length function has to be consistent with the assignment function via the relationship

$$\mathcal{S} \int_{y}^{\bar{y}} \tilde{\lambda}(\tilde{y}) s(\tilde{y}) d\tilde{y} = \int_{\mu(y)}^{\bar{x}} b(\tilde{x}) d\tilde{x} \tag{13}$$

where $\mathcal{S}$ is the measure of sellers (recall that the measure of buyers is normalized to one). This yields

$$\tilde{\lambda}(y) = \frac{b(\mu(y))}{\mathcal{S} s(y)} \mu'(y). \tag{14}$$

An equilibrium is a solution to (9) - (12) and (14). Totally differentiating (9) yields

$$U'(\mu(y))\mu'(y) = m''(\tilde{\lambda}(y))\tilde{\lambda}'(y) f(\mu(y), y) + m'(\tilde{\lambda}(y)) \left[f_x(\mu(y), y)\mu'(y) + f_y(\mu(y), y)) \right].$$

The derivative $U'(\mu(y))$ has to coincide with the expression in (10). That is, we get the equation

$$\begin{aligned} & m''(\tilde{\lambda}(y))\tilde{\lambda}'(y) f(\mu(y), y) \\ & + [m'(\tilde{\lambda}(y)) - q(\tilde{\lambda}(y))]\mu'(y) f_x(\mu(y), y) \\ & + m'(\tilde{\lambda}(y)) f_y(\mu(y), y)) \\ = \; & 0 \end{aligned} \tag{15}$$

which gives us an expression for $\tilde{\lambda}'(y)$. Differentiating (14) we get the second order differential equation, and after substituting for $\tilde{\lambda}'(y)$ we get

$$\begin{aligned} & \mu''(y) \\ = \; & -\frac{\mathcal{S}}{m''(\tilde{\lambda}(y)) f(\mu(y), y) b(\mu(y))} \\ & \times \left[\frac{1}{\mathcal{S} s(y)} \left(b'(\mu(y)) s(y) \mu'(y) - b(\mu(y)) s'(y) \right) \mu'(y) + [m'(\tilde{\lambda}(y)) - q(\tilde{\lambda}(y))]\mu'(y) f_x(\mu(y), y) + m'(\tilde{\lambda}(y)) f_y(\mu(y), y)) \right. \end{aligned}$$

The right hand side involves $\mu(y)$ and $\mu'(y)$. Clearly the right hand side is positive when $\mu'(y) = 0$, so $\mu'(y)$ is always strictly increasing. We have two initial conditions on $\mu(.)$. First, $\mu(\bar{x}) = \bar{y}$. Second, we guess on $\bar{U}$ for the utility $U(\bar{x})$. Clearly $\bar{U} \in [0, f(\bar{x}, \bar{y})]$. Since $\mu(\bar{y}) = \bar{x}$, $\bar{U}$ determines by (9) $\tilde{\lambda}(\bar{y})$ uniquely, which we call $\tilde{\lambda}(\bar{y})|_{\bar{U}}$, which in turn by (14) determines $\mu'(\bar{x})$ uniquely, which we call $\mu'(\bar{x})|_{\bar{U}}$. Given the initial conditions, this differential equation exists, is unique and varies continuously with the initial condition for y in $(\check{y}, \bar{y}]$ for $\check{y}$ high enough, delivering unique queue length $\tilde{\lambda}(y)|_{\bar{U}}$ and assignment $\mu(y)|_{\bar{U}}$.

Let $U(x)|_{\bar{U}}$ be the utility induced under the assignment and queue length associated with $\bar{U}$, and let $\pi(y)|_{\bar{U}}$ be the associated utilities. Denote the lower bound on the domain of integration by $\check{y}$. We can push $\check{y}$ down as long as $U(\mu(\check{y}))|_{\bar{U}} > 0$, $\pi(\check{y})|_{\bar{U}} > 0$, $\check{y} > \underline{y}$ and $\mu(\check{y})|_{\bar{U}} > \underline{x}$. Let $\widehat{y}|_{\bar{U}}$ be the infimum that meets these criteria. It is continuos in $\bar{U}$. For $\bar{U}$ close to zero, $U(\mu(\check{y}))|_{\bar{U}} > 0$ is the only binding constraint because buyer types are matched too quickly. Raising $\bar{U}$ will lead to a point where two of the conditions are binding, as for $\bar{U}$ large only $\pi(\check{y})|_{\bar{U}} > 0$ is binding as seller types are matched quickly. Since $\mu(\check{y})|_{\bar{U}} > \underline{x}$ is only binding when $U(\mu(\check{y}))|_{\bar{U}} > 0$ is not, and $\check{y} > \underline{y}$ is only binding when $\pi(\check{y})|_{\bar{U}} > 0$ is not, we are able to find a constellation fulfilling the equilibrium requirements.

Conditions (9) - (12) are necessary for equilibrium. Moreover, some tedious algebra shows that equation (15) ensures that every buyer is at his first order condition, which can be shown to be the global maximum. For the buyer, a negative definit Hessian to his maximization problem is ensured exactly by root-supermodularity and $1/q(\lambda)$ convex (Assumption **A1**). ■

4 Concluding Remarks

The pricing mechanism plays a key role in the allocation of heterogeneous agents. In decentralized Walrasian markets, market frictions naturally arise from the coordination problem that buyers face in response to prices posted by sellers. The main thesis in this paper is that in the presence of those frictions and complementarities in the match value function of heterogeneous agents, the pricing mechanism facilitates sorting compared to random matching.

In a frictionless Walrasian world, supermodularity is sufficient for positive assortative matching. At the other extreme, in a world with completely random meetings, a strong degree of complementarity, log-supermodularity, is required to ensure positive assortative matching. In our economy with decentralized Walrasian prices, the price mechanism mitigates some of the those frictions. Sellers of different types use their prices to orient buyers where to buy from. With the impact of the frictions reduced by prices relative to random matching, the complementarities needed to offset the natural tendency of frictions to induce negative sorting, is of a lesser degree. Root-supermodularity is a weaker requirement than log-supermodularity and it ensures positive assortative matching.

Because buyer types are not observable, sorting endogenously reveals information. The pricing mechanism induces buyers to choose a seller type thus revealing the buyer's type. Price competition therefore leads to ex ante sorting. This is in sharp contrast with the literature on competing auctions where heterogeneous buyers randomize over sellers offering mechanism and the mechanism then screens ex post by selecting the higher buyer type in the auction.

5 Appendix

Proof that for the standard unball matching function $m(\lambda) = 1 - e^{-\lambda}$ the term $\lambda/m(\lambda)$ is convex:

Proof. The first derivative of $1/q$ is $-q'/q^2$. Since $q' = -(1 - e^{-\lambda} - \lambda e^{-\lambda})/\lambda^2$, we have $\left(\frac{1}{q}\right)' = \frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{(1-e^{-\lambda})^2}$. The second derivative is

$$\left(\frac{1}{q}\right)'' = \frac{e^{-\lambda}(1-e^{-\lambda})}{(1-e^{-\lambda})^4}\left[\lambda(1-e^{-\lambda}) - 2(1-e^{-\lambda}-\lambda e^{-\lambda})\right].$$

The term in square brackets is positive, since it is equal to zero at $\lambda = 0$ and is increasing in λ as $1 - e^{-\lambda} - \lambda e^{-\lambda} > 0$. ■

Proof of Lemma 8:

Proof. For the first statement, observe that $a(\lambda) = \frac{m'(\lambda)q'(\lambda)}{m''(\lambda)q(\lambda)} \leq 1/2$ is equivalent to $\frac{q''(\lambda)}{q'(\lambda)} \leq \frac{m''(\lambda)}{m'(\lambda)}$. To see this, observe that:

$$\begin{aligned}
\frac{m'(\lambda)q'(\lambda)}{m''(\lambda)q(\lambda)} &\leq 1/2 \qquad (16)\\
&\Leftrightarrow -2q'(\lambda) \leq -\frac{m''(\lambda)q(\lambda)}{m'(\lambda)}\\
&\Leftrightarrow m''(\lambda) - 2q'(\lambda) \leq m''(\lambda) - \frac{m''(\lambda)q(\lambda)}{m'(\lambda)}\\
&\Leftrightarrow \frac{1}{\lambda}\left[m''(\lambda) - 2q'(\lambda)\right] \leq \frac{m''(\lambda)}{\lambda m'(\lambda)}\left[m'(\lambda) - q(\lambda)\right]\\
&\Leftrightarrow q''(\lambda) \leq \frac{m''(\lambda)}{m'(\lambda)}q'(\lambda)\\
&\Leftrightarrow \frac{q''(\lambda)}{q'(\lambda)} \geq \frac{m''(\lambda)}{m'(\lambda)}.
\end{aligned}$$

For the second statement, recall $1/q$ convex iff $q''q - 2(q')^2 \leq 0$, which implies its second part. From Proposition 6 we know that assumption **A1** coincides with $\frac{m'(\lambda)}{m''(\lambda)}\frac{q'(\lambda)}{q(\lambda)} \leq \frac{1}{2}$ for all $\lambda \in [0, \infty)$, which together with $\mathcal{E}_m - \mathcal{E}_q = 1$ (which is a general property of constant returns to scale functions) which implies its first part.

For the forth statement, since $x(m(\lambda)) = q(\lambda)$ we have $q'(\lambda) = x'(m)m'(\lambda)$ and $q''(\lambda) = x'(m)m''(\lambda) + x''(m)m'(\lambda)$. Therefore $x''(m) \leq 0$ is equivalent to

$$\begin{aligned}
&x''(m)m'(\lambda) \leq 0\\
\Leftrightarrow\ &q''(\lambda) - x'(m)m''(\lambda) \leq 0\\
\Leftrightarrow\ &q''(\lambda) - \frac{q'(\lambda)}{m'(\lambda)}m''(\lambda) \leq 0
\end{aligned}$$

which has been shown in (16) to be equivalent with $a(\lambda) \leq 1/2$. ■

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